Multipoint Galerkin Asymptotic Waveform Evaluation for Model Order Reduction of Frequency Domain FEM Electromagnetic Radiation Problems

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Abstract—In this paper, a model order reduction technique is presented. This technique, known as Galerkin asymptotic waveform evaluation (GWE), or multipoint Galerkin asymptotic waveform evaluation (MGWE) if multiple expansion points are considered simultaneously, can be used to reduce matrices describing electromagnetic (EM) phenomena generated through the finite element method (FEM) to a smaller space while still accurately approximating the characteristics of the original responses. The resulting solution procedure of using GWE or MGWE to solve FEM equations allows for wideband frequency simulations with a reduction in total computation time. Numerical simulations using these methods are shown along with traditional methods such as using an LU decomposition at each frequency point of interest and asymptotic waveform evaluation (AWE). Comparisons in accuracy as well as computation time are also given.

Index Terms—Asymptotic waveform evaluation, computer aided engineering, model order reduction, Padé approximants, perfectly matched layer (PLM).

I. INTRODUCTION

O
VER the past several years, much research has been devoted to the areas of model order reduction (MOR). Essentially, MOR is a process in which the number of unknowns (order) of a mathematical representation (model) of a problem of interest is decreased. Although there are many reasons that may motivate the application of a MOR procedure, all are ultimately related to obtaining the problem solution more quickly while still maintaining accuracy. One popular MOR technique is asymptotic waveform evaluation (AWE) in which a low order approximation of the system response is formulated [7]. As the name indicates, as the order of the approximation is increased, the approximate response begins to “asymptotically” approach that of the original system response [8]. Once the smaller, reduced order model is generated, it can then be solved inexpensively for many different values of frequency. In this way, a broadband fast frequency sweep response can be calculated more efficiently than if the original, large circuit matrices were solved at each frequency of interest.

In AWE analysis, to reduce the original problem to a smaller model, the system transfer functions are expanded into Taylor series whose coefficients are called moments. Moment matching is then used to approximate the models with lower-order transfer functions. Using AWE, the transfer functions of system models are approximated by matching the first 2\(q\) moments (i.e., matching terms up to and including order \(2q-1\)) of exact solutions to lower \(q\)-order models [1], [7], [10]. AWE then uses a Padé approximation to provide an estimation of the system response. However, AWE is known to have issues involving accuracy and/or efficiency. Some attempts used to maintain the accuracy includes techniques such as complex frequency hopping (CFH) [4], [5], and scaling [11]. In [3], a multipoint Padé approximation was given to compute the poles and residues of microwave circuits. This method was more accurate than CFH because all expansion points were considered simultaneously.

The computational electromagnetic (EM) community has recently started developing MOR algorithms to decrease the time required for simulations. In [12], AWE is applied to solve scattering problems modeled using the finite element method (FEM). The spectral Lanczos decomposition method (SLDM) is used in [14] to solve cavity problems modeled with FEM. The SLDM is limited in applicability, however, because open region problems can not be treated. Propagation and radiation cases are treated using Padé via Lanczos (PVL) on a finite-difference time-domain (FDTD) grid in [2], where unknowns for both the electric and magnetic vector fields must be solved for. In [2], the FDTD grid is terminated with a perfectly matched layer (PML). Since PVL requires the matrix equation to be linear in frequency, solving a matrix equation resulting from using a PML backed FDTD grid with PVL is not trivial. This problem is addressed in [13] with a solution that requires introducing an auxiliary variable and thereby increasing the number of unknowns. Finally, open domain problems are also treated using PVL on a finite element mesh in [9] where only the electric field is solved for. However, to conform to the linear frequency requirement of PVL, the procedure in [9] requires doubling the number of unknowns.

To overcome some of the computational difficulties described above, a Galerkin asymptotic waveform evaluation (GWE) is presented in [6]. Unlike PVL, GWE can solve a matrix equation that has a polynomial frequency variation. In this paper, a procedure for modeling an open region domain
with a FEM mesh terminated with an anisotropic PML and solved using GAWE is introduced. This procedure does not require increasing the number of unknowns. In addition, a new interpretation of the derivation of GAWE is given. This derivation uses a different approach to show the moment matching procedure of GAWE. Finally, this procedure is extended to a multipoint Galerkin asymptotic waveform evaluation (MGAWE) in which more than one expansion point is considered simultaneously. The remainder of this paper is organized as follows.

In Section II, the equations will be given to show how antenna radiation problems can be solved using FEM with an anisotropic PML and GAWE. Then Section III contains a new way of looking at the theoretical developments of GAWE. These developments clearly show the moment matching procedure of GAWE. Numerical examples are then presented in Section IV. Next, MGAWE is given in Section V followed by more numerical examples in Section VI. The summary and conclusions as well as some possible areas of future research are given in Section VII.

