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Convergence of Successive Approximations for Nonexpansive Operators †

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Abstract

Iterative algorithms are of wide interest in engineering computation. However, demonstration of convergence and characterization of fixed points can present difficulties in analysis. In this paper we survey properties of nonexpansive, nonlinear mappings that are relevant to establishing the existence of fixed points and the convergence of iterative schemes to compute them. Application of these properties is illustrated in an alternative treatment of the popular successive projections algorithm. In addition, an iterative scheme is presented for computing the projection onto a finite intersection of closed convex sets by a simple cyclic sequence of projections onto the individual sets.

Key Words

nonexpansive operator, successive approximations, nearest-point projection, fixed point

I. Introduction

A *fixed point* of an operator f is an element x in the domain of f satisfying $f(x) = x$. Iterative solution for a fixed point is the basis for computation in applications ranging from polynomial rooting [20] to image reconstruction [2, 18, 28, 30]. Likewise, a dual approach to constrained optimization problems often gives rise to fixed point problems [5, 14, 25, 27, 29]. Commonly employed results require that f is a contraction [28], is linear [2], or has compact domain [30]. However, convergence analysis often requires more general operators and may become problematic.

In this paper we survey several properties of nonexpansive mappings relevant to establishing the existence of fixed points and the convergence of iterative schemes to compute them. Averaged mappings, firmly nonexpansive maps, and nearest-point projections onto convex sets are considered as important special cases. The application of these properties in engineering computation is illustrated by an alternative formulation of the

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popular successive projection algorithm [34]. Additionally, an algorithm is presented for computing the projection onto the intersection of m closed convex sets, given only the m projection operators onto the constituent sets.

Perhaps the simplest and most useful result in fixed point theory is the Banach fixed point theorem, or contraction mapping principle, which asserts that if f is a self-mapping of a complete normed linear space $\mathcal{H}$ satisfying

$$\|f(x) - f(y)\| \leq L\|x - y\|, \quad L < 1 \quad \forall\, x, y \in \mathcal{H}$$

then f has a unique fixed point, and, moreover, for every $x \in \mathcal{H}$ the sequence of *successive approximations*, or *Picard iterates*, $\{f^n(x)\}$, converges to the fixed point. Within the less restrictive class of nonexpansive operators ($L \leq 1$), the most frequently cited results require boundedness. For example, the Brouwer fixed point theorem assures that for any continuous self-mapping of the unit sphere in $\mathbb{R}^n$ there exists a fixed point. First published in 1910, the result was previously proven by Poincaré (1886) in a similar form. In Hilbert spaces, an analogous result is given by Browder:

Property 1 ([3]) *If $\mathcal{K}$ is a bounded closed convex set in a Hilbert space $\mathcal{H}$ and $f : \mathcal{K} \mapsto \mathcal{K}$ is nonexpansive, then there exists some $x \in \mathcal{K}$ such that $f(x) = x$.*

II. Fixed Points of Nonexpansive Maps

Often in application the operator f is not a contraction, so the Banach fixed point theorem is not applicable. Likewise, the domain of f may be neither closed nor bounded, so the well-known Brouwer fixed point results for nonexpansive operators [17, 30] are not applicable. This section explores the asymptotic behavior of Picard iterates, $\{f^n(x)\}$, for a general nonexpansive operator f. Let $\mathcal{H}$ be a real Hilbert space[1] with inner product $\langle x, y\rangle$ and norm $\|x\| = \langle x, x\rangle^{1/2}$, $x, y \in \mathcal{H}$. Suppose $\mathcal{D} \subset \mathcal{H}$. An operator $f : \mathcal{D} \mapsto \mathcal{H}$ is said to be *nonexpansive* if

$$\|f(x) - f(y)\| \leq \|x - y\| \quad \forall\, x, y \in \mathcal{D}.$$

The class of nonexpansive mappings is more than just a trivial generalization of a Lipschitz condition; this class of operators is intimately connected with differential equations, monotone operators, and the geometry of Banach spaces. Nonexpansive mappings may be fixed point free, and when such a mapping has a fixed point it need not be unique (e.g., the identity mapping). The following Properties characterize the existence of fixed points and provide a necessary and sufficient condition for convergence of the Picard iterations. In the sequel, let $\mathcal{K}$ denote a nonempty, closed and convex subset of $\mathcal{H}$.

