Stability Analysis of One-Dimensional Asynchronous Swarms *

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Abstract

Coordinated dynamical swarm behavior occurs when certain types of animals forage for food or try to avoid predators. Analogous behaviors can occur in engineering systems (e.g. in groups of autonomous mobile robots or air vehicles). In this paper we characterize swarm "cohesiveness" as a stability property and provide conditions under which collision-free convergence can be achieved for an asynchronous swarm with finite-size swarm members that have proximity sensors and neighbor position sensors that only provide delayed position information. Moreover, we give conditions under which an asynchronous mobile swarm following (pushed by) an "edge-leader" can maintain cohesion during movements even in the presence of sensing delays and asynchronism. Such stability analysis is of fundamental importance if one wants to understand the coordination mechanisms for groups of autonomous vehicles or robots where inter-member communication channels are less than perfect and collisions must be avoided.

1 Introduction

A variety of organisms have the ability to cooperatively forage for food while trying to avoid predators and other risks. For instance, when a school of fish searches for prey, or if it encounters a predator, the fish often make coordinated maneuvers as if the entire group were one organism [1]. Analogous behavior is seen in flocks of birds, herds of wildebeests, groups of ants, and swarms of social bacteria [2, 3]. We call this kind of aggregate motion "swarm behavior." A high-level view of a swarm suggests that the organisms are cooperating to achieve some purposeful behavior and achieve some goal. Naturalists and biologists have studied such swarm behavior for decades. Moreover, computer scientists in the field of "artificial life" have studied how to model and simulate biological swarms to understand how such "social animals" interact, achieve goals, and evolve [4, 5].

Recently, there has been a growing interest in biomimicry of the mechanisms of foraging and swarming for use in engineering applications since the resulting swarm intelligence can be applied in optimization (e.g., in telecommunication systems) [2, 3], robotics [6, 7], traffic patterns in intelligent transportation systems [8, 9, 10], and military applications [11]. For instance, there has been a growing interest in groups (swarms) of flying vehicles [12, 13]. Moreover, it has been proposed that swarms of robots may provide the possibility of enhanced task performance, high reliability (fault tolerance), low unit complexity, and decreased cost over traditional systems. Also, it has been argued that a swarm of robots can accomplish some tasks that would be impossible for a single robot to achieve. Particular research includes that

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We include an appendix with some simulations that provide some insights into swarm behavior. It is proposed that the appendix will not be published, but that a URL will be provided to the interested readers.

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of Beni [7] who introduced the concept of cellular robotic systems, and the related study in [14]. The behavior-based control strategy put forward by Brooks [15] is quite well known and it has been applied to collections of simple independent robots, usually for simple tasks. Mataric [16] describes experiments with a homogeneous population of robots acting under different communication constraints. Suzuki [17] considered a number of two-dimensional problems of formation of geometric patterns with distributed anonymous mobile swarm robots. Other approaches and results in this area are summarized in [6, 18].

In this paper, we are interested in mathematical modeling and analysis of stability properties of swarms. We think of stability as characterizing the cohesiveness of the swarm as it moves. Stability is a basic qualitative property of swarms since if it is not present, then it is typically impossible for the swarm to achieve any other group objective. Stability analysis of swarms is still an open problem but there have been several areas of relevant progress. In biology, researchers have used "continuum models" for swarm behavior based on non-local interactions, and have studied stability properties [19]. Jin et al. in [20] studied stability of synchronized distributed control of one-dimensional and two-dimensional swarm structures. Interestingly, the model and stability analysis in [20] are quite similar to the model and proof of stability for the load balancing problem in computer networks [21, 22]. Next, we would note that there have been several investigations into the stability of inter-vehicle distances in "platoons" in intelligent transportation systems (e.g., in [23, 24] or of the "slinky effect" in [25, 26], and traffic flow in [8, 10]). Finally, we would note that the study of stability properties of aircraft (spacecraft) formations is a relevant and active research area [12, 13].

This paper presents a single finite-size swarm member model, and then builds a one-dimensional asynchronous swarm model by putting many single swarm members together. In contrast to related existing results, the contribution of this paper lies in that we provide conditions under which the swarm will keep its cohesiveness even in the presence of sensing delays and asynchronism. In particular, we will conclude that for one-dimensional stationary edge-member asynchronous swarms, total asynchronism leads to asymptotic convergence and partial asynchronism leads to finite time convergence (both collision-free). Furthermore, we present conditions under which one-dimensional asynchronous mobile swarms following, or pushed by, an "edge-leader" can maintain collision-free cohesion during movements even with sensing delays and asynchronism. Our desire to consider collision-free cohesion for finite-size vehicles significantly complicates the analysis compared to the case where point-size vehicles are studied and collisions are allowed (e.g., as in [17]). Our study uses a discrete time discrete event dynamical system [21] approach and unlike the studies of platoon stability in intelligent transportation systems we avoid detailed characteristics of low level "inner-loop control" and vehicle dynamics in favor of focusing on high level mechanisms underlying qualitative swarm behavior when there are imperfect communications. Swarm stability for the $M \geq 2$ dimensional case will be studied in a forthcoming paper.

2 Modeling

First we explain the capabilities of a single swarm member and then we provide a mathematical model for a one-dimensional N-member asynchronous swarm. Moreover, we give mathematical models for N-member asynchronous mobile swarms following (being pushed by) an "edge-leader."

2.1 Single Swarm Member Model

A one-dimensional swarm is a set of N swarm members that moves along the real line. Here, we present a single swarm member model as shown in Figure 1, where we use a square to represent it. Assume the swarm member has a physical size (width) $w > 0$ and its position is the center of the square. It has a "proximity sensor" for both sides. This sensor has a sensing range $\varepsilon > w$, which means that once another swarm member reaches a distance of ε from it, the sensor *instantaneously* indicates the position of the other member. However, if its neighbors are not in its sensing range, the proximity sensor for the left neighbor will return $-\infty$ (or, practically, some large negative number), and the one for the right neighbor will return ∞. The proximity sensor is used to help avoid swarm member collisions and ensures that our framework allows for finite-size vehicles, not just points. Each swarm member also has a "neighbor position sensor," which can sense the positions of neighbors to its left and right if they are present. We assume that there is no restriction on how close a neighbor must be for the neighbor position sensor to provide a sensed value of its position. The sensed position information may be subjected to random delays (i.e., each swarm member's knowledge about its neighbors' positions may be outdated). We assume that each swarm member knows its own position with no delay. Notice that we define the position, distance and sensor sensing range of the finite-size swarm member with respect to its center, not its left or right edge.

Swarm members like to be close to each other, but not too close. Suppose $d > 0$ is the desired "comfortable distance" between two adjacent swarm neighbors, and that $d > \varepsilon$. Each swarm member senses the inter-swarm member distance via both neighbor position and proximity sensors and makes decisions for movements according to the error between the sensed distance and the comfortable distance d via its "decision-making" mechanism. And then, the decisions are inputed to its "driving device," which provides locomotion for it (for the one-dimensional case, movement either to the left or right). Each swarm member will try to move to maintain a comfortable distance to its neighbors. This will tend to make the group move together in a cohesive "swarm."

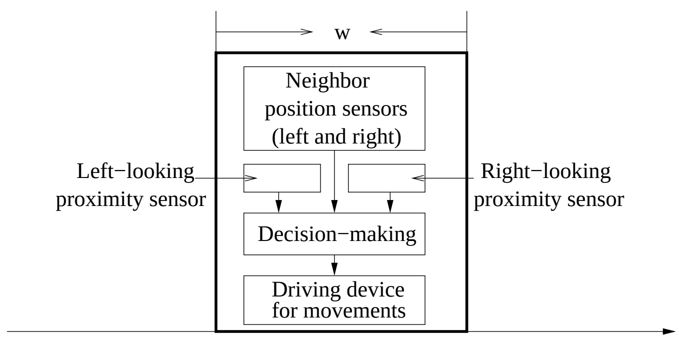

Figure 1: Single swarm member with a finite size w, one-dimensional case.

2.2 One-Dimensional Asynchronous Swarm Model

A one-dimensional swarm is formed by putting many of the above single swarm members together on the real line as shown in Figure 2. Let $x^i(t)$ denote the position of swarm member i at time t. We have $x^i(t) \in R$, $i = 1, 2, ..., N$, and if $N \geq 2$, we assume that $x^{i+1}(0) - x^i(0) > \varepsilon$, for $i = 1, 2, ..., N-1$ initially so that there are no overlapping in finite-size swarm members. Let $x_{pl}^{i-1}(t)$ ($x_{pr}^{i+1}(t)$) denote member i's left-neighbor (right-neighbor) position information sensed by its left-looking (right-looking) proximity sensor at time t. From above assumptions, we have

$$x_{pl}^{i-1}(t) = \begin{cases} x^{i-1}(t) & \text{if } x^i(t) - x^{i-1}(t) \leq \varepsilon \\ -\infty & \text{otherwise} \end{cases}, \text{ for } i = 2, 3, ..., N \quad (1)$$

$$x_{pr}^{i+1}(t) = \begin{cases} x^{i+1}(t) & \text{if } x^{i+1}(t) - x^i(t) \leq \varepsilon \\ \infty & \text{otherwise} \end{cases}, \text{ for } i = 1, 2, ..., N-1 \quad (2)$$

We assume that every swarm member knows d, and there is a set of times $T = \{0, 1, 2, ...\}$ at which one or more swarm members update their positions. Let $T^i \subseteq T, i = 1, 2, ..., N$, be a set of times at which the i^{th} member's position $x^i(t), t \in T^i$, is updated. Notice that the elements of T^i should be viewed as the indices of the sequence of physical times at which updates take place, not the real times. These time indices are non-negative integers and can be mapped into physical times. The T^i, $i = 1, 2, ..., N$, are independent of each other for different i. However, they may have intersections (i.e., it could be that $T^i \cap T^j \neq \emptyset$ for $i \neq j$, so two or more swarm members may move simultaneously). Here, our model assumes that swarm members sense their neighbor positions and update their positions only at time indices $t \in T^i$ and at all times $t \notin T^i$, $x^i(t)$ is left unchanged. A variable $\tau^i_{i-1}(t) \in T$ (respectively, $\tau^i_{i+1}(t) \in T$) for $i = 2, 3, ..., N$ ($i = 1, 2, ..., N-1$) is used to denote the time index of the real time where position information about neighbor $i-1$ ($i+1$) was obtained by member i at $t \in T^i$ and it satisfies $0 \leq \tau^i_{i-1}(t) \leq t$ ($0 \leq \tau^i_{i+1}(t) \leq t$) for $t \in T^i$. Of course, while we model the times at which neighbor position information is obtained as being the same times at which one or more swarm members decide where to move and actually move, it could be that the *real time* at which such neighbor position information is obtained is earlier than the real time where swarm members moved. The difference $t - \tau^i_{i-1}(t)$ ($t - \tau^i_{i+1}(t)$) between current time t and the time $\tau^i_{i-1}(t)$ ($\tau^i_{i+1}(t)$) can be viewed as a form of communication delay (of course the actual length of the delay depends on what real times correspond to the indices t, $\tau^i_{i-1}(t)$, or $\tau^i_{i+1}(t)$). Moreover, it is important to note that we assume that $\tau^i_{i-1}(t) \geq \tau^i_{i-1}(t')$ (respectively, $\tau^i_{i+1}(t) \geq \tau^i_{i+1}(t')$) if $t > t'$ for $t, t' \in T^i$. This ensures that member i will use the most recently obtained neighbor position information. Furthermore, we assume swarm member i will use the real-time neighbor position information $x^{i-1}(t)$ and/or $x^{i+1}(t)$ provided by its proximity sensors instead of information from its neighbor position sensors $x^{i-1}(\tau^i_{i-1}(t))$ and/or $x^{i+1}(\tau^i_{i+1}(t))$ if their neighbors are inside the sensing range of its proximity sensors. This information will be used for position updating until member i gets more recent information, for example, from its neighbor position sensor.

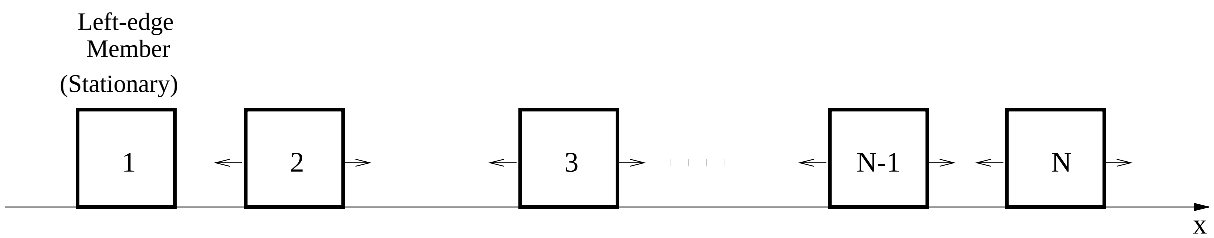

Figure 2: One-dimensional asynchronous swarm, all members moving to be adjacent to the stationary edge member.

Next, based on [22] we specify two assumptions that we use to characterize asynchronism for swarms.

Assumption 1. (Total Asynchronism): *Assume the sets T^i, $i = 1, 2, ..., N$, are infinite, and if for each k, $t_k \in T^i$ and $t_k \to \infty$ as $k \to \infty$, then $\lim_{k\to\infty} \tau^i_j(t_k) = \infty$ for $j = i-1, i+1$.*

This assumption guarantees that each swarm member moves infinitely often and the old position information of neighbors of each swarm member is eventually purged from it. More precisely, given any time t_1, there exists a time $t_2 > t_1$ such that $\tau^i_j(t) \geq t_1$, $\forall i, j$ and $j = i-1, i+1$ and $t \geq t_2$. On the other hand, the delays $t - \tau^i_j(t)$ in obtaining position information of neighbors of member i can become unbounded as t increases. Next, we specify a more restrictive type of asynchronism, but one which is usually easy to implement in practice.

Assumption 2. (Partial Asynchronism): *There exists a positive integer B (i.e., $B \in Z^+$, where Z^+ represents the set of positive integers) such that:*

(a) For every i and $t \geq 0$, $t \in T$, at least one of the elements of the set $\{t, t+1, ..., t+B-1\}$ belongs to T^i.

