Finite element analysis of bodies of revolution using the measured equation of invariance

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Abstract. This paper is concerned with the finite element analysis of electromagnetic scattering from an axially symmetric perfectly conducting body of revolution with an arbitrary cross section in free space. The coupled azimuthal potential formulation is used to express the solution in terms of the two field components, \( E_\phi \) and \( H_\phi \). The use of a Fourier series expansion reduces the three-dimensional problem to a series of two dimensional problems. At the truncation boundary the measured equation of invariance (MEI) is used. The MEI method uses information about the scatterer to find an equation for the boundary nodal values in terms of the values at neighboring nodes. The MEI produces a boundary condition which allows the matrix to retain its sparse structure. This method also allows the finite element mesh to be truncated very close to the body. Numerical examples are given for spheres, finite cylinders, and a cone.

Introduction

This paper is concerned with the application of the finite element method (FEM) to the problem of scattering from perfectly conducting bodies of revolution. One of the major problems with using the finite element method in an unbounded region is the truncation of the mesh. At the truncation surface a boundary condition must be specified which accurately accounts for the Sommerfeld radiation condition. Various methods to handle this have been proposed by researchers including Silvester and Hsieh [1971], Engquist and Majda, [1977], Mei, [1974], and Cangellaris and Lee, [1990]. The application of the method proposed by Mei, [1974] to the body of revolution was done by Morgan et al., [1977], where they used the coupled azimuthal potential (CAP) formulation to reduce the three dimensional problem into a series of two dimensional ones.

The above methods all have some weakness. Either they are inefficient or inaccurate at modeling electrically large geometries. Recently, the measured equation of invariance (MEI) has been proposed by Mei et al., [1994]. This method has shown itself to be both accurate and efficient for two-dimensional geometries when compared to the other methods described above. In this paper we extend the MEI method to bodies of revolution. The CAP formulation is based on the use of the field components, \( F_\phi \) and \( \eta H_\phi \) (\( \eta \) is the impedance of free space), as the unknowns. In general, FEM can be applied to lossy isotropic inhomogeneous scatterers, but in this paper we will only consider perfectly conducting scatterers to simplify the development of the MEI to bodies of revolutions.

Numerical results are given for several conducting spheres, finite length cylinders, and a
cone. Comparisons of the results are made regarding nodal density, number of nodal layers, and the coupling between neighboring nodes. A sample mesh is shown in Figure 1. The shaded area is the perfect electric conductor, which is the scatterer. The three dimensional scatterer is generated by revolving the shaded area around the z axis. The mesh shown in Figure 1 has three layers of nodes.

Formulation

For the body of revolution problem the geometry of the scatterer is invariant with respect to $\phi$; however, the fields are functions of $\phi$. It is useful to express the electric and magnetic fields in their Fourier modes in $\phi$, because we will then have a series of modes which are invariant in $\phi$ as given by

$$E_{\phi}(\rho, \phi, z) = \sum_{m=-\infty}^{\infty} e^{m}(\rho, z)e^{im\phi}$$  \hspace{1cm} (1)

$$H_{\phi}(\rho, \phi, z) = \sum_{m=-\infty}^{\infty} h^{m}(\rho, z)e^{im\phi}$$  \hspace{1cm} (2)

where the $e^{j\omega t}$ time variation has been suppressed. Instead of solving a three-dimensional FEM problem we are able to solve a series of two dimensional problems for $e^{m}(\rho, z)$ and $h^{m}(\rho, z)$ from (1) and (2), since both these terms are independent of $\phi$. In this work, the CAP formulation is used to perform the finite element analysis. This analysis is described in the papers given in the introduction and will not be considered here. Instead, the remainder of the paper will be devoted to the description of the boundary conditions.