Property 2 ([11, 12]) *For a nonexpansive mapping $f : \mathcal{K} \mapsto \mathcal{K}$, the following are equivalent:*

[1] Results are reported here for Hilbert spaces; most can be directly applied in any uniformly convex Banach space.

(i) *f has a fixed point.*

(ii) $\{\|f^n(x)\|\}$ *is a bounded sequence for some* x *in* $\mathcal{K}$.

(iii) *There is a bounded sequence* $\{y_n\}$ *in* $\mathcal{K}$ *such that* $\lim_n(y_n - f(y_n)) = 0$.

(iv) *the origin is in the range of* $(I - f)$, *where* I *is the identity map.*

Property 3 **([12])** *For a nonexpansive mapping* $f : \mathcal{K} \mapsto \mathcal{K}$, *the set of fixed points of a nonexpansive operator is closed and convex.*

Property 4 **([12])** *Let* $f : \mathcal{K} \mapsto \mathcal{K}$ *be a nonexpansive operator with a fixed point, and let* x *be a point in* $\mathcal{K}$. *Then,* $\{f^n(x)\}$ *converges weakly to a fixed point of* f *if and only if* f *is* weakly asymptotically regular[2] *at* x, *i.e.,*

$$\textit{weak} \lim_n (f^n(x) - f^{n+1}(x)) = 0.$$

An operator f that is (weakly) asymptotically regular for all $x \in \mathcal{K}$ is said to be (weakly) asymptotically regular. As an example of a nonexpansive operator $\mathbb{R}$ to $\mathbb{R}$ with a fixed point and not asymptotically regular, consider $f(x) = -(x + 1)$.

For nonexpansive mappings without a fixed point, the Picard iterations still yield useful information. In particular, we have the following positive result.

Property 5 **([24])** *Let* $f : \mathcal{K} \mapsto \mathcal{K}$ *be a nonexpansive mapping. Then* $\lim_n \frac{1}{n} f^n(x) = -v$, *where* v *is the unique element of minimum norm in the closure of the range of* $(I - f)$.

A. Averaged maps

Although a nonexpansive f with a fixed point may not be asymptotically regular, the averaged mapping $f_\lambda = \lambda f + (1 - \lambda)I$, where $0 < \lambda < 1$, shares the same fixed point set, and the successive approximations $\{f_\lambda^n(x)\}$ are guaranteed to converge to a fixed point from any initial iterate.

Property 6 *Let* f *be a nonexpansive mapping on* $\mathcal{H}$, *and define* $f_\lambda = \lambda f + (1 - \lambda)I$, *where* $\lambda \in (0, 1)$. *Then,*

(i) f_λ *has the same fixed point set as* f.

(ii) f_λ *is nonexpansive for all* $\lambda \in (0, 1)$.

Property 7 **([21])** *Let* $f : \mathcal{K} \mapsto \mathcal{K}$ *be a nonexpansive operator with at least one fixed point. Although* f *itself may not be weakly asymptotically regular, the averaged mapping* f_λ, $\lambda \in (0, 1)$, *is weakly asymptotically regular, and* $\{f_\lambda^n(x)\}$ *is therefore weakly convergent to a fixed point of* f *for every* $x \in \mathcal{K}$.

The asymptotic behavior of the averaged mapping f_λ also provides insight into the range of f, even when f has no fixed point.