II. FEM AND ANISOTROPIC PML WITH GAWE

Consider the $TM_z$ case for the generalized two-dimensional (2-D) Helmholtz equation applied to an antenna radiation problem where the permeability ($\mu$) and permittivity ($\epsilon$) are anisotropic

$$\left( \frac{\partial}{\partial x} \frac{1}{\mu_0 \mu_x} \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \frac{1}{\mu_0 \mu_y} \frac{\partial}{\partial y} + \omega^2 \epsilon_x \right) E_z = j\omega J_z.$$  

The angular frequency is given by $\omega$ and the problem is driven by an electric current source $J_z$. The problem domain of interest is shown in Fig. 1, where $\Omega$ is the FEM region, which is the union of the regions $\Omega_k$ for $k = 1, 2, \ldots, 4$, and $\partial \Omega$ is the perfect electric conductor (PEC) outer boundary of the FEM region. The region between $\Omega$ and the parallel dashed region contains anisotropic PML. In particular, let

$$\mu_x = \frac{\alpha_x - j\frac{\beta_x}{\omega}}{\alpha_x - j\frac{\beta_x}{\omega} \mu_0}, \quad \mu_y = \frac{\alpha_y - j\frac{\beta_y}{\omega}}{\alpha_y - j\frac{\beta_y}{\omega} \mu_0}.$$  

Apply the method of weighted residuals (and multiply through by $\mu_0$) to obtain

$$\int \int_{\Omega} \left( \frac{\alpha_y - j\frac{\beta_y}{\omega}}{\alpha_x - j\frac{\beta_x}{\omega} \mu_0} \frac{\partial}{\partial x} E_z \frac{\partial}{\partial x} \phi_k + \frac{\alpha_x - j\frac{\beta_x}{\omega}}{\alpha_y - j\frac{\beta_y}{\omega} \mu_0} \frac{\partial}{\partial y} E_z \frac{\partial}{\partial y} \phi_k \right) d\Omega + \int \int_{\partial \Omega} \left( \frac{\alpha_y - j\frac{\beta_y}{\omega}}{\alpha_x - j\frac{\beta_x}{\omega} \mu_0} \frac{\partial}{\partial x} E_z \frac{\partial}{\partial y} \phi_k \right) n_x d\ell + \int \int_{\partial \Omega} \left( \frac{\alpha_x - j\frac{\beta_x}{\omega}}{\alpha_y - j\frac{\beta_y}{\omega} \mu_0} \frac{\partial}{\partial x} E_z \frac{\partial}{\partial y} \phi_k \right) n_y d\ell = \int \int_{\partial \Omega} \left( \alpha - j\frac{\beta}{\omega} \right) \frac{\partial}{\partial x} E_z \frac{\partial}{\partial x} \phi_k d\Omega.$$  

where $\phi_i$ are the weighting functions. Enforce the boundary conditions on tangential $H$ at the interface between elements as shown in Fig. 2 to give

$$\frac{1}{\mu_{x1}} \frac{\partial E_{z1}}{\partial y} = -j\omega H_{y1} = -j\omega H_{y2} = \frac{1}{\mu_{x2}} \frac{\partial E_{z2}}{\partial y}$$

and

$$\frac{1}{\mu_{y1}} \frac{\partial E_{z1}}{\partial x} = -j\omega H_{x1} = -j\omega H_{x2} = \frac{1}{\mu_{y2}} \frac{\partial E_{z2}}{\partial x}.$$  

Therefore, all contributions from interelement boundaries cancel each other. In addition, since $\phi_i \equiv 0$ on PEC (i.e., do not test there), the term in (2) that contains $\int_{\partial \Omega} d\ell$ is equal to...
TABLE I
VALUES FOR $\alpha_x$, $\alpha_y$, $\beta_x$, $\beta_y$, AND $C_j(\omega)$ FOR $j = 1 \ldots 4$ FOR EACH REGION $\Omega_k$ FOR $k = 1 \ldots 4$

