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Output Feedback Control for Stabilizable and Incompletely Observable Nonlinear Systems*

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Abstract

This paper introduces a new approach for output feedback stabilization of SISO systems which, unlike most of the techniques found in the literature, does not use high-gain observers and control input saturation to achieve separation between the state feedback and observer designs. Rather, we show that by using nonlinear observers, together with a projection algorithm, the same kind of separation principle is achieved for a larger class of systems, namely stabilizable and incompletely observable plants. Furthermore, this new approach avoids using knowledge of the inverse of the observability mapping, which is needed by most techniques in the literature when controlling general stabilizable systems.

1 Introduction

The area of nonlinear output feedback control has received much attention after the publication of the work [1], in which the authors developed a systematic strategy for the output feedback control of input-output linearizable systems with full relative degree, which employed two basic tools: an high-gain observer to estimate the derivatives of the outputs (and hence the system states in transformed coordinates), and control input saturation to isolate the peaking phenomenon of the observer from the system states. Essentially the same approach has later been applied in a number of papers by various researchers (see, e.g., [2, 3, 4, 5, 6, 7, 8]) to solve different problems in output feedback control. In most of the papers found in the literature, (see,

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e.g., [1, 2, 3, 5, 4, 6]) the authors consider input-output feedback linearizable systems with either full relative degree or minimum phase zero dynamics. The work in [9] showed that for nonminimum phase systems the problem can be solved by extending the system with a chain of integrators at the input side. However, the results contained there are local. In [10], by putting together this idea with the approach found in [1], the authors were able to show how to solve the output feedback stabilization problem for general stabilizable and uniformly completely observable systems, provided that the inverse of the observability mapping is explicitly known. The recent work in [8] unifies all these approaches to prove a separation principle for a very general class of nonlinear systems. It appears that the largest class of nonlinear SISO systems for which the output feedback stabilization problem has been solved is that of locally stabilizable and completely observable systems. Moreover, when dealing with systems which are not feedback linearizable, the works [9, 10, 8] require the explicit knowledge of the inverse of the observability mapping, thus somewhat restricting the variety of problems to which their algorithm can be applied.

The objective of this paper is to relax the two restrictions above, by developing a new output feedback strategy for nonlinear SISO locally or globally stabilizable systems which are only observable on regions of the state space. Furthermore, for the implementation of our controller, the inverse of the observability mapping is not needed. These two features are achieved by means of a nonlinear observer instead of the standard high-gain observer found in the literature, and of a new projection algorithm which *eliminates* the peaking phenomenon in the observer states, thus avoiding the need to use control input saturation. To the best of our knowledge, this work, besides introducing a new methodology for output feedback control design, enlarges the class of SISO systems considered in the literature of the field so far.

2 Problem Formulation and Assumptions

Consider the following dynamical system,

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n, u, y \in \mathbb{R}$, f and h are known smooth functions, and $f(0,0) = 0$. Our control objective is to construct a stabilizing controller for (1) without the availability of the system states x. In order to do so, we need an observability assumption. Define

$$y_e \triangleq \begin{bmatrix} y \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \mathcal{H}\left(x, u, \ldots, u^{(n_u-1)}\right) \triangleq \begin{bmatrix} h(x,u) \\ \varphi_1(x,u,u^{(1)}) \\ \vdots \\ \varphi_{n-1}\left(x,u,\ldots,u^{(n_u-1)}\right) \end{bmatrix} \tag{2}$$

($y^{(n-1)}$ is the $n-1$-th derivative) where

$$
\begin{aligned}
\varphi_1(x,u,u^{(1)}) &= \frac{\partial h}{\partial x} f(x,u) + \frac{\partial h}{\partial u} u^{(1)} \\
\varphi_2(x,u,u^{(1)},u^{(2)}) &= \frac{\partial \varphi_1}{\partial x} f(x,u) + \frac{\partial \varphi_1}{\partial u} u^{(1)} + \frac{\partial \varphi_1}{\partial u^{(1)}} u^{(2)} \\
&\vdots \\
\varphi_{n-1}\left(x,u,\dots,u^{(n_u-1)}\right) &= \frac{\partial \varphi_{n-2}}{\partial x} f(x,u) + \frac{\partial \varphi_{n-2}}{\partial u} u^{(1)} + \dots + \frac{\partial \varphi_{n-2}}{\partial u^{(n_u-2)}} u^{(n_u-1)}
\end{aligned}
\tag{3}
$$

where $0 \le n_u \le n$ ($n_u = 0$ indicates that there is no dependence on u). In the most general case, $\varphi_i = \varphi_i(x,u,\dots,u^{(i)})$, $i = 1,2,\dots,n-1$. In some cases, however, we may have that $\varphi_i = \varphi_i(x,u)$ for all $i = 1,\dots,r-1$ and some integer $r > 1$. This happens in particular when system (1) has a well-defined relative degree r. Here, we do not require the system to be input-output feedback linearizable, and hence to possess a well-defined relative degree. In the case of systems with well-defined relative degree, $n_u = 0$ corresponds to having $r \ge n$, while $n_u = n$ corresponds to having $r = 0$. Now, we are ready to state our first assumption.

Assumption A1. System (1) is observable over the set $\mathcal{X} \times \mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$ containing the origin, i.e., the mapping

$$
y_e = \mathcal{H}\left(x,u,\dots,u^{(n_u-1)}\right) \tag{4}
$$

is invertible with respect to x and its inverse is smooth, for all $x \in \mathcal{X}$, $[u,u^{(1)},\dots,u^{(n_u-1)}]^\top \in \mathcal{U}$.

Remark 1: In the existing literature, an assumption similar to A1 can be found in [9] and [10]. It is worth stressing, however, that in that work the authors adopt a global observability assumption, i.e., the set $\mathcal{X} \times \mathcal{U}$ is taken to be $\mathbb{R}^n \times \mathbb{R}^{n_u}$. In many practical applications the system under consideration may be observable in some subset of $\mathbb{R}^n \times \mathbb{R}^{n_u}$ only, thus preventing the use of most of the output feedback techniques found in the literature, including the ones found in [9], [10], and [8].

Next, augment the system with n_u integrators on the input side, which corresponds to using a compensator of order n_u. System (1) can be rewritten as follows,

$$
\begin{aligned}
\dot{x} &= f(x,z_1) \\
\dot{z}_1 &= z_2 \\
&\vdots \\
\dot{z}_{n_u} &= v
\end{aligned}
\tag{5}
$$

Notice that the differential equation (5) is now affine in the new control v. Define the extended state variable $\chi = [x^\top, z^\top]^\top \in \mathbb{R}^{n+n_u}$, and the associated *extended system*

$$
\begin{aligned}
\dot{\chi} &= f_e(\chi) + g_e v \\
y &= h_e(\chi)
\end{aligned}
\tag{6}
$$

where $f_e(\chi) = [f^\top(x,z_1),\ z_2, \dots, z_{n_u}, 0]^\top$, $g_e = [0, \dots, 1]^\top$, and $h_e(\chi) = h(x,z_1)$.

Assumption A2. The origin of (1) is locally stabilizable (stabilizable) by a static function of x, i.e., there exists a smooth function $\bar{u}(x)$ such that the origin is an asymptotically stable (globally asymptotically stable) equilibrium point of $\dot{x} = f(x, \bar{u}(x))$.

Remark 2: Assumption A2 implies that the origin of the extended system (6) is locally stabilizable (stabilizable) by a function of χ as well. A proof of the local stabilizability property for (6) may be found, e.g., in [11], while its global counterpart is a well known consequence of the integrator backstepping lemma (see, e.g., Theorem 9.2.3 in [12] or Corollary 2.10 in [13]). Therefore we conclude that for the extended system (6) there exists a smooth control $\bar{v}(\chi)$ such that its origin is asymptotically stable under closed-loop control. Let $\mathcal{D}$ be the domain of attraction of the origin of (6), and notice that, when A2 holds globally, $\mathcal{D} = \mathbb{R}^n \times \mathbb{R}^{(n_u)}$.

Remark 3: In [9] the authors consider affine systems and use a feedback linearizability assumption in place of our A2. Here, we consider the more general class of non-affine systems for which the origin is locally stabilizable (stabilizable). In this respect, our assumption A2 relaxes also the stabilizability assumption found in [10], while it is essentially equivalent to Assumption 2 in [8]. In conclusion, the class of systems satisfying A1 and A2 is, to the best of our knowledge, the largest class of SISO systems considered in the output feedback literature so far.