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[2] A sequence $\{x_n\} \in \mathcal{H}$ is said to converge weakly to x if $\lim_{n \to \infty} \langle x_n, y \rangle = \langle x, y \rangle$ for every $y \in \mathcal{H}$. Weak convergence implies strong convergence in finite dimensional spaces.

Property 8 ([1]) *Let $f : \mathcal{K} \mapsto \mathcal{K}$ be nonexpansive and define the averaged mapping $f_\lambda = \lambda f + (1-\lambda)I$, $\lambda \in (0,1)$. Let v be the unique point of least norm in the closure of the range of $(I - f)$. Then for all x in $\mathcal{K}$*

(i) $\lim_{n\to\infty} \frac{1}{n} f_\lambda^n(x) = -v$.

(ii) $\lim_{n\to\infty}[f_\lambda^n(x) - f_\lambda^{n+1}(x)] = v$.

(iii) f has no fixed point if and only if $\lim_{n\to\infty} \|f_\lambda^n(x)\| = \infty$ *for all x in $\mathcal{K}$.*

In relation to the asymptotic regularity condition of Property 4, observe that $\{f_\lambda^n(x)\}$ is a Cauchy sequence if and only if v is the zero vector and f has a fixed point. For successive approximations, the averaged mapping f_λ is often used in place of f for an enhanced rate of convergence; this technique is known as *overrelaxation*.

B. Firmly nonexpansive maps

Next we consider the special subclass of firmly nonexpansive mappings.

Property 9 ([11, 12]) *Let $\mathcal{D} \subset \mathcal{H}$. For a mapping $f : \mathcal{D} \mapsto \mathcal{H}$ the following are equivalent.*

(i) f is firmly nonexpansive[3].

(ii) $2f - I$ is nonexpansive.

(iii) $f = \frac{1}{2}(g + I)$ for some nonexpansive g.

(iv) $\|f(x) - f(y)\|^2 \leq \langle f(x) - f(y),\, x - y\rangle \;\; \forall\, x, y \in \mathcal{K}$.

Property 10 *A composition of firmly nonexpansive mappings is likewise firmly nonexpansive.*

Moreover, the averaged mapping $\lambda f + (1-\lambda)I$ is firmly nonexpansive for nonexpansive f and $0 \leq \lambda \leq \frac{1}{2}$ [4].

In general, the iterates $\{f^n(x)\}$ of a nonexpansive mapping with a fixed point do not converge. However, as cited in the next Property, for firmly nonexpansive mappings having a fixed point the Picard iterates are guaranteed to converge to a fixed point.

Property 11 ([12]) *Let $f : \mathcal{K} \mapsto \mathcal{K}$ be a firmly nonexpansive mapping with a fixed point. Then, for every $x \in \mathcal{K}$, the sequence of iterates $\{f^n(x)\}$ converges weakly to a fixed point of f.*

Next, the asymptotic behavior is considered for the more general case in which f may be fixed point free. The following two Properties characterize when the sequence of Picard iterates $\{f_\lambda^n(x)\}$ is *unbounded* for all initial iterates x and guarantee that the successive difference in iterates converges.

[3]In a Banach space, a single-valued operator f mapping is said to be *firmly nonexpansive* if $\|f(x) - f(y)\| \leq \|\lambda(x-y) + (1-\lambda)(f(x) - f(y))\|$ for all x, y in the domain of f and all $\lambda > 0$.

Property 12 ([12]) *Let f be a firmly nonexpansive mapping of a closed convex subset $\mathcal{K}$ of a Hilbert space. Then f is fixed point free if and only if $\lim_{n\to\infty} \|f^n(x)\| = \infty$ for all $x \in \mathcal{K}$.*

Necessity is lost if f is merely nonexpansive: consider $f(x) = |x-2| - 1$, $x \in \mathbb{R}$.