<table>
<thead>
<tr>
<th>$\alpha_x$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\beta$</th>
<th>$\omega^2 - j\beta\omega$</th>
<th>$\omega^2 - j\beta\omega$</th>
<th>$\omega^2 - j\beta\omega$</th>
<th>$\omega^2 - j\beta\omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_y$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\omega^2 - j2a\beta\omega - \beta^2$</td>
<td>$\omega^2 - j2a\beta\omega - \beta^2$</td>
<td>$\omega^2 - j2a\beta\omega - \beta^2$</td>
<td>$\omega^2 - j2a\beta\omega - \beta^2$</td>
</tr>
<tr>
<td>$\beta_x$</td>
<td>$\beta$</td>
<td>0</td>
<td>0</td>
<td>$\omega^2 - j3\alpha\beta^2\omega$</td>
<td>$\omega^2 - j3\alpha\beta^2\omega$</td>
<td>$\omega^2 - j3\alpha\beta^2\omega$</td>
<td>$\omega^2 - j3\alpha\beta^2\omega$</td>
</tr>
<tr>
<td>$\beta_y$</td>
<td>0</td>
<td>$\beta$</td>
<td>0</td>
<td>$\omega^2 - j3\alpha\beta^2\omega$</td>
<td>$\omega^2 - j3\alpha\beta^2\omega$</td>
<td>$\omega^2 - j3\alpha\beta^2\omega$</td>
<td>$\omega^2 - j3\alpha\beta^2\omega$</td>
</tr>
<tr>
<td>$C_1$</td>
<td>$\omega^2 - j\beta\omega$</td>
<td>$\omega^2$</td>
<td>$\omega^2 - j2a\beta\omega - \beta^2$</td>
<td>$\omega^2 - j2a\beta\omega - \beta^2$</td>
<td>$\omega^2 - j2a\beta\omega - \beta^2$</td>
<td>$\omega^2 - j2a\beta\omega - \beta^2$</td>
<td></td>
</tr>
<tr>
<td>$C_2$</td>
<td>$\omega^2 - j\beta\omega$</td>
<td>$\omega^2$</td>
<td>$\omega^2 - j2a\beta\omega - \beta^2$</td>
<td>$\omega^2 - j2a\beta\omega - \beta^2$</td>
<td>$\omega^2 - j2a\beta\omega - \beta^2$</td>
<td>$\omega^2 - j2a\beta\omega - \beta^2$</td>
<td></td>
</tr>
<tr>
<td>$C_3$</td>
<td>$\omega^4 - j3\alpha\beta^2\omega$</td>
<td>$\omega^2$</td>
<td>$\omega^2 - j3\alpha\beta^2\omega$</td>
<td>$\omega^2 - j3\alpha\beta^2\omega$</td>
<td>$\omega^2 - j3\alpha\beta^2\omega$</td>
<td>$\omega^2 - j3\alpha\beta^2\omega$</td>
<td></td>
</tr>
<tr>
<td>$C_4$</td>
<td>$\omega^4 - j3\alpha\beta^2\omega$</td>
<td>$\omega^2$</td>
<td>$\omega^2 - j3\alpha\beta^2\omega$</td>
<td>$\omega^2 - j3\alpha\beta^2\omega$</td>
<td>$\omega^2 - j3\alpha\beta^2\omega$</td>
<td>$\omega^2 - j3\alpha\beta^2\omega$</td>
<td></td>
</tr>
</tbody>
</table>

zero for all interelement and outer boundaries. Equation (2) now becomes

$$\int_{\Omega} C_1(\omega) \frac{1}{\mu_0} \frac{\partial E_x}{\partial x} \phi_i + C_2(\omega) \frac{1}{\mu_0} \frac{\partial E_y}{\partial y} \phi_i - C_3(\omega) \epsilon_0 \epsilon_r E_x \phi_i dS = -\int_{\Omega} C_4(\omega) j J_x \phi_i dS \quad (3)$$

where $C_j(\omega)$ for $j = 1 \ldots 4$ is given in Table I for each region $\Omega_k$ for $k = 1 \ldots 4$.

Since (3) is composed of terms which depend on polynomial orders of $\omega$, the terms in (3) with common polynomial orders can be grouped together. In addition, $E_z$ can be expanded in terms of the FEM basis functions with the number of basis functions being equal to the number of weighting functions, resulting in $N \times N$ matrices. The final equation is of the form

$$(B_0 + s B_1 + s^2 B_2 + s^3 B_3 + s^4 B_4) \chi(s) = z_0 + s z_1 + s^2 z_2 + s^3 z_3 \quad (4)$$

where $s = j\omega$ and $\chi(s)$ is the unknown solution vector containing the unknown values of $E_z$ at the nodes of the grid. The matrices $B_l$ for $l = 0 \ldots 4$ and vectors $z_m$ for $m = 0 \ldots 3$ are all independent of $s$. Now the method shown in Section III can be used to solve the above equation.

