# Mapped Equations

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## 1 The mappings

The evolving interface between air and water makes the design of numerical methods difficult. The approach adopted here is to map the deformed geometry into a rectangular shape in new, logical coordinates at the cost of changing the details of the equations of motion. The design of the numerical method then hinges on the details of these equations.

Let's introduce the new coordinates, $(X, Y, Z, \tau)$, as follows,

$$x = X \tag{1.1a}$$
$$y = Y \tag{1.1b}$$
$$z = F(X, Y, Z, \tau) \tag{1.1c}$$
$$t = \tau \tag{1.1d}$$

with the property that when $Z = 0$

$$z = h(x, y, t) \tag{1.1e}$$

marks the location of the boundary. A specific mapping worth considering is

$$F(X, Y, Z, \tau) = Z + h(X, Y, \tau)\,\exp(-\alpha Z) \tag{1.2}$$

where $\alpha$ controls the nature of the mapping far from the interface. When $\alpha \neq 0$, then the coordinate planes $Z =$ constant become horizontal far from the interface, allowing appropriate boundary conditions to be applied there.

To transform the partial differential equations into the new coordinates, we need the transformed derivatives.

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + F_\tau \frac{\partial}{\partial z} \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - \frac{F_\tau}{F_Z}\frac{\partial}{\partial Z} \tag{1.3a}$$
$$\frac{\partial}{\partial X} = \frac{\partial}{\partial x} + F_X \frac{\partial}{\partial z} \qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial X} - \frac{F_X}{F_Z}\frac{\partial}{\partial Z} \tag{1.3b}$$
$$\frac{\partial}{\partial Y} = \frac{\partial}{\partial y} + F_Y \frac{\partial}{\partial z} \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial Y} - \frac{F_Y}{F_Z}\frac{\partial}{\partial Z} \tag{1.3c}$$
$$\frac{\partial}{\partial Z} = F_Z \frac{\partial}{\partial z} \qquad \frac{\partial}{\partial z} = \frac{1}{F_Z}\frac{\partial}{\partial Z} \tag{1.3d}$$

We also need the second derivatives, but first define

$$G_0 = \frac{F_\tau}{F_Z} = \frac{h_\tau \exp(-\alpha Z)}{1 - \alpha h \exp(-\alpha Z)} \tag{1.4a}$$

$$G_1 = \frac{F_X}{F_Z} = \frac{h_X \exp(-\alpha Z)}{1 - \alpha h \exp(-\alpha Z)} \tag{1.4b}$$

$$G_2 = \frac{F_Y}{F_Z} = \frac{h_Y \exp(-\alpha Z)}{1 - \alpha h \exp(-\alpha Z)} \tag{1.4c}$$

$$G_3 = \frac{1}{F_Z} = \frac{1}{1 - \alpha h \exp(-\alpha Z)} \tag{1.4d}$$

Then,

$$\frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial X} - G_1 \frac{\partial}{\partial Z}\right)\left(\frac{\partial}{\partial X} - G_1 \frac{\partial}{\partial Z}\right) \tag{1.5a}$$

$$\frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial Y} - G_2 \frac{\partial}{\partial Z}\right)\left(\frac{\partial}{\partial Y} - G_2 \frac{\partial}{\partial Z}\right) \tag{1.5b}$$

$$\frac{\partial^2}{\partial z^2} = G_3 \frac{\partial}{\partial Z}\left(G_3 \frac{\partial}{\partial Z}\right) \tag{1.5c}$$

Finally, let us define the Laplacian in the new variables as

$$\mathcal{L}\{\phi\} = \left[\left(\frac{\partial}{\partial X} - G_1 \frac{\partial}{\partial Z}\right)^2 + \left(\frac{\partial}{\partial Y} - G_2 \frac{\partial}{\partial Z}\right)^2 + G_3 \frac{\partial}{\partial Z}\left(G_3 \frac{\partial}{\partial Z}\right)\right] \phi \tag{1.6}$$

In what follows, it is useful to note that the following derivative operators commute (see Appendix A):

$$\left(\frac{\partial}{\partial X} - G_1 \frac{\partial}{\partial Z}\right) G_3 \frac{\partial}{\partial Z} = G_3 \frac{\partial}{\partial Z}\left(\frac{\partial}{\partial X} - G_1 \frac{\partial}{\partial Z}\right) \tag{1.7a}$$

$$\left(\frac{\partial}{\partial X} - G_1 \frac{\partial}{\partial Z}\right)\left(\frac{\partial}{\partial \tau} - G_0 \frac{\partial}{\partial Z}\right) = \left(\frac{\partial}{\partial \tau} - G_0 \frac{\partial}{\partial Z}\right)\left(\frac{\partial}{\partial X} - G_1 \frac{\partial}{\partial Z}\right) \tag{1.7b}$$

$$G_3 \frac{\partial}{\partial Z}\left(\frac{\partial}{\partial \tau} - G_0 \frac{\partial}{\partial Z}\right) = \left(\frac{\partial}{\partial \tau} - G_0 \frac{\partial}{\partial Z}\right) G_3 \frac{\partial}{\partial Z} \tag{1.7c}$$