There are three different boundaries on the finite element mesh for a perfectly conducting body of revolution, namely, the $z$ axis, the surface of the scatterer, and the truncation boundary. For the field values on the $z$ axis there are two different cases which need to be considered [Morgan and McE, 1979]. If $|m| \neq 1$, then we need to satisfy the homogeneous Dirichlet boundary conditions for both $e^{m}(\rho, z)$ and $h^{m}(\rho, z)$. If $|m| = 1$, we must satisfy the homogeneous Neumann boundary conditions. On the surface of the perfect electrical conductor we must satisfy the homogeneous Dirichlet boundary condition for $e^{m}(\rho, z)$, and the homogeneous Neumann boundary condition for $h^{m}(\rho, z)$.

At the truncation boundary the MEI boundary conditions are used. These boundary conditions are based on the postulate that for each node on the truncation boundary there exists a linear relation,

$$\sum_{i=0}^{N-1} (a_i e_i^{m,s} + b_i \eta h_i^{m,s}) = 0$$  \hspace{1cm} (3)

where $e_i^{m,s}$ and $h_i^{m,s}$ are the scattered electric and magnetic fields, respectively, for mode $m$, where $i = 0$ is the subscript of the node of interest and $i = 1$ to $i = N - 1$ are those of neighboring nodes. This means that a nodal value on the truncation boundary will be coupled only to values at the neighboring nodes. The free space impedance $\eta$ is placed explicitly in (3) to properly scale $b_i$ relative to $a_i$ for better conditioned matrices.
To define the MEI equation, we still need to find the coefficients $a_i$ and $b_i$ for each nodal value. To obtain geometry specific measuring functions a surface current density $\mathbf{J}^n(\mathbf{r}')$ on the scatterer surface is used. This current density is called a metron. Note that in (3) we are free to multiply by a constant and thus choose one of the coefficients. Thus $2N - 1$ unknowns must be determined. By choosing $2N - 1$ linearly independent metrons, we can form a $2N - 1$ by $2N - 1$ matrix equation to solve for the unknowns. It would also be possible to use more than $N$ metrons and then find a solution for the overdetermined system, but this is not done here.

To find $a_i$ and $b_i$ we will now need measuring functions for both the electric and magnetic fields. Consider the electric vector potential $\mathbf{A}$ due to a metron $\mathbf{J}^n$ ($n = 1, \ldots, 2N - 1$),

$$\mathbf{A}^{m,n} = \int_{\partial S} \int_0^{2\pi} \frac{e^{-j[k\mathbf{r} - \mathbf{r}']}}{4\pi |\mathbf{r} - \mathbf{r}'|} \mathbf{J}^n(\mathbf{r}') e^{j\mathbf{im}\phi} \, d\phi \, d\theta'$$

(4)

where $\mathbf{r}$ is the observation point and $\mathbf{r}'$ is the source point. We can now use the vector potential to solve for the measuring functions $e^{m,n}$ and $h^{m,n}$, which are the field values at the nodes on, and just inside, the truncation boundary due to the metron $\mathbf{J}^n$. From Harrington [1961], we get

$$e^{m,n} = \hat{\mathbf{e}} \cdot \left( \frac{\eta}{jk} \nabla \times \nabla \times \mathbf{A}^{m,n} \right)$$

(5)

$$\eta h^{m,n} = \hat{\mathbf{h}} \cdot \left( \nabla \times \mathbf{A}^{m,n} \right).$$

(6)

Substituting these measuring functions back into the MEI equation, we can solve for the coefficients $a_i$ and $b_i$. The metrons used are given by

$$\mathbf{J}^n = \begin{cases} \frac{\hat{\mathbf{e}} \sin \left( \frac{(n+1)\theta}{2} \right)}{\left( \hat{n} \times \hat{\mathbf{e}} \right) \sin \left( \frac{n\theta}{2} \right)} & \text{n odd} \\ \left( \hat{n} \times \hat{\mathbf{e}} \right) \sin \left( \frac{n\theta}{2} \right) & \text{n even} \end{cases}$$

(7)

where $\theta$ is a mapping of $\partial S$ to $[0^\circ, 180^\circ]$ (see Figure 1). In the case where the geometry is spherical, $\theta$ becomes one of coordinates in the spherical coordinate system. The metrons are chosen so that there is no net flow of current into the single point on the axis. For the numerical results the lowest values of $n$ are used. For example, when four nodes (eight unknowns total for $H_\phi$ and $E_\phi$) are coupled together, we choose $n = 1, \ldots, 7$ since one coefficient is specified arbitrarily.

