A Partitioning Technique for the Finite Element Solution of
Electromagnetic Scattering from Electrically Large
Dielectric Cylinders

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Abstract—A finite element partitioning scheme has been developed to reduce the computational costs of modeling electrically large geometries. In the partitioning scheme, the cylinder is divided into many sections. The finite element method is applied to each section independent of the other sections, and then the solutions in each section are coupled through the use of the tangential field continuity conditions between adjacent sections. Since the coupling matrix is significantly smaller than the original finite element matrix, it is expected that both the CPU time and memory costs can be significantly reduced. The partitioning scheme is coupled to the bisection method to account for the boundary truncation. Numerical results are presented to demonstrate the efficiency and accuracy of the method.

I. INTRODUCTION

The finite element method (FEM) is being used extensively to perform electromagnetic modeling, because it is efficient in modeling geometries that are both penetrable and inhomogeneous. The efficiency is largely due to the fact that the resulting matrix is sparse. Banded matrix solvers [1] have traditionally been used to solve the FEM stiffness matrix. However, when the electrical size of the geometry of interest is large, the number of unknowns required to obtain an accurate solution may be so large that the use of banded solvers becomes computationally infeasible. The development of direct sparse matrix solvers such as the nested dissection method [2] has been a major area of research. These methods are capable of handling multiple excitations with only a small increase in computation costs. The computational complexity of sparse matrix methods is usually less than that of banded matrix methods. For example, the computation cost of the nested dissection method is proportional to $N^{1.5}$, while the computation cost for the banded matrix methods is proportional to $N^2$ for two-dimensional geometries. The direct sparse matrix methods rely heavily upon the use of matrix partitioning or reordering schemes to reduce the number of nonzero entries in the factorization of the resulting matrices.

In numerical electromagnetics, several researchers have developed methods that have characteristics very similar to those used in direct sparse matrix methods. However, rather than formulating it from a physical viewpoint where the geometry is divided into subsections, within each section, a numerical solution is generated independent of the other sections. The coupling between the sections is performed in an iterative manner [3], [4] or by employing rigorous and more computationally expensive procedures [5], [6]. It should be noted that all these techniques have been developed for integral equation methods. In this paper, a partitioning technique is presented for the FEM to solve the problem of electromagnetic scattering from a two-dimensional cylinder. The integral equation methods, the geometry is divided into subsections in which a set of finite element solutions is generated for each section independent of the other sections. The sections are then coupled together to form a sparse matrix that is significantly smaller than the original FEM stiffness matrix. This partitioning technique can be coupled to a boundary truncation method such as the bisection method [7] to solve the scattering problem. During this research, it was discovered that another group was employing a very similar method for waveguiding geometries [8]. It is expected that both methods will produce the same efficiency.

II. FORMULATION

Let us consider an infinitely long isotropic dielectric cylinder whose shape and material properties are invariant along the $z$ axis, as shown in Fig. 1. The incident wave is also assumed to be invariant in $z$; therefore, the fields around the cylinder can be decoupled into $TM(E_z, H_y, H_x)$ and $TE(H_z, E_y, E_x)$ polarizations. In this paper, the $TM$ polarization is described. With some slight modifications, the formulation for the $TE$ polarization can also be obtained. To simplify the notation, let us define $E$ to be $E_z$. $E$ satisfies the generalized Helmholtz equation, which is given by

$$\nabla \cdot \left( \frac{1}{\mu} \nabla E \right) + \omega^2 \epsilon^* E = 0 \quad (1)$$

where $\epsilon^* = \epsilon - j(\sigma/\omega)$. It is assumed that the $\exp(j\omega t)$ time dependence has been suppressed.

Initially, the formulation is presented to solve a boundary value problem in which the Neumann boundary condition is specified on $dS_{ext}$ in Fig. 1. Then the procedure for coupling the partitioning method to the bisection method is described to solve the scattering problem. In the partitioning method, the cylinder is divided into $M$ sections as shown in Fig. 2, where $dS_1, \ldots, dS_{m-1}, dS_M$ denote the interior boundaries that are created from the partitioning operation. The segments $dS_0$ and $dS_M$ are artificial constructs that allow us to represent the first and last sections as four-sided sections.

