Robust Output Feedback Control of Incompletely Observable Nonlinear Systems Without Input Dynamic Extension*

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Abstract

We introduce a new output feedback controller for general MIMO nonlinear systems which are only observable on regions of the state and input spaces. Unlike previous approaches, we do not add integrators at the input side of the system, and thus avoid the need to design a stabilizing control law for a higher order system. Robustness with respect to a class of time-varying disturbances is guaranteed, and the performance of any state feedback controller designed to achieve closed-loop stability with respect to a set (e.g., a robust controller) is recovered.

1 Introduction

The past decade has witnessed a growing interest in the output feedback control problem and the generalization and exploitation of a nonlinear separation principle. Given a stabilizing state feedback controller and a state observer designed independent of each other, under which conditions will the interconnection of these two guarantee closed-loop stability? The work by Esfandiari and Khalil [1] laid the foundations for a new approach to solve this problem that was later employed and extended by other researchers and by Khalil and colleagues ([2, 3, 4, 5], to cite a few) to the point of formulating a separation principle for a rather general class of nonlinear systems in [2, 6]. The main ingredients of this approach can be summarized as follows:

1. A chain of integrators is added at the input side of the system, and the state feedback control law is redesigned to stabilize the "extended system."

2. A high-gain observer is designed to estimate the derivatives of the system output and, together with the integrator states, these are used to obtain an estimate of the state of the system.

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3. This estimate is employed in the state feedback control law and control saturation is used to guarantee its global boundedness.

Unfortunately, this approach cannot be used when the system is not uniformly completely observable (UCO), i.e., when the system is observable only on regions of the state and input spaces. For example, the Moore-Greitzer model describing the dynamics of stall and surge in jet engine compressors is not observable when there is no mass flow through the engine, and there currently seems to be no solution for this engine output feedback control problem based on a separation principle.

In [7, 8] we introduced a new methodology for solving this class of more general output feedback control problems. This technique replaces points 2 and 3 above with the following:

2'. A nonlinear state observer that depends on the integrators' states is designed to directly estimate the system states.

3'. The state estimate is modified via a projection method so that it lies within an observable region and guarantees global boundedness when it is employed in the closed-loop.

In [7, 8] we proved that by using this technique a separation principle similar to the one formulated in [6] is achieved and, in [9, 10], this methodology was applied to derive a solution to the stall and surge control problem described above.

In this work we extend the theory developed in [7, 8] to deal with MIMO nonlinear systems affected by time-varying disturbances, and for which a state feedback control law can be designed that robustly stabilizes the system (hence, a suitable compact set $\mathcal{N}$ is made attractive and positively invariant by the state feedback control law; notice the similarity with the work in [11]). A major difference between this work and other approaches, including our previous results, is the substitution of point 1 above with the following:

1'. A high-gain (or, alternatively, sliding mode) observer is employed for the estimation of the control input derivatives (or, equivalently, the states of the integrators used in the input dynamic extension in point 1).

Essentially, rather than adding a chain of integrators at the input side of the system, we use an observer to estimate the control input derivatives (hence, our observer has higher order but the overall controller order is the same). This modification results in a simplification of the control design problem in that the original state feedback controller can be *directly* employed by the output feedback controller (i.e., , we eliminate the need to design a control law for the higher-order "extended system" in point 1 above). We prove that, in the presence of disturbances, this approach guarantees uniform ultimate boundedness (UUB) of the closed-loop system trajectories with respect to $\mathcal{N}$ (in complete analogy with the result of Theorem 4 in [11], but here we do not investigate conditions needed on the disturbance so that asymptotic stability is recovered). When no disturbance affects the system, it is shown that a design that is based on a sliding mode observer recovers the asymptotic stability of the closed-loop system with respect to $\mathcal{N}$, while the high-gain observer achieves UUB.

In summary, via the results in [7, 8] and here we replace the procedure given by 1,2 and 3 that is studied in [2, 6] by the procedure given by 1', 2', and 3'. In this way we can consider non-UCO systems, avoid the explicit use of the inverse of the observability mapping, and simplify the design process for the state feedback controller needed by all these approaches that are based on a separation principle.

2 Problem Formulation

Consider the following dynamical system,

$$\begin{aligned} \dot{x} &= f(x,u,d) \\ y &= h(x,u) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$, f and h are sufficiently smooth functions, and $d \in \Delta \subset \mathbb{R}^q$ is an unknown time-varying disturbance contained in a compact set Δ. When no disturbance acts on the plant we replace $d(t)$ by $d^0 \in \Delta$, a known constant "nominal value" of $d(t)$. Our objective is to construct a robust output feedback controller which makes a known compact set uniformly asymptotically stable with a desired region of attraction. The observability properties of system (1) are characterized by the observability mapping

$$y_e \triangleq \begin{bmatrix} y_1 \\ \dot{y}_1 \\ \vdots \\ y_1^{(k_1-1)} \\ y_2 \\ \vdots \\ y_2^{(k_2-1)} \\ \vdots \\ y_p \\ \vdots \\ y_p^{(k_p-1)} \end{bmatrix} = \begin{bmatrix} h_1(x,u) \\ \varphi_1^1(x,u,u^{(1)},d) \\ \vdots \\ \varphi_1^{k_1-1}\left(x,u,\ldots,u^{(k_1-1)},d,\ldots,d^{(k_1-2)}\right) \\ h_2(x,u) \\ \vdots \\ \varphi_2^{k_2-1}\left(x,u,\ldots,u^{(k_2-1)},d,\ldots,d^{(k_2-2)}\right) \\ \vdots \\ h_p(x,u) \\ \vdots \\ \varphi_p^{k_p-1}\left(x,u,\ldots,u^{(k_p-1)},d,\ldots,d^{(k_p-2)}\right) \end{bmatrix} \triangleq \mathcal{H}\left(x,z,d,\ldots,d^{(n_d-1)}\right) \tag{2}$$

where $z \triangleq [u_1,\ldots,u_1^{(n_1-1)},\ldots,u_m,\ldots,u_m^{(n_m-1)}]^\top \in \mathbb{R}^{n_u}$, $\sum_{i=1}^p k_i = n$, $n_u \triangleq n_1+\ldots+n_m$, $0 \le n_i \le \max\{k_1,\ldots,k_p\}$, $0 \le n_d \le \max\{k_1,\ldots,k_p\}-1$ (when $\mathcal{H}$ does not depend on u_i or d then, respectively, we set $n_i = 0$, or $n_d = 0$). Note that the vector z contains only the derivatives of u that end up appearing in the mapping $\mathcal{H}$ for the application at hand. The functions φ_i^j are defined as follows,

$$\begin{aligned} \varphi_i^j(x,u,\ldots,u^{(j)},d,\ldots,d^{(j-1)}) &= \frac{\partial \varphi_i^{j-1}}{\partial x} f(x,u,d(t)) + \sum_{k=0}^{j-1} \frac{\partial \varphi_i^{j-1}}{\partial u^{(k)}} u^{(k+1)} + \sum_{k=0}^{j-2} \frac{\partial \varphi_i^{j-1}}{\partial d^{(k)}} d^{(k+1)} \\ &\quad \text{for } i=1,\ldots,p,\ j=1,\ldots,k_i-1 \\ \varphi_i^0(x,u) &= h_i(x,u), \ \text{ for } i=1,\ldots,p \end{aligned}$$

We are now ready to state our main assumptions.

Assumption A1 (*Disturbance*). The vector fields f, h have the property that

$$\frac{\partial \varphi_i^j}{\partial d} = 0, \text{ for } i = 1, \dots, p, \; j = 1, \dots, k_i - 1, \tag{3}$$

on the domain of definition of φ_i^j. Therefore, (2) can be written as $y_e = \mathcal{H}(x, z)$.

Remark 1: If we view $d(t)$ as an input to system (1), this assumption implies that d has a relative degree (with respect to y) greater than or equal to n. Therefore, for this class of disturbances, $d(t)$ will not affect the output y_i and its $k_i - 1$ derivatives, for $i = 1, \dots, p$.

Assumption A2 (*Observability*). System (1) is observable over the set $\mathcal{X} \times \mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$, i.e., there exists a set of indices $\{k_1, \dots, k_p\}$ such that the mapping $y_e = \mathcal{H}(x, z)$ is invertible with respect to x and its inverse is smooth, for all $x \in \mathcal{X}$, $z \in \mathcal{U}$.

Remark 2: This assumption is a direct generalization of the observability assumption found in [7, 8] to the case of MIMO nonlinear systems. Notice that A2 relaxes analogous assumptions found in the literature (e.g., [12, 2, 6]), where the authors require UCO, i.e., $\mathcal{X} \times \mathcal{U} = \mathbb{R}^n \times \mathbb{R}^{n_u}$. Note that, in general, there may be more that one choice of $\{k_1, \dots, k_p\}$ satisfying Assumption A2. Different choices of these indices will affect the dependence of $\mathcal{H}$ on the derivatives of u differently. In practical applications we may want to find $\{k_1, \dots, k_p\}$ for which A1 holds and other desirable properties are achieved (e.g., the size of z is minimum so the dynamic order of the controller is reduced).

Assumption A3 (*Robust Dynamic Stabilizability*). There exists a dynamic state feedback controller

$$\begin{aligned} \dot{x}_c &= f_c(x_c, x) \\ u &= h_c(x_c, x) \end{aligned} \tag{4}$$

where $x_c \in \mathbb{R}^{n_c}$, f_c, h_c are sufficiently smooth, and such that $\mathcal{N} \subset \mathbb{R}^{n+n_c}$ is a positively invariant, uniformly asymptotically stable compact set for the closed-loop system

$$\begin{aligned} \dot{x}_c &= f_c(x_c, x) \\ \dot{x} &= f(x, h_c(x_c, x), d) \end{aligned} \tag{5}$$

for all $d(t) \in \Delta$. Let $\mathcal{D}$ be the domain of attraction of $\mathcal{N}$.