We have also experimented with decoupling $e^{m,n}$ and $h^{m,n}$ in the MEI equation. To find an equation for $e^{m,n}$, we assume that all the $b_i$ terms in (3) are zero. Thus the equation for $e^{m,n}$ at the boundary is written in terms of $e^{m,n}_i$ ($i = 1, \ldots, N - 1$). Similarly, to find an equation for $h^{m,n}_0$, we assume that all the $a_i$ terms in (3) are zero, and we find an equation in terms of only $h^{m,n}_i$ ($i = 1, \ldots, N - 1$). The effects of this decoupling are discussed in the numerical results section.

For the special case of the spherical conducting scatterer, instead of using the vector potential, two scalar radial potentials $A_r$ and $F_r$ are used. To absorb outgoing spherical waves, we choose the potentials used to produce the measuring functions to be

$$A_r^{m,n} = \frac{1}{k\eta} \mathbf{\hat{H}}^{(2)}_n(kr) F_r^{m,n}(\cos(\theta)) e^{j\mathbf{m}\phi}$$

(8)

$$F_r^{m,n} = \frac{1}{k} \mathbf{\hat{H}}^{(2)}_n(kr) F_r^{m,n}(\cos(\theta)) e^{j\mathbf{m}\phi}$$

(9)

where

$$\mathbf{\hat{H}}^{(2)}_n(R) = R \mathbf{\hat{h}}^{(2)}_n(R).$$

The spherical Bessel function of the third kind $h^{(2)}_n(R)$ is defined by Abramowitz and Stegun, [1972]. Note that $A_r^{m,n}$ excites modes which are transverse magnetic (TM) to $\hat{r}$, and $F_r^{m,n}$ excites modes which are transverse electric (TE) to $\hat{r}$. Thus we have a complete set of modes for the source free region outside the scatterer. The measuring functions excited by these two potentials are

$$e^{m,n} = \left( -\nabla \times \hat{F}^{m,n} + \frac{\eta}{jk} \nabla \times \nabla \times \mathbf{A}^{m,n} \right)$$

(10)

$$\eta h^{m,n} = \left( \eta \nabla \times \hat{A}^{m,n} + \frac{1}{jk} \nabla \times \nabla \times \hat{F}^{m,n} \right).$$

(11)
Again, by substituting these results back into the MEI equation we can solve for the needed coefficients $a_i$ and $b_i$. Note that the index $n$ is varied to find the coefficients. The index $m$ is determined by the azimuthal harmonic under consideration.

The MEI boundary condition equation given in (3) is written in terms of the scattered fields $e_i^{m,s}$ and $h_i^{m,s}$. However, the finite element equations are written in terms of the total fields $e_i^m$ and $h_i^m$. If we make the substitutions

$$e_i^{m,s} = e_i^m - e_i^{m,\mathrm{inc}}$$  \hspace{1cm} (12)

and

$$h_i^{m,s} = h_i^m - h_i^{m,\mathrm{inc}}$$  \hspace{1cm} (13)

into (3), we get the expression

$$\sum_{i=0}^{n-1} (a_i e_i^m + b_i \eta h_i^m) - \sum_{i=0}^{n-1} (a_i e_i^{m,\mathrm{inc}} + b_i \eta h_i^{m,\mathrm{inc}})$$  \hspace{1cm} (14)

where $e_i^{m,\mathrm{inc}}$ and $h_i^{m,\mathrm{inc}}$ are the modes of the incident fields (in the absence of the scatterer) for mode $m$ at node $i$. The left side of (14) is now in terms of the total fields so it is ready to be placed into the finite element stiffness matrix. All the terms on the right side of (14) are known, so they can be placed in the forcing vector (the right side of the finite element system of equations).