## 2 The Mapped Equations

By applying the transformation rules for the derivatives directly to the equations of motion, we obtain

$$\rho u_\tau - \rho G_0 u_Z + \rho u\,(u_X - G_1 u_Z) + \rho v\,(u_Y - G_2 u_Z) + \rho w G_3 u_Z = - p_X + G_1 p_Z + \mu\mathcal{L}\{u\} \tag{2.1a}$$

$$\rho v_\tau - \rho G_0 v_Z + \rho u\,(v_X - G_1 v_Z) + \rho v\,(v_Y - G_2 v_Z) + \rho w G_3 v_Z = - p_Y + G_2 p_Z + \mu\mathcal{L}\{v\} \tag{2.1b}$$

$$\rho w_\tau - \rho G_0 w_Z + \rho u\,(w_X - G_1 w_Z) + \rho v\,(w_Y - G_2 w_Z) + \rho w G_3 w_Z = - G_3\, p_Z + \mu\mathcal{L}\{w\} - \rho g \tag{2.1c}$$

and the incompressibility condition becomes

$$u_X - G_1 u_Z + v_Y - G_2 v_Z + G_3 w_Z = 0 \tag{2.1d}$$

At the interface, $Z = 0$, we have continuity of the velocity,

$$u^{(1)} = u^{(2)} = u^{(I)}\,, \quad v^{(1)} = v^{(2)} = v^{(I)}\,, \quad w^{(1)} = w^{(2)} = w^{(I)} \tag{2.2}$$

and the kinematic condition

$$h_\tau + u^{(I)} h_X + v^{(I)} h_Y = w^{(I)} \tag{2.3}$$

The dynamic interfacial conditions are:

$$\begin{aligned}\left(h_X^2 - 1\right) &\left[\mu^{(1)}\left(G_3^{(1)} u_Z^{(1)} + w_X^{(1)} - G_1^{(1)} w_Z^{(1)}\right) - \mu^{(2)}\left(G_3^{(2)} u_Z^{(2)} + w_X^{(2)} - G_1^{(2)} w_Z^{(2)}\right)\right] \\ &+ h_Y\left[\mu^{(1)}\left(u_Y^{(1)} - G_2^{(1)} u_Z^{(1)} + v_X^{(1)} - G_1^{(1)} v_Z^{(1)}\right) - \mu^{(2)}\left(u_Y^{(2)} - G_2^{(2)} u_Z^{(2)} + v_X^{(2)} - G_1^{(2)} v_Z^{(2)}\right)\right] \\ &+ h_X h_Y\left[\mu^{(1)}\left(G_3^{(1)} v_Z^{(1)} + w_Y^{(1)} - G_2^{(1)} w_Z^{(1)}\right) - \mu^{(2)}\left(G_3^{(2)} v_Z^{(2)} + w_Y^{(2)} - G_2^{(2)} w_Z^{(2)}\right)\right] \\ &+ 2h_X\left[\mu^{(1)}\left(u_X^{(1)} - G_1^{(1)} u_Z^{(1)} - G_3^{(1)} w_Z^{(1)}) - \mu^{(2)}(u_X^{(2)} - G_1^{(2)} u_Z^{(2)} - G_3^{(2)} w_Z^{(2)}\right)\right] = 0\end{aligned} \tag{2.4a}$$

$$\begin{aligned}\left(h_Y^2 - 1\right) &\left[\mu^{(1)}(G_3^{(1)} v_Z^{(1)} + w_Y^{(1)} - G_2^{(1)} w_Z^{(1)}) - \mu^{(2)}(G_3^{(2)} v_Z^{(2)} + w_Y^{(2)} - G_2^{(2)} w_Z^{(2)})\right] \\ &+ h_X\left[\mu^{(1)}(u_Y^{(1)} - G_2^{(1)} u_Z^{(1)} + v_X^{(1)} - G_1^{(1)} v_Z^{(1)}) - \mu^{(2)}(u_Y^{(2)} - G_2^{(2)} u_Z^{(2)} + v_X^{(2)} - G_1^{(2)} v_Z^{(2)})\right] \\ &+ h_X h_Y\left[\mu^{(1)}(G_3^{(1)} u_Z^{(1)} + w_X^{(1)} - G_1^{(1)} w_Z^{(1)}) - \mu^{(2)}(G_3^{(2)} u_Z^{(2)} + w_X^{(2)} - G_1^{(2)} w_Z^{(2)})\right] \\ &+ 2h_Y\left[\mu^{(1)}(v_Y^{(1)} - G_2^{(1)} v_Z^{(1)} - G_3^{(1)} w_Z^{(1)}) - \mu^{(2)}(v_Y^{(2)} - G_2^{(2)} v_Z^{(2)} - G_3^{(2)} w_Z^{(2)})\right] = 0\end{aligned} \tag{2.4b}$$