Remark 3: In [11] the authors use a similar assumption, with the additional requirement that f_c and h_c be globally bounded functions of x, a requirement that is not restrictive since it can be fulfilled by using saturation functions. On the other hand, owing to the structure of our output feedback controller, this restriction is not needed in our framework. Notice that, while in [11] $d(t)$ is a measurable exogenous signal, we assume $d(t)$ to be an unmeasurable disturbance (indeed we do not include $d(t)$ as an input to the control law (4)), and hence our control objective is somewhat different.

Next, define the closed-loop system

$$\begin{aligned} \dot{\chi} &= F(\chi, u, d(t)) \\ y &= H(\chi) \end{aligned} \tag{6}$$

where $\chi = [x_c^\top, x^\top]^\top$, the $\dot{\chi}$ equation is defined by (5), and $H(\chi) = h(x, h_c(x_c, x))$. Assumption A3 implies that $\mathcal{N}$ is positively invariant and a uniformly asymptotically stable set of (6) when $u = h_c(\chi)$.

3 Cascade Observer Designs

The output feedback controller design for UCO systems involves adding integrators at the input side and redesigning a stabilizing control law $v = \phi(\chi, z)$ for the extended system (z being the state of the chain of integrators). This idea was first introduced by Tornambé in [12] and later employed in several other works [2, 5, 13, 14, 7, 8]. Essentially the same idea is used in [11, 6], where the authors consider classes of systems represented in a special form, and show how general UCO systems can be transformed into this form provided the dynamics are extended with integrators at the input side. This technique has the advantage of providing the exact knowledge of the input derivatives which are needed for observer design. However, such a dynamic extension may introduce a significant complication in the design of the control law for the extended system (e.g., when using integrator backstepping on u, one may observe an explosion of nonlinear terms in v, particularly when n_u is high). Consider, for instance, the Moore-Greitzer model mentioned in the introduction. A simple state feedback controller for this model is given in [9, 10] as the sum of two linear terms and one nonlinear term. The observability mapping of the plant depends on u and $\dot{u}$, hence $n_u = 2$. By applying the integrator backstepping lemma two times one gets a control law v for the "extended system" which is the sum of around 30 nonlinear terms. This general problem is well-known in the backstepping literature and, in some cases, it may be overcome by carefully avoiding needless cancellations during the control design. Clearly, avoiding the use of integrators at the input side of the system would introduce a significant simplification in the design of a state feedback controller.

In [15], two dynamically decoupled high-gain observers are employed to estimate the derivatives of y (vector y_e) *and* those of u (vector z). By assuming the *explicit* knowledge of $\mathcal{H}^{-1}$ an estimate of x is then calculated as $\hat{x} = \mathcal{H}^{-1}(\hat{y}_e, \hat{z})$ (the hats are used to denote the estimates); see Figure 1. The dynamic decoupling between the two observers allows for a separate convergence analysis, and it is shown that the state estimation error converges with arbitrarily fast rate to a small neighborhood of the origin. In [15], the output feedback control problem was not taken in consideration and, to the best of our knowledge, this idea has not been further investigated in the literature. In previous work [7, 8] we showed how to avoid the need for the knowledge of $\mathcal{H}^{-1}$ by introducing a nonlinear observer for the state x. Furthermore, by projecting the observer estimates onto an appropriate compact set, we showed how one can achieve a separation principle for incompletely observable nonlinear systems. By following the same methodology, and using an idea similar to the one found in [15], we now aim at simplifying the output feedback controller design, particularly the construction of the stabilizing state feedback control law, by avoiding the use of integrators at the input side of the system. In order to do that we have to deal with two problems. First, by working in the original

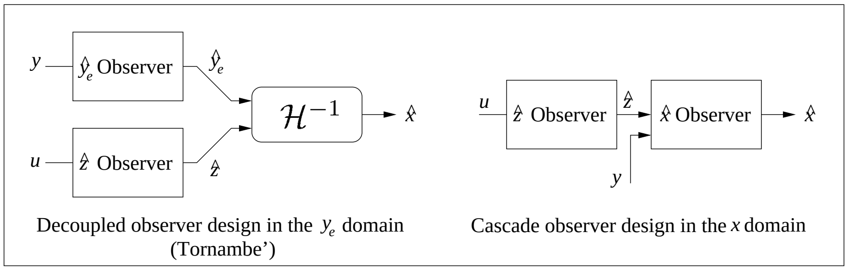

Figure 1: Different observer structures resulting from different working domains

state domain (rather than the y_e domain) the two observers are in cascade, and therefore their dynamics, unlike in [15], are coupled (see Figure 1). Second, by employing a high-gain observer to estimate the input derivatives, we introduce a small asymptotic estimation error, whose effects on the closed-loop stability must be analyzed. To this end, we introduce two observer designs, both involving modifications to the original observer structure introduced in [7, 8].

3.1 Cascade Design 1: High-Gain Observer For The Input Derivatives

Consider the following nonlinear observer,

$$
\begin{aligned}
\dot{\hat{x}} &= f(\hat{x}, \hat{u}, d^0) + \left[\frac{\partial \mathcal{H}(\hat{x}, \hat{z})}{\partial \hat{x}}\right]^{-1} \left[(\mathcal{E}^x)^{-1} L\,(y - \hat{y}) - \frac{\partial \mathcal{H}(\hat{x}, \hat{z})}{\partial \hat{z}} (\mathcal{E}^z)^{-1} K C(u - \hat{u})\right] \\
\hat{y} &= h(\hat{x}, \hat{u})
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\dot{\hat{z}} &= A\hat{z} + (\mathcal{E}^z)^{-1} K(u - \hat{u}) \\
\hat{u} &= C\hat{z}
\end{aligned}
\tag{8}
$$

where $L = \textit{block-diag}[L^1, \ldots, L^p], K = \textit{block-diag}[K^1, \ldots, K^m]$, and L^i, K^j are Hurwitz vectors of dimension $k_i \times 1$ and $n_j \times 1$, respectively, for $i = 1, \ldots, p$, $j = 1, \ldots, m$. Analogously, we let $A = \textit{block-diag}[A^1, \ldots, A^m]$, $B = \textit{block-diag}[B^1, \ldots, B^m]$, $C = \textit{block-diag}[C^1, \ldots, C^m]$, where A^i, B^i,and C^i are in controllable/observable canonical form and have dimensions $n_i \times n_i$, $n_i \times 1$, and $1 \times n_i$, respectively. Finally, $\mathcal{E}^x = \textit{block-diag}[\mathcal{E}^x_1, \ldots, \mathcal{E}^x_p]$, where $\mathcal{E}^x_i = \textit{diag}[\rho, \rho^2, \ldots, \rho^{k_i}]$ and $\rho \in \mathbb{R}$, and $\mathcal{E}^z = \textit{block-diag}[\mathcal{E}^z_1, \ldots, \mathcal{E}^z_p]$, where $\mathcal{E}^z_j = \textit{diag}[\epsilon, \epsilon^2, \ldots, \epsilon^{n_j}]$ and $\epsilon \in \mathbb{R}$. Let $U \triangleq [u_1^{(n_1)}, \ldots, u_m^{(n_m)}]^\top$, then the vector $\dot{z}$ can be expressed as $\dot{z} = Az + BU$, and $u = Cz$.

Theorem 1 *Consider system (1) and suppose A2 is satisfied for $\mathcal{X} = \mathbb{R}^n$, $\mathcal{U} = \mathbb{R}^{n_u}$, x and z are confined to within a compact invariant set Ω, and U is bounded for all $t \geq 0$. Then, the cascaded observers (7) and (8) guarantee that $\hat{x}$ and $\hat{z}$ are bounded for all $t \geq 0$, and the estimation error converges arbitrarily fast to an arbitrarily small neighborhood of the origin, i.e., for all $\delta, T > 0$, there exist $\bar{\rho}, \bar{\epsilon}$, $0 < \bar{\rho}, \bar{\epsilon} \leq 1$, such that $\|\hat{x} - x\| \leq \delta$, $\|\hat{z} - z\| \leq \delta$, for all $t \geq T$, whenever $\rho \in (0, \bar{\rho}), \epsilon \in (0, \bar{\epsilon})$.*

Proof. Define the following coordinate transformations: $\xi = \mathcal{H}(x, z)$ and $\hat{\xi} = \mathcal{H}(\hat{x}, \hat{z})$. By A2, $x = \mathcal{H}^{-1}(\xi, z)$, $\hat{x} = \mathcal{H}^{-1}(\hat{\xi}, \hat{z})$ are well-defined, unique, and smooth.