Results

The code used to generate the results in this section is based on a four-node coupling of the MEI boundary condition except at the corners, where only three nodes are coupled. Figure 2 shows the case where the three closest neighbors are used to find the MEI boundary condition. The location where we are trying to find an equation for either $e^m$ or $h^m$ is denoted by the circled node. The neighboring nodes marked by solid circles are used to find the MEI equations. Recall that when $m \neq \pm 1$, the boundary condition on the $z$ axis is given by $e^m = h^m = 0$. In this case the "corner" moves one node away from the $z$ axis along the truncation boundary.

A mesh generator called FASTQ [Blacker, 1991] is used to create all the meshes for the examples considered here.

Spheres

We will now consider some results for the special case of the perfectly conducting sphere. Figure 3 shows a finite element mesh used for the perfectly conducting sphere. This mesh has a nodal density of 30 nodes per wavelength along the truncation boundary and along the $z$ axis.
This mesh has six layers of nodes (0.167λ) including the layers on the conductor and the truncation boundary.

To compare the results of the finite element program to the analytic solution given by Harrington, [1961], we will plot the magnitude of \( \eta H_\phi \) (or for the sphere \( \eta J_\phi \)) on the conducting surface. The results plotted are in the plane defined by \( \phi = 0 \). The TM\(_z\) mode is used in the following plots, because the TE\(_z\) modes result in an \( H_\phi \) of zero in the \( \phi = 0 \) plane. The direction from which the plane wave excitation is incident is defined by \( \theta^{inc} \), and the radius of the sphere is given by \( R \). Note that \( \lambda \) represents one wavelength in free space.

Figure 4 shows the effect of nodal density on the finite element solution. The excitation is a plane wave which is incident at an angle of \(-45^\circ\). This incident wave was chosen because it converges fairly rapidly, but it still has modes excited other than just \( m = \pm 1 \). The finite element solutions are truncated at \( m = \pm 7 \). As expected, we get more accurate results with a higher nodal density. The obvious trade-off is that when a higher nodal density is used the computation time increases. There is also a problem that can occur with the MEI method when the nodal density is increased. As the nodes on the boundary move closer together, the matrix used to solve for the MEI coefficients becomes ill-conditioned. For nodal densities greater than 30 nodes per wavelength we have not been able to obtain accurate coefficients with a traditional matrix solver. In addition, we have observed that the matrix becomes more ill-conditioned as its size increases. On the other hand, when the nodal density is low, there is no problem in finding the boundary condition, but then significant errors appear due to the coarseness of the finite element mesh. The nodal density required also depends on the size of the scatterer. If the scatterer is small, then more nodes per wavelength are needed to model the surface of the scatterer accurately.

Figure 5 shows the results for the case where the plane wave is incident at \(-90^\circ\); that is, the wave is traveling in the positive \( z \) direction. The incident field has significant components for a

![Figure 4](image)

**Figure 4.** Magnetic field on the surface of a conducting sphere, with spherical harmonics used to find the measuring functions (\( \theta^{inc} = -45^\circ, R = 0.7\lambda; \) mesh, six layers of nodes). The effect of changing the nodal density is considered.
large number of modes; the finite element solution is truncated at $m = \pm 7$. The results when the solution is truncated at $m = \pm 5$ and $m = \pm 3$ are also shown to examine the convergence of the solution. The mesh used is shown in Figure 3. Near the $z$ axis the various finite element solutions seem to converge very quickly, but away from the axis, particularly around $\theta = 90^\circ$, the convergence is much slower. This makes sense if we recall the boundary condition on the $z$ axis. The nodal values on the $z$ axis are unrestricted if $m = \pm 1$ and are forced to zero otherwise. Thus we would expect the higher-order modes (i.e., modes with $|m| > 1$) to be small near the $z$ axis. Although the series of finite element solutions has not fully converged, even in the case where the solution is truncated at $m = \pm 3$ the overall behavior of the fields is modeled fairly accurately.