(b) There holds $t - B < \tau_j^i(t) \leq t$ for all i and $j = i-1, i+1$, and all $t \geq 0$ belonging to T^i.

Notice that for the partial asynchronism assumption, each member moves at least once within B time indices and the delays $t - \tau_j^i(t)$ in obtaining position information of neighbors of member i is bounded by B, i.e., $0 \leq t - \tau_j^i(t) < B$.

Let $e^i(t) = x^{i+1}(t) - x^i(t)$, $i = 1, 2, ..., N-1$ denote the distance between adjacent swarm members. Let the function $g(e^i(t) - d)$ denote the attractive and repelling relationship between two swarm neighbors with respect to the error between $e^i(t)$ and the comfortable distance d. We define two different types of g functions below, $g_a(e^i(t) - d)$ and $g_f(e^i(t) - d)$, to denote two different kinds of attractive and repelling relationships that will be used to establish different swarm convergence properties.

Assume that for a scalar $\beta > 1$, $g_a(e^i(t) - d)$ is such that

$$\frac{1}{\beta}(e^i(t) - d) < g_a(e^i(t) - d) < (e^i(t) - d), \text{ if } (e^i(t) - d) > 0; \tag{3}$$

$$g_a(e^i(t) - d) = (e^i(t) - d) = 0, \text{ if } (e^i(t) - d) = 0; \tag{4}$$

$$(e^i(t) - d) < g_a(e^i(t) - d) < \frac{1}{\beta}(e^i(t) - d), \text{ if } (e^i(t) - d) < 0. \tag{5}$$

Equation (3) indicates that if $(e^i(t) - d) > 0$, then swarm member position x^{i+1} is too far away from the position x^i so there is an attractive relationship between swarm members $i+1$ and i (i.e., swarm member $i+1$ wants to move toward swarm member i). In addition, the low bound $\frac{1}{\beta}(e^i(t) - d)$ for $g_a(e^i(t) - d)$ guarantees that swarm member's moving step cannot be infinitely small during its movements to its desired position if $(e^i(t) - d)$ is not infinitely small. The constraint $g_a(e^i(t) - d) < (e^i(t) - d)$ ensures that it will not "over-correct" for the inter-swarm member distance. Equation (4) indicates that if $(e^i(t) - d) = 0$, then swarm member position x^{i+1} is at a comfortable distance d from the position x^i so there are no attractive or repelling relationship between swarm members $i+1$ and i (i.e., swarm member $i+1$ remains stationary). Equation (5) indicates that if $(e^i(t) - d) < 0$, then swarm member position x^{i+1} is too close to the position x^i, so member $i+1$ tries to move away from member i.

Assume that for some scalars β and η, such that $\beta > 1$, and $\eta > 0$, $g_f(e^i(t) - d)$ satisfies

$$\frac{1}{\beta}(e^i(t) - d) < g_f(e^i(t) - d) < (e^i(t) - d), \text{ if } (e^i(t) - d) > \eta; \tag{6}$$

$$g_f(e^i(t) - d) = (e^i(t) - d), \text{ if } -\eta \leq (e^i(t) - d) \leq \eta; \tag{7}$$

$$(e^i(t) - d) < g_f(e^i(t) - d) < \frac{1}{\beta}(e^i(t) - d), \text{ if } (e^i(t) - d) < -\eta. \tag{8}$$

As shown in Figure 3, these relationships are similar to those for g_a except if $-\eta < (e^i(t) - d) < \eta$, the two swarm members can move to be at the comfortable inter-swarm member distance within one move.

Assume that initially each member i, $i = 1, 2, ..., N$, does not have knowledge of its neighbors' positions. We will suppose that it *guesses* that its neighbors are in the comfortable distance to it. Hence each member remains stationary until it first obtains position information about *both* its neighbors. Then it will update its position according to this information. Therefore, for a N-member swarm, the desired converging position of all members is $(x^1(0), x^1(0) + d, x^1(0) + 2d, ..., x^1(0) + (N-1)d)$, where $x^1(0)$ is the initial position of the left-edge member. Notice that in this section we assume that member 1 remains stationary and in the next section we will show how to treat the case where member 1 is mobile.

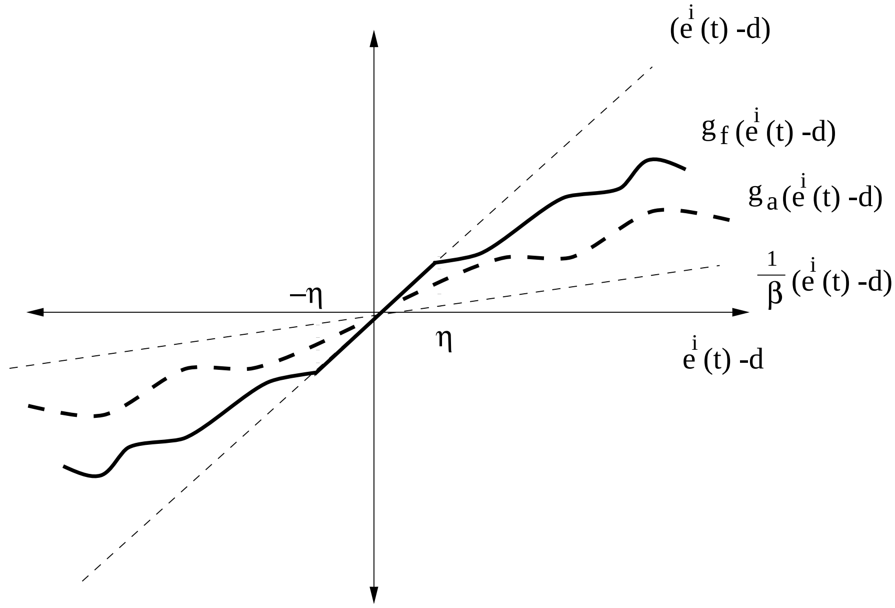

Figure 3: The function $g_a(e^i(t)-d)$ (dashed) and $g_f(e^i(t)-d)$ (solid).

Assume in Figure 2 that member 1 remains stationary and all other members calculate their "predicted update step size" by using their left neighbor information and compare this with the right neighbor information obtained from their proximity sensors in order to make movements without collisions at their update time indices. At $t \in T^i$, the proximity sensors of member i provide its neighbors information $x_{pl}^{i-1}(t)$ and $x_{pr}^{i+1}(t)$ according to Equations (1) and (2) (note that this information may include the real-time position information of its neighbors if its neighbors are inside its ε-range). At the same time, member i also obtains the most recent information of its left neighbor $x^{i-1}(\tau_{i-1}^i(t))$ via its neighbor position sensor. We assume that member i, $i = 2, 3, ..., N$, calculates its "predicted update step size" $\phi^i(t)$ according to its left neighbor information at time $t \in T^i$ by

$$\phi^i(t) = \begin{cases} \min\{|g(x^i(t) - x_{pl}^{i-1}(t) - d)|, (\varepsilon - w)/2\} sgn(x^i(t) - x_{pl}^{i-1}(t) - d), \\ \quad \text{if } x^i(t) - x_{pl}^{i-1}(t) \leq \varepsilon; \\ \min\{|g(x^i(t) - x^{i-1}(\tau_{i-1}^i(t)) - d)|, (\varepsilon - w)/2\} sgn(x^i(t) - x^{i-1}(\tau_{i-1}^i(t)) - d), \\ \quad \text{otherwise.} \end{cases} \tag{9}$$

The update size is computed based on the g function according to the sensed position of member i's left neighbor, where g can be either g_a or g_f, depending on the type of convergence properties we study. The "sgn" function ($sgn(z) = 1$ if $z \geq 0$, $sgn(z) = -1$ if $z < 0$) is used to model the moving direction, where "$-$" represents moving to the left and "$+$" represents moving to the right. Member i uses the real-time information of its left neighbor sensed by its proximity sensor in case of $x^i(t) - x_{pl}^{i-1}(t) \leq \varepsilon$, where $x_{pl}^{i-1}(t)$ is from Equation (1). Otherwise, it uses the information $x^{i-1}(\tau_{i-1}^i(t))$ from the neighbor position sensor. Notice that the update step size is forced to be bounded by $(\varepsilon - w)/2$, where $\varepsilon > 0$ is the sensing range of proximity sensors and w is the size of swarm members. With this step size choice, collisions between swarm neighbors with a physical size w can be avoided by proximity sensors even if two swarm members simultaneously move towards each other due to the delayed information. When g_f is used, we assume $\varepsilon - w > 2\eta$, where $\eta > 0$ is a finite positive number so that swarm neighbors can move to be at the comfortable distance d within one move when their distance is already inside η-neighborhood of the comfortable distance.

A mathematical model for the above swarm, assuming the left-edge member is stationary, is given by

$$\begin{aligned}
x^1(t+1) &= x^1(t), \forall t \in T^1 \\
x^2(t+1) &= \min\{x^2(t) - \phi^2(t), x^3_{pr}(t) - w\}, \forall t \in T^2 \\
\vdots &= \vdots \\
x^{N-1}(t+1) &= \min\{x^{N-1}(t) - \phi^{N-1}(t), x^N_{pr}(t) - w\}, \forall t \in T^{N-1} \\
x^N(t+1) &= x^N(t) - \phi^N(t), \forall t \in T^N \\
\\
x^i(t+1) &= x^i(t),\ \forall t \notin T^i,\ i = 1, 2, ..., N
\end{aligned} \tag{10}$$

where $\phi^i(t)$ is defined in Equation (9), and $x^{i+1}_{pr}(t)$ is defined in Equation (2). Clearly, in the above model, swarm member i, $i = 2, 3, ..., N-1$, makes decisions for its new position by comparing the right neighbor information obtained from proximity sensors with the predicted position computed from $\phi^i(t)$, where "min" is used to model the avoidance of collisions with a right-neighbor via its right-looking proximity sensor when a swarm member moves to the right (notice that member N moves only according to ϕ^N since it does not have a right neighbor). With this and the choice of initial conditions, we have

$$|x^{i+1}(t) - x^i(t)| > w, \text{ for } i = 1, 2, ..., N-1 \tag{11}$$

since we bound the step size with $(\varepsilon - w)/2$ and the proximity sensor with a sensing range ε is used for collision avoidance. Equation (11) means that swarm members with a finite size w in the above swarm model have no collisions during movements.

On the basis of the above model, we summarize the position updating algorithm followed by swarm member i, $i = 2, 3, ..., N$ at $t \in T^i$ as follows:

1. At $t \in T^i$, swarm member i makes measurements for the positions of its neighbors via its proximity sensors. It obtains the actual position information of the neighbor which is inside its sensing range and returns $-\infty$ for $x^{i-1}_{pl}(t)$ if its left neighbor is beyond its sensing range and ∞ for $x^{i+1}_{pr}(t)$ if its right neighbor is beyond its sensing range.

2. In the same time, it receives the most recent left-neighbor position information $x^{i-1}(\tau^i_{i-1}(t))$ (which may be delayed) via its neighbor position sensor and calculates the predicted update step size $\phi^i(t)$ according to Equation (9).

3. It makes decisions by comparing the right neighbor information obtained from proximity sensors with the predicted new position from the last step and instantaneously moves to the new position.

4. It sends its new position information to its neighbors and remains stationary until the next update time index $t \in T^i$.

Notice that this algorithm is only for the case where the left-edge member is stationary. For other cases, the model needs to be modified as we explain next.

2.3 One-Dimensional Asynchronous Mobile Swarm Model Following an Edge-Leader

Now assume that the left-edge member (member 1) of the swarm in Figure 2 moves to the left as a edge-leader (we assume that it does not change directions). All other members will follow it to move to the left. Notice that by symmetry, the case where the right-edge member leads the mobile swarm to the right is the same. Assume a swarm member will consider itself to be an edge-member if its neighbor position sensors only indicate the existence of one neighbor, and a middle member if its neighbor position sensors indicate the existence of both left and right neighbors. The edge-leader is the left edge member if the swarm moves to the left and is the right edge member if the swarm moves to the right.

For some $\gamma > 0$, we assume $[d-\gamma, d+\gamma]$ is a "comfortable distance neighborhood" relative to $x^i(t)$ and $x^{i+1}(t)$ (i.e., when $x^{i+1}(t) - x^i(t) \in [d-\gamma, d+\gamma]$, we say that they are in the comfortable distance neighborhood), where 2γ is the comfortable distance neighborhood size. Assume that $0 < \varepsilon < d - \gamma$ so that we do not consider swarm member $i+1$ to be at a comfortable distance to member i if it is too close to it, where ε is the sensing range of swarm members' proximity sensors.

Consider the two assumptions about asynchronism we specified earlier. Clearly only Assumption 2 (partial asynchronism) will result in cohesiveness for a mobile swarm since the delays in Assumption 1 (total asynchronism), which could be unbounded, will make swarm members lose track of their edge-leader or their neighbors during movements, i.e., the distance between swarm members could become unbounded just because swarm members use arbitrarily old sensed information. Hence, we will build our model based on Assumption 2 (partial asynchronism), which has a positive integer B which we view as an *asynchronism measure*.

For convenience, assume that $x^{i+1}(0) - x^i(0) = d$, for $i = 1, 2, ..., N-1$ initially, i.e., at the beginning all the swarm members are at a comfortable distance to their neighbors. Assume the edge-leader moves with a step $s(t)$ when $t \geq 0$, where $0 < s(t) \leq r$, i.e., $s(t)$ is bounded by a finite positive number r. So we have

$$x^1(t+1) = x^1(t) - s(t), \forall t \in T^1$$

Member 2 remains stationary until it gets the leader's new position information. Then this member tries to follow the leader and update its position according to the new position information since it wants to be at the comfortable distance to the edge-leader. Similarly, all the other swarm members will begin to move and follow their moving neighbors. They all try to be at the comfortable distance to their neighbors. We think of the swarm as maintaining the cohesiveness if all the swarm members are in the comfortable distance neighborhood to their neighbors during the moving process. Notice that the leader's moving step bound r can be used as a measure of how *fast* a cohesive asynchronous swarm moves.