Let $A_c = \textit{block-diag}[A_c^1, \dots, A_c^p]$, $B_c = \textit{block-diag}[B_c^1, \dots, B_c^p]$, $C_c = \textit{block-diag}[C_c^1, \dots, C_c^p]$, where (A_c^i, B_c^i, C_c^i) are in controllable/observable canonical form, and have dimension $k_i \times k_i$, $k_i \times 1$, and $1 \times k_i$, respectively, then it is easy to show that in new coordinates one has

$$\begin{aligned}\dot{\xi} =& \frac{\partial \mathcal{H}}{\partial x} f(x, u, d) + \frac{\partial \mathcal{H}}{\partial z}\left[Az + BU\right] \\ =& A_c \xi + B_c \left[\alpha(\xi, z, d) + \beta(\xi, z, U)\right]\end{aligned} \tag{9}$$

where $\alpha = [\alpha_1, \dots, \alpha_p]^\top$, $\beta = [\beta_1, \dots, \beta_p]^\top$,

$$\begin{aligned}\alpha_i(\xi, z, d) &= \left[\frac{\partial \varphi_i^{k_i-1}}{\partial x} f(x, u, d) + \sum_{k=0}^{k_i-2} \frac{\partial \varphi_i^{k_i-1}}{\partial u^{(k)}} u^{(k+1)}\right]_{x=\mathcal{H}^{-1}(\xi,z)} \\ \beta_i(\xi, z, U) &= \left[\frac{\partial \varphi_i^{k_i-1}}{\partial u^{(k_i-1)}} u^{(k_i)}\right]_{x=\mathcal{H}^{-1}(\xi,z)}\end{aligned}$$

for all $i = 1, \dots, p$. Notice that, due to A1, the disturbance $d(t)$ affects only the term $\alpha(\xi, z, d)$.

In a similar manner, after noting that A1 implies that $\mathcal{H}(\hat{x}, \hat{z}, d) = \mathcal{H}(\hat{x}, \hat{z}, d^0) = \mathcal{H}(\hat{x}, \hat{z})$, with some tedious algebra it can be shown that

$$\begin{aligned}\dot{\hat{\xi}} =& \frac{\partial \mathcal{H}}{\partial \hat{x}} f(\hat{x}, \hat{u}, d^0) + \frac{\partial \mathcal{H}}{\partial \hat{z}} A\hat{z} + \left[\frac{\partial \mathcal{H}}{\partial \hat{x}}\right]\left[\frac{\partial \mathcal{H}}{\partial \hat{x}}\right]^{-1} \left\{(\mathcal{E}^x)^{-1} L C_c(\xi - \hat{\xi}) - \frac{\partial \mathcal{H}}{\partial \hat{z}} (\mathcal{E}^z)^{-1} K C(z - \hat{z})\right\} + \\ &+ \frac{\partial \mathcal{H}}{\partial \hat{z}} (\mathcal{E}^z)^{-1} K C(z - \hat{z}) \\ =& A_c \hat{\xi} + B_c \alpha(\hat{\xi}, \hat{z}, d^0) + (\mathcal{E}^x)^{-1} L C_c(\xi - \hat{\xi})\end{aligned} \tag{10}$$

Define the observer error in the new coordinates, $\tilde{\xi} = \hat{\xi} - \xi$. Then, using (9) and (10) the observer error dynamics are given by

$$\dot{\tilde{\xi}} = (A_c - (\mathcal{E}^x)^{-1} L C_c)\tilde{\xi} + B_c[\alpha(\hat{\xi}, \hat{z}, d^0) - \alpha(\xi, z, d) - \beta(\xi, z, U)]. \tag{11}$$

Introduce the scaling

$$\tilde{\nu} = \bar{\mathcal{E}}^x \tilde{\xi}, \qquad \bar{\mathcal{E}}^x \triangleq \textit{block-diag}[\bar{\mathcal{E}}_1^x, \dots, \bar{\mathcal{E}}_p^x], \text{ where } \bar{\mathcal{E}}_i^x \triangleq \textit{diag}\left[\frac{1}{\rho^{k_i-1}}, \dots, 1\right], \; i = 1, \dots, p$$

After scaling, the observer error equation becomes

$$\dot{\tilde{\nu}} = \frac{1}{\rho}(A_c - L C_c)\tilde{\nu} + B_c\left[\alpha(\hat{\xi}, \hat{z}, d^0) - \alpha(\xi, z, d) - \beta(\xi, z, U)\right] \tag{12}$$

Now, consider the observer for the variable z. After defining the observation error $\tilde{z} = \hat{z} - z$ and the scaling

$$\tilde{z}' = \bar{\mathcal{E}}^z \tilde{z}, \qquad \bar{\mathcal{E}}^z \triangleq block\text{-}diag[\bar{\mathcal{E}}^z_1, \ldots, \bar{\mathcal{E}}^z_m], \text{ where } \bar{\mathcal{E}}^z_i \triangleq diag\left[\frac{1}{\epsilon^{n_i-1}}, \ldots, 1\right], i = 1, \ldots, m$$

we obtain the error dynamics for the input derivatives observer,

$$\dot{\tilde{z}}' = \frac{1}{\epsilon}(A - KC)\tilde{z}' - BU \tag{13}$$

Recall that $[x^\top, z^\top]^\top \in \Omega$, $d(t) \in \Delta$, and U is bounded for all $t \geq 0$, where Ω and Δ are compact sets. Next, consider the Lyapunov function candidate

$$V_o(\tilde{\nu}, \tilde{z}') = \tilde{\nu}^\top P_1 \tilde{\nu} + \tilde{z}'^\top P_2 \tilde{z}'$$

where P_1 and P_2 are solutions of the Lyapunov equations

$$P_1(A_c - LC_c) + (A_c - LC_c)^\top P_1 = -I \tag{14}$$

$$P_2(A - KC) + (A - KC)^\top P_2 = -I \tag{15}$$

(P_1 and P_2 exist and are positive definite because of the way L and K are defined). Without loss of generality, let $[\hat{x}(0)^\top, \hat{z}(0)^\top]^\top \in \Omega$ and define the compact set

$$\tilde{\mathcal{K}} \triangleq \left\{\tilde{\xi} \in \mathbb{R}^n, \tilde{z}' \in \mathbb{R}^{n_u} \mid \hat{\xi}, \xi \in \mathcal{H}(\Omega), z, \hat{z} \in \Omega^z\right\},$$

where Ω^z is the projection of Ω onto the z subspace. By definition $\tilde{\mathcal{K}}$ contains the initial conditions $\tilde{\xi}(0), \tilde{z}(0)$. Next, define the set

$$\Lambda_\kappa \triangleq \left\{\tilde{\xi} \in \mathbb{R}^n, \tilde{z}' \in \mathbb{R}^{n_u} \mid V_o\left(\bar{\mathcal{E}}^x \tilde{\xi}, \bar{\mathcal{E}}^z \tilde{z}\right) \leq \kappa\right\}$$

where κ is chosen so that $\tilde{\mathcal{K}} \subset \Lambda_\kappa$. In the following, we will prove that Λ_κ is invariant for (12) and (13), and that all the trajectories originating in Λ_κ converge to an arbitrarily small neighborhood of the origin. Recalling that $\hat{\xi} = \xi + \tilde{\xi}$, let

$$\hat{\mathcal{K}} \triangleq \left\{\hat{\xi} \in \mathbb{R}^n, \hat{z} \in \mathbb{R}^{n_u} \mid \xi \in \mathcal{H}(\Omega), z \in \Omega^z, [\tilde{\xi}^\top, \tilde{z}^\top]^\top \in \Lambda_\kappa\right\},$$

and notice that $[\mathcal{H}(\Omega), \Omega^z] \subset \hat{\mathcal{K}}$, and that $[\tilde{\xi}^\top, \tilde{z}^\top]^\top \in \Lambda_\kappa$ implies $[\hat{\xi}^\top, \hat{z}^\top]^\top \in \hat{\mathcal{K}}$. Due to the smoothness of α, β, and the boundedness of U we have that there exists a positive scalar C_1 such that

$$\left\|\alpha(\hat{\xi}, \hat{z}, d^0) - \alpha(\xi, z, d) - \beta(\xi, z, U)\right\| \leq C_1, \text{ for all } [\xi^\top, z^\top]^\top, [\hat{\xi}^\top, \hat{z}^\top]^\top \in \hat{\mathcal{K}}, \text{ for all } d(t) \in \Delta \tag{16}$$

Calculate the time derivative of V_o along the $\tilde{\nu}, \tilde{z}'$ trajectories and use the fact that $\|B_c\| = \|B\| = 1$,

$$\begin{aligned}\dot{V}_o &= -\frac{\tilde{\nu}^\top \tilde{\nu}}{\rho} - \frac{\tilde{z}'\tilde{z}'}{\epsilon} + 2\tilde{\nu}^\top P_1 B_c \left[\alpha(\hat{\xi}, \hat{z}, d^0) - \alpha(\xi, z, d) - \beta(\xi, z, U)\right] + 2\tilde{z}' P_2 B U \\ &\leq -\frac{\|\tilde{\nu}\|^2}{\rho} - \frac{\|\tilde{z}'\|^2}{\epsilon} + 2\|P_1\| C_1 \|\tilde{\nu}\| + 2\|P_2\| C_2 \|\tilde{z}'\| \end{aligned} \tag{17}$$

for all $[\tilde{\xi}^\top, \tilde{z}'^\top]^\top \in \tilde{\Lambda}_\kappa$, where C_2 is the upper bound of U, for all $t \geq 0$. Let $\bar{C} = \max\{C_1, C_2\}$, by applying Young's inequality to the two positive terms in (17) we get

$$\begin{aligned}\dot{V}_o &\leq -\frac{\|\tilde{\nu}\|^2}{\rho} - \frac{\|\tilde{z}'\|^2}{\epsilon} + \|P_1\|\bar{C}\|\tilde{\nu}\|^2 + \|P_2\|\bar{C}\|\tilde{z}'\|^2 + \|P_1\|\bar{C} + \|P_2\|\bar{C} \\ &\leq -\left(\frac{1}{\rho} - \|P_1 \bar{C}\|\right)\|\tilde{\nu}\|^2 - \left(\frac{1}{\epsilon} - \|P_2 \bar{C}\|\right)\|\tilde{z}'\|^2 + \|P_1\|\bar{C} + \|P_2\|\bar{C} \end{aligned} \tag{18}$$