A. FEM Solution in a Single Section

Let us consider the $m$th section of the partitioned cylinder as shown in Fig. 3. The solution in this section can be obtained from FEM if
the correct boundary conditions are specified. However, although the Neumann boundary condition on $dS_{ext}$ is assumed to be known, the boundary conditions on $dS_{m-1}$ and $dS_m$ are not known a priori. Let us define $h_{mL}$ and $h_{mR}$ to be the tangential magnetic field on $dS_{m-1}$ and $dS_m$, respectively, for the $m$th element. These two quantities can also be written in terms of a derivative of the electric field $E$ where their orientation is shown in Fig. 3. The tangential magnetic fields can be written in terms of an infinite sum of linearly independent functions multiplied by unknown coefficients $a_i^{(m)}$ and $b_i^{(m)}$.

$$h_{mL}(t) = \frac{1}{j \mu \omega} \frac{\partial E(t)}{\partial n} = \sum_{i=1}^{\infty} a_i^{(m)} \psi_i(t) \quad t \in [0, d_L]$$

$$h_{mR}(t) = \frac{1}{j \mu \omega} \frac{\partial E(t)}{\partial n} = \sum_{i=1}^{\infty} b_i^{(m)} \psi_i(t) \quad t \in [0, d_R]$$

where $\partial/\partial n$ represents a directional derivative whose direction is outward normal from a point on the surface of the $m$th section. The variable $t$ represents a parametric mapping of the $(x, y)$ coordinates along $dS_{m-1}$ to the line defined by the endpoints $t = 0$ and $t = d_L$, where $d_L$ is the length of $dS_{m-1}$. A similar parametric mapping is made for $dS_m$, where $d_R$ is the length of $dS_m$.

In the numerical implementation of (2) and (3) the infinite sums are truncated to $I_{mL}$ and $I_{mR}$ for $dS_{m-1}$ and $dS_m$, respectively. Using the properties of linearity, we can determine the electric field $E_m$ in the $m$th section by superimposing the three boundary value problems shown in Fig. 4, where $E_R^m$, $E_L^m$, and $E_{cm}^m$ represent the electric field solution for the corresponding boundary value problems in the figure. Note that the boundary conditions for $E_R^m$ and $E_L^m$ are given by (2) and (3); therefore, these field solutions are given by

$$E_{cm}^m(x, y) = \sum_{i=1}^{I_{mL}} a_i^{(m)} \Lambda_i^L_m(x, y)$$

$$E_R^m(x, y) = \sum_{i=1}^{I_{mR}} b_i^{(m)} \Lambda_i^R_m(x, y)$$

where $\Lambda_i^L_m$ is the finite element solution for the boundary value problem in Fig. 4 associated with $E_{cm}^m$ except that $\psi_i$ replaces $h_{mL}(t)$ as the boundary condition. Similarly, $\Lambda_i^R_m$ is generated from the replacement of $h_{mR}(t)$ by $\psi_i(t)$ in the boundary value problem associated with $E_R^m$. The total field in the $m$th section can thus be written as

$$E_m(x, y) = E_R^m + \sum_{i=1}^{I_{mL}} a_i^{(m)} \Lambda_i^L_m + \sum_{i=1}^{I_{mR}} b_i^{(m)} \Lambda_i^R_m.$$  

Equation (6) is true for all sections except the first and last ones. For the first section, the Neumann boundary condition on $dS_L$ is known. Thus, the second term in (6) can be directly incorporated into $E_R^1$. Similarly, for the last section, the third term in (6) can be incorporated into $E_R^m$.