Next, we will show that in the above mobile swarm, when swarm members only update their positions according to the g function, they will never have collisions even without proximity sensors. Accordingly, we define the predicted update step size $\phi^i(t)$ of swarm member i according to its left neighbor information for this case as

$$\phi^i(t) = g(x^i(t) - x^{i-1}(\tau^i_{i-1}(t)) - d), \text{ for } i = 2, 3, ..., N$$

Here, "g" can be either g_a or g_f, and it denotes the attractive and repelling relationship between two swarm neighbors. Moreover, we will introduce another such function later and analyze the swarm cohesiveness properties that it achieves.

At the beginning when $t = 0$, $e^i(0) = x^{i+1}(0) - x^i(0) = d$, for $i = 1, 2, ..., N-1$. When $t \geq 0, t \in T^1$, the edge-leader begins to move and we then have $e^1(t) = x^2(t) - x^1(t) > d$. Member 2, the edge-leader's right neighbor, remains stationary until it gets the leader's new position information $x^1(\tau^2_1(t))$. From Assumption 2 (partial asynchronism), we know

$$t - B < \tau^2_1(t) \leq t \tag{12}$$

and we have $x^1(\tau^2_1(t)) \geq x^1(t)$. And so

$$x^2(t) - x^1(\tau^2_1(t)) - d \leq x^2(t) - x^1(t) - d \tag{13}$$

According to the definition of g (i.e., both g_a and g_f) and Equation (13), we have

$$g(x^2(t) - x^1(\tau^2_1(t)) - d) \leq x^2(t) - x^1(\tau^2_1(t)) - d \leq x^2(t) - x^1(t) - d \tag{14}$$

At the beginning, member 2's proximity sensors cannot sense its neighbors since at this time $e^1(t)$ is greater than or equal to d. So member 2 updates its position only according to $x^1(\tau_1^2(t))$ and the update step is equal to $g(x^2(t) - x^1(\tau_1^2(t)) - d)$. From Equation (14), we know the update step of member 2 is always less than or equal to the error between the real distance from member 2 to the edge-leader and d (i.e, $x^2(t) - x^1(t) - d$). Hence, the inter-member distance between members 1 and 2 is always greater than or equal to d. Clearly, a similar result holds for all swarm members, so we have

$$e^i(t) \geq d, \forall t \in T^i, i = 1, 2, ..., N-1 \tag{15}$$

Equation (15) implies that all the mobile swarm members' proximity sensors will never sense their neighbors during movements. This also implies that members will never have collisions in the above swarm, even without proximity sensors. Thus, we can write a model in the below for the above mobile swarm,

$$\begin{aligned}
x^1(t+1) &= x^1(t) - s(t), \forall t \in T^1 \\
x^2(t+1) &= x^2(t) - g(x^2(t) - x^1(\tau_1^2(t)) - d), \forall t \in T^2 \\
\vdots &= \quad \vdots \\
x^{N-1}(t+1) &= x^{N-1}(t) - g(x^{N-1}(t) - x^{N-2}(\tau_{N-2}^{N-1}(t)) - d), \forall t \in T^{N-1} \\
x^N(t+1) &= x^N(t) - g(x^N(t) - x^{N-1}(\tau_{N-1}^N(t)) - d), \forall t \in T^N \\
\\
x^i(t+1) &= x^i(t), \ \forall t \notin T^i, \ i = 1, 2, ..., N
\end{aligned} \tag{16}$$

Remark 1: Notice that for a one-dimensional N-member collision-free mobile swarm pushed by an edge-leader, where member N is the edge-leader moving to the left with a step $s(t)$ when $t \geq 0$, $0 < s(t) \leq r$, and $0 < r < \varepsilon - w$, all other members move according to the position of their right neighbor except that they may use position information of their left neighbor for avoiding collisions, Assumption 2 (partial asynchronism) holds, and $x^{i+1}(0) - x^i(0) = d$, for $i = 1, 2, ..., N-1$ initially, a mathematical model of the above swarm is given by

$$\begin{aligned}
x^1(t+1) &= x^1(t) + \phi^1(t), \forall t \in T^1 \\
x^2(t+1) &= \max\{x_{pl}^1(t) + w, x^2(t) + \phi^2(t)\}, \forall t \in T^2 \\
\vdots &= \quad \vdots \\
x^{N-1}(t+1) &= \max\{x_{pl}^{N-2}(t) + w, x^{N-1}(t) + \phi^{N-1}(t)\}, \forall t \in T^{N-1} \\
x^N(t+1) &= \max\{x_{pl}^{N-1}(t) + w, x^N(t) - s(t)\}, \forall t \in T^N \\
\\
x^i(t+1) &= x^i(t), \ \forall t \notin T^i, \ i = 1, 2, ..., N
\end{aligned} \tag{17}$$

where the predicted update step size $\phi^i(t)$ of swarm member i according to its right neighbor information is defined as

$$\phi^i(t) = \begin{cases} \min\{|g(x_{pr}^{i+1}(t) - x^i(t) - d)|, \varepsilon - w\} sgn(x_{pr}^{i+1}(t) - x^i(t) - d), \\ \quad \text{if } x_{pr}^{i+1}(t) - x^i(t) \leq \varepsilon; \\ \min\{|g(x^{i+1}(\tau_{i+1}^i(t)) - x^i(t) - d)|, \varepsilon - w\} sgn(x^{i+1}(\tau_{i+1}^i(t)) - x^i(t) - d), \\ \quad \text{otherwise.} \end{cases} \tag{18}$$

and $x_{pl}^{i-1}(t)$ and $x_{pr}^{i+1}(t)$ are defined in Equations (1) and (2). Notice that here we bound the step size by $\varepsilon - w$ instead of $(\varepsilon - w)/2$ since all swarm members only move to the left.

3 Convergence Analysis of One-Dimensional Asynchronous Swarms

In this section, we will study stability properties of one-dimensional asynchronous swarms on the basis of mathematical models we built earlier and provide conditions under which the swarm will obtain and keep the cohesiveness even in the presence of sensing delays and asynchronism. First we will consider stationary edge-member asynchronous N-member swarms, and then we will investigate N-member asynchronous mobile swarms following, or pushed by an edge-leader.

3.1 Convergence Analysis of Stationary Edge-Member Asynchronous Swarms

Here, we will provide conditions under which the swarm in Figure 2 will converge to be adjacent to a stationary left-edge member. By symmetry, the case for convergence to a stationary right-edge member is the same. We begin with the two-member case, then consider the general N-member case where the proofs will depend on the $N = 2$ case.

3.1.1 Convergence for a Two-Member Swarm

Suppose there is a one-dimensional two-member swarm, which has member i and $i-1$. Assume at the beginning we have $x^i(0) - x^{i-1}(0) > \varepsilon$ and member $i-1$ remains stationary so that proximity sensors of member i will never sense member $i-1$. Therefore, we define the update step size $\phi^i(t)$ of member i in such a two-member swarm as follows,

$$\phi^i(t) = g(x^i(t) - x^{i-1}(\tau^i_{i-1}(t)) - d) \tag{19}$$

where the step size only depends on the g function.

Lemma 1. *For an $N = 2$ totally asynchronous swarm modeled by*

$$\begin{aligned} x^{i-1}(t+1) &= x^{i-1}(t), \forall t \in T \\ x^i(t+1) &= x^i(t) - \phi^i(t), \forall t \in T^i \\ x^i(t+1) &= x^i(t), \forall t \notin T^i \end{aligned} \tag{20}$$

where $x^{i-1}(t) = x^{i-1}_c$, x^{i-1}_c is a constant, $x^i(0) - x^{i-1}(0) > \varepsilon$, and $\phi^i(t)$ is defined in Equation (19) with $g = g_a$, it is the case that for any γ, $0 < \gamma < d - \varepsilon$, there exists a time t' such that $x^i(t') \in [x^{i-1}_c + d - \gamma, x^{i-1}_c + d + \gamma]$ and also $\lim_{t\to\infty} x^i(t) = x^{i-1}_c + d$.

Proof. First, for the model in Equation (20), note that at the beginning member i remains stationary until it senses member $i-1$'s information $x^{i-1}(\tau^i_{i-1}(t))$, which is equal to x^{i-1}_c since member $i-1$ always stays at the position of x^{i-1}_c. Hence, we can write Equation (20) as follows,

$$\begin{aligned} x^{i-1}(t+1) &= x^{i-1}_c, \forall t \in T \\ x^i(t+1) &= x^i(t) - g_a(x^i(t) - x^{i-1}_c - d), \forall t \in T^i \\ x^i(t+1) &= x^i(t), \forall t \notin T^i \end{aligned} \tag{21}$$

Next, define a Lyapunov-like function (note that it is not a Lyapunov function since $x = [x^1, ..., x^N]$ is *not* the state of the system in Equation (10)),

$$V_i(t) = |x^i(t) - (x^{i-1}_c + d)|, \forall t \in T^i,$$

that measures how close swarm member i is to the comfortable distance from member $i-1$. Notice that

$$\begin{aligned} V_i(t+1) &= |x^i(t+1) - (x_c^{i-1} + d)| \\ &= |x^i(t) - g_a(x^i(t) - (x_c^{i-1} + d)) - (x_c^{i-1} + d)| \\ &= |(x^i(t) - x_c^{i-1} - d) - g_a(x^i(t) - x_c^{i-1} - d)|, \forall t \in T^i. \end{aligned}$$

We will do a case analysis. First, for any $t \in T^i$ such that $x^i(t) - x_c^{i-1} = d$, we have

$$g_a(x^i(t) - (x_c^{i-1} + d)) = g_a(0) = 0$$

according to Equation (4), so we get

$$x^i(t+1) = x^i(t) - 0 = x_c^{i-1} + d,$$

which means member i will stay at the position $(x_c^{i-1} + d)$. Next, we will analyze the cases where at some $t \in T^i$, $x^i(t) - x_c^{i-1} > d$ or $x^i(t) - x_c^{i-1} < d$.

Note that if at any time $t \in T^i$, $x^i(t) - x_c^{i-1} > d$ ($x^i(t) - x_c^{i-1} < d$), it will be the case that $\forall t' \geq t, t' \in T^i, x^i(t') - x_c^{i-1} > d$ ($x^i(t') - x_c^{i-1} < d$). Due to this, in our analysis of bounds on the time required to reach the comfortable distance neighborhood, we can treat the cases of $x^i(t) - x_c^{i-1} > d$ and $x^i(t) - x_c^{i-1} < d$ separately.

Case 1: If at some $t \in T^i$, $x^i(t) - x_c^{i-1} > d$, then

$$g_a(x^i(t) - x_c^{i-1} - d) > 0 \tag{22}$$

and

$$V_i(t) = x^i(t) - (x_c^{i-1} + d). \tag{23}$$

According to Equation (3), we have for this $t \in T^i$

$$g_a(x^i(t) - x_c^{i-1} - d) \leq (x^i(t) - x_c^{i-1} - d)$$

so for this $t \in T^i$

$$V_i(t+1) = x^i(t) - x_c^{i-1} - d - g_a(x^i(t) - x_c^{i-1} - d), \tag{24}$$

and then using Equations (22), (23), and (24),

$$\Delta V_i = V_i(t+1) - V_i(t) = -g_a(x^i(t) - x_c^{i-1} - d) < 0. \tag{25}$$

Moreover, if at some $t \in T^i$, $x^i(t) - x_c^{i-1} > d$, from Equations (3) and (21), we have $x^i(t+1) - x_c^{i-1} > d$. So Equation (25) always holds in this case.

From Equations (3) and (25), we know that if member i is beyond the $\gamma-$range of position $(x_c^{i-1} + d)$ (i.e., $x^i(t) - x_c^{i-1} > d + \gamma$), we get

$$\frac{\gamma}{\beta} < \frac{1}{\beta}(x^i(t) - x_c^{i-1} - d) < g_a(x^i(t) - x_c^{i-1} - d)$$

So it will move toward this range with a moving step at least larger than $\frac{\gamma}{\beta}$. Hence, member i needs at most

$$\frac{\beta}{\gamma}(x^i(0) - x_c^{i-1} - d - \gamma)$$

update time steps, and at least one update time step, to reach the $\gamma-$range of position $(x_c^{i-1}+d)$, where $x^i(0)$ is the initial position of member i (in this case, we have $(x^i(0)-x_c^{i-1}-d) \geq (x^i(t)-x_c^{i-1}-d)$, for $t \geq 0$). So there exists a time t' such that $x^i(t') \in (x_c^{i-1}+d, x_c^{i-1}+d+\gamma]$.

Case 2: If at some $t \in T^i$, $x^i(t)-x_c^{i-1} < d$, then

$$g_a(x^i(t)-x_c^{i-1}-d) < 0 \tag{26}$$

and

$$V_i(t) = (x_c^{i-1}+d) - x^i(t). \tag{27}$$

According to Equation (5), we have

$$(x^i(t)-x_c^{i-1}-d) \leq g_a(x^i(t)-x_c^{i-1}-d)$$

so

$$V_i(t+1) = g_a(x^i(t)-x_c^{i-1}-d) - (x^i(t)-x_c^{i-1}-d), \tag{28}$$

and then using Equations (26), (27), and (28), for this $t \in T^i$

$$\Delta V_i = V_i(t+1) - V_i(t) = g_a(x^i(t)-x_c^{i-1}-d) < 0. \tag{29}$$

Similar to Case 1, from Equations (5) and (21), if at some $t \in T^i$, $x^i(t)-x_c^{i-1} < d$, we have $x^i(t+1)-x_c^{i-1} < d$. So Equation (29) always holds in this case. Also, similar to Case 1, from Equation (5) and (29) we know that if $x^i(t)-x_c^{i-1} < d-\gamma$, we get

$$g_a(x^i(t)-x_c^{i-1}-d) < \frac{1}{\beta}(x^i(t)-x_c^{i-1}-d) < -\frac{\gamma}{\beta}$$

So it will move toward this range with a moving step at least larger than $\frac{\gamma}{\beta}$. Hence, we can show that member i needs at most

$$\frac{\beta}{\gamma}(x_c^{i-1}+d-x^i(0)-\gamma)$$

update time steps, and at least one update time step, to reach the $\gamma-$range of position $(x_c^{i-1}+d)$, where $x^i(0)$ is the initial position of member i (in this case, we have $(x_c^{i-1}+d-x^i(0)) \geq (x_c^{i-1}+d-x^i(t))$, for $t \geq 0$). So there exists a time t' such that $x^i(t') \in [x_c^{i-1}+d-\gamma, x_c^{i-1}+d)$.