3 Nonlinear Observer: Its Need and Stability Analysis

Assumption A2 allows us to design a stabilizing state feedback control $v = \phi(x, z)$. In order to perform output feedback control x should be replaced by its estimate. Many researchers adopted an input-output feedback linearizability assumption ([1], [2], [5], [4], [6]) and transformed the system into normal form

$$\begin{aligned} \dot{\pi}_i &= \pi_{i+1}, \qquad 1 \le i \le r-1 \\ \dot{\pi}_r &= \bar{f}(\pi, \Pi) + \bar{g}(\pi, \Pi)\, u \\ \dot{\Pi} &= \Phi(\pi, \Pi), \qquad \Pi \in \mathbb{R}^{n-r} \\ y &= \pi_1 \end{aligned} \tag{7}$$

In this framework the problem of output feedback control finds a very natural formulation, as the first r derivatives of y are equal to the states of the π-subsystem (i.e., the linear subsystem). The works [1, 2, 4] solve the output feedback control problem for systems with no zero dynamics (i.e., $r = n$), so that the first $n-1$ derivatives of y provide the entire state of the system. In the presence of zero dynamics (Π-subsystem), the use of input-output feedback linearization to put the system into normal form (7) forces the use of a minimum phase assumption (e.g, [5]) since the states of the Π-subsystem cannot be estimated from the derivatives of the output and, hence, cannot be controlled by output feedback. It is for this reason that output feedback control of nonminimum phase systems is regarded as a particularly challenging problem. Researchers who have addressed this problem (e.g., [9], [10]) relied on the explicit knowledge of the inverse of the mapping $\mathcal{H}$ in (4)

$$x = \mathcal{H}^{-1}\left(y_e, z_1, \dots, z_{n_u}\right)$$

so that estimation of the first $n-1$ derivatives of y (the vector y_e) provides an estimate of x

$$\hat{x} = \mathcal{H}^{-1}\left(\hat{y}_e, z_1, \dots, z_{n_u}\right)$$

since the vector z, being the state of the controller, is known. Next, to estimate the derivatives of y, they employed an high-gain observer. Both the works [9] and [10] (the latter dealing with the larger class of stabilizable systems) rely on the knowledge of $\mathcal{H}^{-1}$ to prove closed-loop stability. In addition to this, the recent work [8] proves that a separation principle holds for a quite general class of nonlinear systems which includes (1) provided that $\mathcal{H}^{-1}$ is explicitly known and that the system is uniformly completely observable. In order to develop a practical output feedback control algorithm, however, $\mathcal{H}^{-1}$ cannot be assumed to be explicitly known. Hence, rather than designing an high-gain observer to estimate y_e and using $\mathcal{H}^{-1}(\cdot,\cdot)$ to get x, the approach adopted here is to estimate x directly using a nonlinear observer for system (1) and using the fact that the z-states are known. In other words, we can regard our problem as that of building a reduced order observer for the closed-loop system states[1]. The observer has the form

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}, z_1) + \left[\frac{\partial \mathcal{H}(\hat{x}, z)}{\partial \hat{x}}\right]^{-1} \mathcal{E}^{-1} L \left[y(t) - \hat{y}(t)\right] \\ \hat{y}(t) &= h(\hat{x}, z_1) \end{aligned} \tag{8}$$

where L is a $n \times 1$ vector, $\mathcal{E} = \operatorname{diag}\left[\rho, \rho^2, \dots, \rho^n\right]$, and $\rho \in (0, 1]$ is a fixed design constant.

Notice that (8) does not require any knowledge of $\mathcal{H}^{-1}$ and has the advantage of operating in x-coordinates. The observability assumption A1 implies that the Jacobian of the mapping $\mathcal{H}$ with respect to x is invertible, and hence the inverse of $\partial\mathcal{H}(\hat{x}, z)/\partial\hat{x}$ in (8) is well defined. In the work [14], the authors used an observer structurally identical to (8), for the more restrictive class of input-output feedback linearizable systems with full relative degree. Here, by modifying the definition of the mapping $\mathcal{H}$, we considerably relax these conditions by just requiring the general observability assumption A1 to hold. Furthermore, we propose a different proof than the one found in [14] which, besides being easier (in our view), clarifies the relationship among (8) and the high-gain observers commonly found in the output feedback literature. Next, we state the result and its proof.

Theorem 1 *Consider system (5) and assume A1 is satisfied for $\mathcal{X} = \mathbb{R}^n$ and $\mathcal{U} = \mathbb{R}^{n_u}$, the state χ belongs to a compact invariant set Ω, and that $|v(t)| \leq M$ for all $t \geq 0$, with M a positive constant. Choose L such that $A_c - LC_c$, where (A_c, B_c, C_c) is the controllable/observable canonical realization, is Hurwitz.*

Under these conditions and using observer (8), the following two properties hold

- *(i) Asymptotic stability of the estimation error: There exists $\bar{\rho}$, $0 < \bar{\rho} \leq 1$, such that for all $\rho \in (0, \bar{\rho})$, $\hat{x} \to x$ as $t \to +\infty$.*

[1] Throughout this section we assume A1 to hold globally, since we are interested in the ideal convergence properties of the state estimates. In the next section we will show how to modify the observer equation in order to achieve the same convergence properties when A1 holds over the set $\mathcal{X} \times \mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$.

(ii) Arbitrarily fast rate of convergence: For each positive T, ϵ, there exists ρ^*, $0 < \rho^* \leq 1$, such that for all $\rho \in (0, \rho^*]$, $\|\hat{x} - x\| \leq \epsilon \; \forall t \geq T$.

Proof. Consider the filtered transformation

$$\xi = \mathcal{H}(x, z) = \begin{bmatrix} h(x, z_1) \\ \varphi_1(x, z_1, z_2) \\ \vdots \\ \varphi_{n-1}(x, z_1, \ldots, z_{n_u}) \end{bmatrix} \tag{9}$$

A1 guarantees that $x = \mathcal{H}^{-1}(\xi, z)$ is well-defined, unique, and smooth. Let us express system (5) in new coordinates. By definition, $\xi = [y, \dot{y}, \ldots, y^{(n-1)}]^\top$ and, with φ_{n-1} defined in (3),

$$\begin{aligned} y^{(n)} &= \left[\frac{\partial \varphi_{n-1}}{\partial x} f\left(\mathcal{H}^{-1}(\xi, z), z\right) + \sum_{k=1}^{n_u - 1} \frac{\partial \varphi_{n-1}}{\partial z_k} \left(\mathcal{H}^{-1}(\xi, z), z\right) \; z_{k+1} \right] + \left[\frac{\partial \varphi_{n-1}}{\partial z_{n_u}} \left(\mathcal{H}^{-1}(\xi, z), z\right) \right] v \\ &\triangleq \alpha(\xi, z) + \beta(\xi, z) v \end{aligned}$$

Hence, in the new coordinates (5) becomes

$$\dot{\xi} = A_c \xi + B_c \left[\alpha(\xi, z) + \beta(\xi, z) v\right] \tag{10}$$

Next, transform the observer (8) to new coordinates $\hat{\xi} = \left[\hat{y}, \dot{\hat{y}}, \ldots, \hat{y}^{(n-1)}\right]^\top = \mathcal{H}(\hat{x}, z)$ so that

$$\begin{aligned} \dot{\hat{\xi}}_1 &= \frac{\partial h}{\partial \hat{x}} f(\hat{x}, z_1) + \frac{\partial h}{\partial \hat{x}} \left[\frac{\partial \mathcal{H}}{\partial \hat{x}}\right]^{-1} \mathcal{E}^{-1} L \left[y - h(\hat{x}, z_1)\right] + \frac{\partial h}{\partial z_1} \dot{z}_1 \\ &= \hat{\xi}_2 + \frac{\partial h}{\partial \hat{x}} \left[\frac{\partial \mathcal{H}}{\partial \hat{x}}\right]^{-1} \mathcal{E}^{-1} L \left[y - h(\hat{x}, z_1)\right] \end{aligned} \tag{11}$$

Similarly, for $i = 2, \ldots, n-1$

$$\dot{\hat{\xi}}_i = \frac{\partial \varphi_{i-1}}{\partial \hat{x}} (\hat{x}, z_1, \ldots, z_i) \left\{ f(\hat{x}, z_1) + \left[\frac{\partial \mathcal{H}}{\partial \hat{x}}\right]^{-1} \mathcal{E}^{-1} L \left(y - h(\hat{x}, z_1)\right) \right\} + \sum_{k=1}^{i} \frac{\partial \varphi_{i-1}}{\partial z_k} z_{k+1} \tag{12}$$

By definition,

$$\hat{\xi}_{i+1} = \varphi_i(\hat{x}, z_1, \ldots, z_{i+1}) = \frac{\partial \varphi_{i-1}}{\partial \hat{x}} f(\hat{x}, z_1) + \sum_{k=1}^{i} \frac{\partial \varphi_{i-1}}{\partial z_k} z_{k+1}$$

Hence, we conclude that

$$\dot{\hat{\xi}}_i = \hat{\xi}_{i+1} + \frac{\partial \varphi_{i-1}}{\partial \hat{x}} \left[\frac{\partial \mathcal{H}}{\partial \hat{x}}\right]^{-1} \mathcal{E}^{-1} L \left(y - h(\hat{x}, z_1)\right), \qquad i = 2, \ldots, n-1 \tag{13}$$

Finally,

$$\dot{\hat{\xi}}_n = \alpha(\hat{\xi}, z) + \beta(\hat{\xi}, z)v + \frac{\partial \varphi_{n-1}}{\partial \hat{x}} \left[\frac{\partial \mathcal{H}}{\partial \hat{x}}\right]^{-1} \mathcal{E}^{-1} L\left[y - h(\hat{x}, z_1)\right] \tag{14}$$