**B. Coupling the Solution Between Sections**

Once the solutions have been generated for every section as described above, it is necessary to couple them together in order to determine the coefficients $a_i^{(m)}$ and $b_i^{(m)}$. The coupling is accomplished by enforcing the continuity of the tangential magnetic and electric fields at the interface between adjacent sections. From (2) and (3), the continuity of the tangential magnetic field is specified by

$$\sum_{i=1}^{I_{mR}} b_i^{(m)} \psi_i(t) = - \sum_{i=1}^{I_{mL}} a_i^{(m)} \psi_i(t) \quad m = 1, 2, \cdots, M - 1. \quad (7)$$

The negative sign on the right hand side of (7) is due to the orientation of the tangential magnetic field quantities in Fig. 3. Due to this continuity condition, it is appropriate to assume that $I_{mR} = I_{m+1L} \equiv I_m$. The sum in (7) can be equated term by term to obtain

$$a_i^{(m+1)} = -b_i^{(m)} \quad m = 1, 2, \cdots, M - 1. \quad (8)$$

The use of (8) in (6) eliminates half of the unknowns. We determine the remaining unknowns by enforcing the continuity of the tangential electric field on $dS_m$.

$$E_m(x, y) = E_{m+1}(x, y) \quad (x, y) \in dS_m. \quad (9)$$

In general, it is not possible to choose the coefficients $a_i^{(m)}$ and $b_i^{(m)}$ to satisfy (9) exactly. Instead, (9) is enforced in a weak sense, i.e.,

$$\int_{dS_m} E_m(x, y) w_{j,m} \, dl = \int_{dS_m} E_{m+1}(x, y) w_{j,m} \, dl \quad (10)$$

where $w_{j,m}$ ($j = 1, 2, \cdots, I_m$) represents a set of linearly independent weighting functions on $dS_m$. Substituting (8) into (6) and then
The number of FEM solutions is equal to the number of boundary conditions. Thus, the partitioning technique can be used to generate each of the finite element solutions. The actual implementation of the bymoment method can significantly affect the efficiency. In the current scheme, the partitioning technique is used to find the FEM solution for the entire computational domain due to each Neumann boundary condition. This scheme is the simplest to implement, but it is computationally inefficient because the finite element solutions associated with the interior boundaries, $A_{km}$ and $A_{km}^R$, must be recomputed each time the partitioning technique is applied. A much more efficient scheme would be to generate FEM solutions for the bymoment method on each individual section. Then the coefficients associated with the bymoment method on each section can be placed in the coupling matrix. The solution of this coupling matrix provides a complete solution to the scattering problem. We are currently implementing and testing this scheme.

### III. Numerical Considerations

To determine the efficiency of the partitioning technique, we perform an asymptotic analysis to determine the number of floating point operations required for this method and compare the results to the standard method, in which the FEM solutions are computed with a banded matrix solver based on LU decomposition. In addition, the memory requirements are compared. The analysis is performed for an electrically large two-dimensional square grid consisting of bilinear square elements with $N$ nodes (one unknown per node) in the grid. To simplify the asymptotic analysis, we assume Neumann boundary conditions are enforced. For the standard method, the number of floating point operations for the LU decomposition is proportional to $N^3$, while the backsolve is proportional to $N^{3/2}$. In the partitioning technique, the square grid is partitioned into $M$ uniform sections. In this analysis, it is assumed that as the electrical size of the square increases, the number of sections $M$ increases in such a way as to maintain a constant width for each section. The LU decomposition time associated with each section is greatly reduced because the bandwidth of the FEM matrix remains constant. Consequently, the sum of the decomposition times for all the sections is proportional to $N$. The partitioning method requires performing numerous backsolve operations on each section. The number of floating point operations required to perform the backsolves for all the sections is proportional to $N^{3/2}$. The solution of the final coupling matrix shown in Fig. 5 requires an operation count that is proportional to $M N^{3/2}$. Since $M$ is proportional to $N^{1/2}$, the complexity of the partitioning method is equal to that of the standard method, but $M$ is usually chosen to have values of $N^{1/2}/20$ to $N^{1/2}/40$. Thus, the operation count can be reduced by a large multiplicative factor. However, in practice, the cylinders must be very large—over 50 wavelengths in both directions—for the computation time associated with the coupling matrix to dominate the overall computation time. For smaller cylinders, we expect the backsolve time to be the dominant factor.