Finally, for asymptotic convergence, from Equations (25) and (29), we know that for $t \in T^i$, $V_i(t)$ will asymptotically tend to zero (i.e., $\lim_{t\to\infty} x^i(t) = x_c^{i-1}+d$). **Q.E.D.**

Corollary 1. *For a two-member swarm in Lemma 1, if member $i-1$, instead of remaining stationary at x_c^{i-1}, monotonically approaches the position x_c^{i-1} as $t \to \infty$ (i.e., $x_c^{i-1} \leq x^{i-1}(t+1) < x^{i-1}(t)$ or $x^{i-1}(t) < x^{i-1}(t+1) \leq x_c^{i-1}$, $\forall t \in T^{i-1}$), and $|x^{i-1}(0)-x_c^{i-1}| < \gamma$, where $0 < \gamma < d-\varepsilon$, the conclusion of Lemma 1 still holds.*

Proof. First, we define $\delta_{i-1}(t) = |x^{i-1}(t)-x_c^{i-1}|$, which denotes member $i-1$'s distance to its desired position x_c^{i-1} at time $t \in T^{i-1}$. Since $x^{i-1}(t)$ monotonically goes to x_c^{i-1} as $t \to \infty$, we have

$$\lim_{t\to\infty} \delta_{i-1}(t) = 0 \tag{30}$$

And also we know $|x^{i-1}(0) - x_c^{i-1}| < \gamma$. Hence, we have $|x^{i-1}(t) - x_c^{i-1}| < \gamma$, i.e.,

$$0 \le \delta_{i-1}(t) < \gamma \tag{31}$$

Next, let

$$\delta_i(t) = x^i(t) - x_c^{i-1} - d, t \in T^i \tag{32}$$

which is member i's distance to its desired position $x_c^{i-1} + d$ at time $t \in T^i$. Note that $\delta_i(t)$ can be negative, where a negative value means member i stays on the left side of $x_c^{i-1} + d$. A positive value means it is on the right side of $x_c^{i-1} + d$. Member i will move according to

$$x^i(t+1) = x^i(t) - g_a(x^i(t) - x^{i-1}(\tau_{i-1}^i(t)) - d), \forall t \in T^i \tag{33}$$

Hence, we have

$$\begin{aligned} \delta_i(t+1) &= x^i(t) - x_c^{i-1} - d - g_a(x^i(t) - x^{i-1}(\tau_{i-1}^i(t)) - d) \\ &= \delta_i(t) - g_a(x^i(t) - x^{i-1}(\tau_{i-1}^i(t)) - d) \end{aligned} \tag{34}$$

So

$$\Delta\delta_i(t) = \delta_i(t+1) - \delta_i(t) = -g_a(x^i(t) - x^{i-1}(\tau_{i-1}^i(t)) - d) \tag{35}$$

Since member $i-1$ monotonically approaches its desired position x_c^{i-1} from the left,

$$x^{i-1}(t) = x_c^{i-1} - \delta_{i-1}(t), \forall t$$

and due to the definition of the delays, so

$$x^{i-1}(\tau_{i-1}^i(t)) = x_c^{i-1} - \delta_{i-1}(\tau_{i-1}^i(t)) \tag{36}$$

From Equations (32) and (35), we have

$$\begin{aligned} \Delta\delta_i(t) &= -g_a(x^i(t) - (x_c^{i-1} - \delta_{i-1}(\tau_{i-1}^i(t))) - d) \\ &= -g_a(\delta_i(t) + \delta_{i-1}(\tau_{i-1}^i(t))) \end{aligned} \tag{37}$$

We will study different possible cases when member $i-1$ monotonically approaches its desired position x_c^{i-1} from the left (i.e, $x^{i-1}(t) < x^{i-1}(t+1) \le x_c^{i-1}$, $\forall t \in T^{i-1}$). By symmetry, the cases where member $i-1$ monotonically approaches x_c^{i-1} from the right (i.e, $x_c^{i-1} \le x^{i-1}(t+1) < x^{i-1}(t)$, $\forall t \in T^{i-1}$) are the same.

Case 1: If $x^i(0) \ge x_c^{i-1} + d$, we have $\delta_i(0) = x^i(0) - x_c^{i-1} - d \ge 0$. This case is shown in Figure 4(a). As we know, $x^{i-1}(0) < x_c^{i-1}$, so we have $x^i(0) - x^{i-1}(0) > d$. At the beginning, member i remains stationary until it gets member $i-1$'s position information. At this time,

$$\delta_i(t) \ge 0 \text{ and } \delta_{i-1}(\tau_{i-1}^i(t)) > 0$$

From the definition of g_a, we know from Equation (37)

$$\Delta\delta_i(t) = -g_a(\delta_i(t) + \delta_{i-1}(\tau_{i-1}^i(t))) < 0$$

This means $\delta_i(t)$ will keep decreasing until at some time $t^b \in T^i$,

$$\delta_i(t^b) = -\delta_{i-1}(\tau_{i-1}^i(t^b)), \tag{38}$$

This decreasing process is due to the attractive relationship since the distance from member i to $i-1$ sensed by member i is larger than d (note that this sensed information may include delay). We can think of this process as seeking a "balance" between $\delta_i(t)$ and $\delta_{i-1}(\tau_{i-1}^i(t))$. The balance is achieved when Equation (38) is satisfied. At the time $t^b \in T^i$, from Equations (32), (36) and (38), we have

$$x^i(t^b) - x^{i-1}(\tau_{i-1}^i(t^b)) = x_c^{i-1} + d + \delta_i(t^b) - (x_c^{i-1} - \delta_{i-1}(\tau_{i-1}^i(t^b))) = d$$

Which means the distance from member i to $i-1$ sensed by member i is equal to d.

However, at the same time, we know from Assumption 1 and Equation (30) that

$$\lim_{t\to\infty} \delta_{i-1}(\tau_{i-1}^i(t)) = 0 \tag{39}$$

So the balance just achieved will be lost when $t > t^b$ due to the gradual decreasing of $\delta_{i-1}(\tau_{i-1}^i(t))$. Since

$$\delta_i(t) < -\delta_{i-1}(\tau_{i-1}^i(t)), \text{ for } t > t^b, t \in T^i$$

we have

$$\Delta\delta_i(t) = -g_a(\delta_i(t) + \delta_{i-1}(\tau_{i-1}^i(t))) > 0, \text{ for } t > t^b, t \in T^i$$

which means that $\delta_i(t)$ will keep increasing after $t > t^b$ until it achieves balance again (i.e., until Equation (38) is satisfied). Analogous to above, this is due to the repelling inter-member relationship. As $\delta_{i-1}(\tau_{i-1}^i(t))$ tends to zero gradually, this balancing process will make $\delta_i(t)$ also tend to zero gradually, i.e., $\lim_{t\to\infty} x^i(t) = x_c^{i-1}$. Member i's moving trail is shown by dashed line in Figure 4(a).

Case 2: If $x^i(0) < x_c^{i-1} + d$, we have $\delta_i(0) = x^i(0) - x_c^{i-1} - d < 0$.

Case 2a: If $\varepsilon < x^i(0) - x^{i-1}(0) < d$ at the beginning, as shown in Figure 4(b), we have

$$x^i(t) - (x_c^{i-1} - \delta_{i-1}(\tau_{i-1}^i(t))) < d$$

Therefore, from Equation (32), we have

$$\delta_i(t) + \delta_{i-1}(\tau_{i-1}^i(t)) < 0 \text{ and } \delta_i(t) < 0$$

Thus, from Equation (37)

$$\Delta\delta_i(t) = -g_a(\delta_i(t) + \delta_{i-1}(\tau_{i-1}^i(t))) > 0$$

which means $\delta_i(t)$ will keep increasing until it achieves balance with $\delta_{i-1}(\tau_{i-1}^i(t))$, i.e., Equation (38) is satisfied. Similar to Case 1, $\delta_i(t)$ will go to zero gradually in the process of keeping balance with $\delta_{i-1}(\tau_{i-1}^i(t))$, which tends to zero asymptotically. The moving trail of member i in this case is shown by dashed line in Figure 4(b).

Case 2b: If $x^i(0) - x^{i-1}(0) > d$ at the beginning, as shown in Figure 4(c), we have

$$\delta_{i-1}(\tau_{i-1}^i(t)) + \delta_i(t) > 0$$

Thus, from Equation (37) $\Delta\delta_i(t) < 0$. So $\delta_i(t)$ will keep decreasing until it achieves balance with $\delta_{i-1}(\tau_{i-1}^i(t))$, i.e., Equation (38) is satisfied. However, similar to Case 1, the balance just obtained will be lost as $\delta_{i-1}(\tau_{i-1}^i(t))$ tends to zero asymptotically. Since $\Delta\delta_i(t) > 0$ at this time, $\delta_i(t)$ will begin to increase in order to get balance with $\delta_{i-1}(\tau_{i-1}^i(t))$ again. Finally, $\delta_i(t)$ will go to zero gradually in the process of keeping balance with $\delta_{i-1}(\tau_{i-1}^i(t))$, which tends to zero asymptotically. Member i's moving trail in this case is shown by dashed line in Figure 4(c). **Q.E.D.**

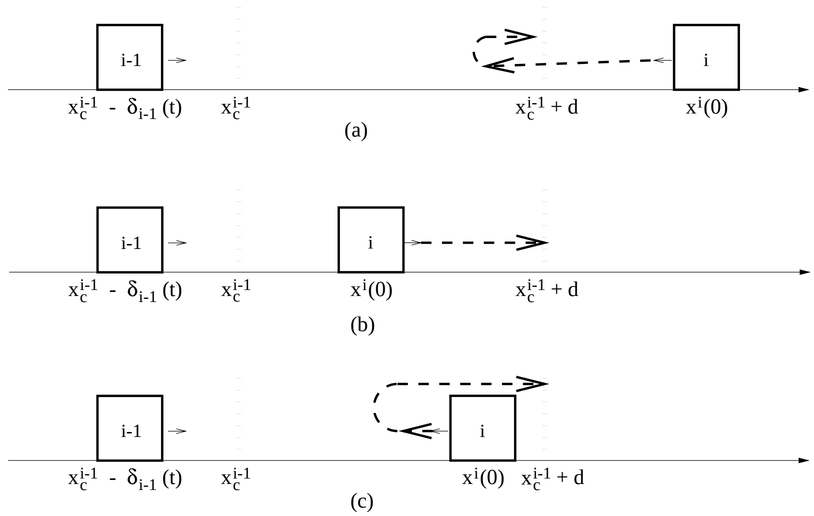

Figure 4: Two-member swarm, member $i-1$ approaches to its desired position x_c^{i-1} from the left.

Lemma 2. *For an $N = 2$ partially asynchronous swarm modeled by Equation (20) but with $g = g_f$, where $x^{i-1}(t) = x_c^{i-1}$, x_c^{i-1} is a constant, and $x^i(0) \geq x^{i-1}(0)+\varepsilon$, for any η, $0 < \eta < d-\varepsilon$, swarm member i will converge to the position of $(x_c^{i-1}+d)$ in some finite time, that is bounded by $B[\frac{\beta}{\eta}(|x^i(0)-x_c^{i-1}-d|-\eta)+2]$.*

Proof. According to Assumption 2, we know that at most after time B from the beginning, member i will sense member $i-1$'s position information x_c^{i-1}. Then we get the results from the proof of Lemma 1 after replacing g_a with g_f and choosing $\gamma = \eta$.

For the case $-\eta \leq (x^i(t) - x_c^{i-1} - d) \leq \eta$ and $x^i(t) - x_c^{i-1} \neq d$, swarm member i will converge to the position of $(x_c^{i-1} + d)$ in *one* time step according to Equation (7). Similar to the proof of Lemma 1 we can prove that if $x^i(t) - x_c^{i-1} > d + \eta$ (here $\eta = \gamma$), it will move toward this $\eta-$range with a moving step at least larger than $\frac{\eta}{\beta}$. Hence, member i needs at most $\frac{\beta}{\eta}(x^i(0) - x_c^{i-1} - d - \eta)$ update time steps, and at least one update time step, to reach the $\eta-$range of position $(x_c^{i-1} + d)$. From Assumption 2, we know that for a partially asynchronous swarm, the maximum update time interval is B. Also, according to Equation (7), member i will arrive at the position $(x_c^{i-1} + d)$ in the next update time step. So the total time, including delay time and moving time (time to move within the range plus up to B steps more to move to the position $(x_c^{i-1} + d)$), needed to achieve convergence is bounded by

$$B + B[\frac{\beta}{\eta}(x^i(0) - x_c^{i-1} - d - \eta)] + B = B[\frac{\beta}{\eta}(x^i(0) - x_c^{i-1} - d - \eta) + 2] \tag{40}$$

Similar to the case of $x^i(t) - x_c^{i-1} < d - \eta$, we obtain the total time, including delay time and moving time, needed to achieve convergence is bounded by

$$B + B[\frac{\beta}{\eta}(x_c^{i-1} + d - x^i(0) - \eta)] + B = B[\frac{\beta}{\eta}(x_c^{i-1} + d - x^i(0) - \eta) + 2] \tag{41}$$

Combining Equations (40) with (41), we get the time needed to achieve convergence is bounded by $B[\frac{\beta}{\eta}(|x^i(0) - x_c^{i-1} - d| - \eta) + 2]$. **Q.E.D.**

3.1.2 Convergence Analysis for an N-Member Swarm

Here, we will show that all members in an N-member swarm converge to be at a comfortable distance d from their neighbors on the basis of the analysis of two-member swarms.

Theorem 1. **(Total Asynchronism, Asymptotic Convergence):** *For an N-member swarm which is modeled by Equation (10) with $g = g_a$, $N > 1$, Assumption 1 (total asynchronism) holds, and $x^{i+1}(0) - x^i(0) > \varepsilon$, $i = 1, 2, ..., N-1$, the swarm members' positions $(x^1, x^2, ..., x^N)$ will asymptotically converge to $(x^1(0), x^1(0)+d, x^1(0)+2d, ..., x^1(0)+(N-1)d)$, where $x^1(0)$ is the initial position of the stationary left-edge member.*

Proof. We will use a mathematical induction method, where our induction hypothesis will be that k of N swarm members asymptotically converge to $(x^1(0), x^1(0)+d, x^1(0)+2d, ..., x^1(0)+(k-1)d)$ and from this we will show that $k+1$ of N members converge.

