and integrates to

\[ I \approx \text{Im} \left( k l \hat{n} \cdot \hat{x} \right) \eta_i \left( k h \hat{n} \cdot \hat{z} \right) e^{2ikz} \]

where

\[ \eta_i(x) = \begin{cases} \sin(x)/x, & x \neq 0 \\ 1, & x = 0. \end{cases} \]

The result in (6) differs from that in (3) by the insertion of the two factors that account for the inclination of the rectangular plate with respect to the viewer. It is these inclination factors that were not properly accounted for in [1].

We express this result in spherical coordinates so that we can compare it to the corresponding expression in [1] more easily. With the origin of coordinates translated to the point \( (x', y', z') \), we write \( \hat{n} = \hat{x} \sin \theta \cos \varphi + \hat{y} \sin \theta \sin \varphi + \hat{z} \cos \theta \),

\[ 0 \leq \theta < \pi/2, \quad 0 \leq \varphi < 2\pi \]

with \( \theta \) measured from the positive z-axis toward the negative one and \( \varphi \) from the positive x-axis toward the positive y-axis. The strict inequality on \( \theta \) is none other than the PO condition \( \hat{n} \cdot \hat{z} > 0 \). From this and (6) we get that

\[ I \approx \text{Im} \left( k l \tan \theta \cos \varphi \right) \eta_i \left( k h \tan \theta \sin \varphi \right) e^{2ikz}. \]

We compare this result to (7) in [1]. If in (9) we set \( \varphi = 0 \) or \( \varphi = \pi \), we get

\[ I \approx \text{Im} \left( k l \tan \theta \right) e^{2ikz}. \]

while, if we set \( \varphi = \pi/2 \) or \( \varphi = 3\pi/2 \), we get

\[ I \approx \text{Im} \left( k h \tan \theta \right) e^{2ikz}. \]

These are identical to (7) in [1] provided the pixel’s area factor \( lb \) is removed, the taper factor \( \cos^2 \theta \) is inserted, and the sinc function is defined according to (7) above [the original, and prevailing, definition of the sinc function is (4) \( \sin \pi x / \pi x \). Setting \( \varphi = 0 \) or \( \varphi = \pi \) in (8) means that we have a flat plate rotated through an angle \( \theta \) about the y-axis. Similarly, setting \( \varphi = \pi/2 \) or \( \varphi = 3\pi/2 \) means that we have a plate rotated about the x-axis. Accordingly, formula (7) in [1] interprets surfaces at the pixel level as flat plates rotated either about the vertical (y-axis) or the horizontal (x-axis). It ignores all other orientations.

III. CONCLUSION

For the hardwired graphics approach to PO we have obtained a formula that we consider an improvement over that of Rius et al [1]. We did this by using not only z-buffer information at the pixel level but also information about the normal. In effect, we have approximated the visible scattering surface by flat panels, each tangent to the surface at the pixel center. Since each panel is defined only over one pixel, adjacent panels do not necessarily intersect one another. Thus, as with the one-term (z-buffer) approximation of the surface at the pixel level, the original surface is broken up into a number of disconnected flat panels. This may be unpalatable to some, but it is the best approximation of the surface for the given information.

APPENDIX

From one of the reviewers we have learned that XPATCHI [5] (Version 5.4, dated Feb. 22, 1993), a code that also takes advantage of computer graphics, employs an expression in agreement with ours. The reviewer mentions that this formula “greatly enhances the accuracy of the PO solution.” We have also been in communication with Dr. Rius, one of the authors of [1], and he has informed us that he will present his views on this matter in a forthcoming submission to this journal.

REFERENCES


A Perfectly Matched Anisotropic Absorber for Use as an Absorbing Boundary Condition

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Abstract—An alternative formulation of the “perfectly matched layer” (PML) technique is based on using a layer of diagonally anisotropic material to absorb outgoing waves from the computation domain. The material properties can be chosen such that the interface between the absorbing material and free space is reflection-less for all frequencies, polarizations, and angles of incidence. This approach does not involve a modification of Maxwell’s equations and is easy to implement in codes that allow the use of anisotropic material properties.