In addition to improving the computational efficiency, the partitioning technique can substantially reduce the memory storage from the standard method. The memory requirements of a banded matrix algorithm is $N^{3/2}$. In the partitioning technique, each section can be handled in a sequential manner; therefore, the same memory locations can be used for each section, resulting in a memory requirement of $N^{3/2}/M^2$. The extra factor of $M$ in the denominator is due to the fact that the half-bandwidth of the FEM matrix for each section has been reduced by a factor of $M$. For the coupling matrix in Fig. 5, the memory requirement is proportional to $NM$, where the exact value is dependent upon the type of basis functions used in (2) and (3). For the problems that we consider in this paper, a factor of ten reduction in the memory is typical.

There are several other possible advantages of the partitioning technique. For problems in which the material properties change abruptly, the FEM grid must transition from one grid density to another at the material interface in order to maintain a constant nodal density per wavelength of the material. This transition is not easy to achieve for traditional finite element techniques. However, in the partitioning method each material can be placed in a separate section. Since the grid for each section is independent of the other sections, there is no need to provide a transition between the sections.

In partitioning the geometry into sections, there are many instances where two or more of the sections are geometrically identical. In these instances, the FEM solution need only be computed once for each set of identical sections, thereby further reducing the computational costs. A further advantage of the partitioning technique is that it is...
IV. NUMERICAL RESULTS

To demonstrate the efficiency of the partitioning technique, we consider two boundary value problems and two associated scattering problems. The excitation for all the geometries is assumed to be a TM polarized plane wave of the form

\[ E_{TM}^\infty = E_0 e^{-j k (x \sin \phi - y \cos \phi)} \]  

(12)

The geometry for the boundary value problems consists of a rectangular region of free space. This geometry was chosen because the Neumann boundary condition for a free space region is well known. For the scattering problems, rectangular dielectric cylinders are considered. The parameter of interest is the echo width per wavelength \( L_e / \lambda \) [11], and it is given by

\[ L_e = \lim_{\phi \to \infty} \frac{2 \pi p}{\lambda_0} \frac{|E_{TM}^\infty|^2}{|E_{TM}^\infty|^2} \]  

(13)

where \( E_{TM}^\infty \) is the scattered electric field and \( \lambda_0 \) is the free space wavelength. The echo width is computed as a function of \( \theta \) where \( \theta \) is the angle defined from the positive \( x \) axis.

A finite element code based on the eight-node quadrilateral [1] element has been written to test this method. A nodal density of approximately 20 nodes per wavelength has been maintained for the numerical computations. This choice of nodal density is expected to produce accurate results [12]. The geometries of interest are electrically large; therefore, the FEM solutions may be subject to errors produced by the interior resonances of the computational domain [13]. To remove these errors, a small loss has been introduced in the computation domain. Thus, we set \( \epsilon_r' = \epsilon_r - j \epsilon_r^\infty \), where \( \epsilon_r \) is the relative permittivity given by the geometry. The computation times shown throughout this section are for the Cray YMP supercomputer. The basis and weighting functions \( \Psi \) and \( \phi_{j,m} \) may be either entire domain functions such as sinusoidal functions with support over \( dS_m \) (Fig. 1) or else subdomain functions such as triangle functions with support over a subsection of \( S_m \). For the numerical results presented in this paper, triangle functions with support over two elements (5 nodes) are used.