First, for $k = 1$, which is the left-edge member, it converges to $x^1(0)$ since it remains stationary. Next we must show that given the induction hypothesis, the first $k+1$ members in the N-member swarm will asymptotically converge to the positions $(x^1(0), x^1(0)+d, x^1(0)+2d, ..., x^1(0)+kd)$.

According to our induction hypothesis we know that there exists a time t^* such that swarm member i, for $i = 2, 3, ...k$, will stay in the range between $x^1(0)+(i-1)d-\gamma$ and $x^1(0)+(i-1)d+\gamma$, where $0 < \gamma < d - \varepsilon$. Assume $\delta_i(t) = x^i(t) - (x^1(0)+(i-1)d)$ for $i = 2, 3, ...k$, which represents member i's distance to its desired position, where $\delta_i(t) \in [-\gamma, \gamma]$ for $t \geq t^*$.

Therefore, after $t \geq t^*$, we have

$$x^i(t+1) = x^1(0) + (i-1)d + \delta_i(t), \forall t \in T^i, \text{ for } i = 2, 3, ..., k$$

for the first k members in the swarm except the stationary member 1 and from member $k+1$ to member N, we have

$$\begin{aligned} x^{k+1}(t+1) &= \min\{x^{k+1}(t) - \phi^{k+1}(t), x^{k+2}_{pr}(t) - w\}, \\ \vdots &= \quad \vdots \\ x^{N-1}(t+1) &= \min\{x^{N-1}(t) - \phi^{N-1}(t), x^{N}_{pr}(t) - w\}, \forall t \in T^{N-1} \\ x^{N}(t+1) &= x^N(t) - \phi^N(t), \forall t \in T^N \end{aligned} \tag{42}$$

where g in $\phi^i(t)$ is g_a.

Now considering member $k+1$'s movements after $t \geq t^*$, we will do a case analysis.

Case 1: If for some $t \in T^{k+1}$, $t \geq t^*$, member $k+1$'s proximity sensor senses member k, i.e.,

$$x^k_{pl}(t) = x^k(t) = x^1(0) + (k-1)d + \delta_k(t)$$

from Equation (9), member $k+1$ will move away from member k according to

$$\begin{aligned} x^{k+1}(t+1) = \min\Big\{ & x^{k+1}(t) - \min\{|g_a(x^{k+1}(t) - (x^1(0) + (k-1)d + \delta_k(t)) - d)|, (\varepsilon - w)/2\} \cdot \\ & sgn(x^{k+1}(t) - (x^1(0) + (k-1)d + \delta_k(t)) - d), x^{k+2}_{pr}(t) - w \Big\}, \forall t \in T^{k+1} \end{aligned} \tag{43}$$

Case 2: If member $k+1$'s proximity sensor cannot sense member k, it will update its position by

$$\begin{aligned} x^{k+1}(t+1) = \min\Big\{ & x^{k+1}(t) - \min\{|g_a(x^{k+1}(t) - x^k(\tau^{k+1}_k(t)) - d)|, (\varepsilon - w)/2\} \cdot \\ & sgn(x^{k+1}(t) - x^k(\tau^{k+1}_k(t)) - d), x^{k+2}_{pr}(t) - w \Big\}, \forall t \in T^{k+1} \end{aligned} \tag{44}$$

Case 2a: If member $k+1$ moves towards to member k, due to its outdated information $x^k(\tau^{k+1}_k(t))$ about member k, such that after some time its proximity sensor senses member k, then it will be the same as Case 1.

Case 2b: If member $k+1$ moves according to Equation (44), but its proximity sensor never senses member k, then according to Assumption 1, given time t^*, there exists a time $t^c > t^*$ such that $\tau_k^{k+1}(t) \geq t^*$, $\forall k$ and $t \geq t^c$. So after $t \geq t^c$, member $k+1$ knows member k is at the position $x^1(0) + (k-1)d + \delta_k(t)$. It will move according to Equation (43).

According to our induction hypothesis, we already know

$$\lim_{t\to\infty} \delta_i(t) = 0, \text{ for } i = 2, 3, ...k$$

So if member $k+2$ does not prevent its movements (i.e., $x^{k+1}(t+1)$ is never equal to $x_{pr}^{k+2}(t) - w$ in Equation (43)), as $\delta_k(t)$ goes to zero, member $k+1$ will move to its desired position as $\delta_k(t)$ by

$$\begin{aligned} x^{k+1}(t+1) &= x^{k+1}(t) - \min\{|g_a(x^{k+1}(t) - (x^1(0) + (k-1)d + \delta_k(t)) - d)|, (\varepsilon - w)/2\} \cdot \\ & \quad sgn(x^{k+1}(t) - (x^1(0) + (k-1)d + \delta_k(t)) - d), \forall t \in T^{k+1} \end{aligned} \tag{45}$$

If $|g_a(x^{k+1}(t) - (x^1(0) + (k-1)d + \delta_k(t)) - d)|$ is always less than $(\varepsilon - w)/2$ in Equation (45), member $k+1$ will move by

$$x^{k+1}(t+1) = x^{k+1}(t) - g_a(x^{k+1}(t) - (x^1(0) + (k-1)d + \delta_k(t)) - d), \forall t \in T^{k+1}$$

so that it will asymptotically converge to $(x^1(0) + (k-1)d + d) = (x^1(0) + kd)$ by Corollary 1. If $|g_a(x^{k+1}(t) - (x^1(0) + (k-1)d + \delta_k(t)) - d)|$ is larger than $(\varepsilon - w)/2$ member $k+1$ will move to the direction where $|x^{k+1}(t) - (x^1(0) + (k-1)d + \delta_k(t)) - d|$ becomes smaller with a step $(\varepsilon - w)/2$. Therefore, there exists some time t^s that after $t > t^s$, $|g_a(x^{k+1}(t) - (x^1(0) + (k-1)d + \delta_k(t)) - d)|$ is always less than $(\varepsilon - w)/2$ so that it will be the same as the above.

If member $k+1$'s proximity sensor finds member $k+2$ nearby at time $t^p \in T^{k+1}$, we have

$$x^{k+1}(t^p + 1) = x^{k+2}(t^p) - w$$

Note that at the same time member $k+2$ also gets member $k+1$'s current position via its proximity sensor since their current adjacent distance is w. There exists a time $t^u \geq t^p + 1$, $t^u \in T^{k+2}$ that member $k+2$ will update its position according to

$$\begin{aligned} x^{k+2}(t+1) &= \min\Big\{x^{k+2}(t) - \min\{|g_a(x^{k+2}(t) - (x^{k+2}(t^p) - w) - d)|, (\varepsilon - w)/2\} \cdot \\ & \quad sgn(x^{k+2}(t) - (x^{k+2}(t^p) - w) - d), x_{pr}^{k+3}(t) - w\Big\}, \forall t \in T^{k+2}, t \geq t^u \end{aligned}$$

Member $k+2$'s temporary destination is the position of $(x^{k+2}(t^p) - w + d)$ (we say "temporary" since as swarm member $k+1$ converges the ultimate desired position may be somewhere else). Similarly, from Equation (42) member $k+3$ could prevent member $k+2$'s further moving, and member $k+4$ could prevent member $k+3$, etc. In the end, member N could prevent member $N-1$. However, member N is free to move to the right.

Since we assume the sets T^i are infinite, and each swarm member moves infinitely often, we know that in the above cases after some time, member N could move away from member $N-1$, and member $N-1$ could move away from member $N-2$, etc. So after member $k+2$ moves away from member $k+1$, member $k+1$ will continue moving to the position $(x^1(0) + kd)$.

However, member $k+2$ may prevent member $k+1$ again after $t > t^u$ due to the total asynchronism. It is similar for member $k+3$, $k+4$, and so on. In that case, it will repeat the above process. We know that member $k+2$'s distance to the position $(x^1(0) + kd)$ is finite, and there are only a finite number of members in the swarm, and swarm member's movements cannot be infinitely small if the distance to its desired position is not infinitely small via the definition of g_a. So there exists a time $t^{np} > t^p$ such that member $k+2$ will have moved beyond the position $(x^1(0)+kd)$. After $t > t^{np}$, member $k+2$ will never prevent member $k+1$ again. From Corollary 1 and our induction hypothesis, member $k+1$ will asymptotically converge to the position $(x^1(0)+kd)$. This ends the induction step. **Q.E.D.**

Theorem 2. (Partial Asynchronism, Finite Time Convergence): *For an N-member swarm which is modeled by Equation (10) with $g = g_f$, $N > 1$, Assumption 2 (partial asynchronism) holds, and $x^{i+1}(0) - x^i(0) > \varepsilon$, $i = 1, 2, ..., N-1$, for any η, $0 < \eta < (\varepsilon - w)/2$, the swarm members' positions $(x^1, x^2, ..., x^N)$ will converge to $(x^1(0), x^1(0)+d, x^1(0)+2d, ..., x^1(0)+(N-1)d)$ in some finite time, that is bounded by*

$$B2^{N-2}[\frac{\beta}{\eta}(\max_i(|x^i(0) - x^1(0) - (i-1)d|) - \eta) + 2] \tag{46}$$

for $i = 2, 3, ..., N$, where $x^1(0)$ is the initial position of the stationary left-edge member.

Proof. We use the same induction method as the proof of Theorem 1 except that the swarm is partially asynchronous now. So we will use Assumption 2 and Lemma 2, instead of Assumption 1 and Corollary 1, to deduce member $k+1$ will arrive at the position $(x^1(0)+kd)$ in some finite time, instead of its desired position neighborhood after some time.

Next, we will try to bound the amount of converging time for the N-member swarm. In Lemma 2 we deduce that for a two-member swarm the time needed to achieve convergence is bounded by

$$B[\frac{\beta}{\eta}(|x^i(0) - x_c^{i-1} - d| - \eta) + 2]$$

For the N-member swarm, it is possible that blockades never occur among swarm members (i.e., members do not hinder their neighbors' movements). As we know, swarm members move to their desired positions with a step at least larger than $\frac{\eta}{\beta}$ when they are beyond η-range of their desired positions due to the definition of g_f and $0 < \frac{\eta}{\beta} < \eta < (\varepsilon - w)/2$. Therefore, for this special case, similar to Lemma 2 we can bound the total converging time by

$$BN[\frac{\beta}{\eta}(\max_i(|x^i(0) - x^1(0) - (i-1)d|) - \eta) + 2], \text{ for } i = 2, 3, ..., N.$$

This is the bound for the case that blockade never happens. As we know, blockades will prolong the converging time. In order to get the bound for the blockade case, we go to another extremely special worse case, that all middle swarm members will be blocked by their neighbors in every moving step and cannot move until the blockades are released. Moreover, we know that the first member (left-edge member) remains stationary, and the Nth member (right-edge member) is free to move to the right. In this special case, we know from analysis in Theorem 1 that each middle swarm member's distance to its desired position, which is $|x^i(0) - x^1(0) - (i-1)d|$ is finite, and they are blocked by their neighbors and can only move after their neighbors do not prevent them any more. This process repeats until they arrive at their desired positions. We know that each of their moving steps is at least larger than $\frac{\eta}{\beta}$. In this special case, for member 2, there are $N-2$ members preventing it. Similarly for member 3, $N-3$ members prevent it; for member 4, $N-4$ members prevent it, and so on, until for member $N-1$, only one member prevent it. Since we know $(N-2)+(N-3)+(N-4)+...+1 < 2^{N-2}$, we can estimate that the maximum possible total update time steps will be $2^{N-2}[\frac{\beta}{\eta}(\max_i(|x^i(0) - x^1(0) - (i-1)d|) - \eta) + 2]$. Then we can bound the total converging time by Equation (46) for this special case since the maximum update time interval is B. In fact, we know that there are many possibilities on how the N-member swarm moves. But all the moving cases can be seen as one or different combinations of the above two special cases, and their bounds lies in between the above two bounds, which are for two special cases. So we can bound the amount of converging time for the N-member swarm by the largest bound in Equation (46). **Q.E.D.**

Remark 2: Note that for an N-member swarm, if $\tau_{i-1}^i(t) = t$ (respectively, $\tau_{i+1}^i(t) = t$) for $i = 2, 3, ..., N$ ($i = 1, 2, ..., N-1$), for all $t \in T^i$, which means member i obtains position information about its neighbor $i-1$ ($i+1$) without delay, then we can get the same results as in Theorem 1 and 2 by the swarm model without proximity sensors.

3.2 Convergence Analysis of Asynchronous Mobile Swarms Following an Edge-Leader

Next, we will study cohesiveness of a mobile swarm. First, we will study the case of using the g_f function, and then what happens if a different g function is used that does not require a swarm member to move to be adjacent to its neighbor in one step if it gets very close to it. While we only consider a left-edge member leading a swarm to the left, by symmetry, the case where the right-edge member leads the mobile swarm to the right is the same (only the model is different).

3.2.1 Convergence for an N-Member Asynchronous Mobile Swarm

First, we choose g_f as the g function in Equation (16) and assume $\gamma = \eta$ (η is used in the definition of g_f). We will show that all members in a N-member mobile swarm will be in a comfortable distance neighborhood from their neighbors during movements if there are constraints on the leader's moving step bound, the partial asynchronism measure, and the comfortable distance neighborhood size.

Theorem 3. *For an N-member asynchronous mobile swarm modeled by Equation (16), where g is g_f, $N > 1$, Assumption 2 (partial asynchronism) holds, and $x^{i+1}(0) - x^i(0) = d$, $i = 1, 2, ..., N-1$, if*

$$0 < r \leq \frac{\gamma}{NB - 1} \tag{47}$$

for a given γ, all the swarm members will be in the comfortable distance neighborhood $[d, d+\gamma]$ of their neighbors during the moving process, where r is the upper bound of the edge-leader's moving step $s(t)$, $B \in Z^+$ is the partial asynchronism measure and γ (choose $\gamma = \eta$) is the comfortable distance neighborhood size.