I. INTRODUCTION

Recently, Berenger introduced a novel absorbing boundary condition for truncating two-dimensional finite difference-time domain (FDTD) meshes [1]. His “perfectly matched layer” (PML) technique is based on using a layer of lossy material to absorb outgoing radiation from the computation domain. In the PML layer, the Cartesian field components are split into two subcomponents (i.e., \( H_{xy} = H_{x} + H_{y} \)). Another formulation of the PML was given by Chew and Weedon [2]. Their approach is based on introducing complex coordinate stretching variables. In both approaches, Maxwell’s equations were modified to add additional degrees-of-freedom. The modifications allow the specification of a lossy material layer such that a planar interface between the PML material and free space...
The following condition is required to match the intrinsic impedance of the medium to free space

\[ \frac{\varepsilon}{\varepsilon_0} = \frac{\mu}{\mu_0} \]  

Thus, \([\varepsilon]\) and \([\mu]\) can be written as

\[ [\mu] = \mu_0 [\Lambda] = \mu_0 \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \]  

\[ [\varepsilon] = \varepsilon_0 [\Lambda] = \varepsilon_0 \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \]  

Consequently, Maxwell's equations in the medium reduce to

\[ \nabla \cdot [\Lambda] \vec{E} = 0 \]  

\[ \nabla \cdot [\Lambda] \vec{H} = 0 \]  

\[ \nabla \times \vec{E} = -j\omega [\Lambda] \vec{H} \]  

\[ \nabla \times \vec{H} = j\omega [\varepsilon_0] [\Lambda] \vec{E} \]  

It can be shown that plane waves are eigensolutions of Maxwell's equations in the diagonally anisotropic medium. General solutions for \( \vec{E} \) and \( \vec{H} \) can be constructed from plane waves of the form

\[ \vec{E}(\vec{r}, t) = \vec{E} e^{-j(\vec{k} \cdot \vec{r} - \omega t)} \]  

\[ \vec{H}(\vec{r}, t) = \vec{H} e^{-j(\vec{k} \cdot \vec{r} - \omega t)} \]  

where \( \vec{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z} \), and \( \vec{E} \) and \( \vec{H} \) are constant vectors. Substituting (6) into (5) results in

\[ \vec{k} \cdot [\Lambda] \vec{E} = \vec{k} \cdot [\Lambda] \vec{H} = 0 \]  

\[ \vec{k} \times \vec{E} = \omega \mu_0 [\Lambda] \vec{H} \]  

\[ \vec{k} \times \vec{H} = -\omega \varepsilon_0 [\Lambda] \vec{E} \]  

The dispersion relation which determines the form of the propagation vector \( \vec{k} \) can be determined easily using the following variable transformations

\[ \vec{E}' = [\Lambda]^{\frac{1}{2}} \vec{E} \]  

\[ \vec{H}' = [\Lambda]^{\frac{1}{2}} \vec{H} \]  

\[ \vec{k}' = \frac{1}{\sqrt{abc}} [\Lambda]^{\frac{1}{2}} \vec{k} \]  

Applying the transformations to (7) results in

\[ \vec{k}' \cdot \vec{E}' = \vec{k}' \cdot \vec{H}' = 0 \]  

\[ \vec{k}' \times \vec{E}' = \omega \mu_0 \vec{H}' \]  

\[ \vec{k}' \times \vec{H}' = -\omega \varepsilon_0 \vec{E}' \]  

Since \( \vec{k}' \) is perpendicular to both \( \vec{E}' \) and \( \vec{H}' \), the dispersion relation can be obtained from (7) and is given by

\[ \vec{k}' \cdot \vec{k}' = k_x^2 = \omega^2 \mu_0 \varepsilon_0 \]  