Choose $\rho < 1/(\|P_1\|\bar{C}), \epsilon < 1/(\|P_2\|\bar{C})$, then it can be verified that when $V_o > 2\bar{V}(\rho, \epsilon)$, where

$$\bar{V}(\rho, \epsilon) = \frac{\max\{\|P_1\|, \|P_2\|\}}{\min\left\{\frac{1}{\rho} - \|P_1\|\bar{C}, \frac{1}{\epsilon} - \|P_2\|\bar{C}\right\}}(\|P_1\|\bar{C} + \|P_2\|\bar{C}),$$

we have that

$$\dot{V}_o < -\left(\frac{\min\left\{\frac{1}{\rho} - \|P_1\|\bar{C}, \frac{1}{\epsilon} - \|P_2\|\bar{C}\right\}}{\max\{\|P_1\|, \|P_2\|\}}\right)\frac{V_o}{2} < 0. \tag{19}$$

Observe that we can choose small enough ρ_1 and ϵ_1 so that $2\bar{V}(\rho, \epsilon) \leq \kappa$ whenever $\rho \in (0, \rho_1]$ and $\epsilon \in (0, \epsilon_1]$. Then, from (19) we have that by choosing

$$\rho \in \left(0, \min\left\{1/(\|P_1\|\bar{C}),\ \rho_1\right\}\right), \qquad \epsilon \in \left(0, \min\left\{1/(\|P_2\|\bar{C}),\ \epsilon_1\right\}\right), \tag{20}$$

the set $\{\tilde{\nu} \in \mathbb{R}^n, \tilde{z}' \in \mathbb{R}^{n_u} \,|\, V_o(\tilde{\nu}, \tilde{z}') \leq 2\bar{V}(\rho, \epsilon)\}$ is contained in $\{\tilde{\nu} \in \mathbb{R}^n, \tilde{z}' \in \mathbb{R}^{n_u} \,|\, V_o(\tilde{\nu}, \tilde{z}') \leq \kappa\}$ (on which the previous inequalities hold), is positively invariant, and it is reached in finite time. This shows the boundedness of the observer states. Similarly, we can conclude that the set $\{\tilde{\nu} \in \mathbb{R}^n, \tilde{z}' \in \mathbb{R}^{n_u} \,|\, V_o \leq 2\bar{V}\}$ can be made arbitrarily small by choosing sufficiently small ρ and ϵ satisfying the bounds (20). Finally, from (19) it follows that the rate of convergence of $\tilde{\nu}, \tilde{z}$ can be made as fast as desired by decreasing ρ and ϵ. From these observations, the fact that $\tilde{x} \triangleq \hat{x} - x = \mathcal{H}^{-1}(\hat{\xi}, \hat{z}) - \mathcal{H}^{-1}(\xi, z)$, and the continuity of $\mathcal{H}^{-1}$ with respect to both arguments, we conclude the proof.

■

Remark 4: Note that even when $d(t) = d^0$, i.e., no disturbance acts on the system, the observers (7), (8) achieve UUB of the estimation error, only. In other words, in order to simplify the controller design by avoiding the need for a dynamic extension at the input side, we pay the price of loosing the asymptotic stability of the estimation error relative to the result in [7, 8]. On the other hand, however, when $d(t) \neq d^0$, we achieve the same result one would get by adding integrators at the input side of the system (see [6]).

3.2 Cascade Design 2: Sliding Mode Observer For The Input Derivatives

As we pointed out in Remark 4, using an high-gain observer to estimate the input derivatives destroys the asymptotic properties of the ideal case when z is perfectly known and $d(t) = d^0$. One way to recover these properties is to use a sliding mode observer based on the equivalent control method (see [16, 17]) to estimate z because of its peculiar property of providing the derivatives of a signal in arbitrarily small finite time. In order to do so, we will have to estimate not only z, but also the vector U. To this end, we let $\bar{z} \stackrel{\triangle}{=} [u_1, \dot{u}_1, \ldots, u_1^{(n_1)}, \ldots, u_m, \ldots, u_m^{(n_m)}]^\top$, and we build a sliding mode observer for $\bar{z}$, rather than z. The nonlinear observer has the form

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}, \hat{u}, d^0) + \left[\frac{\partial \mathcal{H}(\hat{x}, \hat{z})}{\partial \hat{x}}\right]^{-1} \left[(\mathcal{E}^x)^{-1} L\left(y(t) - \hat{y}(t)\right) + \frac{\partial \mathcal{H}(\hat{x}, \hat{z})}{\partial \hat{z}} \mathrm{sgn}(\mathcal{V}) K\right] \\ \hat{y}(t) &= h(\hat{x}, \hat{u}) \end{aligned} \tag{21}$$

$$\begin{aligned} \dot{\hat{\bar{z}}} &= \bar{A}\hat{\bar{z}} - \mathrm{sgn}(\mathcal{V})K \\ \hat{u} &= \bar{C}\hat{\bar{z}} \end{aligned} \tag{22}$$

where $\hat{z}$ is obtained from $\hat{\bar{z}}$ in an obvious way, $\mathcal{E}^x$ and L are defined as before, K and $\mathcal{V}$ are defined as follows,

$$K = [K^{1\top}, \ldots, K^{m\top}]^\top, \qquad K^i = [K_1^i, \ldots, K_{n_i+1}^i]^\top,\ i = 1, \ldots, m \tag{23}$$

$$\mathcal{V} = \mathit{block\text{-}diag}[\mathcal{V}^1, \ldots, \mathcal{V}^m], \qquad \mathcal{V}^i = \mathit{diag}[\mathcal{V}_1^i, \ldots, \mathcal{V}_{n_i+1}^i],\ i = 1, \ldots, m \tag{24}$$

$$\mathcal{V}_1^i = \hat{u}_i - u_i, \qquad \mathcal{V}_{j+1}^i = [K_j^i \,\mathrm{sgn}(\mathcal{V}_j^i)]_{\mathrm{eq}},\ j = 1, \ldots, n_i - 1,\ i = 1, \ldots, m \tag{25}$$

and $\bar{A} = \mathit{block\text{-}diag}[\bar{A}^1, \ldots, \bar{A}^m]$, $\bar{C} = \mathit{block\text{-}diag}[\bar{C}^1, \ldots, \bar{C}^m]$, with $\bar{A}^i \in \mathbb{R}^{n_i+1 \times n_i+1}$, $\bar{C}^i \in \mathbb{R}^{n_i+1}$, and $(\bar{A}^i, \bar{C}^i)$ in canonical observable form. Moreover, $\mathrm{sgn}\left(\mathit{diag}[x_1, \ldots, x_n]\right) \stackrel{\triangle}{=} \mathit{diag}\left[\mathrm{sgn}(x_1), \ldots, \mathrm{sgn}(x_n)\right]$, where $\mathrm{sgn}(x)$ denotes the discontinuous sign function (i.e., if $x > 0$ $\mathrm{sgn}(x) = 1$, if $x < 0$ $\mathrm{sgn}(x) = -1$). The notation $[\ldots]_{\mathrm{eq}}$ is the standard way to denote the equivalent control value of the bracketed term (see [18] for details about the equivalent control method).

Theorem 2 *Consider system (1) and suppose A2 is satisfied for $\mathcal{X} = \mathbb{R}^n$, $\mathcal{U} = \mathbb{R}^{n_u}$, x and z are confined to within a compact invariant set Ω, and U is bounded for all $t \geq 0$. Then, observers (21), (22) guarantee that the following properties hold*

- *(i) Existence and uniqueness of the observer estimates: The differential equations (21), (22) have bounded and unique solutions $\hat{x}(t)$ and $\hat{\bar{z}}(t)$, respectively, defined for all $t \geq 0$.*

- *(i) Arbitrarily fast convergence of the estimation error to an arbitrarily small neighborhood of the origin: For all $\delta, T > 0$, there exist $\bar{\rho}$, $0 < \bar{\rho} \leq 1$, and $\bar{K} > 0$ (component-wise) such that $\|\hat{x} - x\| \leq \delta$, $\|\hat{z} - z\| = 0$, for all $t \geq T$, whenever $\rho \in (0, \bar{\rho})$, $K > \bar{K}$ (component-wise)*

(ii) *Asymptotic convergence when $d(t) = d^0$: For all $\delta, T > 0$, there exist ρ^*, $0 < \rho^* \leq 1$, and $K^* > 0$ (component-wise) such that $\|\hat{x} - x\| \leq \delta$, $\|\hat{z} - z\| = 0$, for all $t \geq T$, and $\hat{x} \to x$ as $t \to \infty$, whenever $\rho \in (0, \rho^*)$, $K > K^*$ (component-wise)*

Proof. By using the equivalent control method (see [18]), we have that the motion of system (22) on the discontinuity surfaces $\mathcal{V}_j^i = 0$, $i = 1, \ldots, m$, $j = 1, \ldots, n_i - 1$, is well-defined and unique and thus $\hat{\bar{z}}(t)$ is well-defined and unique. Furthermore, note that the observer (22) is a slight variation of the one found in [17, 16], but its convergence proof is identical to that found therein. Hence, we have that there exists a scalar $C_2 > 0$ such that $\|\hat{\bar{z}}(t) - \bar{z}(t)\| \leq C_2$ for all $t \geq 0$ and, for any $T_0 > 0$, there exists $\bar{K} > 0$ such that $\|\tilde{\bar{z}}\| = 0$ for all $t \geq T_0$, whenever $K > \bar{K}$ (component-wise). For any $T > 0$, Choose $0 < T_0 < T$, and define the same coordinate transformation used in the proof of Theorem 1, $\xi = \mathcal{H}(x, z)$, $\hat{\xi} = \mathcal{H}(\hat{x}, \hat{z})$. Recall, from the proof of Theorem 1, that

$$\dot{\xi} = A_c\xi + B_c\left[\alpha(\xi, z, d) + \beta(\xi, z, U)\right] \tag{26}$$