$$\left(p^{(1)} - p^{(2)}\right) + h_X \left[\mu^{(1)}(G_3^{(1)} u_Z^{(1)} + w_X^{(1)} - G_1^{(1)} w_Z^{(1)}) - \mu^{(2)}(G_3^{(2)} u_Z^{(2)} + w_X^{(2)} - G_1^{(2)} w_Z^{(2)})\right]$$
$$+ h_Y \left[\mu^{(1)}(G_3^{(1)} v_Z^{(1)} + w_Y^{(1)} - G_2^{(1)} w_Z^{(1)}) - \mu^{(2)}(G_3^{(2)} v_Z^{(2)} + w_Y^{(2)} - G_2^{(2)} w_Z^{(2)})\right]$$
$$- 2\left[\mu^{(1)} G_3^{(1)} w_Z^{(1)} - \mu^{(2)} G_3^{(2)} w_Z^{(2)}\right] = 2T\kappa \quad (2.4c)$$

Note that we allow a jump in the mapping function (1.2) as well. Clearly, $\alpha$ must be positive for $Z > 0$ and negative for $Z < 0$. The curvature is given by

$$2\kappa = \frac{(1 + h_X^2)\ h_{YY} - 2h_X h_Y h_{XY} + (1 + h_Y^2)\ h_{XX}}{(1 + h_X^2 + h_Y^2)^{3/2}} \quad (2.4d)$$

The easiest way to design methods for these equations is to consider the simpler, two-dimensional flow where no dependency on $Y$ is allowed (and $v = 0$).

# 3 Two-dimensional motion

The equations for the fluid motion are now:

$$\rho u_\tau - \rho G_0 u_Z + \rho u\,(u_X - G_1 u_Z) + \rho w G_3 u_Z = -p_X + G_1 p_Z + \mu \mathcal{L}\{u\} \quad (3.1a)$$
$$\rho w_\tau - \rho G_0 w_Z + \rho u\,(w_X - G_1 w_Z) + \rho w G_3 w_Z = -G_3\, p_Z + \mu \mathcal{L}\{w\} - \rho g \quad (3.1b)$$

and the incompressibility condition becomes

$$u_X - G_1 u_Z + G_3 w_Z = 0 \quad (3.1c)$$

At the interface, $Z = 0$, we have continuity of the velocity,

$$u^{(1)} = u^{(2)} = u^{(I)}, \quad w^{(1)} = w^{(2)} = w^{(I)} \quad (3.2)$$

and the kinematic condition

$$h_\tau + u^{(I)} h_X = w^{(I)} \quad (3.3)$$

The dynamic interfacial conditions are:

$$\left(h_X^2 - 1\right) \left[\mu^{(1)} \left(G_3^{(1)} u_Z^{(1)} + w_X^{(1)} - G_1^{(1)} w_Z^{(1)}\right) - \mu^{(2)} \left(G_3^{(2)} u_Z^{(2)} + w_X^{(2)} - G_1^{(2)} w_Z^{(2)}\right)\right]$$
$$+ 2h_X \left[\mu^{(1)} \left(u_X^{(1)} - G_1^{(1)} u_Z^{(1)} - G_3^{(1)} w_Z^{(1)}\right) - \mu^{(2)}(u_X^{(2)} - G_1^{(2)} u_Z^{(2)} - G_3^{(2)} w_Z^{(2)}\right)\right] = 0 \quad (3.4a)$$

$$\left(p^{(1)} - p^{(2)}\right) + h_X \left[\mu^{(1)}(G_3^{(1)} u_Z^{(1)} + w_X^{(1)} - G_1^{(1)} w_Z^{(1)}) - \mu^{(2)}(G_3^{(2)} u_Z^{(2)} + w_X^{(2)} - G_1^{(2)} w_Z^{(2)})\right]$$
$$- 2\left[\mu^{(1)} G_3^{(1)} w_Z^{(1)} - \mu^{(2)} G_3^{(2)} w_Z^{(2)}\right] = 2T\kappa \quad (3.4b)$$