Finally, by using the inverse of the variable transformation in (8), the dispersion relation becomes

\[ \frac{k_x^2}{bc} + \frac{k_y^2}{ac} + \frac{k_z^2}{ab} = k_0^2 \]  

The dispersion relation (11) is the equation of an ellipsoid whose solutions are of the form

\[ k_x = k_0 \sqrt{bc} \sin \theta \cos \phi \]  

\[ k_y = k_0 \sqrt{ac} \sin \theta \sin \phi \]  

\[ k_z = k_0 \sqrt{ab} \cos \theta \]  

This result suggests that the individual components of the propagation vector can be manipulated through the choice of \( a \), \( b \), and \( c \).
Fig. 2. Oblique incidence on a z-plane interface. In general, $\theta_T$ is complex.

III. REFLECTION COEFFICIENT

To determine the reflection coefficient of the interface in Fig. 1, a plane wave incidence problem must be solved. The propagation vectors are restricted to the $zz$-plane, so the dispersion relation (11) reduces to

$$k_z = k_0 \sqrt{bc \sin \theta}$$

$$(13)$$

The geometry of the plane wave incidence problem is shown in Fig. 2. Any arbitrarily polarized plane wave can be decomposed into a near combination of TE, (E has only a y-component) and TM, (H has only a y-component) modes. For the TE case, the incident, reflected, and transmitted waves are given by

$$E_x (r) = E_0 e^{-jk_0(sin \theta_x.x + \cos \theta_x.z)}$$

$$E_x (r) = R^{TE} E_0 e^{-jk_0(sin \theta_x.x - \cos \theta_x.z)}$$

$$E_x (r) = T^{TE} E_0 e^{-jk_0(\sqrt{bc \sin \theta_x.x + \sqrt{ab \cos \theta_x.z})}$$

$$H_z (r) = \frac{\varepsilon_0}{\mu_0} (-\cos \theta_x.x + \sin \theta_x.z)$$

$$H_z (r) = R^{TE} \frac{\varepsilon_0}{\mu_0} (\cos \theta_x.x + \sin \theta_x.z)$$

$$H_z (r) = T^{TE} \frac{\varepsilon_0}{\mu_0} \left( -\sqrt{\frac{a}{\varepsilon}} \cos \theta_x.x + \sqrt{\frac{b}{\varepsilon}} \sin \theta_x.z \right)$$

$$H_z (r) = e^{-jk_0(\sqrt{bc \sin \theta_x.x + \sqrt{ab \cos \theta_x.z})}$$

where $R^{TE}$ and $T^{TE}$ are the reflection and transmission coefficients for TE$\theta$ polarization.

Enforcing the tangential continuity of $E$ and $H$ on the interface involves matching the magnitude and phase of the $E_y$- and $H_x$-components. Magnitude matching results in

$$1 + R^{TE} = T^{TE} \cos \theta_i - R^{TE} \cos \theta_i = T^{TE} \sqrt{\frac{b}{a}} \cos \theta_i.$$  

$$(15)$$

The phase matching condition is a generalization of Snell's law and is given by

$$\sin \theta_i = \sin \theta_r$$

$$\sqrt{bc \sin \theta_i} = \sin \theta_i.$$  

$$(16)$$

Solving for $R^{TE}$ in (15) and (16) gives

$$R^{TE} = \frac{\cos \theta_i - \sqrt{\frac{b}{a}} \cos \theta_i}{\cos \theta_i + \sqrt{\frac{b}{a}} \cos \theta_i}$$

$$(17)$$

For the TM case, the same procedure can be followed to determine $R^{TM}$, resulting in

$$R^{TM} = \frac{\sqrt{\frac{b}{a}} \cos \theta_i - \cos \theta_i}{\cos \theta_i + \sqrt{\frac{b}{a}} \cos \theta_i}$$

$$(18)$$

Imposing the requirement $\sqrt{bc} = 1$ simplifies the phase matching condition (16). It follows that $\theta_i = \theta_r$ and the reflection coefficients are not a function of incident angle. By also requiring $a = b$, the interface will be perfectly reflectionless for any frequency, angle of incidence, and polarization.