By using (11), (13), and (14) we can write, in compact form,

$$\begin{aligned} \dot{\hat{\xi}} &= A_c\hat{\xi} + B_c[\alpha(\hat{\xi}, z) + \beta(\hat{\xi}, z)v] + \left[\frac{\partial \mathcal{H}}{\partial \hat{x}}\right] \left[\frac{\partial \mathcal{H}}{\partial \hat{x}}\right]^{-1} \mathcal{E}^{-1} L\left[y - h(\hat{x}, z_1)\right] \\ &= A_c\hat{\xi} + B_c[\alpha(\hat{\xi}, z) + \beta(\hat{\xi}, z)v] + \mathcal{E}^{-1} L\left[\xi_1 - \hat{\xi}_1\right] \end{aligned} \tag{15}$$

Define the observer error in the new coordinates, $\tilde{\xi} = \hat{\xi} - \xi$. Then, the observer error dynamics are given by

$$\dot{\tilde{\xi}} = (A_c - \mathcal{E}^{-1} L\, C_c)\tilde{\xi} + B_c \left[\alpha(\hat{\xi}, z) + \beta(\hat{\xi}, z)v - \alpha(\xi, z) - \beta(\xi, z)v\right] \tag{16}$$

Next, define the coordinate transformation

$$\tilde{\nu} = \mathcal{E}'\tilde{\xi}, \qquad \mathcal{E}' \triangleq \operatorname{diag}\left[\frac{1}{\rho^{n-1}}, \frac{1}{\rho^{n-2}}, \dots, 1\right] \tag{17}$$

In the new domain the observer error equation becomes

$$\dot{\tilde{\nu}} = \frac{1}{\rho}(A_c - LC_c)\tilde{\nu} + B_c \left[\alpha(\hat{\xi}, z) + \beta(\hat{\xi}, z)v - \alpha(\xi, z) - \beta(\xi, z)v\right] \tag{18}$$

where, by assumption, $A_c - LC_c$ is Hurwitz. Let P be the solution to the Lyapunov equation

$$P(A_c - LC_c) + (A_c - LC_c)^\top P = -I \tag{19}$$

and consider the Lyapunov function candidate $V_o(\tilde{\nu}) = \tilde{\nu}^\top P \tilde{\nu}$. Calculate the time derivative of V_o along the $\tilde{\nu}$ trajectories:

$$\dot{V}_o = -\frac{\tilde{\nu}^\top \tilde{\nu}}{\rho} + 2\tilde{\nu}^\top P B_c \left[\alpha(\hat{\xi}, z) + \beta(\hat{\xi}, z)v - \alpha(\xi, z) - \beta(\xi, z)v\right] \tag{20}$$

Next, we seek to provide a bound to the bracketed term in (20). Without loss of generality let $[\hat{x}(0)^\top, z(0)^\top]^\top \in \Omega$ and define the compact set $\mathcal{K}_{\tilde{\xi}} \triangleq \left\{\tilde{\xi} \in \mathbb{R}^n \mid \hat{\xi}, \xi \in \mathcal{H}(\Omega)\right\}$. By definition, $\mathcal{K}_{\tilde{\xi}}$ contains the initial condition $\tilde{\xi}(0)$. Next, define the set $\Lambda_\kappa \triangleq \left\{\tilde{\xi} \in \mathbb{R}^n \mid V_o(\mathcal{E}'\tilde{\xi}) \leq \kappa\right\}$ where κ is chosen so that $\mathcal{K}_{\tilde{\xi}} \subset \Lambda_\kappa$. In the following we will prove that Λ_κ is invariant under (16), and that all trajectories originating in Λ_κ converge asymptotically to the origin. Recalling that $\hat{\xi} = \xi + \tilde{\xi}$, let $\mathcal{K}_{\hat{\xi}} \triangleq \left\{\hat{\xi} \in \mathbb{R}^n \mid \xi \in \mathcal{H}(\Omega), \tilde{\xi} \in \Lambda_\kappa\right\}$, and notice that $\mathcal{H}(\Omega) \subset \mathcal{K}_{\hat{\xi}}$.

Due to the smoothness of $\alpha + \beta v$ and the boundedness of v, the following inequality holds true over the compact set $\mathcal{K}_{\hat{\xi}}$:

$$\begin{aligned} &\sup_{|v| \leq M} \left\|\alpha(\xi^1, z) + \beta(\xi^1, z)v - \alpha(\xi^2, z) - \beta(\xi^2, z)v\right\| \leq \gamma\|\xi^1 - \xi^2\|, \\ &\forall \xi^1, \xi^2 \in \mathcal{K}_{\hat{\xi}}, \forall z \in \Omega^z \triangleq \{z \in \mathbb{R}^{n_u} \mid \chi \in \Omega\} \end{aligned} \tag{21}$$

Using this inequality in (20)

$$\dot{V}_o \leq -\frac{\|\tilde{\nu}\|^2}{\rho} + 2\|P\|\gamma\|\tilde{\xi}\|\|\tilde{\nu}\| \leq -\frac{\|\tilde{\nu}\|^2}{\rho} + 2\|P\|\gamma\|\tilde{\nu}\|^2 \tag{22}$$

Defining $\bar{\rho} = \min\{1/(2\|P\|\gamma), 1\}$, we conclude that for all $\rho < \bar{\rho}$ the $\tilde{\xi}$ trajectories starting in $\mathcal{K}_{\tilde{\xi}}$ will converge asymptotically to the origin, and hence part (i) of the theorem is proved.

As for part (ii), note that $\lambda_{\min}(\mathcal{E}'P\mathcal{E}') \geq \lambda_{\min}(\mathcal{E}')^2\lambda_{\min}(P) = \lambda_{\min}(P)$, since $\lambda_{\min}(\mathcal{E}') = 1$. Next, $\lambda_{\max}(\mathcal{E}'P\mathcal{E}') \leq \lambda_{\max}(\mathcal{E}')^2\lambda_{\max}(P) = 1/(\rho^{2(n-1)})\lambda_{\max}(P)$, since $\lambda_{\max}(\mathcal{E}') = 1/\rho^{(n-1)}$. Therefore

$$\lambda_{\min}(P)\|\tilde{\xi}\|^2 \leq V_o = \tilde{\xi}^\top \mathcal{E}'P\mathcal{E}'\tilde{\xi} \leq \frac{1}{\rho^{2(n-1)}}\lambda_{\max}(P)\|\tilde{\xi}\|^2 \tag{23}$$

Define $\bar{\epsilon}$ so that $\|\tilde{\xi}\| \leq \bar{\epsilon}$ implies that $\|\hat{x} - x\| \leq \epsilon$ (the smoothness of $\mathcal{H}^{-1}$ guarantees that $\bar{\epsilon}$ is well defined). By inequality (23) we have that $V_o \leq \bar{\epsilon}^2\lambda_{\min}(P)$ implies that $\|\tilde{\xi}\| \leq \bar{\epsilon}$, and $V_o(0) \triangleq V_o(\tilde{\nu}(0)) \leq (1/\rho^{2(n-1)})\lambda_{\max}(P)\|\tilde{\xi}(0)\|^2$. Moreover, from (22)

$$\dot{V}_o(t) \leq -\left(\frac{1}{\rho} - 2\|P\|\gamma\right)\|\tilde{\nu}\|^2 \leq -\frac{1}{\lambda_{\min}(P)}\left(\frac{1}{\rho} - 2\|P\|\gamma\right)V_o(t) \tag{24}$$

Therefore, by the Comparison Lemma (see, e.g., [15]), $V_o(t)$ satisfies the following inequality

$$\begin{aligned} V_o(t) &\leq V_o(0)\exp\left\{-\frac{1}{\lambda_{\min}(P)}\left(\frac{1}{\rho} - 2\|P\|\gamma\right)t\right\} \\ &\leq \frac{1}{\rho^{2(n-1)}}\lambda_{\max}(P)\|\tilde{\xi}(0)\|^2\exp\left\{-\frac{1}{\lambda_{\min}(P)}\left(\frac{1}{\rho} - 2\|P\|\gamma\right)t\right\} \end{aligned} \tag{25}$$

which, for sufficiently small ρ, can be written as

$$V_o(t) \leq \frac{a_1}{\rho^{2n}}\exp\left\{-\frac{a_2}{\rho}t\right\}, \qquad a_1, a_2 > 0$$

An upper estimate of the time T such that $\|\hat{x} - x\| \leq \epsilon$ for all $t \geq T$, is calculated as follows

$$\frac{a_1}{\rho^{2n}}\exp\left\{-\frac{a_2}{\rho}t\right\} \leq \bar{\epsilon}^2\lambda_{\min}(P) \text{ for all } t \geq T = \frac{2n\rho}{a_2}\log\left(\frac{a_1}{\bar{\epsilon}\rho}\right)$$

Noticing that $T \to 0$ as $\rho \to 0$, we conclude that T can be made arbitrarily small by choosing a sufficiently small ρ^*, thus concluding the proof of part (ii).

■

Remark 4: Part (ii) of Theorem 1 implies that the observer convergence rate can be made arbitrarily fast. This property is essential for closed-loop stability.