In a similar manner, it can be shown that

$$\begin{aligned}
\dot{\hat{\xi}} &= A_c\hat{\xi} + B_c[\alpha(\hat{\xi}, \hat{z}, d^0) + \beta(\hat{\xi}, \hat{z}, \hat{U})] + \left[\frac{\partial \mathcal{H}}{\partial \hat{x}}\right]\left[\frac{\partial \mathcal{H}}{\partial \hat{x}}\right]^{-1}\left\{(\mathcal{E}^x)^{-1}LC_c(\xi - \hat{\xi}) - \frac{\partial \mathcal{H}}{\partial \hat{z}}\mathrm{sgn}(\mathcal{V})K\right\} - \frac{\partial \mathcal{H}}{\partial \hat{z}}\mathrm{sgn}(\mathcal{V})K \\
&= A_c\hat{\xi} + B_c[\alpha(\hat{\xi}, \hat{z}, d^0) + \underline{\beta(\hat{\xi}, \hat{z}, \hat{U})}] - (\mathcal{E}^x)^{-1}LC_c(\xi - \hat{\xi})
\end{aligned} \tag{27}$$

Notice that the underlined term arises from the fact that the sliding mode observer is of order $n_u + m$. Since (27) does not contain any discontinuity and is locally Lipschitz, we conclude that the solution $\hat{\xi}(t)$ exists and is unique. Since $\hat{x} = \mathcal{H}^{-1}(\hat{\xi}, \hat{z})$, $\mathcal{H}^{-1}$ is smooth, and $\hat{z}(t)$ is well-defined and unique, we conclude that $\hat{x}(t)$ is also well-defined and unique, thus proving the existence and uniqueness part of the theorem. In order to prove that $\hat{x}(t)$ is bounded for all $t \geq 0$ we consider the error dynamics for ξ

$$\dot{\tilde{\xi}} = (A_c - (\mathcal{E}^x)^{-1}LC_c)\tilde{\xi} + B_c\left[\alpha(\hat{\xi}, \hat{z}, d^0) + \beta(\hat{\xi}, \hat{z}, \hat{U}) - \alpha(\xi, z, d) - \beta(\xi, z, U)\right] \tag{28}$$

As in the proof of Theorem 1, introduce the scaling $\tilde{\nu} = \bar{\mathcal{E}}^x\tilde{\xi}$, where $\bar{\mathcal{E}}^x$ is defined as before, so that the error dynamics become

$$\dot{\tilde{\nu}} = \frac{1}{\rho}(A_c - LC_c)\tilde{\nu} + B_c\left[\alpha(\hat{\xi}, \hat{z}, d^0) + \beta(\hat{\xi}, \hat{z}, \hat{U}) - \alpha(\xi, z, d) - \beta(\xi, z, U)\right] \tag{29}$$

Consider the Lyapunov function candidate $V_o(\tilde{\nu}) = \tilde{\nu}^\top P_1\tilde{\nu}$, where P_1 is defined in (14). Next, in complete analogy with the proof of Theorem 1, let $\tilde{\mathcal{K}} \triangleq \left\{\tilde{\xi} \in \mathbb{R}^n \,|\, \hat{\xi}, \xi \in \mathcal{H}(\Omega)\right\}$, so that $\tilde{\xi}(0) \in \tilde{\mathcal{K}}$, define the set $\Lambda_\kappa \triangleq \left\{\tilde{\xi} \in \mathbb{R}^n \,|\, V_o\left(\bar{\mathcal{E}}^x\tilde{\xi}\right) \leq \kappa\right\}$, where κ is chosen so that $\tilde{\mathcal{K}} \subset \Lambda_\kappa$, and let $\hat{\mathcal{K}} \triangleq \left\{\hat{\xi} \in \mathbb{R}^n \,|\, \xi \in \mathcal{H}(\Omega), \tilde{\xi} \in \Lambda_\kappa\right\}$, and notice that $\mathcal{H}(\Omega) \subset \hat{\mathcal{K}}$. Due to the smoothness of α, β with respect to to their arguments, and the boundedness of U, for all $t \geq 0$, we have that

$$\sup_{U\in\Omega^z,\,\hat{U}\in\hat{\Omega}^z} \left\|\alpha(\hat{\xi},\hat{z},d^0)+\beta(\hat{\xi},\hat{z},\hat{U})-\alpha(\xi,z,d)-\beta(\xi,z,U)\right\| \leq \gamma\left(\|\tilde{\xi}\|+\|\tilde{z}\|+\|\tilde{d}\|\right) \tag{30}$$

for all ξ, $\hat{\xi} \in \hat{\mathcal{K}}$, $z \in \Omega^z$, and $\hat{z} \in \hat{\Omega}^z$, where $\tilde{d} = d^0 - d(t)$, Ω^z is the projection of Ω onto the z subspace, and $\hat{\Omega}^z$ is the compact set containing $\hat{z}(t)$ for all $t \geq 0$ and for all $\hat{z}(0) \in \Omega^z$. The existence of $\hat{\Omega}^z$ is guaranteed by the boundedness of $\hat{\tilde{z}}(t)$, for all $t \geq 0$. Now, by calculating the time derivative of V_o, using (30) and the fact that $\|\tilde{\xi}\| \leq \|\tilde{\nu}\|$,

$$\begin{aligned}
\dot{V}_o &= -\frac{\tilde{\nu}^\top\tilde{\nu}}{\rho} + 2\tilde{\nu}^\top P_1 B_c\left[\alpha(\hat{\xi},\hat{z},d^0)+\beta(\hat{\xi},\hat{z},\hat{U})-\alpha(\xi,z,d)-\beta(\xi,z,U)\right]\\
&\leq -\frac{\|\tilde{\nu}\|^2}{\rho} + 2\|P_1\|\|\tilde{\nu}\|\gamma\left(\|\tilde{\xi}\|+\|\tilde{z}\|+\|\tilde{d}\|\right)\\
&\leq -\frac{\|\tilde{\nu}\|^2}{\rho} + 2\|P_1\|\|\tilde{\nu}\|\gamma\left(\|\tilde{\nu}\|+\|\tilde{z}\|+\|\tilde{d}\|\right),
\end{aligned} \tag{31}$$

for all $\tilde{\xi} \in \Lambda_\kappa$ (and hence ξ, $\hat{\xi} \in \hat{\mathcal{K}}$), $z \in \Omega^z$, and $\hat{z} \in \hat{\Omega}^z$. Using the fact $\|\tilde{z}\| \leq C_2$ and applying Young's inequality, after a few manipulations we get

$$\dot{V}_o \leq -\left[\frac{1}{\rho} - 2\|P_1\|\gamma - \|P_1\|\gamma\left(C_2 + \sup_{t\geq 0}\|\tilde{d}(t)\|\right)\right]\|\tilde{\nu}\|^2 + \|P_1\|\gamma\left(C_2 + \sup_{t\geq 0}\|\tilde{d}(t)\|\right), \tag{32}$$

where $\tilde{d}(t)$ is bounded for all $t \geq 0$. In complete analogy with the proof of Theorem 1 we have that, when $V_o > 2\bar{V}(\rho)$, where

$$\bar{V}(\rho) = \frac{\|P_1\|^2\gamma\left(C_2 + \sup_{t\geq 0}\|\tilde{d}(t)\|\right)}{\frac{1}{\rho} - 2\|P_1\|\gamma - \|P_1\|\gamma\left(C_2 + \sup_{t\geq 0}\|\tilde{d}(t)\|\right)},$$

we have that

$$\dot{V}_o \leq -\left(\frac{\frac{1}{\rho} - 2\|P_1\|\gamma - \|P_1\|\gamma\left(C_2 + \sup_{t\geq 0}\|\tilde{d}(t)\|\right)}{\|P_1\|}\right)\frac{V_o}{2} < 0. \tag{33}$$

Choose $\bar{\rho}$ such that $\bar{\rho} < \left[\|P_1\|\gamma(2 + C_2 + \sup_{t\geq 0}\|\tilde{d}(t)\|)\right]^{-1}$ and $2\bar{V}(\rho) \leq \kappa$, then for any $\rho \in (0, \bar{\rho})$ we have that Λ_κ is a positively invariant set and hence $\tilde{\nu}(t)$ is bounded for all $t \geq 0$. This concludes the proof of boundedness of the observer states. Next, from the properties of the sliding mode observer (22) and our choice of $\bar{K}$, we have that for all $t \geq T_0$, and for any $K > \bar{K}$, (31) becomes

$$\dot{V}_o \leq -\left(\frac{1}{\rho} - 2\|P_1\|\gamma\right)\|\tilde{\nu}\|^2 + 2\gamma\,\|P_1\|\,\|\tilde{d}\|\,\|\tilde{\nu}\| \tag{34}$$

Using similar arguments to those in Theorem 1 (compare (34) to (17)), we can choose ρ sufficiently small so that $\|\tilde{x}\| \leq \delta$, for all $t \geq T$, thus concluding the proof of part (ii) of the theorem. Part (iii) is readily proven once it is observed that, when $\tilde{d}(t) = 0$, (34) becomes

$$\dot{V}_o \leq -\left(\frac{1}{\rho} - 2\|P_1\|\gamma\right)\|\tilde{\nu}\|^2 \tag{35}$$

and hence $\tilde{\xi}(t)$ converges exponentially to zero with arbitrarily fast rate.