III. GALERKIN ASYMPTOTIC WAVEFORM EVALUATION

Assume it is desired to solve the problem (4) where each of the $B_l$ is a given $N \times N$ complex constant matrix, each $z_m$ is a given complex constant $N$-vector, and $\chi(s)$ is the desired unknown vector that is explicitly shown to be a function of $s$. Furthermore, assume that $s = j\omega$ where $\omega = 2\pi f$ for the frequency of excitation $f$. To solve (4), expand about some complex constant point $s_0$ and define a complex variable $\sigma$ such that $\sigma = s - s_0$. Then define the matrices and vectors

$$A_0 = B_0 + s_0 B_1 + s_0^2 B_2 + s_0^3 B_3 + s_0^4 B_4$$
$$A_1 = B_1 + 2s_0 B_2 + 3s_0^2 B_3 + 4s_0^3 B_4$$
$$A_2 = B_2 + 3s_0 B_3 + 6s_0^2 B_4$$
$$A_3 = B_3 + 4s_0 B_4$$
$$A_4 = B_4$$
$$y_0 = z_0 + s_0 z_1 + s_0^2 z_2 + s_0^3 z_3$$
$$y_1 = z_1 + 2s_0 z_2 + 3s_0^2 z_3$$
$$y_2 = z_2 + 3s_0 z_3$$
$$y_3 = z_3$$

and use (5) in (4) to give

$$\left( A_0 + \sigma A_1 + \sigma^2 A_2 + \sigma^3 A_3 + \sigma^4 A_4 \right) \chi(s + s_0) = y_0 + \sigma y_1 + \sigma^2 y_2 + \sigma^3 y_3 \quad (6)$$

Assume that there is a collection of linearly independent vectors $v_i$ for $i = 1 \ldots N$, each of which has length $N$. Define an $N \times N$ matrix $V_N$, such that the $i$th column of $V_N$ is $v_i$. Then the solution $\chi$ can be represented as

$$\chi(s + s_0) = V_N g_N(s) = \sum_{i=1}^{N} v_i \gamma_i(s) \quad (7)$$

where $\gamma_i(s)$ is the $i$th component of some $N$-vector $g_N(s)$. It should be noted that the choice of $v_i$ and $\gamma_i(s)$ is not unique. To make a $q$th-order subset of $V_N g_N(s)$ unique, it is desired to find a set of $v_i$ and associated $\gamma_i(s)$ such that the approximation

$$\chi(s + s_0) \approx V_N g_N(s) = \sum_{i=1}^{q} v_i \gamma_i(s) \quad (8)$$

minimizes the residual

$$r_q(\sigma) = (A_0 + \sigma A_1 + \sigma^2 A_2 + \sigma^3 A_3 + \sigma^4 A_4) \sum_{i=1}^{q} v_i \gamma_i(s) - (y_0 + \sigma y_1 + \sigma^2 y_2 + \sigma^3 y_3) \quad (9)$$

in the sense that if $r_q(\sigma)$ is expressed in a Taylor series as

$$r_q(\sigma) = \sum_{j=0}^{\infty} \sigma^j r_q^j \quad (10)$$

then

$$r_q^j = 0 \quad \text{for} \quad j = 0 \ldots q - 1 \quad (11)$$

and

$$r_q(\sigma) \perp V_q \quad (12)$$

For notational purposes, expand $\gamma_i(s)$ in a Taylor series as

$$\gamma_i(s) = \sum_{j=0}^{\infty} \sigma^j \gamma_i^j \quad (13)$$

and define

$$g_i^j = \begin{bmatrix} \gamma_i^1 \\ \gamma_i^2 \\ \vdots \\ \gamma_i^q \end{bmatrix} \quad (5)$$

The set $V_q$ and $g_i(\sigma)$ can be found as follows.
To start the method, for the step $i = 1$, assume $\mathbf{x}(\sigma + s_0) = \mathbf{0}$ and so $r_0^i(\sigma) = -(\gamma_0 + \sigma y_1 + \sigma^2 y_2 + \sigma^3 y_3)$. Therefore, $r_0^0 = -\gamma_0$. Choose $\mathbf{A}_0 V_1$ in the direction of $r_0^0$ so that there is a component of the solution that can span that part of the residual, thus forcing this component of the residual to zero as required in the definition involving $r_0^0 = 0$ for $j = 0\ldots q - 1$ as given above. Therefore, choose

$$w_0 = -\mathbf{A}_0^{-1} y_0$$

and $v_1 = w_0/\|w_0\|$. Let $V_1 = [v_1]$ and now assume that $\mathbf{x}(\sigma + s_0) = \mathbf{V}_1 g_1(\sigma) = v_1 \gamma_1(\sigma)$, so one can find $r_1(\sigma) = (\mathbf{A}_0 + \sigma \mathbf{A}_1 + \sigma^2 \mathbf{A}_2 + \sigma^3 \mathbf{A}_3 + \sigma^4 \mathbf{A}_4) v_1 \gamma_1(\sigma) - \gamma_0 - \Omega_1 + \sigma^2 y_2 + \sigma^3 y_3$. Therefore, $r_1^1 = \mathbf{A}_0 v_1 \gamma_1^0 - \gamma_0$. Note from how $v_1$ was constructed that $y_0 \in \mathbf{A}_0 V_1$. Therefore, it is possible to force $r_1^0 = 0$ by choosing $\gamma_1^0$ appropriately. Assuming that $\mathbf{V}_1^H \mathbf{A}_0 V_1 \neq 0$, choose

$$\gamma_1^0 = (\mathbf{v}_1^H \mathbf{A}_0 v_1)^{-1} \mathbf{v}_1^H y_0.$$

This is also equivalent to setting $\mathbf{V}_1^H r_1^0 = 0$ (i.e., doing a Galerkin test with $v_1$) so this part of the residual is forced out of the space that is spanned by $v_1$.