Remark 5: Using inequality (25), we find the upper bound for the estimation error in ξ-coordinates

$$\|\tilde{\xi}\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\frac{1}{\rho^{n-1}}\|\tilde{\xi}(0)\|\exp\left\{-\frac{1}{2\lambda_{\min}(P)}\left(\frac{1}{\rho} - 2\|P\|\gamma\right)t\right\} \tag{26}$$

Hence, during the initial transient, $\tilde{\xi}(t)$ may exhibit peaking, and the size of the peak grows larger as ρ decreases and the convergence rate is made faster. This phenomenon and its implications on output feedback control has been studied in the seminal work [1]. The analysis in that paper shows that a way to isolate the peaking of the observer estimates from the system states is to saturate the control input outside of the compact set of interest. The same idea has then been adopted in several other works in the output feedback control literature (see, e.g., [1, 2, 10, 3, 5, 4, 6, 7, 8]). Rather than following this approach, in the next section we will present a new technique for isolating the peaking phenomenon which allows for the use of the weaker Assumption A1.

Remark 6: It is interesting to note that in ξ-coordinates the nonlinear observer (8) is identical to the standard high-gain observer found in the nonlinear output feedback control literature (see, e.g., [9, 1, 2, 10, 3, 5, 4, 6, 7, 8]). Our observer, however, has the advantage of avoiding the knowledge of the inverse of the mapping $\mathcal{H}$, as well as working in x coordinates, directly.

4 Output Feedback Stabilizing Control

Consider system (6), by using assumption A2 and Remark 2 we conclude that there exists a smooth stabilizing control $v = \phi(x,z) = \phi(\chi)$ which makes the origin of (6) an asymptotically stable equilibrium point with domain of attraction $\mathcal{D}$. By the converse Lyapunov theorem found in [16], there exists a continuously differentiable function V defined on $\mathcal{D}$ satisfying, for all $\chi \in \mathcal{D}$,

$$\alpha_1(\|\chi\|) \leq V(\chi) \leq \alpha_2(\|\chi\|) \tag{27}$$

$$\lim_{\chi \to \partial\mathcal{D}} \alpha_1(\|\chi\|) = \infty \tag{28}$$

$$\frac{\partial V}{\partial \chi}\left(f_e(\chi) + g_e\, v\right) \leq -\alpha_3(\|\chi\|) \tag{29}$$

where $\alpha_i,\ i = 1,2,3$ are class $\mathcal{K}$ functions (see [17] for a definition), and $\partial\mathcal{D}$ stands for the boundary of the set $\mathcal{D}$. Define compact sets Ω_{c_1}, Ω_{c_2}, $\Omega^x_{c_2}$, and $\Omega^z_{c_2}$ as follows

$$\Omega_{c_1} \triangleq \{\chi \,|\, V \leq c_1\},\ \Omega_{c_2} \triangleq \{\chi \,|\, V \leq c_2\},\ \Omega^x_{c_2} \triangleq \{x \in \mathbb{R}^n \,|\, \chi \in \Omega_{c_2}\},\ \Omega^z_{c_2} \triangleq \{z \in \mathbb{R}^{n_u} \,|\, \chi \in \Omega_{c_2}\}$$

where $c_2 > c_1 > 0$. Next, the following assumption is needed.

Assumption A3. Assume c_2 can be selected so that the following conditions are satisfied:

1. $\mathcal{H}(\Omega^x_{c_2}, z) \subset C_\xi(z) \subset \mathcal{H}(\mathcal{X}, z)$, for all $z \in \Omega^z_{c_2}$, for some convex compact $C_\xi(z)$
2. $\Omega^z_{c_2} \subset \mathcal{U}$

Remark 7: See Figure 1 for a pictorial representation of the sets in A3. This assumption represents a basic requirement for output feedback control. It is satisfied when there exists a sphere of dimension $n + n_u$, centered at the origin, which is contained in $\mathcal{X} \times \mathcal{U}$; this requirement is satisfied in most practical examples. On the other hand, Assumption A3 fails when, for example, the origin belongs to the boundary of $\mathcal{X} \times \mathcal{U}$,

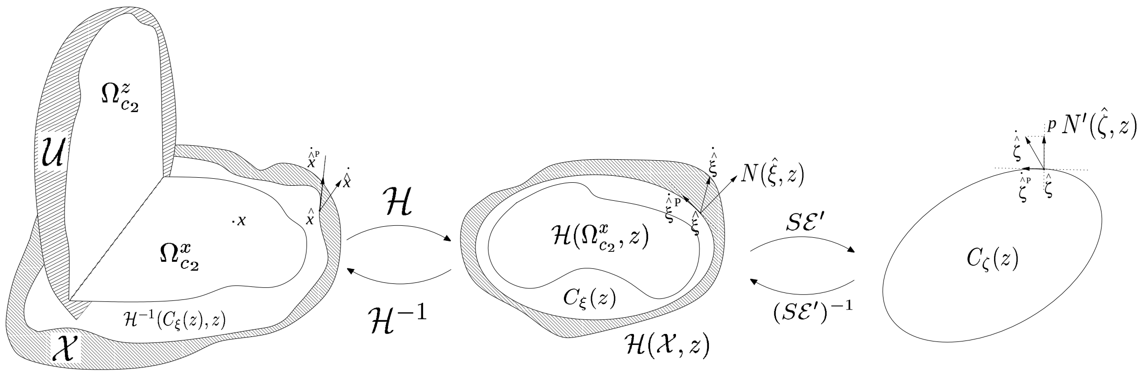


Figure 1: The mechanism behind the observer estimates projection.

and thus there is no neighborhood centered at the origin and contained in $\mathcal{X} \times \mathcal{U}$.

4.1 Observer Estimates Projection

As we already pointed out in Remark 4, in order to isolate the peaking phenomenon from the system states, the approach generally adopted in several papers is to saturate the control input to prevent it from growing above a given threshold. This technique, however, does not eliminate the peak in the observer estimate and, hence, cannot be used to control general systems like the ones satisfying assumption A1, since even when the system state lies in the observable region $\mathcal{X} \times \mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$, the observer estimates may enter the unobservable domain where (8) is not well defined. It appears that in order to deal with systems that are not completely observable, one has to eliminate the peaking from the observer by guaranteeing its estimates to be confined in a prespecified compact set contained in $\mathcal{X}$.

A very common procedure used in the adaptive control literature (see [18]) to confine vectors of parameter estimates within a desired convex set is gradient projection. This idea cannot be directly applied to our problem, mainly because $\dot{\hat{x}}$ is not proportional to the gradient of the observer Lyapunov function and, thus, the projection cannot be guaranteed to preserve the convergence properties of the estimate. Inspired by this idea, however, we propose a way to modify the $\dot{\hat{x}}$ equation which confines $\hat{x}$ to within a prespecified compact set while preserving its convergence properties.

Recall the coordinate transformation defined in (9) and let

$$\xi = \mathcal{H}(x, z), \qquad \hat{\xi} = \mathcal{H}(\hat{x}, z), \qquad \tilde{\xi} = \hat{\xi} - \xi \tag{30}$$

Next, *project*[2] the observer estimate as follows

[2]The projection defined in (31) is discontinuous in the variable $\hat{\xi}$, therefore raising the issue of the existence and uniqueness of its solutions. We refer the reader to Remark 8, were this issue is addressed and a solution is proposed.

$$\dot{\hat{x}}^P = \left[\frac{\partial \mathcal{H}}{\partial \hat{x}}\right]^{-1} \left\{\mathcal{P}\left(\hat{\xi}, \dot{\hat{\xi}}, z, \dot{z}\right) - \frac{\partial \mathcal{H}}{\partial z}\dot{z}\right\}$$

$$\mathcal{P}(\hat{\xi}, \dot{\hat{\xi}}, z, \dot{z}) = \begin{cases} \dot{\hat{\xi}} - \Gamma \dfrac{N(\hat{\xi})\left(N(\hat{\xi}, z)^\top \dot{\hat{\xi}} + N_z(\hat{\xi}, z)^\top \dot{z}\right)}{N(\hat{\xi}, z)^\top \Gamma N(\hat{\xi}, z)} & \text{if } N(\hat{\xi}, z)^\top \dot{\hat{\xi}} + N_z(\hat{\xi}, z)^\top \dot{z} \geq 0 \text{ and } \hat{\xi} \in \partial C_\xi(z) \\ \dot{\hat{\xi}} & \text{otherwise} \end{cases} \tag{31}$$

where $\Gamma = (S\mathcal{E}')^{-1}(S\mathcal{E}')^{-1}$, $S = S^\top$ denotes the matrix square root of P (defined in (19)) and $N(\hat{\xi}, z)$, $N_z(\hat{\xi}, z)$ are the normal vectors to the boundary of $C_\xi(z)$ with respect to ξ and z, respectively. The following lemma shows that (31) guarantees boundedness and preserves convergence for $\hat{x}$.

Lemma 1 *: If A3 holds and (31) is used:*

(i) Boundedness: $\hat{x}^P(t) \in \mathcal{H}^{-1}(C_\xi(z), z) \subset \mathcal{X}$ for all t, and for all $z \in \Omega^z_{c_2}$.