where

$$2\kappa = \frac{h_{XX}}{(1 + h_X^2)^{3/2}} \quad (3.4c)$$

### 3.1 Pressure equation

One of the approaches to solving these equations is to use (3.1c)) to derive an equation for the pressure without time derivatives. Also, the terms with the factor $\mu$ are eliminated. Apply the operators

$$\frac{\partial}{\partial X} - G_1 \frac{\partial}{\partial Z}, \quad G_3 \frac{\partial}{\partial Z}$$

to (3.1a),(3.1b) respectively and add the results. Since the derivative operators commute (see (1.7)), we obtain

$$\mathcal{L}\{p\} = 2\rho \left[ (u_X - G_1 u_Z) G_3 w_Z - G_3 u_Z (w_X - G_1 w_Z) \right] \tag{3.5}$$

If we consider the velocity and interface location known at some time $t$, then we may interpret (3.5)) as a Poisson equation for $p$. However, we need boundary conditions. Two far-field conditions are obvious, but we have only one clear interface condition (3.4b)). To obtain another condition, evaluate (3.1b)) on either side of the interface and subtract appropriately,

$$\begin{aligned}\frac{G_3^{(1)}}{\rho^{(1)}} p_Z^{(1)} - \frac{G_3^{(2)}}{\rho^{(2)}} p_Z^{(2)} = G_0^{(1)} w_Z^{(1)} - G_0^{(2)} w_Z^{(2)} + u^{(I)} G_1^{(1)} w_Z^{(1)} - u^{(I)} G_1^{(2)} w_Z^{(2)} \\ - w^{(I)} G_3^{(1)} w_Z^{(1)} + w^{(I)} G_3^{(2)} w_Z^{(2)} + \nu^{(1)} \mathcal{L}\{w^{(1)}\} - \nu^{(2)} \mathcal{L}\{w^{(2)}\}\end{aligned} \tag{3.6}$$

Now, (3.4b),(3.6)) give jump conditions at the interface for both $p$ and $p_Z$.

Once $p$ is determined, (3.1a),(3.1b)) may be updated using (3.2) as one interface condition. We also need jump conditions for the vertical derivatives of the velocities. We may combine (3.1c) with (3.4) to obtain

$$2\left(h_X^2 + 1\right)\left(\mu^{(1)} G_3^{(1)} w_Z^{(1)} - \mu^{(2)} G_3^{(2)} w_Z^{(2)}\right) = \left(h_X^2 - 1\right)\left(p^{(1)} - p^{(2)} - 2T\kappa\right) \tag{3.7}$$

Once $w$ has been updated, then (3.1c) provides the jump in $u_Z$.

### 3.2 Streamfunction and vorticity

The vorticity is defined as $\nabla \times \mathbf{u}$. For two-dimensional flow, there is only one component, $\omega \mathbf{j}$, given by

$$\omega = G_3 u_z - w_X + G_1 w_Z \tag{3.8}$$

An evolution equation for the vorticity arises by applying the derivative operators,

$$G_3 \frac{\partial}{\partial Z}, \quad \frac{\partial}{\partial X} - G_1 \frac{\partial}{\partial Z}$$

to (3.1a),(3.1b) respectively and subtracting the results. The relationships (1.7) come in handy. In particular, the pressure is eliminated from the result. The result can be simplified by noting

$$G_3\frac{\partial}{\partial Z}\left[u\left(\frac{\partial u}{\partial X}-G_1\frac{\partial u}{\partial Z}\right)\right]-\left(\frac{\partial}{\partial X}-G_1\frac{\partial}{\partial Z}\right)\left[u\left(\frac{\partial w}{\partial X}-G_1\frac{\partial w}{\partial Z}\right)\right]= \\ u\left(\frac{\partial \omega}{\partial X}-G_1\frac{\partial \omega}{\partial Z}\right)+\left(\frac{\partial u}{\partial X}-G_1\frac{\partial u}{\partial Z}\right)\omega \tag{3.9a}$$

$$G_3\frac{\partial}{\partial Z}\left(wG_3\frac{\partial u}{\partial Z}\right)-\left(\frac{\partial}{\partial X}-G_1\frac{\partial}{\partial Z}\right)\left(wG_3\frac{\partial w}{\partial Z}\right)=wG_3\frac{\partial \omega}{\partial Z}+G_3\frac{\partial w}{\partial Z}\omega \tag{3.9b}$$

By adding (3.9a) and (3.9b) and using (3.1c), we may derive the simple evolution equation,

$$\rho\omega_\tau+\rho u\left(\omega_X-G_1\omega_Z\right)+\rho wG_3\omega_Z=\mu\mathcal{L}\{\omega\} \tag{3.10}$$