For the step $i = 2$, again assume $\mathbf{x}(\sigma + s_0) = \mathbf{V}_1 g_1(\sigma) = v_1 \gamma_1(\sigma)$, so $r_2^0 = \mathbf{A}_0 V_1 v_1^0 - \gamma_0$. To find $v_2$, consider $r_2^1$. Choose $\mathbf{A}_0 V_2$ in the direction of $r_2^1$ so there is a component of the solution that can span that part of the residual, thus forcing this component of the residual to zero as required in the definition involving $r_2^0$ given above. Note that $\mathbf{A}_0^{-1} r_2^1 = v_2 \gamma_2^0 - \gamma_0$. Since $V_1$ already spans $v_1$, there is no need to consider $v_2 \gamma_2^0$ when constructing $v_2$. Therefore, choose

$$w_1 = \mathbf{A}_0^{-1} (\mathbf{A}_1 \mathbf{V}_1 \gamma_1^0 - \gamma_0).$$

orthogonalize $w_1$ against $v_1$ and denote this new vector $\mathbf{w}_1$. Then $v_2 = \mathbf{w}_1/\|\mathbf{w}_1\|$. Now $V_2 = [v_1, v_2]$. Also now assume $\mathbf{x}(\sigma + s_0) = \mathbf{V}_2 g_2(\sigma) = v_2 \gamma_2(\sigma) + \gamma_2 y_2$. Following the same procedure as before, $r_2^1 = \mathbf{A}_0 V_2 v_2^0 + \mathbf{A}_0 v_2 \gamma_2^0 - \gamma_2 y_2$. Therefore, if $\gamma_2$ is chosen to be zero, then $r_2^1 = 0$. In addition, $r_3^0 = \mathbf{A}_0 \mathbf{V}_3 g_3^0$. If $\gamma_3^0$ is chosen to be zero, then $r_3^0 = 0$. Therefore, it is possible to force $r_3^0 = 0$ by choosing $\gamma_3^0$ appropriately. Assuming that $\det(\mathbf{V}_3^H \mathbf{A}_0 V_3) \neq 0$, choose

$$\gamma_3^0 = (\mathbf{v}_3^H \mathbf{A}_0 v_3)^{-1} \mathbf{v}_3^H y_0.$$

Again, this is equivalent to setting $\mathbf{V}_3^H r_3^0 = 0$ (i.e., doing a Galerkin test with $v_3$) so this part of the residual is forced out of the space that is spanned by $v_3$.

For the step $i = 3$, again assume $\mathbf{x}(\sigma + s_0) = \mathbf{V}_3 g_3(\sigma) = v_3 \gamma_3(\sigma) + \gamma_3 y_3$. To find $v_3$, consider $r_3^1$. Choose $\mathbf{A}_0 V_3$ in the direction of $r_3^1$ so there is a component of the solution that can span that part of the residual, again forcing this component of the residual to zero as required. Note that $\mathbf{A}_0^{-1} r_3^1 = \mathbf{V}_3 ^H g_3^0 + (\mathbf{A}_1 \mathbf{V}_3 \gamma_3^0 + \mathbf{A}_2 \mathbf{V}_3 \gamma_2^0 + \mathbf{A}_3 \mathbf{V}_3 \gamma_1^0 - \gamma_3 y_3)$. Since $V_3$ is already spanned, there is no need to consider $\mathbf{V}_3 ^H g_3^0$ when constructing $v_3$. Therefore, choose

$$w_2 = \mathbf{A}_0^{-1} (\mathbf{A}_1 \mathbf{V}_2 \gamma_2^0 + \mathbf{A}_2 \mathbf{V}_1 \gamma_1^0 - \gamma_2).$$