■

Remark 5: In practical implementation, the values of the equivalent controls may be obtained by low-pass filtering the discontinuous terms in the observer (22). Such a filtering may, in general, introduce small estimation errors (see [18] for a complete analysis of this problem), with a consequent impact on the observer error $\tilde{z}$, which then may not become *identically* zero, but rather remain small. In such a case, result (iii) of Theorem 2 would be lost. Hence, apart from its theoretical interest, the practical relevance of Theorem 2 remains limited. Moreover, the lack of Lyapunov analysis in the literature for the convergence of this observer will force us, in what follows, to restrict the observability Assumption A2 when this observer is employed in closed-loop control (see Section 4.2).

4 Robust Output Feedback Stabilizing Control

Our main objective in this section is to extend the projection idea found in [7, 8] to the case under consideration. Recall that the observers (7), (21) exhibit peaking; hence, when A2 is not satisfied globally, the observer states $\hat{x}$, $\hat{z}$ may exit the observable region $\mathcal{X} \times \mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$, thus rendering the observer equations (7), (21) undefined. Here, we will modify the $\dot{\hat{x}}, \dot{\hat{z}}$ equations to guarantee that $\hat{x}$ and $\hat{z}$ are confined to within $\mathcal{X} \times \mathcal{U}$, while preserving their convergence properties.

Considering system (6) and using Assumption A3 we conclude that $u = h_c(\chi)$ makes the compact set $\mathcal{N}$ positively invariant and uniformly asymptotically stable with domain of attraction $\mathcal{D}$, for all $d(t) \in \Delta$. If $\mathcal{D}$ is not all of $\mathbb{R}^{n+n_c}$, we can use the converse Lyapunov theorem introduced in [11] (Theorem 1) to conclude that there exists a smooth Lyapunov function in $\mathcal{D}$ with respect to $\mathcal{N}$, i.e., there exist two $\mathcal{K}_\infty$ functions α_1 and α_2 and a positive function α_3 such that,

$$\alpha_1(\omega_{\mathcal{N}}(\chi)) \leq V(\chi) \leq \alpha_2(\omega_{\mathcal{N}}(\chi)), \qquad \forall \chi \in \mathcal{D} \tag{36}$$

$$\dot{V}(\chi) \leq -\alpha_3(\omega_{\mathcal{N}}(\chi)), \qquad \forall \chi \in \mathcal{D}/\mathcal{N} \tag{37}$$

where $\omega_{\mathcal{N}}(\chi)$ is a positive definite function with respect to $\mathcal{N}$, continuous and proper in $\mathcal{D}$ (i.e., it tends to infinity on the boundary of $\mathcal{D}$). See [11] for more details.

On the other hand, when the validity of A3 is local or global, converse theorems for this type of stability can be found in [19], or deducted from the theory of stability preserving mappings found in [20] (in the local case, apply Theorem 4.6.5 to the differential inclusion associated to (1); its global counterpart is a straightforward generalization). Now, define the compact sets Ω_{c_1}, Ω_{c_2}, $\Omega^x_{c_2}$, and Ω^z as follows

$$\Omega_{c_1} \triangleq \{\chi \,|\, V \le c_1\},\ \Omega_{c_2} \triangleq \{\chi \,|\, V \le c_2\},\ \Omega^x_{c_2} \triangleq \{x \in \mathbb{R}^n \,|\, \chi \in \Omega_{c_2}\},\ c_2 > c_1 > 0$$

and Ω^z is the compact set which is invariant with respect to the z trajectories (its existence follows from the smoothness of $u = h_c(\chi)$ which implies the boundedness of its derivatives). Notice that for all $c_2 > 0$ we have $\mathcal{N} \subset \Omega^x_{c_2} \subset \mathcal{D}$. Now consider the following mapping,

$$\mathcal{F} : \mathbb{R}^n \times \mathbb{R}^{n_u} \longrightarrow \mathbb{R}^n \times \mathbb{R}^{n_u}, \qquad \mathcal{F}(x, z) \triangleq \begin{bmatrix} \mathcal{H}(x, z) \\ z \end{bmatrix}$$

then, it is readily seen that $\mathcal{F}$ is a diffeomorphism on $\mathcal{X} \times \mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$. Next, the following assumption is needed.

Assumption A4 (*Topology of* $\mathcal{X} \times \mathcal{U}$). Assume that c_2 can be selected so that the following condition is satisfied:

$$\mathcal{F}\left(\Omega^x_{c_2}, \Omega^z\right) \subset C_\xi \subset \mathcal{F}\left(\mathcal{X}, \mathcal{U}\right), \text{ for some convex compact } C_\xi \tag{38}$$

Remark 6: This assumption consists of three basic requirements. First, it requires that there exists an invariant set (a neighborhood of $\mathcal{N}$), such that its projection onto the x subspace ($\Omega^x_{c_2}$) is contained in $\mathcal{X}$. Second, it requires that the control input $u = h_c(\chi)$ has been designed so that its derivatives (vector z) are confined to within a compact set (Ω^z), which is contained in $\mathcal{U}$. Lastly, it requires that the two compact sets $\Omega^x_{c_2}$ and Ω^z can be inscribed in an observable domain whose image under $\mathcal{F}$ (the set C_ξ) is convex. The latter requirement imposes a restriction on the topology of the observable region (e.g., it cannot have "holes" where we want to stabilize the system at). In conclusion, A4 requires that the state feedback control law be designed so that the state and input trajectories travel inside a "topologically well-behaved" observable region. It appears to be a rather basic condition for output feedback control.

4.1 Observer Estimates Projection: First Design

Recall the coordinate transformation used in the proofs of Theorem 1 and 2, and let

$$\xi = \mathcal{H}(x, z), \qquad \hat{\xi} = \mathcal{H}(\hat{x}, z), \qquad \tilde{\xi} = \hat{\xi} - \xi \tag{39}$$

Note that $\dot{\hat{\xi}} = \left[\dfrac{\partial \mathcal{H}(\hat{x}, \hat{z})}{\partial \hat{x}}\right] \dot{\hat{x}} + \left[\dfrac{\partial \mathcal{H}(\hat{x}, \hat{z})}{\partial \hat{z}}\right] \dot{\hat{z}}$, which is well-defined when $\hat{x} \in \mathcal{X}$, $\hat{z} \in \mathcal{U}$. Next, project[1]

[1]The projections defined in (40) and (41) are discontinuous in the variables $\hat{\xi}$, $\hat{z}$, respectively, therefore raising the issue of the existence and uniqueness of their solutions. We refer the reader to Remark 7, were this issue is addressed and a solution is proposed.

$$\dot{\hat{x}}^P = \left[\frac{\partial \mathcal{H}}{\partial \hat{x}}\right]^{-1} \left\{\mathcal{P}_1\left(\hat{\xi}, \dot{\hat{\xi}}, \hat{z}, \dot{\hat{z}}\right) - \frac{\partial \mathcal{H}}{\partial \hat{z}}\dot{\hat{z}}^P\right\}$$

$$\mathcal{P}_1\left(\hat{\xi}, \dot{\hat{\xi}}, \hat{z}, \dot{\hat{z}}\right) = \begin{cases} \dot{\hat{\xi}} - \Gamma_1 \dfrac{N_\xi\left(N_\xi^\top \dot{\hat{\xi}} + N_z^\top \dot{\hat{z}}\right)}{N_\xi^\top \Gamma_1 N_\xi + N_z^\top \Gamma_2 N_z} & \text{if } N_\xi^\top \dot{\hat{\xi}} + N_z^\top \dot{\hat{z}} \geq 0 \text{ and } [\hat{\xi}^\top, \hat{z}^\top]^\top \in \partial C_\xi \\ \dot{\hat{\xi}} & \text{otherwise} \end{cases} \tag{40}$$

$$\dot{\hat{z}}^P = \mathcal{P}_2\left(\hat{\xi}, \dot{\hat{\xi}}, \hat{z}, \dot{\hat{z}}\right)$$

$$\mathcal{P}_2\left(\hat{\xi}, \dot{\hat{\xi}}, \hat{z}, \dot{\hat{z}}\right) = \begin{cases} \dot{\hat{z}} - \Gamma_2 \dfrac{N_z\left(N_\xi^\top \dot{\hat{\xi}} + N_z^\top \dot{\hat{z}}\right)}{N_\xi^\top \Gamma_1 N_\xi + N_z^\top \Gamma_2 N_z} & \text{if } N_\xi^\top \dot{\hat{\xi}} + N_z^\top \dot{\hat{z}} \geq 0 \text{ and } [\hat{\xi}^\top, \hat{z}^\top]^\top \in \partial C_\xi \\ \dot{\hat{\xi}} & \text{otherwise} \end{cases} \tag{41}$$

where $\Gamma_1 = (S^1\bar{\mathcal{E}}^x)^{-1}(S^1\bar{\mathcal{E}}^x)^{-1}$, $\Gamma_2 = (S^2\bar{\mathcal{E}}^z)^{-1}(S^2\bar{\mathcal{E}}^z)^{-1}$, $S^1 = {S^1}^\top$, $S^2 = {S^2}^\top$ denote the matrix square roots of P_1, P_2 defined in (14),(15); $\bar{\mathcal{E}}^x, \bar{\mathcal{E}}^z$ are the scalings introduced in the proof of Theorem 1, and N_ξ, N_z are the ξ and z components of the normal vector N to ∂C_ξ, the boundary of C_ξ, at $\hat{\xi}$, $\hat{z}$ i.e., $N(\hat{\xi}, \hat{z}) = \left[N_\xi^\top(\hat{\xi}, \hat{z}), N_z^\top(\hat{\xi}, \hat{z})\right]^\top$. The following lemma shows that (40), (41) guarantee boundedness and preserve convergence of the estimates $\hat{x}$, $\hat{z}$.