Proof. For such a N-member mobile swarm, each swarm member follows its left neighbor except that the edge-leader moves by itself. We know from Equation (15) that there are no collisions between members. This decouples the problem so that we can consider each pair of neighboring swarm members individually.

Consider the relationship between member 1, the edge-leader, and member 2. According to Assumption 2, for every $t \geq 0$, $t \in T^i$, swarm member i updates its position at least at a time represented by one of the elements of the set $\{t, t+1, ..., t+B-1\}$, and from Equation (12), we have

$$0 \leq t - \tau_1^2(t) < B, \text{ for } t \in T^2, t \geq 0$$

which means member 2's delay in knowing about member 1 can be as large as $B - 1$.

In order to show the cohesiveness of the first two members, we will show that member 2 can keep the distance from member 1 in the range of comfortable distance neighborhood even in the "worst" case. Here, the worst case means that member 1 will update its position with a maximum possible step r in every time index in T (i.e., when $T^1 = T$), and member 2 only updates its position at the last element of $\{t, t+1, ..., t+B-1\}$ for $t \in T^2, t \geq 0$ (i.e., at $t+B-1$). Moreover, in the worst case, member 2's sensed information about member 1 has the maximum delay $B-1$. So in the worst case we have member 2 senses member 1's position as

$$x^1(\tau_1^2(t)) = x^1(t-B-1), t \in T^2 \tag{48}$$

Assume when $t = 0$, the initial position of member 1 is $x^1(0)$, so member 2's initial position is $x^2(0) = x^1(0) + d$. Considering the worst case above, in the first time set $\{0, 1, ..., B-1\}$ from $t = 0$, the distance between members 1 and 2 increases as the edge-leader moves with a step r. When $t = B-1$, i.e, the last time index of the first time set $\{0, 1, ..., B-1\}$, the leader member 1's position will be

$$x^1(t) = x^1(0) - (B-1)r$$

since it updates its position with a step r in every time index. From Equation (48), member 2's sensed information about member 1 at $t = B-1$ is

$$x^1(\tau_1^2(B-1)) = x^1(0)$$

Hence, member 2 remains stationary at $x^1(0) + d$ at this time index since it thinks it is still at the comfortable distance to member 1. In fact, their actual distance at this time is

$$e^1(B-1) = x^2(B-1) - x^1(B-1) = x^1(0) + d - (x^1(0) - (B-1)r) = d + (B-1)r$$

Next, in the worst case, member 2 will move at the last possible time and have the oldest possible information about the position of member 1. Hence, in the second time set $\{B, B+1, ..., 2B-1\}$ from $t = 0$, when $t = B$, member 1's position will be

$$x^1(t) = x^1(0) - Br$$

and member 2 still remains stationary at $x^1(0) + d$ since in the worst case it is not the time for it to update its position. Their distance now is

$$e^1(B) = x^1(0) + d - (x^1(0) - Br) = d + Br$$

From time index B to $2B-1$, the distance between members 1 and 2, $e^1(t)$, will increase by r in each time step since member 1 keeps updating by steps of size r and member 2 remains stationary. When $t = 2B-1$, member 1's position will be

$$x^1(2B-1) = x^1(0) - (2B-1)r$$

Member 2 stays at the position $x^1(0) + d$ at $t = 2B-1$, and will update its position according to

$$x^2(t+1) = x^2(t) - g_f(x^2(t) - x^1(\tau_1^2(t)) - d) \tag{49}$$

and then, the distance between them at $t = 2B-1$ is

$$e^1(2B-1) = d + (2B-1)r$$

which is the maximum distance achieved during the times $\{B, B+1, ..., 2B-1\}$. The minimum distance is $d + Br$ when $t = B$. From Equation (48), we know

$$x^1(\tau_1^2(2B-1)) = x^1(B) = x^1(0) - Br$$

and so, the argument of g_f in Equation (49) is

$$x^2(2B-1) - x^1(\tau_1^2(2B-1)) - d = x^1(0) + d - (x^1(0) - Br) - d = Br \tag{50}$$

From Equation (47) and the fact that $N > 1$, $B \in Z^+$, we have

$$Br \leq (2B-1)r \leq (NB-1)r \leq \gamma \tag{51}$$

Since we choose $\gamma = \eta$, from Equations (50), (51), and the definition of g_f, we then have

$$g_f(x^2(2B-1) - x^1(\tau_1^2(2B-1)) - d) = x^2(2B-1) - x^1(\tau_1^2(2B-1)) - d = Br$$

So

$$x^2(2B) = x^2(2B-1) - Br = x^1(0) + d - Br \tag{52}$$

In the third time set $\{2B, 2B+1, ..., 3B-1\}$, when $t = 2B$, member 1's position will be

$$x^1(t) = x^1(0) - 2Br$$

from Equation (52), we have

$$e^1(2B) = x^2(2B) - x^1(2B) = x^1(0) + d - Br - (x^1(0) - 2Br) = d + Br$$

The inter-member distance decreases back to $d + Br$ due to the position update of member 2, and then it will repeat the above distance increasing process. When $t = 3B - 1$, it reaches the maximum distance $d + (2B-1)r$ in this time set.

We find that in the worst case the distance between members 1 and 2 will change from the minimum $d + Br$ to the maximum $d + (2B-1)r$ periodically and the period is B. Thus, we can conclude that the maximum possible inter-member distance for members 1 and 2 is $d+(2B-1)r$ during the moving process.

Next, we try to find the maximum possible inter-member distance between members 2 and 3 so that a special case (that is different from the worst case for members 1 and 2 above) is considered as follows. As we know, in the above worst case for members 1 and 2, when $t = 2B - 1$,

$$x^1(2B-1) = x^1(0) - (2B-1)r$$

and

$$x^2(2B-1) = x^1(0) + d$$

At this time, member 3 stays at the position of $x^1(0)+2d$. Now different from above, we assume that since $t = 2B-1$, member 2 gets the position information of member 1 without any delay (i.e., $x^1(\tau_1^2(2B-1)) = x^1(2B-1) = x^1(0) - (2B-1)r$) and member 1 still updates its position with a maximum possible step r in every time index in T. Moreover, member 3 only updates its position at the last element of $\{t, t+1, ..., t+B-1\}$ for $t \in T^3, t \geq 0$ (i.e., at $t + B - 1$). And then, when $t = 2B$,

$$x^1(2B) = x^1(0) - 2Br$$

From Equation (47) and the fact that $N > 1$

$$x^2(2B-1) - x^1(\tau_1^2(2B-1)) - d = x^1(0) + d - (x^1(0) - (2B-1)r) - d = (2B-1)r \leq (NB-1)r \leq \gamma$$

(note that this is different from Equation (51) above since for this case we assume that there is no delay in member 2 getting member 1's position information). From the definition of g_f, we then have

$$g_f(x^2(2B-1) - x^1(\tau_1^2(2B-1)) - d) = x^2(2B-1) - x^1(\tau_1^2(2B-1)) - d = (2B-1)r$$

Therefore, from Equation (49) we get

$$x^2(2B) = x^1(0) + d - (2B-1)r$$

and the distance between members 1 and 2 at $t = 2B$ is

$$e^1(2B) = x^2(2B) - x^1(2B) = d + r$$

which is clearly smaller than the worst case when we consider only members 1 and 2 above. Here, we are creating a worst case for members 2 and 3, which is different from the worst case for members 1 and 2. Next, note that member 3 remains at the position of $x^1(0) + 2d$ since the distance between members 2 and 3 is equal to d at time $t = 2B - 1$. And so, their distance at $t = 2B$ is

$$e^2(2B) = x^3(2B) - x^2(2B) = x^1(0) + 2d - (x^1(0) + d - (2B-1)r) = d - (2B-1)r$$

We further assumed members 1 and 2 will update their positions synchronously (i.e., they have the same update time sets and member 2 obtains the position information of member 1 without any delay) after $t = 2B$ so that they will keep their distance as d past the time $t = 2B + 1$. Consequently, in the following time set $\{2B+2, ..., 3B-2\}$, members 1 and 2 will update their positions with a maximum possible step r synchronously while maintaining an inter-member distance d. During this time period, member 3 remains at the position of $x^1(0) + 2d$ since in order to create the worst case, we assume it only updates its position at the last element of $\{2B, 2B+1, ..., 3B-1\}$ (i.e., $t = 3B - 1$). Therefore, the distance between members 2 and 3 will keep increasing until $t = 3B - 1$.

When $t = 3B - 1$,

$$x^1(3B-1) = x^1(0) - (3B-1)r$$

and

$$x^2(3B-1) = x^1(0) + d - (3B-1)r$$

So, the distance between members 1 and 2 is

$$e^1(3B-1) = x^2(3B-1) - x^1(3B-1) = d$$

Furthermore, member 3 remains at the position $x^1(0) + 2d$ and will update its position according to its sensed information of member 2, so that the inter-member distance of members 2 and 3 at this time

$$e^2(3B-1) = x^3(3B-1) - x^2(3B-1) = (x^1(0) + 2d) - (x^1(0) + d - (3B-1)r) = d + (3B-1)r$$

which is the maximum possible distance in this time period. Notice that this maximum possible inter-member distance of members 2 and 3 is larger than the maximum possible inter-member distance for members 1 and 2, which is $d + (2B-1)r$. In fact, this is also the maximum possible inter-neighbor distance of members 2 and 3 in all their updating time sets T^2 and T^3. The reason is that this distance will decrease at $t = 3B$ due to the position updating of member 3, and then in the future time indices, even in the worst case, the maximum possible distance between members 2 and 3 will only be equal to $d + (2B-1)r$. Here, the worst case is similar to that of members 1 and 2 we discussed above, where members 1 and 2 keep updating synchronously their positions with a maximum possible step r in every time index, and member

3 has the maximum sensing delay $B-1$ about the position member 2 and can only updates its position at the last element of $\{t, t+1, ..., t+B-1\}$ for $t \in T^3, t \geq 0$.

In the same way, we can find that the maximum possible inter-member distance between members $N-1$ and N is $d+(NB-1)r$ when $t = NB-1$, which is the largest of all possible inter-neighbor distances in the time set T. Hence, we conclude that the maximum possible inter-neighbor distance for N members is $d+(NB-1)r$.

From Equation (47), we have

$$d + (NB-1)r \leq d + \gamma$$

and from Equation (15), we then have

$$d \leq e^i(t) \leq d + (NB-1)r \leq d + \gamma, \text{ for } i = 1, 2, ..., N-1$$

which means all members will always be in the comfortable distance neighborhood $[d, d+\gamma]$ with their neighbors. So all members can keep the distance from their neighbors in the range of comfortable distance neighborhood even in the worst case. **Q.E.D.**

Remark 3: Note the following about Equation (47):

- For a given B and γ, it provides bound on how fast a N-member swarm can move and still maintain the type of cohesiveness characterized by γ. For example, increases in swarm size, communication delays, or swarm cohesiveness (smaller γ) require decreases in the rate of movement of the leader.
- For a given r and B, it provides the size of the neighborhood that will be maintained and hence specifies a degree of cohesiveness of a N-member swarm.
- For a given r and γ, it provides constraints on how to design a communication system (i.e., what is needed for B) for a N-member swarm between swarm members and indicates how often they must update their positions.

Remark 4: From Theorems 1 and 2, we can see that if the edge-leader stops moving (i.e., $s(t) = 0, t \geq t^1$ for some $t^1 \in T^1$), all other $N-1$ members will converge to be adjacent to the edge-leader with a comfortable distance d.

3.2.2 Alternative Convergence Conditions

Now we consider the case of using the g_a function in Equation (16), as shown in Figure 5. Intuitively, we can see that for some g_a function it is possible that the distance between two swarm neighbors will diverge since the following member's update step g_a cannot keep up with the moving rate of the leading member. Hence, not all g_a functions can make asynchronous mobile swarms maintain their cohesiveness during movements. However, we can add some constraints on g_a function to create a new function, say g_b, which will result in asynchronous mobile swarms maintaining their cohesiveness under some conditions on leader's moving step bound, the partial asynchronism measure, and the comfortable distance neighborhood size.

Assume that for some scalars β and b, such that $\beta > 1$, and $b > 0$, $g_b(e^i(t) - d)$ is such that

$$(e^i(t) - d - b) \leq g_b(e^i(t) - d) < (e^i(t) - d), \text{ if } (e^i(t) - d) > \frac{\beta b}{\beta - 1} \tag{53}$$

$$\frac{1}{\beta}(e^i(t) - d) < g_b(e^i(t) - d) < (e^i(t) - d), \text{ if } 0 < (e^i(t) - d) \leq \frac{\beta b}{\beta - 1} \tag{54}$$

$$g_b(e^i(t)-d) = (e^i(t)-d) = 0, \text{ if } (e^i(t)-d) = 0 \tag{55}$$

$$(e^i(t)-d) < g_b(e^i(t)-d) < \frac{1}{\beta}(e^i(t)-d), \text{ if } -\frac{\beta b}{\beta-1} \leq (e^i(t)-d) < 0 \tag{56}$$

$$(e^i(t)-d) < g_b(e^i(t)-d) \leq (e^i(t)-d+b), \text{ if } (e^i(t)-d) < -\frac{\beta b}{\beta-1} \tag{57}$$

As shown in Figure 5, these relationships are similar to those for g_a except that there are two new bounds for the g_b function in Equations (53) and (57), which guarantee the following members are in the $b-$neighborhood of desired comfortable distance of their leading neighbors after each update step so that the following members can keep up with the movements of their leading neighbors. Similarly, we will show that with the g_b function in Equation (16), all members can also be in a comfortable distance neighborhood from their neighbors during movements under some constraints on leader's moving step bound, the partial asynchronism measure, the comfortable distance neighborhood size, and the parameters of the g_b.