The first example is the problem of a plane wave propagating through a \( 2 \lambda_0 \times 2 \lambda_0 \) region of free space. A finite element mesh of 16,141 nodes (5,200 elements) is used to discretize the entire computation domain. In Table I, a comparison is made between the standard method and the partitioning technique in which the domain has been divided into either six \( (2 \lambda_0 \times 4 \lambda_0) \) or 12 \( (2 \lambda_0 \times 2 \lambda_0) \) sections. Since the geometry of each section is the same, only one section is considered in the FEM computation. From the table, it is evident that the reduction in CPU time is undramatic, especially given the fact that the FEM solution is only computed in one section. The improvements that we expect from the asymptotic analysis are not present in this example because the bandwidth of the FEM matrix for each section is the same as the bandwidth for the entire computation domain. Thus, a reduction in CPU time cannot be expected for elongated geometries that are electrically large only along one direction. The reduction in memory requirements, on the other hand, is significant (a factor of more than eight for the 12-section case).

The second example is a scattering problem based on the first example. The scatterer is a \( 1 \lambda_0 \times 12 \lambda_0 \) dielectric cylinder \((\epsilon_r = 4)\). The mesh is the same as the one used in the first example. Because the geometry is electrically large, 226 expansion functions are required for accurate implementation of the bymoment method, which results in a dramatic increase in computation costs. The echo width solution is shown in Fig. 6 for an incidence angle of 90°. The results from the partitioning technique and the standard method are virtually identical, demonstrating that the partitioning has very little effect on the solution. Because the bandwidth of the FEM matrix is small, the majority of the CPU time is spent on the boundary truncation. From Table I, we see that the partitioning technique is less efficient than the standard FEM solution. This is due to the inefficient manner in which the bymoment method is coupled to the partitioning technique. The efficiency is expected to improve significantly with a better scheme. The memory savings has also been reduced compared to the first example due to the memory requirements of the bymoment method.

For the third example, we consider a plane wave propagating through a \( 9 \lambda_0 \times 9 \lambda_0 \) region of free space. Because the region is a square, it is expected to exhibit the computational savings described in the asymptotic analysis. The computation domain is discretized with a grid consisting of 29,105 nodes (9,580 elements). For the partitioning technique, the domain is divided into both five \((9 \lambda_0 \times 1.8 \lambda_0)\) and ten \((9 \lambda_0 \times 0.9 \lambda_0)\) sections. In Table I the computation times are
tabulated for all three cases. The reduction in CPU time is very significant (more than a factor of 14 for the ten-section case). Again, we took advantage of the fact that the FEM solution only needs to be computed in one section since all the sections are identical. However, if the computation is performed for all the sections, the CPU time increases from 34.5 s (for the five-section case) to approximately 65 s. Thus, for square dielectric cylinders with nonidentical sections, the computational savings is still very large. Furthermore, the memory requirements have been reduced from 26 Mwords for the standard method to 1.1 Mwords for the partitioning method.

In the fourth and final example, the plane wave scattering from a square dielectric ($\varepsilon_r = 4$) cylinder with sides of length $4.5\lambda_0$ is considered. The FEM grid is the same as the one used in the third example, and the bymoment boundary condition is implemented with 180 expansion functions. The echo width solution is shown in Fig. 7 for $\phi_i = 45^\circ$. The two methods produce virtually identical results. A comparison of the memory and CPU requirements for this example is shown in Table I. For the partitioning technique, the computation domain is divided into both five and ten sections. The reduction in CPU time is not as dramatic as in the third example because of the inefficient manner in which the bymoment method is coupled to the partitioning technique. However, the savings are still very significant in both computation time and memory costs.

V. SUMMARY

A new method has been presented to decrease the solution time for the finite element modeling of electromagnetic scattering problems. In this method, the geometry is partitioned into smaller sections. Within each section, a set of finite element solutions is generated that are independent of the other sections. The sections are then coupled by the use of the field continuity conditions. The resulting matrix equation is block tridiagonal and significantly smaller than the original FEM matrix equation. An asymptotic analysis was provided to demonstrate the potential efficiency of this method. The method was coupled to the bymoment method in order to solve the scattering problem. Finally, numerical results were presented to demonstrate both its accuracy and efficiency for some canonical geometries.

REFERENCES