Lemma 1 *If A4 holds and (40), (41) are applied to observers (7), (8), respectively, then*

(i) Boundedness: $[{\hat{x}^P}^\top, {\hat{z}^P}^\top]^\top \in \mathcal{F}^{-1}(C_\xi) \subset \mathcal{X} \times \mathcal{U}$, *for all* $t \geq 0$.

If, in addition, $x \in \Omega^x_{c_2}, z \in \Omega^z$, *then the following is also true*

(ii) Preservation of the original convergence characteristics: the result of Theorem 1 remains valid for $\hat{x}^P, \hat{z}^P$.

Proof. In order to prove part (i), introduce the coordinate transformations $\zeta = S^1\bar{\mathcal{E}}^x\xi$, $\pi = S^2\bar{\mathcal{E}}^z z$ ($\hat{\zeta}$, $\hat{\pi}$ are defined analogously), and let $O = [\zeta^\top, \pi^\top]^\top$ ($\hat{O}$ is defined analogously). Denote by C_O the set C_ξ in O coordinates, and by $N'(O)$ the corresponding normal vector, and note that the convexity of C_ξ implies the convexity of C_O. In the new coordinates we have

$$\dot{\hat{O}}^P = \frac{d}{dt}\left\{\begin{bmatrix} S^1\bar{\mathcal{E}}^x & 0 \\ 0 & S^2\bar{\mathcal{E}}^z \end{bmatrix}\begin{bmatrix} \dot{\hat{\xi}}^P \\ \dot{\hat{z}}^P \end{bmatrix}\right\} = \begin{bmatrix} S^1\bar{\mathcal{E}}^x & 0 \\ 0 & S^2\bar{\mathcal{E}}^z \end{bmatrix}\begin{bmatrix} \frac{\partial \mathcal{H}}{\partial \hat{x}}\dot{\hat{x}}^P + \frac{\partial \mathcal{H}}{\partial \hat{z}}\dot{\hat{z}}^P \\ \dot{\hat{z}}^P \end{bmatrix} = \tag{42}$$

$$= \begin{cases} \dot{\hat{O}} - \dfrac{N'(\hat{O})N'(\hat{O})^\top}{N'(\hat{O})^\top N'(\hat{O})}\dot{\hat{O}} & \text{if } \dot{\hat{O}}^\top N'(\hat{O}) \geq 0 \text{ and } \hat{O} \in \partial C_O \\ \dot{\hat{O}} & \text{otherwise} \end{cases} \tag{43}$$

The last identity follows from (42) and the fact that

$$N'(\hat{O}) = \begin{bmatrix} (S^1\bar{\mathcal{E}}^x)^{-1} & 0 \\ 0 & (S^2\bar{\mathcal{E}}^z)^{-1} \end{bmatrix} N(\hat{\xi}, \hat{z})$$

Now notice that (43) represents the standard gradient projection found in the optimization literature and widely used in adaptive control (see, e.g., [21]). From the fact that (43) is performed over the convex set C_O, we conclude that $\hat{O}^P(t) \in C_O$ for all $t \geq 0$, which in turn implies that $\left[\hat{\xi}^{P^\top}(t), \hat{z}^{P^\top}\right]^\top \in C_\xi$ and, finally, $\left[\hat{x}^{P^\top}(t), \hat{z}^{P^\top}(t)\right]^\top \in \mathcal{F}^{-1}(C_\xi) \subset \mathcal{X} \times \mathcal{U}$ for all $t \geq 0$, thus proving part (i) of Lemma 1. The proof of part (ii) is based on the knowledge of a Lyapunov function for the observer in $\tilde{\nu}$, $\tilde{z}'$ coordinates. Recall from the proof of Theorem 1 that $V_o = \tilde{\nu}^\top P_1 \tilde{\nu} + \tilde{z}'^\top P_2 \tilde{z}'$ which, in O coordinates, can be written as $V_o = \tilde{O}^\top \tilde{O}$. We want to show that, in O coordinates, $\dot{V}_o < 0$ and $\dot{\hat{O}}^P = \begin{bmatrix} S^1\bar{\mathcal{E}}^x & 0 \\ 0 & S^2\bar{\mathcal{E}}^z \end{bmatrix} \begin{bmatrix} \dot{\hat{\xi}}^P \\ \dot{\hat{z}}^P \end{bmatrix}$ implies that $\dot{V}_o^P \leq \dot{V}_o$, where $\tilde{O}^P = \hat{O}^P - O$ and $V_o^P = \tilde{O}^{P^\top} \tilde{O}^P$. From (43), when $\hat{O}$ is in the interior of C_O, or $\hat{O} \in \partial C_O$ and $\dot{\hat{O}}^\top N'(\hat{O}) < 0$, we have that $V_o = V_o^P$. Let us consider the remaining cases, for which we have

$$\dot{V}_o^P = 2\tilde{O}^{P^\top} \dot{\tilde{O}}^P = 2\tilde{O}^\top \left[\dot{\hat{O}} - O - p(\hat{O}, \dot{\hat{O}}) N'(\hat{O})\right] \tag{44}$$

where $p(\hat{O}, \dot{\hat{O}}) = \dfrac{N'(\hat{O})^\top \dot{\hat{O}}}{N'(\hat{O})^\top N'(\hat{O})}$ is nonnegative since, by assumption, $N'(\hat{O})^\top \dot{\hat{O}} \geq 0$. Thus,

$$\dot{V}_o^P = \dot{V}_o - 2\, p(\hat{O}, \dot{\hat{O}})\, \tilde{O}^\top N'(\hat{O})$$

Since, by assumption, $x \in \Omega_{c_2}^x$ and $z \in \Omega^z$, we have that $\hat{O} \in C_O$ and, since $\hat{O}$ lies in the boundary of C_O, the vector $\tilde{O} = \hat{O} - O$ points outside of C_O which, by the convexity of C_O, implies that $\tilde{O}^\top N'(\hat{O}) \geq 0$, thus concluding the proof of part (ii).

■

4.2 Observer Estimates Projection: Second Design

In this section we follow a different approach in the definition of the projection. From the finite time convergence property of the sliding mode observer, we have that after arbitrarily small time $\hat{z} = z$. During this time interval, however, the observer estimates may peak (due to the presence of high-gains in the vector K) and projection should be employed to keep them inside $\mathcal{U}$. Unlike the design in the previous section, however, we do not have the tools to build a projection for $\hat{z}$ that preserves the convergence characteristics outlined in Theorem 2. This is due to the fact that we do not have a Lyapunov function for the sliding mode observer outside of the sliding manifold (to the best of our knowledge, the convergence analyses found in the literature do not provide it), and our projection design relies on this knowledge. In order to avoid this problem we need to restrict Assumption A2.

Assumption A2' (*Observability*). System (1) is observable over the set $\mathcal{X} \times \mathbb{R}^{n_u}$, where $\mathcal{X} \subset \mathbb{R}^n$, i.e., the mapping $y_e = \mathcal{H}(x, z)$ is invertible with respect to x and its inverse is smooth, for all $x \in \mathcal{X}$, $z \in \mathbb{R}^{n_u}$.

Assumption A2' requires the uniform observability with respect to z, but it is still less restrictive than the UCO assumption commonly found in the literature. Assumption A4 can be consequently relaxed as follows:

Assumption A4' (*Topology of* $\mathcal{X} \times \mathcal{U}$). Assume that c_2 can be selected so that the following condition is satisfied:

$$\mathcal{F}\left(\Omega^x_{c_2}, \hat{\Omega}^z\right) \subset C_\xi \subset \mathcal{F}\left(\mathcal{X}, \mathbb{R}^{n_u}\right), \text{ for some convex compact } C_\xi \tag{45}$$

where $\hat{\Omega}^z$ is defined in the proof of Theorem 2.

Using these assumptions, we do not need to project the estimate provided by the sliding mode observer, thus we only need to make sure that $\hat{x}(t) \in \mathcal{X}$ for all $t \geq 0$. We do this by modifying projection (40) as follows,

$$\begin{aligned} \dot{\hat{x}}^P &= \left[\frac{\partial \mathcal{H}}{\partial \hat{x}}\right]^{-1} \left\{\mathcal{P}_1\left(\hat{\xi}, \dot{\hat{\xi}}, \hat{z}, \dot{\hat{z}}\right) - \frac{\partial \mathcal{H}}{\partial \hat{z}} \dot{\hat{z}}^P\right\} \\ \mathcal{P}_1\left(\hat{\xi}, \dot{\hat{\xi}}, \hat{z}, \dot{\hat{z}}\right) &= \begin{cases} \dot{\hat{\xi}} - \Gamma_1 \dfrac{N_\xi\left(N_\xi^\top \dot{\hat{\xi}} + N_z^\top \dot{\hat{z}}\right)}{N_\xi^\top \Gamma_1 N_\xi} & \text{if } N_\xi^\top \dot{\hat{\xi}} + N_z^\top \dot{\hat{z}} \geq 0 \text{ and } [\hat{\xi}^\top, \hat{z}^\top]^\top \in \partial C_\xi \\ \dot{\hat{\xi}} & \text{otherwise} \end{cases} \end{aligned} \tag{46}$$

Lemma 2 *If A4 holds and (46), is applied to observer (21), then*

(i) Boundedness: $\hat{x}^P(t) \in \mathcal{F}^{-1}(C_\xi) \subset \mathcal{X} \times \mathbb{R}^{n_u}$*, for all* $t \geq 0$*.*

If, in addition, $x \in \Omega^x_{c_2}$*, then the following is also true*

(ii) Preservation of the original convergence characteristics: properties (i) and (ii) established by Theorem 2 remain valid for $\hat{x}^P$*.*

Proof. It is sufficient to notice that, when A4' holds, $\dot{\hat{z}}$ does not need to be projected and (40) can be applied to (21), after setting $\Gamma_2 = 0$ (i.e., no projection is applied to $\hat{z}$), thus yielding (46).