The way to determine the velcity from $\omega$ is as follows. Introduce the streamfunction $\psi$ by the requirement that the velocity is given by

$$u=-G_3\psi_Z\,,\quad w=\psi_X-G_1\psi_Z \tag{3.11}$$

and note that (3.1c) is automatically satisfied as a consequence. Further, (3.8) becomes

$$\mathcal{L}\{\psi\}=-\omega \tag{3.12}$$

We have derived a diffusion equation for the vorticity and an elliptic equation for the streamfunction. We need interface conditions for both these equations. From (3.2), we have

$$\psi^{(1)}=\psi^{2)}=\psi^{(I)}\,,\quad \psi_Z^{(1)}=\psi_Z^{2)} \tag{3.13}$$

Unfortunately, we must recast (3.4) into a form that gives interfacial conditions for $\omega$.

The starting point is to substitute (3.1c) into (3.8) to obtain,

$$\left(G_1^2+G_3^2\right)u_Z=G_3\omega+G_1u_X+G_3w_X \tag{3.14a}$$

Then substitute this result back into (3.1c) to obtain

$$\left(G_1^2+G_3^2\right)w_Z=G_1\omega-G_3u_X+G_1w_X \tag{3.14b}$$

Note that we have expressed vertical velocity derivatives in terms of the vorticity and horizontal derivatives Then consider

$$F=\left(h_X^2-1\right)\left(G_3u_Z+w_X-G_1w_Z\right)+2h_X\left(u_X-G_1u_Z-G_3w_Z\right) \tag{3.15}$$

and note that (3.4a) becomes $\mu^{(1)}F^{(1)}=\mu^{(2)}F^{(2)}$. After rearranging terms and using (1.4b,1.4d), we find

$$G_3^2F=-\left(G_1^2+G_3^2\right)G_3u_Z-\left(G_1^2+G_3^2\right)G_1w_Z+\left(G_1^2-G_3^2\right)w_X+2G_1G_3u_X$$

Now substitute (3.14) and (3.15) into this equation to obtain

$$G_3^2 F = -\left(G_1^2 + G_3^2\right)\,\omega - 2G_3^2 w_X + 2G_1 G_3 u_X$$

or

$$F = -\left(h_X^2 + 1\right)\,\omega - 2w_X + 2h_X u_X \tag{3.16}$$

Consequently, (3.16) may be used in place of (3.15) to rewrite (3.4a) as

$$\left(h_X^2 + 1\right)\,\left(\mu^{(1)}\omega^{(1)} - \mu^{(2)}\omega^{(2)}\right) = -2\left(\mu^{(1)} - \mu^{(2)}\right)\,\left(w_X^{(I)} - h_X u_X^{(I)}\right) \tag{3.17}$$

which gives the jump in vorticity across the interface.

To incorporate the other dynamic interface condition (3.4b), consider

$$H = h_X\left(G_3 u_Z + w_X - G_1 w_Z\right) - 2G_3 w_Z \tag{3.18a}$$

or

$$G_3 H = G_1 G_3 u_Z - \left(G_1^2 + 2G_3^2\right) w_Z + G_1 w_X \tag{3.18b}$$

Substitute (3.14) into (3.18b) to obtain

$$G_3 H = -G_1\omega + 2G_3 u_X$$

or

$$H = -h_X\omega + 2u_X \tag{3.19}$$

Since (3.4b) can be written as

$$\left(p^{(1)} - p^{(2)}\right) + \left(\mu^{(1)}H^{(1)} - \mu^{(2)}H^{(2)}\right) = 2T\kappa$$

we use (3.19) to obtain

$$\left(p^{(1)} - p^{(2)}\right) - h_X\left(\mu^{(1)}\omega^{(1)} - \mu^{(2)}\omega^{(2)}\right) = -2\left(\mu^{(1)} - \mu^{(2)}\right) u_X^{(I)} + 2T\kappa \tag{3.20}$$

This equation may be differentiated with respect to $X$, keeping $Z = 0$. Then we use the momentum equations (3.1a) and (3.1b) to eliminate the pressure gradients. In particular, we will multiply (3.1a) by $G_3$ and add it to (3.1b) multiplied by $G_1$.