If, in addition, $x \in \Omega^x_{c_2}$ and the assumptions of Theorem 1 are satisfied, then the following is also true

(ii) Preservation of original convergence characteristics: properties (i) and (ii) established by Theorem 1 remain valid for $\hat{x}^P$.

Proof. In order to prove part (i) we need another coordinate transformation, $\zeta = S\mathcal{E}'\xi$, (similarly, let $\hat{\zeta} = S\mathcal{E}'\hat{\xi}$, $\tilde{\zeta} = S\mathcal{E}'\tilde{\xi}$). Denote by $C_\zeta(z)$ the compact convex set in ζ coordinates, i.e., $C_\zeta(z) \triangleq \left\{\zeta \in \mathbb{R}^n \mid (\mathcal{E}')^{-1}S^{-1}\zeta \in C_\xi(z)\right\}$, and let $N'(\hat{\zeta}, z)$, $N'_z(\hat{\zeta}, z)$ be the normal vectors to the boundary of $C_\zeta(z)$ with respect to ζ and z, respectively ($N'(\hat{\zeta}, z)$ is shown in Figure 1). Hence, it is sufficient to show that the projection (31) will keep $\hat{\zeta}$ inside $C_\zeta(z)$, which in turn guarantees that $\hat{x} = \mathcal{H}^{-1}\left(\hat{\xi}, z\right)$ is contained in the compact set $\mathcal{H}^{-1}\left(C_\xi(z), z\right) \subset \mathcal{X}$. After coordinate transformation (30) we have that

$$\dot{\hat{\xi}}^P = \frac{d}{dt}\left\{\mathcal{H}(\hat{x}^P, z)\right\} = \left[\frac{\partial \mathcal{H}}{\partial \hat{x}}\dot{\hat{x}}^P + \frac{\partial \mathcal{H}}{\partial z}\dot{z}\right] \tag{32}$$

$$= \mathcal{P}(\hat{\xi}, \dot{\hat{\xi}}) \tag{33}$$

In order to relate $N'(\hat{\zeta}, z)$, $N'_z(\hat{\zeta}, z)$ to $N(\hat{\xi}, z)$, $N'_z(\hat{\zeta}, z)$, notice that the boundary of $C_\xi(z)$ can be expressed as the set $\partial C_\xi(z) = \{\xi \in \mathbb{R}^n \mid b(\xi, z) = 0\}$ for some continuous function $b(\xi, z)$, and hence $N(\hat{\xi}, z) = \nabla_\xi b|_{(\xi=\hat{\xi}, z)}$, $N_z(\hat{\xi}, z) = \nabla_z b|_{(\xi=\hat{\xi}, z)}$. Similarly, the boundary of $C_\zeta(z)$ is the set $\partial C_\zeta(z) = \{\zeta \in \mathbb{R}^n \mid b((S\mathcal{E}')^{-1}\zeta, z) = 0\}$ and $N'(\hat{\zeta}, z) = (S\mathcal{E}')^{-1}\nabla_\zeta b|_{(\xi=\hat{\xi}, z)} = (S\mathcal{E}')^{-1}N(\hat{\xi}, z)$, $N'_z(\hat{\zeta}, z) = N_z(\hat{\xi}, z)$. The expression of the projection (31) in ζ coordinates is found by noting that

$$\dot{\hat{\zeta}}^P = S\mathcal{E}'\dot{\hat{\xi}}^P = \begin{cases} S\mathcal{E}'\dot{\hat{\xi}} - (S\mathcal{E}')^{-1}\dfrac{N\left(N^\top \dot{\hat{\xi}} + N_z^\top \dot{z}\right)}{N^\top \Gamma N} & \text{if } N^\top \dot{\hat{\xi}} + N_z^\top \dot{z} \geq 0 \text{ and } \hat{\xi} \in \partial C_\xi(z) \\ S\mathcal{E}'\dot{\hat{\xi}} & \text{otherwise} \end{cases} \tag{34}$$

and then substituting $N' = (S\mathcal{E}')^{-1}N$, $N'_z = N_z$, and $\dot{\hat{\xi}} = (S\mathcal{E}')^{-1}\dot{\hat{\zeta}}$, to find that

$$\dot{\hat{\zeta}}^P = \begin{cases} \dot{\hat{\zeta}} - \dfrac{N'\left(N'^\top \dot{\hat{\zeta}} + N_z'^\top \dot{z}\right)}{N'^\top N'} & \text{if } N'^\top \dot{\hat{\zeta}} + N_z'^\top \dot{z} \geq 0 \text{ and } \hat{\zeta} \in \partial C_\zeta(z) \\ \dot{\hat{\zeta}} & \text{otherwise} \end{cases} \tag{35}$$

Next, we show that the domain $C_\zeta(z) = \{\zeta \in \mathbb{R}^n \,|\, b((S\mathcal{E}')^{-1}\zeta, z) \leq 0\}$ is invariant for (35). In order to do that, consider the candidate Lyapunov function $V_{C_\zeta} = \max\{b\left((S\mathcal{E}')^{-1}\zeta, z\right), 0\}$ which is positive definite with respect to the set $C_\zeta(z)$, and calculate its time derivative along the trajectory of (35) when $\hat{\zeta}^P \in \partial C_\zeta(z)$,

$$\dot{V}_{C_\zeta} = N'(\hat{\zeta}, z)^\top \dot{\hat{\zeta}}^P + N_z'(\hat{\zeta}^P, z)\dot{z} \tag{36}$$

$$= N'^\top \dot{\hat{\zeta}} - \frac{N'^\top N'\left(N'^\top \dot{\hat{\zeta}} + N_z'^\top \dot{z}\right)}{N'^\top N'} + N_z'\dot{z} \tag{37}$$

$$= 0 \tag{38}$$

thus showing that $C_\zeta(z)$ is an invariant set for (35) and, hence, that $\hat{\zeta}^P(t) \in C_\zeta(z)$ for all t, which in turn implies that $\hat{\xi}^P(t) \in C_\xi(z)$ for all t and, finally, $\hat{x}^P(t) \in \mathcal{H}^{-1}(C_\xi(z), z)$ for all t, thus proving part (i) of the theorem.

The proof of part (ii) is based on the knowledge of a Lyapunov function for the observer in $\tilde{\nu}$ coordinates (see (17)). Notice that $\tilde{\zeta} = S\tilde{\nu}$, and $V_o = \tilde{\nu}^\top P \tilde{\nu} = (\tilde{\nu}^\top S)(S\tilde{\nu}) = \tilde{\zeta}^\top \tilde{\zeta}$. We want to show that, in $\tilde{\zeta}$ coordinates, $\dot{V}_o < 0$ and $\dot{\hat{\zeta}}^P = S\mathcal{E}'\dot{\hat{\xi}}^P$ implies that $\dot{V}_o^P \leq \dot{V}_o$, where $\tilde{\zeta}^P = \hat{\zeta}^P - \zeta$, and $V_o^P = \tilde{\zeta}^{P\top} \tilde{\zeta}^P$.

From (35), when $\hat{\zeta}$ is in the interior of $C_\zeta(z)$, or $\hat{\zeta}$ is on the boundary of $C_\zeta(z)$ and $N'^\top \dot{\hat{\zeta}} + N_z'^\top \dot{z} < 0$ (i.e., the update is pointed to the interior of $C_\zeta(z)$), we have that $V_o = V_o^P$. Let us consider all the remaining cases and, since $\hat{\zeta}^P = \hat{\zeta}$ and $\tilde{\zeta}^P = \tilde{\zeta}$ (projection only operates on $\dot{\hat{\zeta}}$), we have

$$\dot{V}_o^P = 2\tilde{\zeta}^P \dot{\tilde{\zeta}}^P = 2\tilde{\zeta}^\top \dot{\tilde{\zeta}}^P = 2\tilde{\zeta}^\top \left[\dot{\hat{\zeta}} - \dot{\zeta} - p(\hat{\zeta}, \dot{\hat{\zeta}}, z, \dot{z})N'(\hat{\zeta})\right] \tag{39}$$

where $p(\hat{\zeta}, \dot{\hat{\zeta}}, z, \dot{z}) = \dfrac{N'^\top \dot{\hat{\zeta}} + N_z'^\top \dot{z}}{N'(\hat{\zeta})^\top N'(\hat{\zeta})}$ is nonnegative since, by assumption, $N'^\top \dot{\hat{\zeta}} + N_z'^\top \dot{z} \geq 0$. Thus,

$$\dot{V}_o^P = \dot{V}_o - 2p\tilde{\zeta}^\top N' \tag{40}$$

Since, by assumption, $x \in \Omega_{c_2}^x$, we have that $\zeta \in C_\zeta(z)$ and, since $\hat{\zeta}$ lies on the boundary of $C_\zeta(z)$, the vector $\hat{\zeta} - \zeta$ points outside of $C_\zeta(z)$ which, by the convexity of $C_\zeta(z)$, implies that $\tilde{\zeta}^\top N' \geq 0$, thus concluding the proof of part (ii).