Now orthogonalize $w_2$ against $v_2$ and denote this new vector $\mathbf{w}_2$. Then $v_3 = \mathbf{w}_2/\|\mathbf{w}_2\|$ and $V_3 = [v_1, v_2, v_3]$. Also assume $\mathbf{x}(\sigma + s_0) = \mathbf{V}_2 g_2(\sigma)$. Again following the same procedure, $r_3^0 = \mathbf{A}_0 \mathbf{V}_3 g_3^0$, and so $r_3^0 = r_3^2 = 0$ if $\gamma_3^0$ is chosen to be zero. In addition, $r_3^1 = \mathbf{A}_0 \mathbf{V}_3 \gamma_3^0 + \mathbf{A}_1 \mathbf{V}_3 \gamma_2^0 + \mathbf{A}_2 \mathbf{V}_3 \gamma_1^0 - \gamma_2 = \mathbf{A}_0 \mathbf{V}_3 g_3^0 + \mathbf{A}_1 \mathbf{V}_3 \gamma_2^0 + \mathbf{A}_2 \mathbf{V}_3 \gamma_1^0 - \gamma_2$. Note from how $v_3$ was constructed that $\mathbf{A}_1 \mathbf{V}_2 \gamma_2^0 + \mathbf{A}_2 \mathbf{V}_1 \gamma_1^0 - \gamma_2 \in \mathbf{A}_0 V_3$. Therefore, it is possible to force $r_3^1 = 0$ by choosing $\gamma_3^0$ appropriately. Assuming that $\det(\mathbf{V}_3^H \mathbf{A}_0 V_3) \neq 0$, choose

$$\gamma_3^0 = (\mathbf{v}_3^H \mathbf{A}_0 v_3)^{-1} \mathbf{v}_3^H y_0.$$

Again, this is equivalent to setting $\mathbf{V}_3^H r_3^0 = 0$ (i.e., doing a Galerkin test with $v_3$) so this part of the residual is forced out of the space that is spanned by $v_3$.
Fig. 3.  $L_U$, AWE and GAWE responses for the magnitude of the response in Example 1. Solid $\rightarrow L_U$ response, dash–dash $\rightarrow$ AWE response, dash–dot $\rightarrow$ GAWE response.

Fig. 4.  $L_U$, AWE and GAWE responses with multiple expansion points for the magnitude of the response in Example 1. Solid $\rightarrow L_U$ response, dash–dash $\rightarrow$ AWE response, dash–dot $\rightarrow$ GAWE response.

Now orthogonalize $\mathbf{w}_{i-1}$ against $\mathbf{V}_{i-1}$ and denote this new vector $\mathbf{w}_{i-1}$. Then $\mathbf{v}_i = \mathbf{w}_{i-1}/||\mathbf{w}_{i-1}||$ and $\mathbf{V}_i = [\mathbf{v}_1 \cdots \mathbf{v}_i]$. Also, now assume $\mathbf{x}(\sigma + \mathbf{s}_0) = \mathbf{V}_i g_0(\sigma)$. As before, $r_1^0 \cdots r_{i-2}^0 = 0$ if $\gamma_0^0 \cdots \gamma_{i-2}^0$ are all chosen.
to be zero. Now, \( r_{i-1}^{-1} = A_0 V_i g_i^{-1} + A_1 V_i g_i^{-2} + A_2 V_i g_i^{-3} + A_3 V_i g_i^{-4} \), \( A_0 V_i g_i^{-1} = A_1 V_i g_i^{-2} = A_2 V_i g_i^{-3} = A_3 V_i g_i^{-4} \), and since \( V_i \) was constructed so that \( A_0 V_i g_i^{-1} = A_2 V_i g_i^{-3} + A_3 V_i g_i^{-4} + A_1 V_i g_i^{-2} \), then it is possible to force \( r_{i-1}^{-1} = 0 \) by choosing \( g_i^{-1} \) appropriately.

Assuming that \( det(V_i^H A_0 V_i) \neq 0 \), choose

\[
g_i^{-1} = (V_i^H A_0 V_i)^{-1} V_i^H \times \begin{pmatrix} -A_1 V_i g_i^{-2} & -A_2 V_i g_i^{-3} \\ -A_3 V_i g_i^{-4} & -A_1 V_i g_i^{-2} \end{pmatrix}.
\]

Again, this is equivalent to setting \( V_i^H r_{i-1}^{-1} = 0 \) (a Galerkin test with \( V_i \)) so this part of the residual is forced out of the space that is spanned by \( V_i \).

Once \( V_q \) is found, then \( g_q(\sigma) \) is found to be

\[
g_q(\sigma) = (V_q^H (A_0 + \sigma A_1 + \sigma^2 A_2 + \sigma^3 A_3 + \sigma^4 A_4) V_q)^{-1} V_q^H (y_0 + \sigma y_1 + \sigma^2 y_2 + \sigma^3 y_3)
\]

by using (12). Then

\[
x(\sigma + \omega_0) \approx V_q g_q(\sigma).
\]

IV. GAWE NUMERICAL EXAMPLES

A nodal-based FEM is used to solve some 2-D, \( T_{Mz} \) problems for \( E_z \). The resulting matrix equation is solved using GAWE. Some examples are given below where all solutions are computed using MATLAB running on a 266 MHz Pentium II with 256 MB of RAM. All examples are benchmarked against an \( LU \) decomposition computed at many sampling points in frequency with a simple linear interpolation between them. Although more sophisticated interpolation schemes (such as Padé interpolation) exist which would require fewer frequency sampling points, they still would require the frequency response at several points. Therefore, several \( LU \) decompositions would still need to be calculated and the methods proposed herein would still be more computationally efficient.