We proceed by breaking up the algebra into separate steps. Consider,

$$\begin{aligned} H_1 &= G_3\left(u_\tau - G_0 u_Z\right) + G_1\left(w_\tau - G_0 w_Z\right) \\ &= G_3 u_\tau + G_1 w_\tau - G_0\omega - G_0 w_X \end{aligned} \tag{3.21}$$

where we have used (3.14) to eliminate the vertical derivatives of $u$ and $w$. Continuing in this way, we have the next terms given by

$$\begin{aligned} H_2 &= G_3\left[u\left(u_X - G_1 u_Z\right) + w G_3 u_Z\right] + G_1\left[u\left(w_X - G_1 w_Z\right) + w G_3 w_Z\right] \\ &= \left(G_3 w - G_1 u\right)\omega + G_3\left(u u_X + w w_X\right) \end{aligned} \tag{3.22}$$

For the viscous terms, we proceed differently.

$$\begin{aligned} \mathcal{L}\{u\} &= \left(\frac{\partial}{\partial X} - G_1\frac{\partial}{\partial Z}\right)\left(\frac{\partial}{\partial X} - G_1\frac{\partial}{\partial Z}\right) u + G_3\frac{\partial}{\partial Z}\left(G_3\frac{\partial u}{\partial Z}\right) \\ &= \left(\frac{\partial}{\partial X} - G_1\frac{\partial}{\partial Z}\right)\left(\frac{\partial}{\partial X} - G_1\frac{\partial}{\partial Z}\right) u + G_3\frac{\partial}{\partial Z}\left[\omega + \left(\frac{\partial}{\partial X} - G_1\frac{\partial}{\partial Z}\right) w\right] \\ &= G_3\omega_Z + \left(\frac{\partial}{\partial X} - G_1\frac{\partial}{\partial Z}\right)\left(\frac{\partial}{\partial X} - G_1\frac{\partial}{\partial Z}\right) u + \left(\frac{\partial}{\partial X} - G_1\frac{\partial}{\partial Z}\right) G_3 w_Z \\ &= G_3\omega_Z \end{aligned} \tag{3.23}$$

where we have used the definition of vorticity (3.8) in the first step; the commutability of operators (1.7a); and the incompressibility condition (3.1c). Similarly,

$$\begin{aligned} \mathcal{L}\{w\} &= \left(\frac{\partial}{\partial X} - G_1\frac{\partial}{\partial Z}\right)\left(\frac{\partial}{\partial X} - G_1\frac{\partial}{\partial Z}\right) w + G_3\frac{\partial}{\partial Z}\left(G_3\frac{\partial w}{\partial Z}\right) \\ &= \left(\frac{\partial}{\partial X} - G_1\frac{\partial}{\partial Z}\right)\left(\frac{\partial}{\partial X} - G_1\frac{\partial}{\partial Z}\right) w - G_3\frac{\partial}{\partial Z}\left(\frac{\partial}{\partial X} - G_1\frac{\partial}{\partial Z}\right) u \\ &= \left(\frac{\partial}{\partial X} - G_1\frac{\partial}{\partial Z}\right)\left(\frac{\partial}{\partial X} - G_1\frac{\partial}{\partial Z}\right) w \\ &\qquad - \left(\frac{\partial}{\partial X} - G_1\frac{\partial}{\partial Z}\right)\left[\omega + \left(\frac{\partial}{\partial X} - G_1\frac{\partial}{\partial Z}\right) w\right] \\ &= -\omega_X + G_1\omega_Z \end{aligned} \tag{3.24}$$

Consequently,

$$\begin{aligned} H_3 &= G_3\mathcal{L}\{u\} + G_1\mathcal{L}\{w\} \\ &= \left(G_1^2 + G_3^2\right)\omega_Z - G_1\omega_X \end{aligned} \tag{3.25}$$

By the definitions of $H_1$, $H_2$ and $H_3$, we have

$$\rho\left(H_1 + H_2\right) = -G_3 p_X + \mu H_3 - G_1\rho g \tag{3.26}$$

Note that at the interface, the quantity

$$\frac{H_1 + H_2}{G_3} = \left(u_\tau^{(I)} + h_x w_\tau^{(I)} - h_\tau w_X^{(I)} + u^{(I)} u_X^{(I)} + w^{(I)} w_X^{(I)}\right) \tag{3.27}$$

is continuous. Thus, by evaluating (3.26) on either side of the interface and subtracting, we obtain

$$p_X^{(1)} - p_X^{(2)} = -\left(\rho^{(1)} - \rho^{(2)}\right)\left[u_\tau^{(I)} + h_X w_\tau^{(I)} + \left(w^{(I)} - h_\tau\right) w_X^{(I)} + u^{(I)} u_X^{(I)} - h_X g\right] + \left(1 + h_X^2\right)\left(\mu^{(1)} G_3^{(1)} \omega_Z^{(1)} - \mu^{(2)} G_3^{(2)} \omega_Z^{(2)}\right) - h_X\left(\mu^{(1)} \omega_X^{(1)} - \mu^{(2)} \omega_X^{(2)}\right) \quad (3.28)$$

where we have used the result $G_1 = h_X G_3$ at the interface, and we replaced $H_3$ by (3.25). By differentiating (3.20), we have

$$p_X^{(1)} - p_X^{(2)} = 2T\kappa_X + h_{XX}\left(\mu^{(1)}\omega^{(1)} - \mu^{(2)}\omega^{(2)}\right) + h_X\left(\mu^{(1)}\omega_X^{(1)} - \mu^{(2)}\omega_X^{(2)}\right) - 2\left(\mu^{(1)} - \mu^{(2)}\right) u_{XX} \quad (3.29)$$