Property 13 ([12]) *Let $\mathcal{K}$ be a closed convex subset of a Hilbert space $\mathcal{H}$ and $f : \mathcal{K} \mapsto \mathcal{K}$ a firmly nonexpansive mapping. Then for all $x \in \mathcal{K}$, $\lim_{n\to\infty}[f^n(x) - f^{n+1}(x)] = v$ where v is the unique point of least norm in the closure of the range of $(I - f)$. In addition, $\lim \frac{1}{n} f^n(x) = -v$.*

C. Projections onto convex sets

In Hilbert space, nearest-point projection operators onto convex sets are an important subclass of firmly nonexpansive operators.

Property 14 *Let $\mathcal{K}$ denote any nonempty, closed, convex subset of a Hilbert space $\mathcal{H}$. Then there exists a unique $y \in \mathcal{K}$ such that $\inf_{z\in\mathcal{K}} \|x - z\| = \|x - y\|$.*

This correspondence is denoted by $y = P_{\mathcal{K}}(x)$, where $P_{\mathcal{K}} : \mathcal{H} \mapsto \mathcal{K}$ is said to be the *nearest-point projection operator*, or simply the *projection*, of $\mathcal{H}$ onto the closed convex set $\mathcal{K}$. The projection is uniquely characterized by the inequality

$$\langle x - P_{\mathcal{K}}(x), y - P_{\mathcal{K}}(x)\rangle \leq 0 \;\forall\, y \in \mathcal{K}.$$

The operator $P_{\mathcal{K}}$ is linear if and only if $\mathcal{K}$ is a subspace.

Property 15 ([12]) *Let $P_{\mathcal{K}}$ be the projection onto a closed convex set $\mathcal{K} \subset \mathcal{H}$. Then, P is firmly nonexpansive, $(I - P_{\mathcal{K}})$ is nonexpansive, and the reflection of $\mathcal{H}$ in $\mathcal{K}$ defined by $2P_{\mathcal{K}} - I$ is nonexpansive.*

III. Application: Successive Projections

The method of successive projections (SP) onto convex sets [34] provides a convergent iterative algorithm to compute a signal in the intersection of convex constraint sets by projecting successive estimates of the signal onto each convex set in turn. The behavior of this technique is easily described in the context of firmly nonexpansive operators established in Section 2. The conceptual basis for SP is the characterization of each known property of a signal as a closed convex subset of a Hilbert space of signals. Thus, m such properties restrict the signal, x, to lie in the intersection $\mathcal{K} = \bigcap_{j=1}^{m} \mathcal{K}_j$. A signal common to all m convex sets is then determined iteratively by the method of successive approximation. Defining f as the composition of the projections P_j onto the respective sets $\mathcal{K}_j$,

$$f = P_m \circ P_{m-1} \circ \cdots \circ P_1, \tag{1}$$

the fixed point set of f is precisely $\mathcal{K}$, the intersection of the m constraint sets, *provided* the intersection $\mathcal{K}$ is nonempty [4, 34]. Hence, the determination of a signal in $\mathcal{K}$ is reduced to the determination of a fixed point of f, and the previous Properties directly provide an iterative scheme.

Theorem 1 *Let $f : \mathcal{H} \mapsto \mathcal{H}$ be the composition $f = P_m \circ P_{m-1} \circ \cdots \circ P_1$, and assume the set $\mathcal{K} = \bigcap_{j=1}^m \mathcal{K}_j$ is nonempty. For every initial iterate $x \in \mathcal{H}$, the sequence of iterates $\{f^n(x)\}$ converges weakly to some $\hat{x} \in \mathcal{K}$.*

Proof. The theorem follows trivially from the Properties enumerated above. By Properties 15 and 10, f is firmly nonexpansive. Further, y is a fixed point of f if and only if $y \in \mathcal{K}$. Then, for $\mathcal{K}$ nonempty, weak convergence follows from Property 11. ∎

Since weakly convergent sequences are bounded, it then follows that strong convergence is assured if the iterates are eventually contained in some compact or finite-dimensional subset of $\mathcal{H}$. Moreover, convergence is strong if and only if at least one subsequence of $f^n(x)$ converges strongly. The following corollary gives the over-relaxed version of the iterations in Equation (1).