■

Remark 8: In order to avoid the discontinuity in the right hand side of (31) introduced by $\mathcal{P}$, one can define $\mathcal{P}$ to be the smooth projection introduced in [19]. In this case, part (i) of Lemma 1 would have to be modified to

$$\hat{x}^P(t) \in \mathcal{H}^{-1}(\bar{C}_\xi(z), z),\, \forall z \in \mathbb{R}^{n_u} \tag{41}$$

where $\bar{C}_\xi(z) \supset C_\xi(z)$ is a convex set which can be made arbitrarily close to $C_\xi(z)$, and condition 1 of A3 would have to be replaced by the following

1′. $\mathcal{H}(\Omega^x_{c_2}, z) \subset C_\xi(z) \subset \bar{C}_\xi(z) \subset \mathcal{H}(\mathcal{X}, z)$, for all $z \in \Omega^z_{c_2}$, for some convex compact sets $C_\xi(z)$ and $\bar{C}_\xi(z)$ (42)

Remark 9: From the proof of Lemma 1 we conclude that (31) performs a projection for $\hat{x}$ over the compact set $\mathcal{H}^{-1}(C_\xi(z), z)$ which, in general, is unknown since we do not know $\mathcal{H}^{-1}$, and is generally *not* convex (see Figure 1). It is interesting to note that applying a standard gradient projection for $\hat{x}$ over an arbitrary convex domain does not necessarily preserve the convergence result (ii) in Theorem 1.

4.2 Closed-Loop Stability

To perform output feedback control we replace the state feedback law $v = \phi(x, z)$ with $\hat{v} = \phi(\hat{x}^P, z)$ which, by the smoothness of ϕ and the fact that $\hat{x}^P$ is guaranteed to belong to the compact set $\mathcal{H}^{-1}(C_\xi(z), z)$, is bounded provided that z is confined to within a compact set. Furthermore, the limit (28) guarantees that for any compact set $\mathcal{D}'$ contained in the region of attraction $\mathcal{D}$, one can choose a large enough c_1 so that $\mathcal{D}' \subset \Omega_{c_1} \subset \Omega_{c_2} \subset \mathcal{D}$. When the observability assumption A1 is satisfied globally, one can choose any compact $\mathcal{D}' \subset \mathcal{D}$; hence, if A1 and A2 hold globally, $\mathcal{D}'$ can be any compact set in $\mathbb{R}^n \times \mathbb{R}^{n_u}$. Taking in account the restriction on c_2 imposed by Assumption A3, choose $\mathcal{D}'$ to be an arbitrary compact set contained in Ω_{c_1}. In the following we will show that $\hat{v}$ makes the origin of (6) asymptotically stable and that $\mathcal{D}'$ is contained in its region of attraction. The proof is divided in three steps:

1. (Lemma 2). *Invariance of* Ω_{c_2} *and ultimate boundedness:* Using the arbitrarily fast rate of convergence of the observer (see part (ii) in Theorem 1), we show that any trajectory originating in Ω_{c_1} cannot exit the set Ω_{c_2} and converges in finite time to an arbitrarily small neighborhood of the origin. Here, Lemma 1 plays an important role, in that it guarantees $\hat{v}$ to be bounded, and thus it allows us to use the same idea found in [1] to prove stability.
2. (Lemma 3). *Asymptotic stability of the origin:* By using Lemma 2 and the exponential stability of the observer estimate, we prove that the origin of the closed-loop system is asymptotically stable.
3. (Theorem 2). *Closed-loop stability:* Finally, by putting together the results of Lemma 2 and 3, we conclude the closed-loop stability proof.

Let $\tilde{x} \triangleq \hat{x} - x$, and $\tilde{x}^P \triangleq \hat{x}^P - x$, and note that part (ii) of Lemma 1 applies and $\tilde{x}^P \to 0$ as $t \to \infty$ with arbitrarily fast rate, as long as $\chi \in \Omega_{c_2}$. Moreover, the smoothness of the control law implies that

$$\|\phi(\hat{x}, z) - \phi(x, z)\| \leq \bar{\gamma}\|\tilde{x}\| \tag{43}$$

for all $x, \hat{x} \in \mathcal{H}^{-1}(C_\xi(z), z)$, for all $z \in \Omega^z_{c_2}$ and some $\bar{\gamma} > 0$. Assume that $\hat{x}(0) \in \Omega^x_{c_2}$, and let A be a positive constant satisfying $\|\partial V/\partial \chi\| \leq A$ for all χ in Ω_{c_2} (its existence is guaranteed by V being continuously differentiable). Now, we can state the following lemma.

Lemma 2 *Suppose that the initial condition $\chi(0)$ is contained in $\mathcal{D}' \subset \Omega_{c_1}$, define the set $\Omega_\epsilon \triangleq \{\chi : V(\chi) \leq d_\epsilon\}$, where $d_\epsilon = \alpha_2 \circ \alpha_3^{-1}(\mu\, A\, \bar{\gamma}\, \epsilon)$, and choose $\epsilon > 0$ and $\mu > 1$ such that $d_\epsilon < c_1$. Then, there exists a positive scalar $\rho^*, 0 < \rho^* \leq 1$, such that, for all $\rho \in (0, \rho^*]$, the closed-loop system trajectories remain confined in Ω_{c_2}, the set $\Omega_\epsilon \subset \Omega_{c_1}$ is positively invariant, and is reached in finite time.*

Proof. Since $V(\chi(0)) \leq c_1 < c_2$, there exists a time $T_1 > 0$ such that $V(\chi(t)) \leq c_2$, for all $t \in [0, T_1)$. Choose T_0 such that $0 < T_0 < T_1$. Then, after noticing that, for all $t \in [0, T_1)$, $z \in \Omega^z_{c_2}$ and $\hat{x}^P \in \mathcal{H}^{-1}(C_\xi(z), z)$ and hence v is bounded, we can apply Theorem 1, part (ii), and conclude that for any positive ϵ there exists a positive $\rho^*, 0 < \rho^* \leq 1$ such that, for all $\rho \leq \rho^*$, $\|\tilde{x}^P\| \leq \epsilon, \forall t \in [T_0, T_1)$. Hence, for all $t \in [T_0, T_1)$, we have that $V(\chi(t)) \leq c_2$ and $\|\tilde{x}^P(t)\| \leq \epsilon$. In order for part (ii) in Theorem 1 to hold for all $t \geq T_0$, χ must belong to Ω_{c_2} for all $t \geq 0$. So far we can only guarantee that $\chi \in \Omega_{c_2}$ for all $t \in [0, T_1)$ and hence the result of Theorem 1 applies in this time interval, only. Next, we will show that $T_1 = \infty$, i.e., Ω_{c_2} is an invariant set, so that the result of Theorem 1 will be guaranteed to hold for all $t \geq 0$.

Consider the Lyapunov function candidate V defined in (27)-(29). Taking its derivative with respect to time and using (43),

$$
\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial \chi}\left[f_e(x,z) + g_e \phi(x,z)\right] + \frac{\partial V}{\partial \chi} g_e \left[\phi(\hat{x}^P, z) - \phi(x,z)\right] \\
&\leq -\alpha_3(\|\chi\|) + \left\|\frac{\partial V}{\partial \chi}\right\| \|\phi(\hat{x}^P, z) - \phi(x,z)\| \\
&\leq -\alpha_3(\|\chi\|) + A\bar{\gamma}\|\tilde{x}^P\| \\
&\leq -\alpha_3(\|\chi\|) + A\bar{\gamma}\epsilon \\
&\leq -\alpha_3 \circ \alpha_2^{-1}(V) + A\bar{\gamma}\epsilon
\end{aligned}
$$

for all $t \in [T_0, T_1)$. When $V \geq d_\epsilon$ we have that

$$\dot{V} \leq -(\mu - 1)A\bar{\gamma}\epsilon$$

hence V decays linearly, which in turn implies that $\chi(t) \in \Omega_{c_2}$ and that Ω_ϵ is reached in finite time.

■

Remark 10: The use of the projection for the observer estimate plays a crucial role in the proof of Lemma 2. As ρ is made smaller, the observer peak may grow larger, thus generating a large control input, which in turn might drive the system states χ outside of Ω_{c_1} in shorter time. The boundedness of the control input makes sure that the exit time T_1 is independent of ϵ, since the maximum size of the $\hat{v}$ will not depend on ϵ, thus allowing us to choose ϵ independently of T_1.

Lemma 2 proves that the all the trajectories starting in Ω_{c_1} will remain confined within Ω_{c_2} and converge to an arbitrarily small neighborhood of the origin in finite time. Now, in order to complete the stability analysis, it remains to show that the origin of the output feedback closed-loop system is asymptotically stable, so that if Ω_ϵ is small enough all the closed-loop system trajectories converge to it.