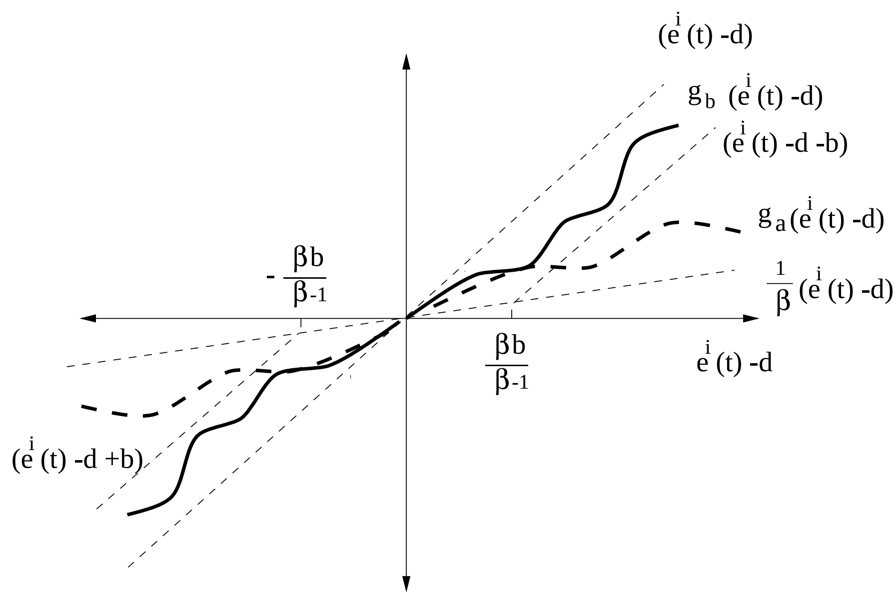

Figure 5: The function $g_a(e^i(t)-d)$ (dashed) and $g_b(e^i(t)-d)$ (solid).

Theorem 4. *For an N-member asynchronous mobile swarm modeled by Equation (16), where g is g_b, $N > 1$, Assumption 2 (partial asynchronism) holds, and $x^{i+1}(0) - x^i(0) = d$, $i = 1, 2, ..., N-1$, if*

$$0 < r \leq \frac{\gamma - b}{NB - 1} \tag{58}$$

for a given γ, all the swarm members will be in the comfortable distance neighborhood $[d, d+\gamma]$ of their neighbors during the moving process, where r is the upper bound of the edge-leader's moving step $s(t)$, $B \in Z^+$ is the partial asynchronism measure, γ is the comfortable distance neighborhood size, and b ($0 < b < \gamma$) is the parameter of g_b function.

Proof. Similar to the proof of Theorem 3, we first analyze the worst case of member 1 and member 2 we defined in the proof of Theorem 3. We find that in the worst case the distance between member 1 and 2 will change between the minimum interval $[d + Br, d + Br + b]$ to the maximum interval $[d + (2B - 1)r, d + (2B-1)r + b]$ periodically and the period is B. Thus, we can conclude that the maximum possible inter-member distance for member 1 and 2 is $d + (2B-1)r + b$ during the moving process.

Next, we find the bound of all possible inter-member distance of members 2 and 3 is $d+(3B-1)r+b$. In the same way, we can find that the bound for possible inter-member distance between members $N-1$ and N is $d+(NB-1)r+b$, which is actually the largest bound of all possible inter-neighbor distances in the time set T. Hence, we conclude that a bound for all possible inter-neighbor distance in a N-member mobile asynchronous swarm is $d+(NB-1)r+b$.

From Equation (58), we have $d+(NB-1)r+b \leq d+\gamma$ and according to Equation (15), we then have

$$d \leq e^i(t) \leq d+(NB-1)r+b \leq d+\gamma, \text{ for } i=1,2,...,N-1$$

which means all members in a N-member mobile swarm will always be in the comfortable distance neighborhood $[d, d+\gamma]$ with their neighbors. So all members can keep the distance from their neighbors in the range of comfortable distance neighborhood even in the worst case. **Q.E.D.**

3.3 Convergence Analysis of Asynchronous Mobile Swarms Pushed by an Edge-Leader

Next, we provide different conditions under which an N-member asynchronous mobile swarm pushed by an edge-leader can maintain cohesiveness during movements on the basis of the model in Equation (17). While we only consider a right-edge member pushing a swarm to the left, by symmetry, the case where the left-edge member pushes the mobile swarm to the right is the same. We can conclude the following two theorems via the similar analysis ideas above on the basis of the model in Equation (17) with the g_f and g_b functions, respectively.

Theorem 5. *For an N-member asynchronous mobile swarm modeled by Equation (17), where g is g_f, $N>1$, Assumption 2 (partial asynchronism) holds, $0<\gamma<d-\varepsilon$, and $x^{i+1}(0)-x^i(0)=d$, $i=1,2,...,N-1$, if*

$$0 < r \leq \frac{\gamma}{NB-1} \tag{59}$$

for a given γ, all the swarm members will be in the comfortable distance neighborhood $[d-\gamma, d]$ of their neighbors during the moving process, where r is the upper bound of the edge-leader's moving step $s(t)$, $B \in Z^+$ is the partial asynchronism measure, γ (choose $\gamma=\eta$, and $0<\eta<\varepsilon-w$) is the comfortable distance neighborhood size, and ε is the sensing range of proximity sensors.

Theorem 6. *For an N-member asynchronous mobile swarm modeled by Equation (17), where g is g_b, $N>1$, Assumption 2 (partial asynchronism) holds, $0<\gamma<\min\{d-\varepsilon, \varepsilon-w\}$, and $x^{i+1}(0)-x^i(0)=d$, $i=1,2,...,N-1$, if*

$$0 < r \leq \frac{\gamma-b}{NB-1} \tag{60}$$

for a given γ, all the swarm members will be in the comfortable distance neighborhood $[d-\gamma, d]$ of their neighbors during the moving process, where r is the upper bound of the edge-leader's moving step $s(t)$, $B \in Z^+$ is the partial asynchronism measure, γ is the comfortable distance neighborhood size, b ($0<b<\gamma$) is the parameter of g_b function, and ε is the sensing range of proximity sensors.

Here, we have $0<\gamma<\varepsilon-w$ so that swarm neighbors move only according to the corresponding g function when their distance is already inside γ-neighborhood of the comfortable distance. The proof of these two theorems is similar to that of Theorems 3 and 4, where we analyze the worst case to find the minimum or the lower bound of all possible inter-member distances and show that this minimum or lower bound is inside the comfortable distance neighborhood $[d-\gamma, d]$ under the constraint in Equation (59) or Equation (60).

4 Conclusions

We construct mathematical models of one-dimensional asynchronous swarms by putting N identical single swarm members together. We allow for finite-size swarm members and ensure collision-free swarming in all of our analysis. We show that for one-dimensional stationary edge-member asynchronous swarms, total asynchronism leads to asymptotic convergence and partial asynchronism leads to finite time convergence. Moreover, we provide conditions under which an asynchronous mobile swarm following (pushed by) a edge-leader can keep the cohesiveness during movements even in the presence of delays and partial asynchronism based on our model with different g functions. In particular, all members in a N-member mobile swarm will be in a comfortable distance neighborhood $[d, d+\gamma]$ ($[d-\gamma, d]$) from their neighbors while the swarm moves if there are constraints on leader's moving step bound r, the partial asynchronism measure B, and the comfortable distance neighborhood size γ. In this way, we have shown conditions under which a one-dimensional asynchronous swarm in different cases can maintain cohesion even in the presence of delays and asynchronism. Notice that the conditions obtained from our analysis provide extra constraints on the low level vehicle dynamics when we consider designing an actual vehicular swarm.

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Appendix: Simulation Studies[1]

Here, we will provide some simulation examples to illustrate convergence properties of the one-dimensional asynchronous swarms we discussed earlier. First, we will simulate the swarm in Figure 2 converging to be adjacent to a stationary left-edge member under the partial asynchronism assumption in some finite time, which is summarized in Theorem 2. Then, we will give an example of a mobile swarm following an edge-leader on the real line and show that it can maintain its cohesiveness during movements under the conditions in Theorems 3 and 4, respectively. In the simulation, let $T = \{0, 1, 2, ...\}$ represent the indices of the sequence of real times. For convenience, we assume it corresponds to the real time set $\{0, 0.1, 0.2, 0.3, 0.4, ...\}$ on a uniform grid of size 0.1 sec at which one or more swarm members update their positions. And we randomly select the time index set $T^i \subseteq T, i = 1, 2, ..., N$, at which the i^{th} member's position $x^i(t), t \in T^i$, is updated. The T^i, $i = 1, 2, ..., N$, are independent of each other for different i. However, they may have intersections so that two or more swarm members may move simultaneously. Moreover, in order to satisfy the partial asynchronism assumption, we assume $B = 4$ and add constraints to the updating time index set T^i to guarantee each member updates at least once in the B time index interval and the index delays in obtaining neighbor positions are bounded by B.

A Stationary Edge Member Asynchronous Swarms Simulation

Assume we have a one-dimensional 5-member asynchronous swarm and initially (i.e, $t = 0$ sec), five swarm members with a physical size $w = 1$, members $1, 2, 3, 4$, and 5 are at the positions of $0, 2, 4, 22$, and 24 from left to right on the real line respectively, as shown in Figure 6. We assume the comfortable distance $d = 5$, and the sensing range of proximity sensors $\varepsilon = 1.75$. All members will update their positions in their updating time sets T^i except the left edge member (member 1) remains stationary at position 0. Assume the partial asynchronism assumption holds for this swarm with $B = 4$ and we choose a g_f function with a $\eta = 0.25$ satisfying Equations (3), (4), and (5) to define the attractive and repelling relationship. In particular, $g_f(e^i(t) - d) = e^i(t) - d$ if $|e^i(t) - d| \leq \eta$, and $g_f(e^i(t) - d) = 0.4(e^i(t) - d)$ if $|e^i(t) - d| > \eta$. Here, we choose $\beta = 10000$ since $|g_f(e^i(t) - d)| = 0.4|(e^i(t) - d)| > \frac{1}{\beta}|(e^i(t) - d)|$ is required if $|e^i(t) - d| > \eta$, where $\beta > 1$ and also we want to allow a very small movement at any step. With all the above conditions, we get the finite-time convergence according to Theorem 2.

The results of the simulation are given by providing six plots of swarm member positions from $t = 0$ sec to $t = 9.9$ sec as shown in Figure 6. In the $t = 0.4$ sec plot, member 4 moves towards member 3 due to their attractive relationship, and member 2 is repelled by member 1, but prevented by member 3. At the same time, member 5 moves to the right because its current distance to member 4 is less than d. In the $t = 1.5$ sec plot, member 3 updates its position to the right due to the repelling relationship with its left neighbor member 2, and member 2 arrives at its desired position 5, which is at a comfortable distance to member 1. In the last two plots, all members are already at a comfortable distance 5 to their neighbors and remain stationary at their desired positions. We provide all inter-member distances of neighbors during the convergence process in Figure 7. Clearly, all inter-neighbor distances are larger than 0 (i.e., there are no collisions) and gradually converge to the comfortable distance 5. Moreover, it is interesting to note that the inter-member distances do *not* asymptotically decrease at each step; sometimes the inter-member distances can increase then later decrease.

[1]This appendix is added simply to provide some additional background. We do not envision publishing this appendix with the rest of the paper due to page length restrictions. We would like to simply post it on the web, along with the code used to generate the examples.

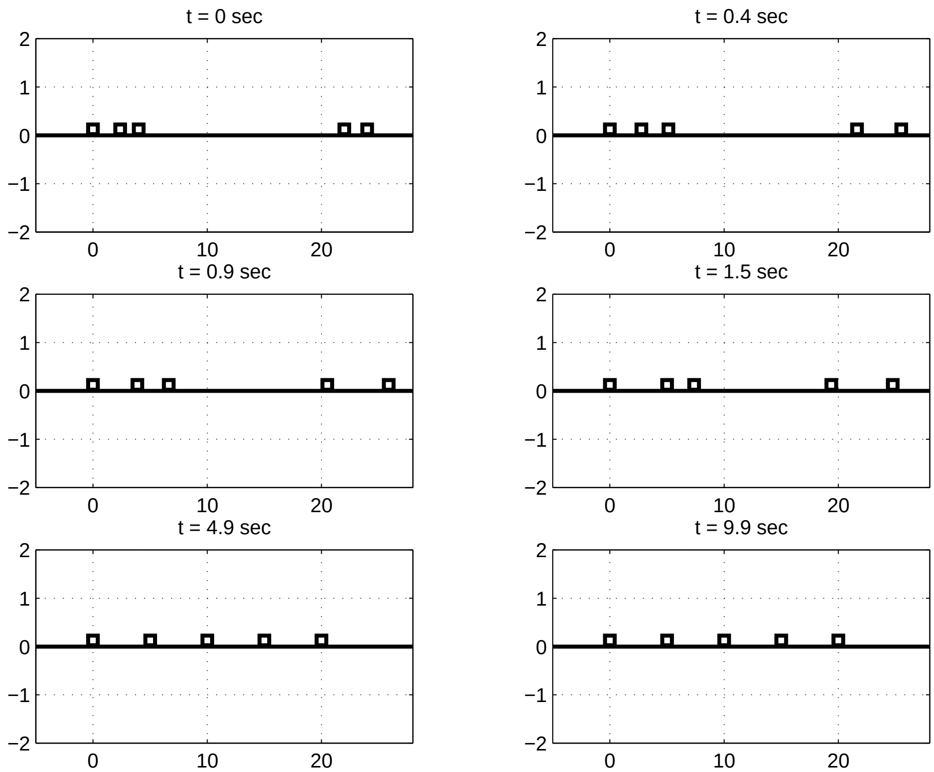

Figure 6: One-dimensional asynchronous 5-member swarm converging behavior simulation.