**Example 1:** The first numerical example is a model of a material cylinder illuminated by a uniform electric line source. The outer boundary is treated with PML backed by a PEC. The size of the resulting \( B_f \) matrix is \( N = 4734 \). The expansion point is \( \omega_0 = j2\pi 2000000 \). The solution is computed up to 500 MHz, which is the frequency where the edge length of a side of an element is about 1/20 of a wavelength. In Fig. 3, the magnitude of the response of \( E_z \) at a node in the FEM model is shown. The response computed using an \( LU \) decomposition at 201 frequency points is shown as a solid curve, traditional AWE with \( q = 30 \) is shown as a dashed curve, and GAWE with \( q = 30 \) is shown as a dash-dot curve. The total time taken by \( LU \) to solve the matrix equations is 130718 s, the time taken by AWE is 730 s, and the time taken by GAWE is 717 s. As can be seen from the figure, GAWE is not only slightly faster than AWE, but also accurate in a wider bandwidth.

**Example 2:** The second numerical example is a 2-D model of a horn antenna. The outer boundary is treated with an absorbing boundary condition instead of a PML. Therefore, the matrix is second order instead of fourth order and the right-hand side is first order instead of third order. In this example, it is desired to find the wave impedance inside the horn. The diagram of the horn is shown in Fig. 5. The dimension of the resulting \( B_f \) matrix is \( N = 3438 \). The expansion point is \( \omega_0 = j2\pi 12000000 \). The expansion point is \( \omega_0 = j2\pi 12000000 \). In Fig. 6, the impedance is shown from 500 MHz to 1.5 GHz. The response computed using an \( LU \) decomposition at 1001 frequency points is shown as a solid curve with a computation time of 40537 s, the AWE solution with \( q = 30 \) is shown as a dashed curve and has a computation time of 101 seconds, and the GAWE solution, also with \( q = 30 \), is shown as a dash–dot curve and has a computation time of 101 s. Again, GAWE is more accurate for essentially the same computation cost.

To cover the entire band, more than one expansion point is needed. In Fig. 4, expansion points for both AWE and GAWE are chosen at \( \omega_{01} = j2\pi 2000000 \) and \( \omega_{02} = j2\pi 4000000 \), with \( q_1 = 30 \) and \( q_2 = 30 \). The total time taken for AWE and GAWE is 1637 and 1612 s, respectively. As can be seen, another expansion point is needed at \( \omega_{03} = j2\pi 5000000 \) for both AWE and GAWE, and AWE also needs yet another expansion point at \( \omega_{04} = j2\pi 10000000 \) to take care of the frequency region below 50 MHz. For \( \omega_{3} = 30 \), an expansion point at \( j2\pi 5000000 \) would increase the time required by AWE and GAWE by 911 and 898 s, respectively. For \( \omega_{4} = 30 \), an expansion point at \( j2\pi 10000000 \) would further increase the time required by AWE by 729 s. It will be shown in Section VI that MGAWE can solve this problem without needing an extra expansion point at either \( j2\pi 5000000 \) or \( j2\pi 10000000 \).
Fig. 6. $L\mu$, AWE and GAWE responses for the impedance of the structure shown in Fig. 5. Solid $\rightarrow L\mu$ response, dash–dash $\rightarrow$ AWE response, dash–dot $\rightarrow$ GAWE response.

$q_1 = 20$ and $q_2 = 20$. The total time taken for AWE and GAWE is 160 and 156 s, respectively. More expansion points are necessary to cover the entire frequency band in this case as well, but it will be shown in Section VI that MGAWE can solve this problem just using the expansion points and number of iterations given above.
V. MULTIPoint GALerkin ASymptotic WAVEform EVALUATION

In Section IV, it was shown that under most cases, more than one expansion point is needed to cover the entire bandwidth of interest. In [3], it is shown that considering expansion points simultaneously can be beneficial compared to considering them independently. Therefore, in this section, a multipoint GAWE is presented.

Consider the process by which GAWE constructs the basis $V_q$. Assume that $num\_pts$ expansion points are desired. Then $q = \sum_{n=1}^{num\_pts} q_n$ where $q_n$ is the number of moments to be generated at the $n$th expansion point. For the first expansion point, the first $q_1$ vectors are exactly the same vectors that would be generated in GAWE.