By eliminating the pressure terms from (3.28) and (3.29), we finally obtain a jump condition on $\omega_Z$.

$$\left(1 + h_X^2\right)\left(\mu^{(1)} G_3^{(1)} \omega_Z^{(1)} - \mu^{(2)} G_3^{(2)} \omega_Z^{(2)}\right) = 2T\kappa_X + 2h_X\left(\mu^{(1)}\omega_X^{(1)} - \mu^{(2)}\omega_X^{(2)}\right) + h_{XX}\left(\mu^{(1)}\omega^{(1)} - \mu^{(2)}\omega^{(2)}\right) - 2\left(\mu^{(1)} - \mu^{(2)}\right) u_{XX} + \left(\rho^{(1)} - \rho^{(2)}\right)\left[u_\tau^{(I)} + h_X w_\tau^{(I)} + \left(w^{(I)} - h_\tau\right) w_X^{(I)} + u^{(I)} u_X^{(I)}\right] \quad (3.30)$$

# A Commuting operators

The following expressions help us establish the results (1.7):

$$(G_0)_\tau = \frac{F_{\tau\tau}}{F_Z} - \frac{F_\tau F_{Z\tau}}{F_Z^2} \qquad (G_0)_X = \frac{F_{X\tau}}{F_Z} - \frac{F_\tau F_{ZX}}{F_Z^2} \qquad (G_0)_Z = \frac{F_{Z\tau}}{F_Z} - \frac{F_\tau F_{ZZ}}{F_Z^2} \quad \text{(A.1a)}$$

$$(G_1)_\tau = \frac{F_{X\tau}}{F_Z} - \frac{F_X F_{Z\tau}}{F_Z^2} \qquad (G_1)_X = \frac{F_{XX}}{F_Z} - \frac{F_X F_{ZX}}{F_Z^2} \qquad (G_1)_Z = \frac{F_{XZ}}{F_Z} - \frac{F_X F_{ZZ}}{F_Z^2} \quad \text{(A.1b)}$$

$$(G_3)_\tau = -\frac{F_{Z\tau}}{F_Z^2} \qquad (G_3)_X = -\frac{F_{ZX}}{F_Z^2} \qquad (G_3)_Z = -\frac{F_{ZZ}}{F_Z^2} \quad \text{(A.1c)}$$

Consider (1.7a) first.

$$\left(\frac{\partial}{\partial X} - G_1\frac{\partial}{\partial Z}\right) G_3\frac{\partial}{\partial Z} = \frac{\partial G_3}{\partial X}\frac{\partial}{\partial Z} + G_3\frac{\partial^2}{\partial Z\partial X} - G_1\frac{\partial G_3}{\partial Z}\frac{\partial}{\partial Z} - G_1 G_3\frac{\partial^2}{\partial Z^2}$$
$$= G_3\frac{\partial}{\partial Z}\left(\frac{\partial}{\partial X} - G_1\frac{\partial}{\partial Z}\right) + \left(G_3\frac{\partial G_1}{\partial Z} + \frac{\partial G_3}{\partial X} - G_1\frac{\partial G_3}{\partial Z}\right)\frac{\partial}{\partial Z}$$

By using relationships in (A.1b),(A.1c), we find

$$G_3 \frac{\partial G_1}{\partial Z} + \frac{\partial G_3}{\partial X} - G_1 \frac{\partial G_3}{\partial Z} = 0$$

and the result (1.7a) follows.

Similarly,

$$\begin{aligned}
\left(\frac{\partial}{\partial X} - G_1 \frac{\partial}{\partial Z}\right)\left(\frac{\partial}{\partial \tau} - G_0 \frac{\partial}{\partial Z}\right) &= \frac{\partial^2}{\partial \tau \partial X} - G_0 \frac{\partial^2}{\partial Z \partial X} - \frac{\partial G_0}{\partial X}\frac{\partial}{\partial Z} - G_1 \frac{\partial^2}{\partial \tau \partial Z} \\
&\quad + G_1 G_0 \frac{\partial^2}{\partial Z^2} + G_1 \frac{\partial G_0}{\partial Z}\frac{\partial}{\partial Z} \\
&= \left(\frac{\partial}{\partial \tau} - G_0 \frac{\partial}{\partial Z}\right)\left(\frac{\partial}{\partial X} - G_1 \frac{\partial}{\partial Z}\right) \\
&\quad + \left(\frac{\partial G_1}{\partial \tau} - \frac{\partial G_0}{\partial X} - G_0 \frac{\partial G_1}{\partial Z} + G_1 \frac{\partial G_0}{\partial Z}\right)\frac{\partial}{\partial Z}
\end{aligned}$$