Corollary 1 *Let $f_\lambda : \mathcal{H} \mapsto \mathcal{H}$ be the composition*

$$f = [\lambda_m P_m + (1 - \lambda_m)I] \circ [\lambda_{m-1} P_{m-1} + (1 - \lambda_{m-1})I] \circ \cdots \circ [\lambda_1 P_1 + (1 - \lambda_1)I],$$

where $0 < \lambda_j < 2$ and assume the intersection, $\mathcal{K}$, is nonempty. For every initial iterate $x \in \mathcal{H}$, the sequence of iterates $\{f_\lambda^n(x)\}$ converges weakly to some $\hat{x} \in \mathcal{K}$.

Proof. By Property 15, $2P - I$ is nonexpansive. Then, by Property 7 the averaged mapping $\delta_j(2P_j - I) + (1 - \delta_j)I = 2\delta_j P_j + (1 - 2\delta_j)I$ is weakly asymptotically regular for $\delta_j \in (0, 1)$. Let $\lambda_j = \frac{1}{2}\delta_j$ to obtain the result. ∎

Convergence rates and finite iteration error bounds are considered in [19, 22, 34]. The difficulties in generalizing to nonconvex constraint sets are explored in [6, 32].

A. *Modified successive projections*

In general, the limit point $\hat{x}$ depends on both the initial iterate x and the ordering of the constituent projections P_j; $\hat{x}$ is not the projection of x onto $\mathcal{K}$. While $\lim_n (P_m \circ \cdots \circ P_1)^n$ is indeed a nonexpansive mapping onto a nonempty $\mathcal{K}$, it is not, in general, the nearest-point projection onto $\mathcal{K}$. (Consider the simple counterexample provided by a disk and a line in the plane.) Yet, a surprisingly simple modification of the SP algorithm of Theorem 1 provides the projection onto a finite intersection of closed convex sets using only a cyclic sequence of projections onto each individual set.

Given a point $d \in \mathcal{H}$, closed convex sets $\mathcal{K}_1, \cdots, \mathcal{K}_m$, and the corresponding nearest point projections $P_1, \cdots, P_m$, initialize

$$y_1^{(0)} = \cdots = y_m^{(0)} = 0 \text{ and } x_m^{(0)} = d.$$

In each iteration compute $2m$ vectors

$$x_1^{(k)},\ y_1^{(k)},\ \cdots,\ x_m^{(k)},\ y_m^{(k)}$$

as follows: set $x_0^k = x_m^{k-1}$ and, for $i = 1, \cdots, m$ calculate

$$\begin{aligned} z &= x_{i-1}^{(k)} + y_i^{(k-1)}, \\ x_i^{(k)} &= P_i(z), \\ y_i^{(k)} &= z - P_i(z). \end{aligned} \tag{2}$$

The modified successive projections (MSP) algorithm in Equation (2) alters the successive projections algorithm of Theorem 1 at each iteration by adding the previous outer normal $y_i^{(k-1)}$ to $x_{i-1}^{(k)}$ before projection onto set $\mathcal{K}_i$. This adjustment forces the z vector into the normal cone of $\bigcap_{i=1}^m \mathcal{K}_i$ at the solution point $x^\star$. The MSP algorithm apparently originates in [8] for the special case of convex cones; for arbitrary closed convex sets in $\mathbb{R}^n$ the algorithm was independently presented in [16]. A streamlined proof of convergence for infinite-dimensional Hilbert spaces is given in [9].