IV. MATERIAL PROPERTIES OF THE PML

The values of $a$, $b$, and $c$ are not independent. They are related by

$$a = b = \frac{1}{c}.$$  

$$(19)$$

Thus, the PML layer can then be characterized by one complex number $\alpha = \alpha - j\beta$. When $\alpha, \beta > 0$, the transmitted wave will be damped in the anisotropic medium

$$E_x (x, z) = E_0 e^{-jk_0 \cos \theta_x.x} e^{-j\beta z} (\sin \theta_i x + a \cos \theta_i x) e^{j\omega t}.$$  

$$(20)$$

From (20), it is easy to see that $\alpha$ determines the wave length in the anisotropic absorber and $\beta$ determines the rate of decay of the transmitted wave.

Now, it is interesting to interpret the material properties required for a perfectly matched interface. Using (4), the material tensor $[A]$ can be written as

$$[A] = \begin{pmatrix} \alpha & \alpha - j\beta & 0 \\ \alpha - j\beta & 0 & 0 \\ 0 & 0 & \frac{a^2 + \beta^2}{a^2 + \beta^2} \end{pmatrix}.$$  

$$(21)$$

The material properties $\epsilon$, $\mu$, $\sigma_T$, $\sigma_M$ are now given by

$$[\epsilon] = \epsilon_0 [P]$$

$$[\mu] = \mu_0 [P]$$

$$[\sigma_T] = \omega \sigma_0 [S]$$

$$[\sigma_M] = \omega \mu_0 [S]$$

where

$$[P] = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \frac{a^2 + \beta^2}{a^2 + \beta^2} \end{pmatrix}.$$  

$$(23)$$

$$[S] = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -\frac{a^2 + \beta^2}{a^2 + \beta^2} \end{pmatrix}.$$  

$$(24)$$

Note that the z-component of the electric and magnetic conductivities is negative, implying the existence of dependent sources within the material.
Fig. 3. Reflection from a finite slab of a perfectly matched anisotropic medium.

Fig. 4. Geometry of the half-wave dipole radiation problem.

V. FEM IMPLEMENTATION

In any practical implementation of the anisotropic PML, the absorbing layer must be truncated as shown in Fig. 3. The backing used to truncate the PML will have a nonzero, angle-dependent reflection coefficient \( Q(\theta) \). The overall reflection coefficient of the anisotropic PML and its backing is given by

\[
|H(\theta)| = |Q(\theta)|e^{-2\Re(k_0 \cos \theta d)}.
\]

The anisotropic PML mesh truncation scheme was implemented in a FEM code using tetrahedral edge elements (lowest-order TVFEM). A perfectly conducting backing was used to truncate a homogeneous layer of the anisotropic PML.

VI. RESULTS

The far-field radiation pattern of a simple half-wave length dipole antenna was computed using a layer of anisotropic PML for mesh truncation. For this problem, the PML layer was placed 0.15\( \lambda \) away from the antenna element. The layer was 0.1\( \lambda \) thick. The geometry of the problem is shown in Fig. 4.

The FEM solution was obtained using a mesh with 4998 tetrahedra, where the mesh density was approximately 10 elements per wave length. The far-field pattern in two observation planes is plotted in Fig. 5. The results obtained using the anisotropic PML for mesh truncation are in good agreement with the exact solutions, even when using a relatively thin absorbing layer. In this case \( (d = 0.1\lambda) \), the absorber was approximately two elements thick.

VII. CONCLUSION

A new formulation of the PML mesh truncation scheme was developed. The formulation is based on using diagonally anisotropic material to construct a lossy absorbing layer. Although this approach is similar to the PML scheme investigated by others, it is not equivalent. Unlike Berenger's PML, the anisotropic PML does not require any modification of Maxwell's equations. The primary benefit of this approach is convenient implementation, especially in frequency-domain finite element codes.

REFERENCES