Lemma 3 *There exists a positive scalar ϵ^* such that for all $\epsilon \in (0, \epsilon^*]$ all the trajectories starting inside* *the compact set* $\Delta_\epsilon \triangleq \{[\chi^\top, \tilde{x}^\top]^\top \mid V \leq d_\epsilon$ *and* $\|\tilde{x}\| \leq \epsilon\}$ *converge asymptotically to the origin.*

Proof. Without loss of generality, assume ϵ is small enough so that $\hat{x} \in \mathcal{H}^{-1}(C_\xi(z), z)$ and, hence, $\tilde{x}^P = \tilde{x}$. From the proof of Theorem 1 recall that $\tilde{x} = \mathcal{H}^{-1}(\hat{\xi}, z) - \mathcal{H}^{-1}(\xi, z)$. Using A1 we have that the mapping $\mathcal{H}^{-1}$ is locally Lipschitz. Hence, there exists a neighborhood $N_{\tilde{\xi}}$ of the origin such that $\|\tilde{x}\| \leq k_0\|\tilde{\xi}\|$, for all $\tilde{\xi} \in N_{\tilde{\xi}}$, and for some positive constant k_0, which, by (26), implies that the origin of the $\tilde{x}$ system is exponentially stable. By the converse Lyapunov theorem we conclude that there exists a Lyapunov function $V_o'(\tilde{x})$ and positive constants $\bar{c}_1, \bar{c}_2, \bar{c}_3$ such that

$$\bar{c}_1\|\tilde{x}\|^2 \leq V_o' \leq \bar{c}_2\|\tilde{x}\|^2$$
$$\dot{V}_o' \leq -\bar{c}_3\|\tilde{x}\|^2$$

Define the positive scalar ϵ^* such that $\|\tilde{x}\| \leq \epsilon^*$ implies $\tilde{\xi} \in N_{\tilde{\xi}}$ (the existence of ϵ^* is a direct consequence of the fact that $\mathcal{H}$ is locally Lipschitz). Next, define the following composite Lyapunov function candidate

$$V_c(\chi, \tilde{x}) = V(\chi) + \lambda\sqrt{V_o'(\tilde{x})}, \qquad \lambda > \frac{2\sqrt{\bar{c}_2}\bar{\gamma}A}{\bar{c}_3}$$

then,

$$\begin{aligned}
\dot{V}_c &\leq -\alpha_3(\|\chi\|) + A\bar{\gamma}\|\tilde{x}\| - \frac{\lambda}{2\sqrt{V_o'(\tilde{x})}}\bar{c}_3\|\tilde{x}\|^2 \\
&\leq -\alpha_3(\|\chi\|) - \left(\frac{\bar{c}_3\lambda}{2\sqrt{\bar{c}_2}} - A\bar{\gamma}\right)\|\tilde{x}\| < 0
\end{aligned}$$

where we have used the fact that $[\chi^\top, \tilde{x}^\top]^\top \in \Delta_\epsilon$ implies that $\chi \in \Omega_{c_2}$ (provided ϵ is small enough), and hence $\left\|\frac{\partial V}{\partial \chi}\right\| \leq A$. Since $\dot{V}_c$ is negative definite, all the $[\chi^\top, \tilde{x}^\top]^\top$ trajectories starting in Δ_ϵ will converge asymptotically to the origin.

■

We are now ready to state the following closed-loop stability theorem.

Theorem 2 *For the closed-loop system (6), (8), (31), satisfying assumptions A1, A2, and A3, the control law* $\hat{v} = \phi(\hat{x}^P, z)$, *guarantees that there exists a scalar* $\rho^*, 0 < \rho^* \leq 1$, *such that, for all* $\rho \in (0, \rho^*]$, *the set* $\mathcal{D}' \times \Omega^x_{c_2}$ *is contained in the region of attraction of the origin* $(x = 0, z = 0, \hat{x} = 0)$.

Proof. By Lemma 3, there exists $\epsilon^* > 0$ such that, for all $\epsilon \in (0, \epsilon^*]$, Δ_ϵ is a region of attraction for the origin. Use Lemma 2 and the fact that $\chi(0) \in \mathcal{D}' \subset \Omega_{c_1}$ to find $\rho^*, 0 < \rho^* \leq 1$, so that for all $\rho \in (0, \rho^*]$ the state trajectories enter Δ_ϵ in finite time. This concludes the proof of the theorem.

■

Remark 11: Theorem 2 proves regional stability of the closed-loop system, since given an observability subspace $\mathcal{X} \times \mathcal{U}$, and provided condition 1 of A3 is satisfied, the control law $\hat{v}$, together with (8) and (31), make the compact set $\mathcal{D}' \times \Omega_{c_2}^x$ a basin of attraction for the origin of the closed-loop system. The size of the region of attraction $\mathcal{D}' \times \Omega_{c_2}^x$ depends on the size of the set $\mathcal{X} \times \mathcal{U}$ (see condition (4)). If A1 is satisfied globally (as in [9, 10, 8]), or $\mathcal{X} \times \mathcal{U}$ is large enough, then Theorem 2 guarantees that the domain of attraction $\mathcal{D}$ of the closed-loop system under state feedback is recovered by the output feedback controller, in that $\mathcal{D}'$ can be chosen to be any arbitrary compact set contained in $\mathcal{D}$. If, besides being completely uniformly observable, system (1) is also stabilizable (and, therefore, A2 holds globally), then $\mathcal{D} = \mathbb{R}^n \times \mathbb{R}^{n_u}$ and the result of Theorem 2 becomes semi-global, in that Ω_{c_1} and Ω_{c_2} can be chosen arbitrarily large, thus achieving the same property of the controller found in [10].

Remark 12: Analogous to the result in [1, 8], Theorem 2 proves a separation principle for nonlinear systems: given a stabilizing state feedback controller, the performance of the output feedback controller recovers the one under state feedback provided that the parameter ρ is chosen small enough. Furthermore, by slight modification of Theorem 3 in [8], it is easy to show that the closed-loop system trajectories under output feedback approach the trajectories under state feedback as $\rho \to 0$. In conclusion, the output feedback controller presented here achieves the same recovery properties of the one in [8] for the more general class of SISO incompletely observable systems. We must point out, however, that we assume to have perfect knowledge of the system dynamics, whereas the results in [8] admit model uncertainties. We opted not to include model uncertainties to better illustrate the underlying principles of our approach; it is an easy exercise to show that analogous results to the ones in [8] hold when the model uncertainty is included.

Remark 13: If system (1) is completely uniformly observable (and hence $\mathcal{X} \times \mathcal{U} = \mathbb{R}^n \times \mathbb{R}^{n_u}$), one can omit the observer estimates projection (31) and use the standard control input saturation in [1, 4, 2, 5, 6, 8, 7, 3] and prove that, after minor modifications in Lemma 2 and 3, Theorem 2 would still be valid. In this case, the theory provided in this paper shows how to achieve output feedback stabilization with a nonlinear observer, thus avoiding the analytical knowledge of $\mathcal{H}^{-1}$.

5 Example

Consider the following state feedback linearizable dynamical system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= (1 + x_1)\exp(x_1^2) + u - 1 \\ y &= (x_2 - 1)^2 \end{aligned} \tag{44}$$

The control input appears in the first derivative of the output:

$$\dot{y} = 2(x_2 - 1)(1 + x_1)\exp(x_1^2) + 2(x_2 - 1)(u - 1)$$

Notice, however, that the coefficient multiplying u vanishes when $x_2 = 1$, and hence system (44) does not have a well-defined relative degree everywhere. The observability mapping $\mathcal{H}$ is given by

$$y_e = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} (x_2 - 1)^2 \\ 2(x_2 - 1)(1 + x_1)\exp(x_1^2) + 2(x_2 - 1)(u - 1) \end{bmatrix} \tag{45}$$

The first equation in (45) is invertible for all $x_2 < 1$, and its inverse is given by $x_2 = 1 - \sqrt{y}$. Substituting x_2 into the second equation in (45) and isolating the term in x_1, we get

$$(1 + x_1)\exp(x_1^2) = \frac{\dot{y} + 2\sqrt{y}(u - 1)}{-2\sqrt{y}} \tag{46}$$

Since $(1+x_1)\exp(x_1^2)$ is a strictly increasing function, it follows that (46) is invertible for all $x_1 \in \mathbb{R}$, however, an analytical solution to this equation cannot be found. In conclusion, Assumption A1 is satisfied on the domain $\mathcal{X} \times \mathcal{U} = \{x \in \mathbb{R}^2 \mid x_2 < 1\} \times \mathbb{R}$, but an analytical inverse $x = \mathcal{H}^{-1}(y_e, u)$ is not known. The fact that system (44) is incompletely observable, together with the non-existence of an analytical inverse to (45), prevents the application of the output feedback control approaches in [1, 9, 2, 10, 3, 5, 4, 6, 7, 8].

From (45) we have that $n_u = 1$, therefore we add one integrator on the input side,

$$\dot{z}_1 = v, \qquad u = z_1$$

The resulting extended system can be linearized by letting $x_3 = (1 + x_1)\exp(x_1^2) + z_1 - 1$ and rewriting the system in new coordinates $x_e \triangleq [x_1, x_2, x_3]^\top$:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_2 \exp(x_1^2)(2x_1^2 + 2x_1 + 1) + v \end{aligned} \tag{47}$$

Choose $v = -x_2 \exp(x_1^2)(2x_1^2 + 2x_1 + 1) - Kx_e$, where $K = [1, 3, 3]$, so that the closed-loop system becomes $\dot{x}_e = (A_c - B_c K)\, x_e$ with poles placed at -1. Then, the origin $x_e = 0$ is a globally asymptotically equilibrium point of (47), and Assumption A2 is satisfied with $\mathcal{D} = \mathbb{R}^3$.