Theorem 2 ([9]) *Let $\mathcal{K}_1, \cdots, \mathcal{K}_m$ be closed convex subsets of $\mathcal{H}$ with nonempty intersection $\mathcal{K} = \bigcap_{j=1}^m \mathcal{K}_j$, and let $d \in \mathcal{H}$. Then for each i the sequence $\{x_i^{(k)}\}$ in (2) converges (strongly) as $k \to \infty$ to $x^\star$, the unique vector minimizing $\|x - d\|$ subject to $x \in \mathcal{K}$.*

If some $\mathcal{K}_j$ is a closed linear variety, then the outer normal $y_j^{(k)}$ need not be computed since

$$x_j^{(k)} = P_j(x_{j-1}^{(k)} + y_j^{(k-1)}) = P_j(x_{j-1}^{(k)}).$$

Thus, if every $\mathcal{K}_j$ is a linear variety, MSP reduces to SP. The result for this special case is due to Halperin [15] and dates to Von Neumann [31] for $m = 2$. As a corollary, the well-known band-limited extrapolation algorithm of Papoulis and Gerchberg [10, 23] converges to the minimum norm element of $\mathcal{K}$ for $d = 0$ since the constraint sets are linear varieties.

Although the m sequences $\{x_i^{(k)}\}$ are convergent, the m sequences of outer normals $\{y_i^{(k)}\}$ may not be bounded. The boundedness of $\{y_i^{(k)}\}$ is certainly necessary for practical computation and is guaranteed by the following regularity condition.

Corollary 2 ([9]) *If there is a $j \in \{1, \cdots, m\}$ such that*

$$\mathcal{K}_j \cap \bigcap_{i \neq j} int(\mathcal{K}_i) \neq \emptyset$$

then

$$\sup_{k,i} \|y_i^{(k)}\| \leq \infty$$

where $int(\mathcal{K}_i)$ denotes the interior of $\mathcal{K}_i$ with respect to the norm topology.

The convergence proof for Theorem 2 may be obtained by considering the Fenchel dual. The dual problem requires the minimization of a functional of m vectors, which is performed by cyclically minimizing with respect to a single vector while holding others constant; this classical approach yields the iterative algorithm in Equation (2).

B. Example projection operators

The MSP algorithm is applicable in numerous computation tasks including medical imaging, filter design, recovery of Euclidean distance matrices, and maximum likelihood estimation. One common framework is the convex quadratic optimization problem in $\mathbb{R}^n$

$$\min_x x'Qx + q'x \text{ subject to } x \in \mathcal{K} = \bigcap_{i=1}^{m} \mathcal{K}_i \tag{3}$$

where Q is a positive definite $n \times n$ matrix, $q \in \mathbb{R}^n$, and q' denotes the transpose of q. If we choose $\mathcal{H} = \mathbb{R}^n$ with inner product $\langle x, y \rangle = x'Qy$, then (3) is equivalent to

$$\min_{x \in \mathcal{K}} \|d - x\|$$

where $d = -\frac{1}{2}Q^{-1}q$. The computational cost of the algorithm depends on the complexity of each projection operator. Below we review [9, 16] three examples of sets $\mathcal{K}_i$ with simple projections under the inner product $\langle x, y \rangle = x'Qy$. First, for the halfspace $\mathcal{K}_i = \{x \in \mathbb{R}^n : a_i'x \leq b_i\}$ the projection is given by

$$P_i(x) = x - (a_i'Q^{-1}a_i)^{-1}(a_i'x - b_i)_+ Q^{-1}a_i$$

where $(t)_+ = \max(t, 0)$ for $t \in \mathbb{R}$. Second, for a linear constraint with both upper and lower bounds,

$$\mathcal{K}_i = \{x \in \mathbb{R}^n : c_i \leq a_i'x \leq b_i\},$$

the projection is given by

$$P_i(x) = x + (a_i'Q^{-1}a_i)^{-1}\{(c_i - a_i'x)_+ - (a_i'x - b_i)_+\}Q^{-1}a_i.$$