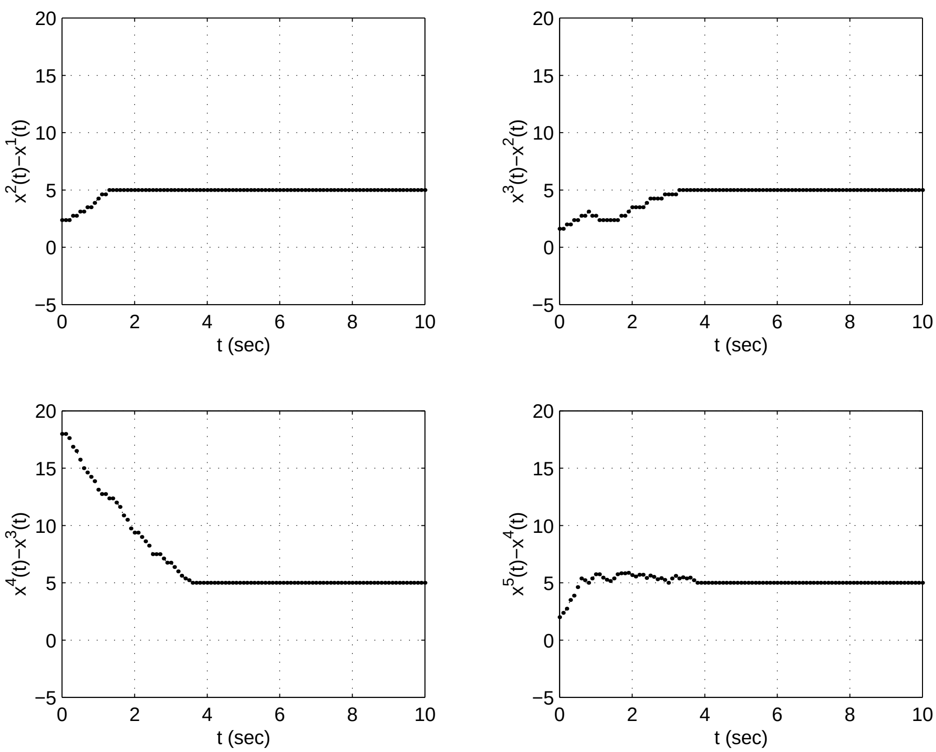

Figure 7: Inter-member distances of neighbors during the convergence process.

B Mobile Asynchronous Swarms Simulation Study

Assume we have a one-dimensional 5-member asynchronous mobile swarm and at the beginning five swarm members with a physical size $w = 1$, members $1, 2, 3, 4$, and 5 are at the positions of $5, 10, 15, 20$, and 25

from left to right on the real line respectively, as shown in Figure 8. We assume the comfortable distance $d = 5$, and the comfortable distance neighborhood size $\gamma = 1$. The left edge member (the leader) keeps moving to the left with a bounded step and all other members follow their left neighbor. We use the same g_f function as above. According to Theorem 3, the edge-leader's moving step is bounded by r, where $0 < r \leq \frac{\gamma}{NB-1} = 0.0526$, in order for the asynchronous mobile swarm to maintain cohesiveness, i.e., all mobile swarm members are at a comfortable distance neighborhood $[5, 6]$ from their neighbors while the swarm moves. Hence, we choose $r = 0.0526$.

The results of the simulation are given by providing six plots of swarm member positions from $t = 0$ sec to $t = 19.9$ sec as shown in Figure 8. We found that all mobile swarm members maintain a distance inside the comfortable neighborhood range $[5, 6]$ from their neighbors in all time indices as shown in Figure 9. Clearly, there are no collisions during the moving process.

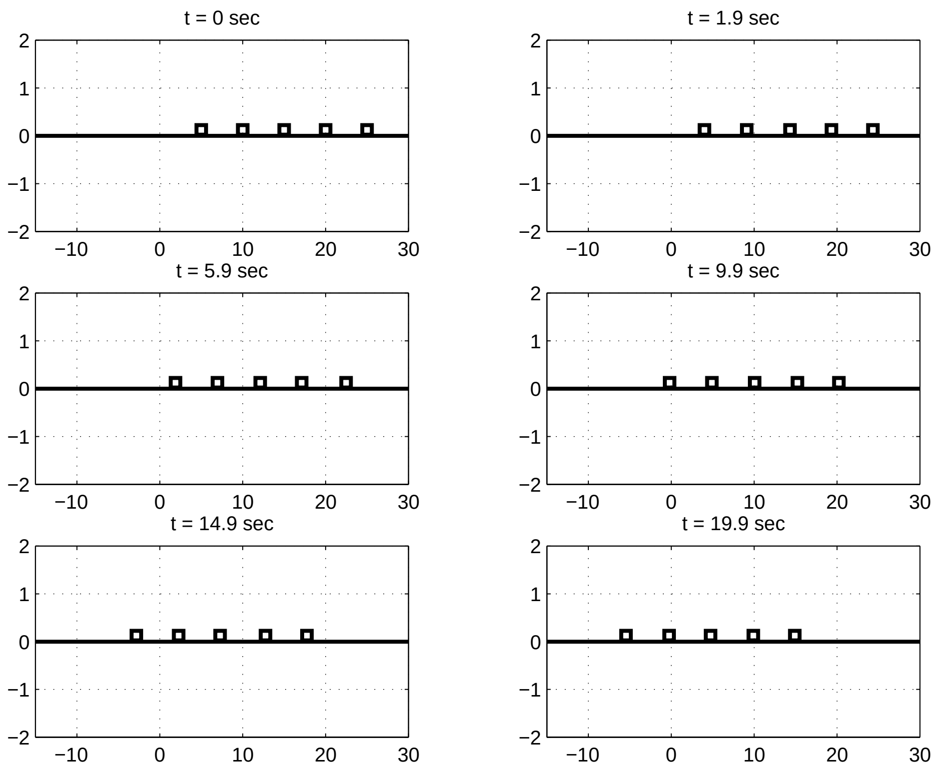

Figure 8: One-dimensional asynchronous 5-member mobile swarm behavior, following a left edge-leader (with convergence conditions in Theorem 3).

In addition, we gave alternative convergence conditions in Theorem 4. We will study the same swarm described above, but use g_b function to represent the attractive and repelling relationship. We choose the g_b function satisfying Equations (53), (54), (55), (56) and (57) to define the inter-member attractive and repelling relationship. In particular, $g_b(e^i(t)-d) = 0.4(e^i(t)-d)$ if $|e^i(t)-d| \leq \frac{\beta b}{\beta-1}$, $g_b(e^i(t)-d) = e^i(t)-d-b$ if $e^i(t) - d > \frac{\beta b}{\beta-1}$, and $g_b(e^i(t) - d) = e^i(t) - d + b$ if $e^i(t) - d < -\frac{\beta b}{\beta-1}$, where $b = 0.5, \beta = 10000$ (we choose $b = 0.5$ since $0 < b < \gamma = 1$ is required; we choose $\beta = 10000$ for the similar reason as above). All other parameters are the same as above. According to Theorem 4, the edge-leader's moving step is bounded by r, where $0 < r \leq \frac{\gamma-b}{NB-1} = 0.0263$. Hence, we choose $r = 0.0263$. Notice that this indicates that the leader must move much slower since now we use the g_b function, which can only keep the following members in the $b-$neighborhood of desired comfortable distance of their leading neighbors.

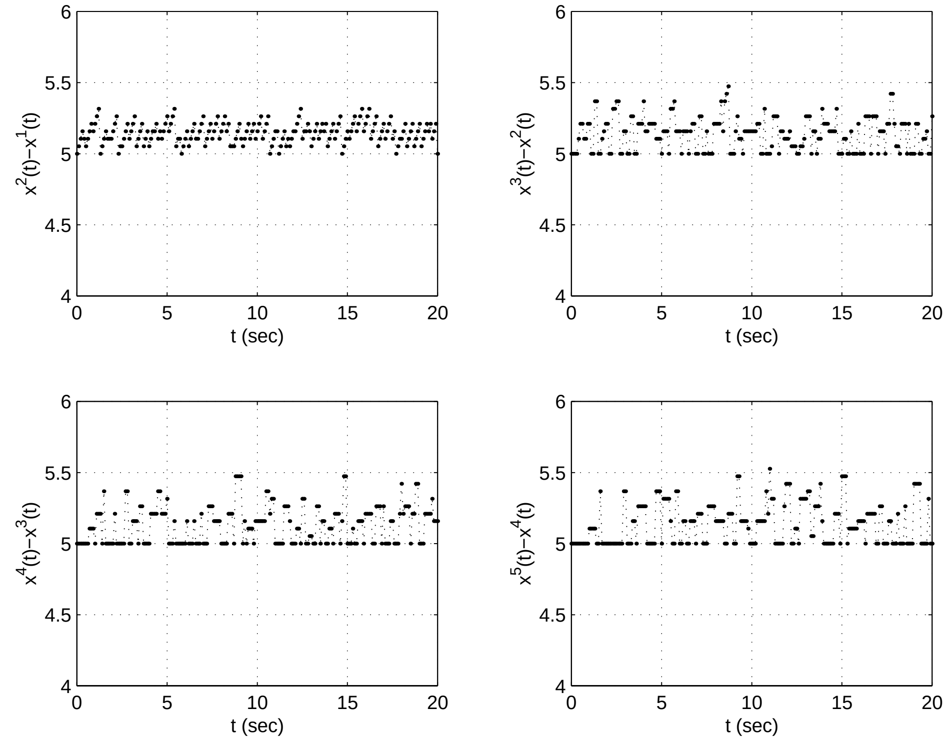

Figure 9: Inter-member distances of neighbors during movements (with convergence conditions in Theorem 3).

Similarly, the results of the simulation are shown in Figure 10 and all mobile swarm members maintain a distance inside the comfortable neighborhood range $[5, 6]$ from their neighbors in all time indices as shown in Figure 11. No collision occurs during the moving process. As we discussed above, the swarm in Figure 10 moves slower than the swarm in Figure 8 and the changing pattern of inter-neighbor distances of two swarms is also different because a different g function is used. Particularly, the g_f function can make the following members keep the comfortable distance with their leading neighbors under some conditions. However, under the same conditions the g_b function only keep the following members in the $b-$neighborhood of comfortable distance of their leading neighbors.

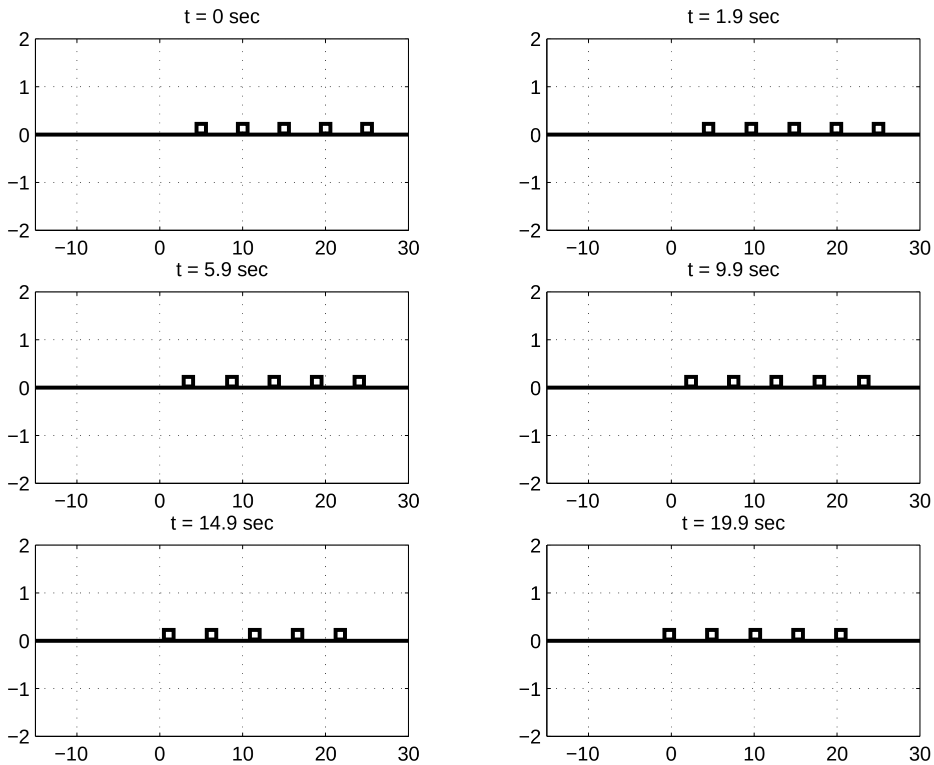

Figure 10: One-dimensional asynchronous 5-member mobile swarm behavior, following a left edge-leader (with alternative convergence conditions in Theorem 4).

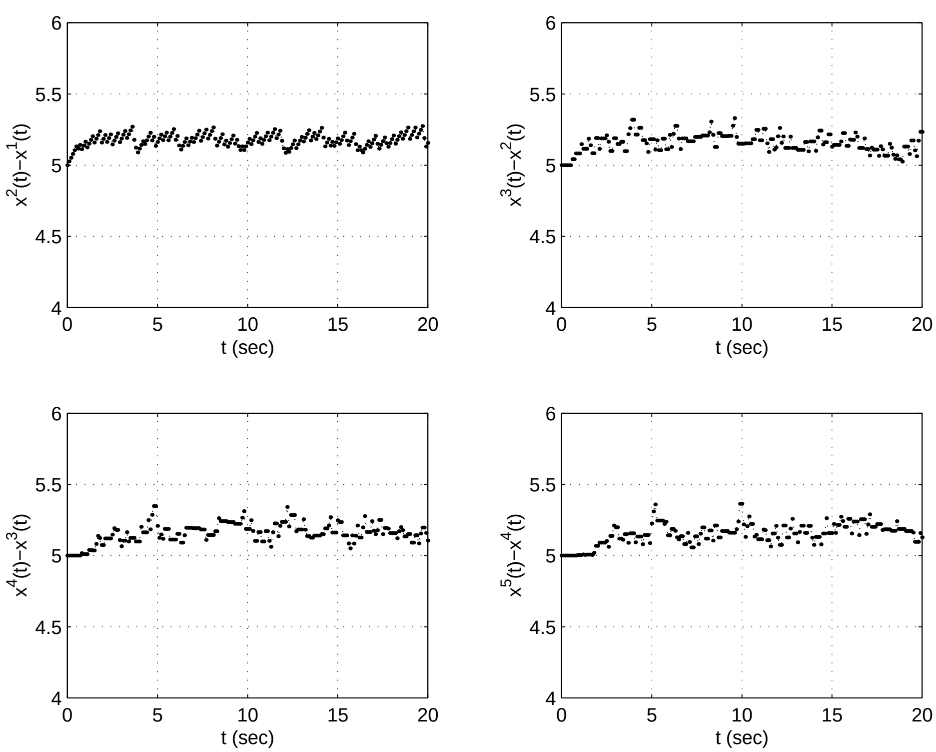

Figure 11: Inter-member distances of neighbors during movements (with alternative convergence conditions in Theorem 4).