When it is desired to start generating vectors from the second expansion point, it is desired that the MGAWE solution still satisfy the requirement that $\mathbf{r}_{q_1} = 0$ for $j = 0, \ldots, q_1 - 1$ at the previous expansion point. In addition, it is desired that at the new expansion point, $\mathbf{r}_{q_2} = 0$ for $j = 0, \ldots, q_2 - 1$. To do this, consider the residual at the new expansion point and generate $g_{q_i}^{i-1}$ as before with the exception that the residual is made perpendicular to all the vectors already generated (i.e., doing a Galerkin test with all the vectors), regardless of which expansion point the vector was generated from. Denote this new $g_{q_i}^{i-1}$ as $g_{q_i}^{ii-1}$ to distinguish it from the old $g_{q_i}^{ii-1}$. The relationship between the integers $i$ and $ii$ is $ii = i - \sum_{n=1}^{i-1} q_n$ where $p$ equals the largest integer that can be taken and still maintain the condition $ii > 0$.

Also note that $g_{q_i}^{ii-1}$ from GAWE and $g_{q_i}^{ii-1}$ from MGAWE are vectors which will have different lengths after the second expansion point is considered. In addition, it should be noted that the $q_l + 1$ to the $\sum_{n=1}^{num\_pts} q_n$ vectors are generated using the same procedure that is used in GAWE with the exception that the new set of $g_{q_i}^{ii-1}$ is used and the vectors should be orthogonalized against all vectors already generated at any expansion point(s) already considered. The above procedure is followed for all vectors generated at all $num\_pts$ expansion points. When $V_q$ is finally constructed, then $g_i(\sigma)$ is found the same way it is found in GAWE using the $\sigma$ corresponding to the last expansion point considered. Therefore

$$g_i(\sigma) = (V_q^{ii} (\mathbf{A}_0 + \sigma \mathbf{A}_1 + \sigma^2 \mathbf{A}_2 + \sigma^3 \mathbf{A}_3 + \sigma^4 \mathbf{A}_4) V_q)^{-1} \times V_q^{ii} (\mathbf{y}_0 + \sigma \mathbf{y}_1 + \sigma^2 \mathbf{y}_2 + \sigma^3 \mathbf{y}_3)$$

and

$$x(\sigma) = x(\sigma + s_{num\_pts}) \approx V_q g_i(\sigma).$$

VI. MGAWE Numerical Examples

The examples used in this section are the same ones used in Section IV.

Example 1: Fig. 8 shows the magnitude of the response of $E_z$ at a node in the FEM model. The response computed using an $LU$ decomposition at 201 frequency points is shown as a solid curve and the response computed using MGAWE with expansion points at $j2\pi 200\mathbf{E}06$ and $j2\pi 400\mathbf{E}06$ with $q_1 = 30$ and $q_2 = 30$ is shown as a dot-dot curve. The total time taken for the MGAWE computations is 1647 s. Although this computation time is 35 s longer than GAWE as shown in Fig. 4, the response is much more accurate. Furthermore, if more expansion points were chosen so that AWE or GAWE would obtain...
the same accuracy as MGAWE, the computation time for AWE or GAWE could then be much more than MGAWE.

**Example 2:** Fig. 9 shows the impedance computed using both an $LU$ decomposition at 1001 frequency points as a solid curve and MGAWE at 1001 frequency points with expansion points at $j2\pi70000E06$ and $j2\pi120000E06$ with $q_1 = 20$ and $q_2 = 20$ as a dot-dot curve. The total computation time for MGAWE is 168 s. For this example, MGAWE is more accurate than the GAWE solution in Fig. 7 at only a cost of 12 more s. Again, if more expansion points were chosen so that AWE or GAWE would obtain the same accuracy as MGAWE, the computation time for AWE or GAWE could then be much more than MGAWE.

**VII. CONCLUSION**

In this paper, methods are described in which matrix equations resulting from a FEM representation of an electromagnetics wave problem are solved using either GAWE or MGAWE. These solutions are compared to each other as well as to solutions computed using an $LU$ decomposition and/or AWE. Some numerical examples, where the outer boundary of the FEM region is treated with either a PML backed by a PEC or with an absorbing boundary condition, are presented to demonstrate that MGAWE can offer superb accuracy for wideband solutions while still maintaining computational efficiency.

Although superb results were obtained in the numerical examples, several issues should be noted. First, where to pick the expansion points remains an open issue. Second, how many moments to generate at each point remains unanswered. Third, a robust way to determine when the solution has converged still needs to be found. Finally, how to handle terms that have an exponential frequency variation in either the matrix or the right-hand side needs to be addressed. These and other issues are subjects of further research by the authors of this paper.

**REFERENCES**


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