Such a constraint would be valuable, for example, in incorporating the physical bounds encountered in an adaptive array system [7]. Third, we review an ellipsoidal constraint

$$\mathcal{K}_i = \{x \in \mathbb{R}^n : (x - \bar{x}_i)'B_i(x - \bar{x}_i) \leq 1\}$$

where B_i is positive definite and $\bar{x}_i \in \mathbb{R}^n$. Let $B_i = L_iL_i'$ and find the spectral decomposition

$$L_i^{-1}Q(L^{-1})' = U_i'D_iU_i$$

where U_i is an orthogonal matrix and $D_i = \operatorname{diag}(d_1, \cdots, d_k)$. Define the functional

$$g(x, \gamma) = \sum_{j=1}^{n} \{(U_iL_i'(x - \bar{x}_i))_j d_j(d_j + \gamma)^{-1}\}^2, \quad \gamma \geq 0$$

where $(\cdot)_j$ denotes the j^{th} component. Define $\gamma(x) = 0$ if $x \in \mathcal{K}$ and otherwise let $\gamma_i(x)$ denote the unique solution $\gamma \geq 0$ of $g_i(x, \gamma) = 1$. Then

$$P_i(x) = (L^{-1})'U_i'(D_i + \gamma_i(x)I)^{-1}D_iQ_iL_i'(x - \bar{x}_i) + \bar{x}_i.$$

C. Empty intersection

The convergence results in Theorems 1 and 2 require that the constraint sets have nonempty intersection; for $\mathcal{K}$ empty, we consider the important case of $m = 2$ convex sets. The following Property relates any fixed points of $P_1 \circ P_2$ to the distance between the sets $\mathcal{K}_1$ and $\mathcal{K}_2$.

Property 16 ([13]) *In a Hilbert space $\mathcal{H}$ let $\mathcal{K}_1$ and $\mathcal{K}_2$ be closed convex sets with projection operators P_1 and P_2, respectively. Then, $\hat{x}$ is a fixed point of $P_1 \circ P_2$ if and only if $\hat{x} \in \mathcal{K}_1$ and*

$$\|\hat{x} - P_2(\hat{x})\| = \inf_{\substack{x \in \mathcal{K}_1 \\ y \in \mathcal{K}_2}} \|x - y\|.$$

However, in general there may not exist a fixed point of $P_1 \circ P_2$; yet, boundedness of $\mathcal{K}_1$ suffices in Hilbert space. Thus, for bounded $\mathcal{K}_1$, the method of alternating projections yields a weakly convergent sequence.

Theorem 3 *Let $\mathcal{H}$ be a Hilbert space and suppose $\mathcal{K}_1$ and $\mathcal{K}_2$ are closed convex sets with $\mathcal{K}_1$ bounded. Let P_1 and P_2 denote the projection operators onto $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. Then for all $x \in \mathcal{H}$, $\{(P_1 \circ P_2)^n(x)\}$ is weakly convergent to a fixed point $\hat{x}$ of the composite mapping $P_1 \circ P_2$. Moreover, the minimum distance between $\mathcal{K}_1$ and $\mathcal{K}_2$ is achieved by $\|\hat{x} - P_2(\hat{x})\|$.*

Strong convergence of $\{P_1 \circ P_2(x)\}$ to a fixed point of $P_1 \circ P_2$ is assured when either (i) $\mathcal{K}_1$ is compact or (ii) $\mathcal{K}_1$ is finite dimensional and the distance between $\mathcal{K}_1$ and $\mathcal{K}_2$ is attained [13]. Moreover, for $\mathcal{K}_1 = \{x : Ax = b\}$ the measurement functionals comprising A may be orthonormalized so that the iterations yield a *least-squares* solution to $Ax = b$ [26]. For the case of $m > 2$ constraint sets (with at least one set bounded), a closed chain of nearest-neighbor elements is formed from the fixed points for the $m!$ compositions of the m projection operators [33].

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