■

Remark 7: In order to avoid the discontinuity in the right hand side of (40), (41) and (46), one can define $\mathcal{P}_1$, $\mathcal{P}_2$ in both designs to be smooth projections, using the idea introduced in [22], provided the smooth convex set $\bar{C}_\xi \supset C_\xi$, arbitrarily close to C_ξ is employed in Assumptions A4 and A4'.

4.3 Closed-Loop Stability

To perform output feedback control, we replace the dynamic state feedback compensator (4) by

$$\begin{aligned} \dot{x}_c &= f_c(x_c, \hat{x}^P) \\ \hat{u} &= h_c(x_c, \hat{x}^P). \end{aligned} \tag{47}$$

By defining $\hat{F}(\chi, \hat{x}^P, u, d(t)) \triangleq \left[f_c^\top(x_c, \hat{x}^P), f^\top(x, u, d(t))\right]$, the closed-loop system is given by

$$\dot{\chi} = \hat{F}(\chi, \hat{x}^P, h_c(x_c, \hat{x}^P, d(t)). \tag{48}$$

By the properness of $V(\chi)$ on $\mathcal{D}$, we have that for any compact set $\mathcal{D}' \subset \mathcal{D}$ there exist c_1, c_2 large enough so that $\mathcal{D}' \subset \Omega_{c_1} \subset \Omega_{c_2} \subset \mathcal{D}$. Taking in account the restriction on c_2 imposed by Assumption A4 (A4'), choose $\mathcal{D}'$ to be an arbitrary compact set contained in Ω_{c_1}. Note that $\hat{F}(\chi, x, u, d(t)) = F(\chi, u, d(t))$, and that, by the smoothness of f_c and f with respect to their arguments, there exists a positive scalar $\bar{\gamma}$ such that

$$\left\|\hat{F}(\chi, \hat{x}^P, h_c(x_c, \hat{x}^P), d) - F(\chi, h_c(x_c, x), d)\right\| \leq \bar{\gamma}\|\hat{x}^P - x\|,$$

for all $\hat{x}^P$, $x \in \Omega^x_{c_2}$. In the following we will state two closed-loop stability theorems for the case when $d(t) \neq d^0$ and $d(t) = d^0$, respectively. The first of the two theorems applies to both of the observer designs introduced in Section 3, while the other applies to the second cascade design, only; throughout the statement and proof of Theorem 3, we will refer to the first design (see Section 3.1), pointing out the differences relative to the second one in square brackets, whenever needed.

Theorem 3 *Suppose that Assumptions A1, A2, [A2'], A3, A4 [A4'] are satisfied and the initial condition* $\chi(0)$ *is contained in* $\mathcal{D}' \subset \Omega_{c_1}$*; define the set* $\Omega_\varepsilon \triangleq \{\chi : V(\chi) \leq d_\varepsilon\}$ *(*$\mathcal{N} \subset \Omega_\varepsilon$*), where* $d_\varepsilon = \alpha_2 \circ \alpha_3^{-1}(2A\varepsilon)$ *and* A *is defined in the proof. Then, for any* $\varepsilon > 0$ *chosen so that* $d_\varepsilon < c_1$*, there exist positive scalars* ρ^**,* $\epsilon^*[K^*]$ *such that, for all* $\rho \in (0, \bar{\rho})$*,* $\epsilon \in (0, \bar{\epsilon})$ *[*$K > K^*$ *component-wise], the output feedback control law (47) (where* $\hat{x}^P$ *is generated by (7), (8), (40), (41) [(21), (22), (46)]) renders* Ω_ε *a uniformly asymptotically stable positively invariant set for (6), with domain of attraction containing* Ω_{c_1}*, and it is reached in finite time; moreover all the internal variables are bounded for all* $t \geq 0$*.*

Sketch of the proof. Here we will use analogous arguments to the ones employed in [7, 8], modified to deal with stability with respect to sets. Since $V(\chi(0)) \leq c_1 < c_2$, there exists a time $T_1 > 0$ such that $V(\chi(t)) \leq c_2$, for all $t \in [0, T_1)$. Choose T_0 such that $0 < T_0 < T_1$. From Lemma 1 [Lemma 2], part (i), we have that $\hat{x}^P(t)$ is well-defined and bounded for all $t \geq 0$, and the bound on its norm does not depend on ρ and ϵ. This in particular implies that T_1 does not depend on ρ and ϵ. Now, use Theorem 1 [Theorem 2] and conclude that for any positive ε, there exist a positive $\bar{\rho}$, $\bar{\epsilon}$ [$\bar{K} > 0$], $0 < \bar{\rho}, \bar{\epsilon} \leq 1$ such that, for all $\rho < \bar{\rho}$, $\epsilon < \bar{\epsilon}$ [$K > \bar{K}$], $\|\tilde{x}^P\| \leq \varepsilon$, for all $t \in [T_0, T_1)$. Next, we show first that χ belongs to Ω_{c_2} for all $t \geq 0$ (and hence $T_1 = \infty$), i.e., Ω_{c_2} is an invariant set. Next, we will show that χ converges to within Ω_ε in finite time.

Consider the Lyapunov function candidate V defined in (36), (37). Taking its derivative with respect to time, by the smoothness of F, $\hat{F}$, h_c, and $V(\chi)$ we get

$$\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial \chi} F\left(\chi, h_c(x_c, x), d\right) + \frac{\partial V}{\partial \chi}\left[\hat{F}\left(\chi, \hat{x}^P, h_c(x_c, \hat{x}^P), d\right) - F\left(\chi, h_c(x_c, x), d\right)\right] \\
&\leq -\alpha_3(\omega_N(\chi)) + \left\|\frac{\partial V}{\partial \chi}\right\| \left\|\hat{F}\left(\chi, \hat{x}^P, h_c(x_c, \hat{x}^P), d\right) - F\left(\chi, h_c(x_c, x), d\right)\right\| \\
&\leq -\alpha_3(\omega_N(\chi)) + A\|\tilde{x}^P\| \\
&\leq -\alpha_3(\omega_N(\chi)) + A\varepsilon \\
&\leq -\alpha_3 \circ \alpha_2^{-1}(V) + A\varepsilon, \qquad (49)
\end{aligned}$$

where A is chosen so that $\|\partial V/\partial \chi\|\bar{\gamma} \leq A$. Inequality (49) holds for all $t \in [T_0, T_1)$ and all $\chi \in \Omega_{c_2}$. From (49) and the fact that ε has been chosen so that $d_\varepsilon < c_1$, we have that $\chi(t)$ cannot cross the boundary of Ω_{c_1}, and therefore χ is bounded for all $t \geq 0$, i.e., $T_1 = \infty$. Moreover, by noticing that, whenever $V > d_\varepsilon$, $\dot{V} \leq -\dfrac{\alpha_3 \circ \alpha_2^{-1}(V)}{2} < 0$, we conclude that Ω_ε is asymptotically stable and is reached in finite time. The boundedness of all the internal variables follows from this result and the fact that (Lemma 1 and 2) the controller states are well-defined and bounded for all $t \geq 0$.

■

Remark 8: Note that, as $\varepsilon \to 0$, Ω_ε becomes arbitrarily close to $\mathcal{N}$. Thus, Theorem 3 implies that the original uniformly asymptotically stable positively invariant set under state feedback is recovered, in the limit as $\varepsilon \to 0$, under output feedback.

Remark 9: Note that the implication of Theorem 3 is essentially identical to one idea found in [11]. This indicates that using an input derivative observer is advantageous in that, besides simplifying the controller design, in the presence of disturbance it yields the same stability results as when dynamic extension is employed at the input side of the system.

Theorem 4 *Suppose that Assumptions A1, A2', A3, A4' are satisfied and the initial condition $\chi(0)$ is contained in $\mathcal{D}' \subset \Omega_{c_1}$. If $d(t) = d^0$, i.e., if no disturbance affects the system, then the output feedback control law (47) (where $\hat{x}^P$ is generated by (21), (22), (46)) renders $\mathcal{N}$ a uniformly asymptotically stable positively invariant set for (6), with domain of attraction containing Ω_{c_1}.*

Proof. When $d(t) = d^0$, from Theorem 2, part (ii) and Lemma 2 we have that $\tilde{x}^P \to 0$ as $t \to \infty$. Arguments completely analogous to the ones used in the proof of Lemma 3 and Theorem 2 in [7, 8] lead to the conclusion that $\mathcal{N}$ is uniformly asymptotically stable.

■

5 Conclusions

Theorems 3 and 4 establish a separation principle for incompletely observable MIMO nonlinear systems with and without disturbances. These results extend and simplify our previous result in [7, 8], but are inspired by the same methodology: the output feedback controller design is entirely carried out in the original system coordinates (hence the analytical knowledge of $\mathcal{H}^{-1}$ is not needed) and an appropriate projection is designed to deal with the incomplete observability of the system and, at the same time, to guarantee separation between the state feedback controller and observer designs. When Assumptions A2 and A3 hold globally, Theorems 3 and 4 show that any compact set in $\mathbb{R}^n \times \mathbb{R}^{n_c}$ can be made a region of attraction for the sets Ω_ε and $\mathcal{N}$, respectively. In other words, the closed-loop system becomes semi-globally stable with respect to Ω_ε, $\mathcal{N}$, respectively. We did not investigate conditions under which $\mathcal{N}$ is asymptotically stable under output feedback even when $d(t) \neq d^0$. The reader may refer to the analysis in [11, 6] for more details on this issue.

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