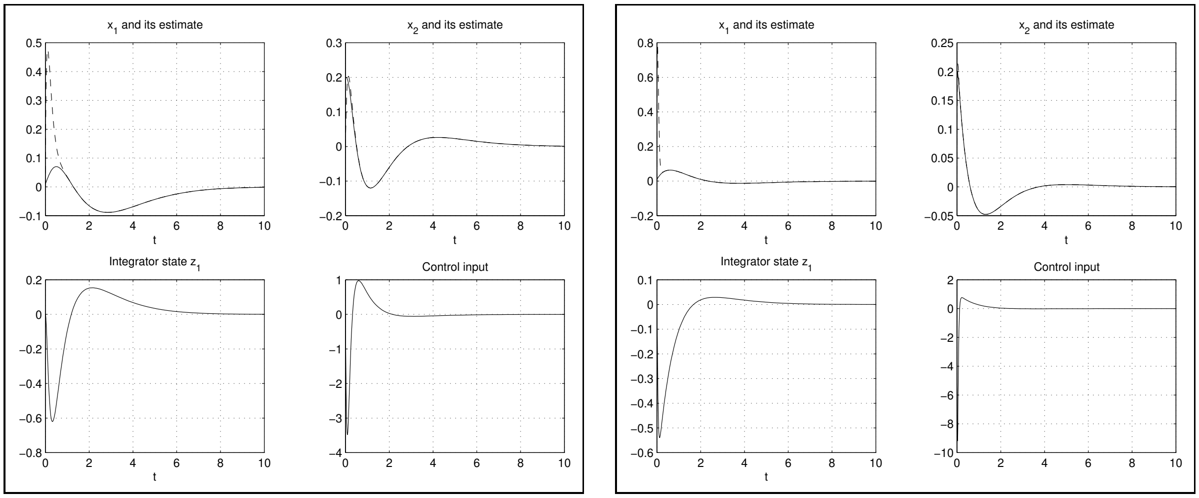
Next, we will seek to find a set $C_\xi(z)$ satisfying Assumption A3. Let $\bar{P}$ be the solution of the Lyapunov equation associated to $A_c - B_c K$, so that a Lyapunov function for system (47) is $V = x_e^\top \bar{P} x_e$, and any set $\Omega_c \triangleq \{x_e \in \mathbb{R}^3 \mid V(x_e) \le c\}$, with $c > 0$, is a region of attraction for the origin. Observe further that $\chi = [x_1, x_2, z_1]^\top = [x_1, x_2, x_3 - (1 + x_1)\exp(x_1^2) + 1]^\top$, and choose $c_2 = \omega^2 \lambda_{\min}(\bar{P})$, where $0 < \omega < 1$. This choice of c_2 makes sure that Ω_{c_2} is contained in the sphere $S_\omega \triangleq \{\chi \in \mathbb{R}^3 \mid \|x_e\| \le \omega\}$ inside which we have $x_2 \le \omega$, implying that $x \in \mathcal{X}$. When $\chi \in S_\omega$ it is easy to show that

$$\begin{gathered} (1 - \omega)^2 \le y \le (1 + \omega)^2 \\ -2\omega(\omega + 1) \le \dot{y} \le 2\omega(\omega + 1) \end{gathered}$$

and hence Assumption A3 is verified with $c_2 = \omega^2 \lambda_{\min}(\bar{P})$ and

$$C_\xi = \left\{\xi \in \mathbb{R}^2 \mid (1 - \omega)^2 \le \xi_1 \le (1 + \omega)^2,\ -2\omega(\omega + 1) \le \xi_2 \le 2\omega(\omega + 1)\right\}$$

Notice that here a set C_ξ is found which is independent of z. In general, we allow C_ξ to depend on z to make condition 1 in A3 less restrictive.



$\rho = 0.2$. $\rho = 0.05$.

Figure 2: Closed-loop trajectories under output feedback.

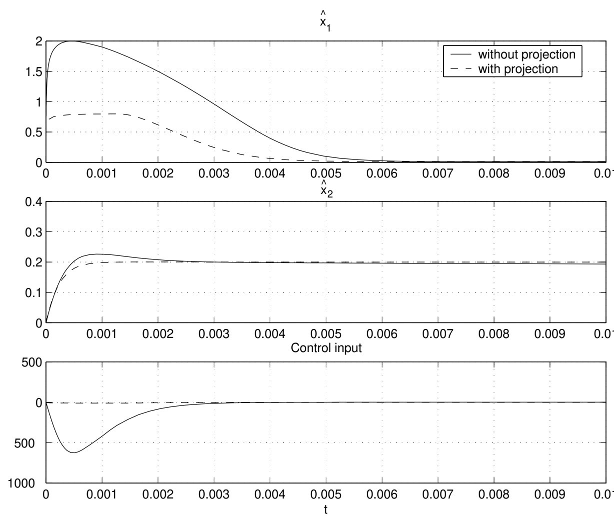


Figure 3: Observer states during the initial peaking phase with and without projection, $\rho = 10^{-3}$.

For our simulations we choose $\omega = 0.9$ and the initial condition of the extended system is set to $x_1(0) = 0.01, x_2(0) = 0.2, z_1(0) = 1.01$, which is contained inside Ω_{c_2}, so that Theorem 2 can be applied. Finally, we choose the observer gain L to be $[4, 4]^\top$, so that its associated polynomial is Hurwitz with both poles placed at -2. We present four different situations to illustrate four features of our output feedback controller:

1. **Arbitrary fast rate of convergence of the observer**. Figure 2 shows the evolution of the χ-trajectory, as well as the control input v, for $\rho = 0.2$ and $\rho = 0.05$. The convergence in the latter case is faster, as predicted by Theorem 1 (see Remark 4).

2. **Observer estimate projection**. Figure 3 shows the evolution of $\hat{x}$ and v for $\rho = 10^{-3}$ with and

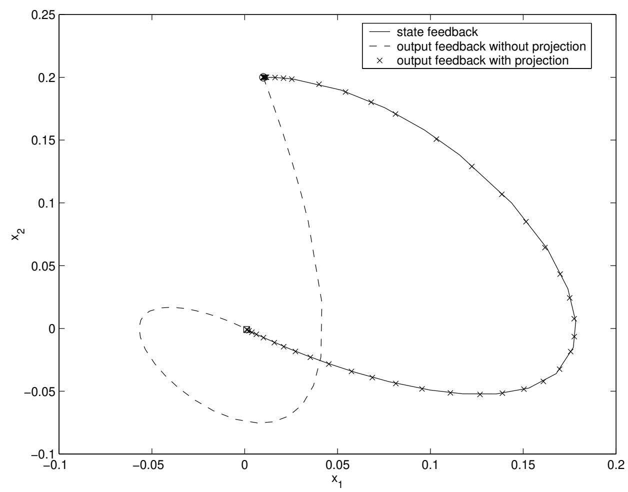


Figure 4: System trajectories in the phase plane with and without projection, $\rho = 10^{-4}$.

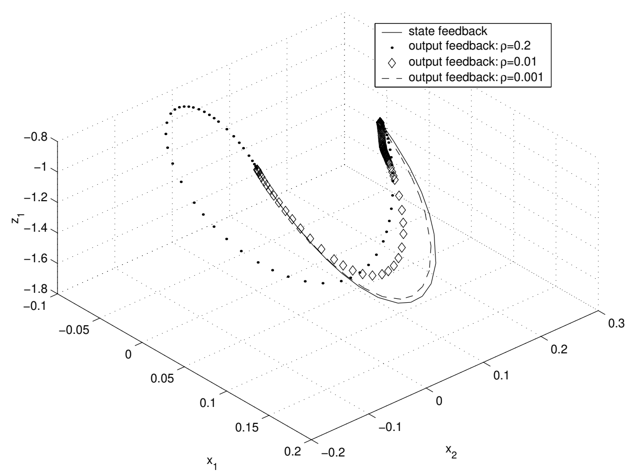


Figure 5: Closed-loop trajectories in the three dimensional space for decreasing values of ρ.

without projection. The projection algorithm successfully eliminates the peak in the observer states, thus yielding a bounded control input, as predicted by the result of Lemma 1.

3. **Observer estimate projection and closed-loop stability**. In Figure 4 a phase plane plot for x is shown with and without observer projection when $\rho = 10^{-4}$. The small value of ρ generates a significant peak which, if projection is not employed, drives the output feedback trajectories away form the state feedback ones and, in general, may drive the system to instability (see Remark 10). On the other hand, using the observer projection, output feedback and state feedback trajectories are almost indistinguishable.

4. **Trajectory recovery**. The evolution of the χ-trajectories for decreasing values of ρ, in Figure 5, shows that the output feedback trajectories approach the state feedback ones as $\rho \to 0$ (see Remark 12).

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