By using relationships from (A.1a),(A.1b), we find

$$\frac{\partial G_1}{\partial \tau} - \frac{\partial G_0}{\partial X} - G_0 \frac{\partial G_1}{\partial Z} + G_1 \frac{\partial G_0}{\partial Z} = 0$$

which establishes (1.7b)

Finally,

$$\begin{aligned}
G_3 \frac{\partial}{\partial Z}\left(\frac{\partial}{\partial \tau} - G_0 \frac{\partial}{\partial Z}\right) &= G_3 \frac{\partial^2}{\partial \tau \partial Z} - G_3 G_0 \frac{\partial^2}{\partial Z^2} - G_3 \frac{\partial G_0}{\partial Z}\frac{\partial}{\partial Z} \\
&= \left(\frac{\partial}{\partial \tau} - G_0 \frac{\partial}{\partial Z}\right) G_3 \frac{\partial}{\partial Z} - \left(\frac{\partial G_3}{\partial \tau} - G_0 \frac{G_3}{\partial Z} + G_3 \frac{\partial G_0}{\partial Z}\right)
\end{aligned}$$

By using relationships from (A.1a,A.1c), we find

$$\frac{\partial G_3}{\partial \tau} - G_0 \frac{G_3}{\partial Z} + G_3 \frac{\partial G_0}{\partial Z} = 0$$

which establishes (1.7c)

# B Velocity Transformation

We need the tangential vectors to the coordinate lines to resolve the velocity into components in the new coordinate system. Since the new system is not orthogonal, we must find the normal to the coordinate curves and then find the tangent as orthogonal to the normal.

The normal to the $Z$ – coordinate curves is given by

$$\nabla \cdot \left(z - Z - h(X, \tau)\mathrm{e}^{-\alpha Z}\right) = -h_X \mathrm{e}^{-\alpha Z}\mathbf{i} + \mathbf{k}$$

So the tangent is

$$\mathbf{t}_x = \frac{1}{D}\mathbf{i} + \frac{h_x \mathrm{e}^{-\alpha Z}}{D}\mathbf{j}, \quad \text{where} \quad D^2 = 1 + h_X^2 \mathrm{e}^{-2\alpha Z} \tag{B.1a}$$

The other tangent is obviously,

$$\mathbf{t}_z = \mathbf{k} \tag{B.1b}$$

The inverse transformation is

$$\mathbf{i} = D\mathbf{t}_x - h_X \mathrm{e}^{-\alpha Z}\mathbf{t}_z \tag{B.2a}$$

$$\mathbf{k} = \mathbf{t}_z \tag{B.2b}$$

Since a particle with trajectory $(x(t), y(t))$ has velocity components,

$$u = \frac{dx}{dt} = \frac{dX}{d\tau} \tag{B.3a}$$

$$w = \frac{dz}{dt} = \left(1 - \alpha h \mathrm{e}^{-\alpha Z}\right)\frac{dZ}{d\tau} + \left(h_\tau + h_X \frac{dX}{d\tau}\right)\mathrm{e}^{-\alpha Z} \tag{B.3b}$$

we may substitute these expressions, along with (B.2), into

$$u\mathbf{i} + w\mathbf{k} = D\frac{dX}{d\tau}\mathbf{t}_x + \left[h_\tau + \left(1 - \alpha h \mathrm{e}^{-\alpha Z}\right)\frac{dZ}{d\tau}\right]\mathbf{t}_z \tag{B.4}$$

Let the velocity be written as $U\mathbf{t}_x + W\mathbf{t}_z$ in the new system. Since

$$u\mathbf{i} + w\mathbf{k} = uD\mathbf{t}_x + \left(w - uh_X \mathrm{e}^{-\alpha Z}\right)\mathbf{t}_z \tag{B.5}$$

we are led directly to the following relationships:

$$U = uD \tag{B.6a}$$

$$W = w - uh_X \mathrm{e}^{-alphaZ} \tag{B.6b}$$

and

$$\frac{dX}{d\tau} = \frac{U}{D} \tag{B.6c}$$

$$\frac{dZ}{d\tau} = \frac{W - h_\tau}{1 - \alpha h \exp(-\alpha Z)} \tag{B.6d}$$