In Figure 6 we try assuming that $E_\phi$ and $H_\phi$ can be decoupled at the truncation boundary. Not surprisingly, we get better results when the equations for $E_\phi$ and $H_\phi$ remain coupled. When the electric and magnetic fields are coupled, the three closest nodes are used to formulate the boundary condition except at the corners where the two closest nodes are used. When the electric and magnetic fields are decoupled, the five closest nodes are used to find the MEI boundary condition (except at the corner where the three closest nodes are used). In this way the MEI equations for both cases have approximately the same number of field variables coupled. It can be seen from the plot in Figure 6 that the results are much better when the electric and magnetic fields are coupled at the truncation boundary.

Figure 7 shows the effect of moving the truncation boundary closer to the scatterer. As in Figure 6, the excitation is a plane wave which is incident at $\theta = -45^\circ$. In Figures 4-6 we have used six layers of nodes ($0.167\lambda$). Now we compare this to using just four ($0.1\lambda$) and three ($0.067\lambda$) layers of nodes. As with most local formulations of the truncation boundary condition, we expect the finite elements results to get worse as the boundary is moved closer to the scatterer. This is exactly what we see in Figure 7. When only three layers of nodes are used the finite element solution does not model the field behavior accurately. Thus it is clear that moving the
Figure 6. Magnetic field on the surface of a conducting sphere, with spherical harmonics used to find the measuring functions \( \theta^{inc} = -45^\circ, R = 0.7\lambda \); mesh, 30 nodes/\( \lambda \), six layers of nodes). The effect of decoupling \( E_\phi \) and \( H_\phi \) on the truncation boundary is considered.

Figure 7. Magnetic field on the surface of a conducting sphere, with spherical harmonics used to find the measuring functions \( \theta^{inc} = -45^\circ, R = 0.7\lambda \); mesh, 30 nodes/\( \lambda \)). The effect of moving the truncation boundary closer to the scatterer is considered.
truncation boundary farther from the scatterer increases the accuracy of the solution.

Figure 8 compares the results of the finite element program with the sinusoidal metrons from (7) to the results of a moment method code, provided by Chuang, [1992]. The method of moments solution is very close to the reference series solution. The overall accuracy of this finite element solution is approximately the same as the solution in Figure 7 with six layers of nodes in which the outgoing spherical harmonics are used to produce the coefficients. Also shown in Figure 8 is the finite element solution for only three layers of nodes with sinusoidal metrons. Although this solution has more error than the six-layer solution, it is significantly more accurate than the three layer solution in Figure 7 in which spherical harmonics are used.

**Cylinders**

We will now consider some geometries other than spheres. Figure 9 shows a finite element mesh used for the perfectly conducting cylinder.

This mesh has a nodal density of 20 nodes per wavelength and has five layers (0.2\(\lambda\)) of nodes including the layers on the conductor and the truncation boundary. Again, the program FASTQ was used to generate the mesh. For the cylinder, accurate results can be obtained with a lower nodal density (or larger elements) in some cases.
One of the reasons we need small elements for the sphere is that the curved surface is being approximated by a series of line segments. For the cylinder the boundary at the conductor surface is not curved so there is no problem with using line segments to define the scatterer.

Figure 10 shows the results for the cylinder defined by the mesh given in Figure 9, where the incident plane wave is traveling in the positive z direction. The finite element method results are compared to moment method results. The largest difference between the two solutions is only about 4% of the maximum of the finite element solution. Figure 11 shows the magnetic field on the surface of the same cylinder. The excitation field is now incident at -45°. This incident field excites higher-order modes, since there is no longer a symmetry in φ. The finite element solution is truncated at modes \( m = \pm 5 \). The finite element solution and the method of moment solution both have the same general shape, but the difference is greater than in Figure 10, where the incident wave excited only the \( m = \pm 1 \) modes. Figure 12 displays the results for another cylinder. The mesh used has five layers of nodes, and a nodal density of 20 nodes per wavelength. Again, the incident field is a plane wave traveling in the positive z direction. The finite element solution and moment method solutions show excellent agreement. Figure 13 shows the results for a larger cylinder, having a height of five wavelengths and a diameter of five wavelengths. The mesh used has a nodal density of 20 nodes per wavelength and has five layers of nodes. The incident field is a plane wave traveling in the positive z direction. Overall, the finite element and moment method solutions show good agreement except near \( \theta = 180° \).

**Cone**

A third type of body of revolution is the cone. The mesh used has a nodal density of 20 nodes per wavelength, has five layers of nodes, and is shown in Figure 14. The results for an incident plane wave traveling in the positive z direction

![Figure 10](image)

**Figure 10.** Magnetic field on the surface of a conducting cylinder, using sinusoidal metrous (\( \theta_{inc} = 180°, R = 0.5\lambda, H = 1.0\lambda \); mesh, 20 nodes/\( \lambda \), five layers of nodes). Comparison of the method of moments (MOM) to the finite element method (FEM)
Figure 11. Magnetic field on the surface of a conducting cylinder, using sinusoidal metrons ($\theta^{nc} = -45^\circ$, $R = 0.5\lambda$, $H = 1.0\lambda$; mesh, 20 nodes/$\lambda$, five layers of nodes). Comparison of MOM to FEM.

Figure 12. Magnetic field on the surface of a conducting cylinder, using sinusoidal metrons ($\theta^{nc} = 180^\circ$, $R = 0.5\lambda$, $H = 0.5\lambda$; mesh, five layers of nodes). Comparison of MOM to FEM.
Figure 13. Magnetic field on the surface of a conducting cylinder, using sinusoidal metrons ($\theta_{inc} = 180^\circ$, $R = 2.5\lambda$, $H = 5.0\lambda$; mesh, 20 nodes/$\lambda$, five layers of nodes). Comparison of MOM to FEM.

is shown in Figure 15. Both the finite element and moment method solutions have the same general shape. There is excellent agreement between the two solutions on the shaded side of the cone. In the illuminated region, both solutions have a trough near 70° and a peak near 130°. However, at the trough there is a relatively large difference in the magnitudes. Figure 16 illustrates the results for the same cone and mesh but with the incident angle of the excitation offset by 30°. The finite element solution is truncated at $m = \pm 5$. Both the magnitude and phase differences are larger than in Figure 15. This is not surprising, because we have regularly seen an increase in the error when the excitation is obliquely incident. In the illuminated region the magnitude of the finite element solution is consistently 10–20% below the method of moments solutions.

Increasing the nodal density used in the mesh for the cone to 32 nodes per wavelength did not improve the overall accuracy of the solution. Thus the error seems to be caused by the MEI method. Also, unlike the sphere geometry, increasing the number of layers by two did not improve upon the results shown. Note that

Figure 14. Mesh for a cone.
Figure 15. Magnetic field on the surface of a conducting cone, using sinusoidal metrons ($\theta^{\text{inc}} = 180^\circ$, $R = 0.5\lambda$, $H = 1.0\lambda$; mesh, 20 nodes/\lambda, five layers of nodes).

Figure 16. Magnetic field on the surface of a conducting cone, using sinusoidal metrons ($\theta^{\text{inc}} = 150^\circ$, $R = 0.5\lambda$, $H = 1.0\lambda$; mesh, 20 nodes/\lambda, five layers of nodes).
a wave scattered from the edge at the base of the cone can impinge upon the truncation boundary at a very steep angle. The MEI method may not be able to accurately model this geometry with sinusoidal metrons.

Conclusions

In this paper, we formulated the measured equation of invariance for the electromagnetic scattering from a body of revolution based on the coupled azimuthal potential method. Results were presented for several canonical geometries. An accurate solution was obtained when the truncation boundary was placed at a sufficient distance from the scatterer. It is interesting to note that the MEI was less accurate for the body of revolution geometries shown here than for similar two-dimensional cases shown in the work by Mei et al., [1994] and Jevtic and Lee, [1994]. In order to recover the same accuracy as in the two dimensional case, we must either use a better choice of metrons or move the truncation boundary farther from the scatterer. Also, two coupling schemes for the MEI coefficients were presented and compared. We demonstrated that more accurate results are obtained when the two azimuthal potentials are coupled in the MEI formulation.

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