Laminar Flow

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Contents

1 Boundary layers

In the presence of viscous effects, boundary layers will form at the interface. We will study their formation under the simplifying assumption that the interface is flat. Subsequently, we will study the influence of small perturbations.

2 Parallel flow

Let us first consider the response of the flow if the wind is suddenly turned on. In other words, we consider a flat interface ($z = 0$), and the wind and water speeds are purely horizontal: $\mathbf{u} = (u(z,t), 0, 0)$. The region above the interface contains air and variables will be designated with a superscript of (1). The water below the interface will have its variables designated with a superscript of (2).

From the notes on the basic equations, we find the following system of equations. In the air, $z > 0$,

$$\rho_1 u_t^{(1)} = \mu_1 u_{zz}^{(1)} \tag{2.1a}$$

$$p_z^{(1)} + \rho_1 g = 0 \tag{2.1b}$$

while in the water $z < 0$,

$$\rho_2 u_t^{(2)} = \mu_2 u_{zz}^{(2)} \tag{2.2a}$$

$$p_z^{(2)} + \rho_2 g = 0 \tag{2.2b}$$

At the interface $z = 0$ ($h = 0$),

$$u^{(1)} = u^{(2)} \tag{2.3a}$$

$$\mu_1 u_z^{(1)} = \mu_2 u_z^{(2)} \tag{2.3b}$$

$$p^{(1)} = p^{(2)} \tag{2.3c}$$

We may solve (2.1b) and (2.2b), and pick the constant of integration so that the pressure vanishes at the interface, ensuring that (2.3c) is satisfied.

$$p^{(1)} + \rho_1 g z = 0 \tag{2.4a}$$

$$p^{(2)} + \rho_2 g z = 0 \tag{2.4b}$$

We may solve (2.1a) and (2.2a) by applying the Laplace transform,

$$U^{(1)}(z,s) = \int_0^\infty u^{(1)}(z,t) \mathrm{e}^{-st}\, dt \tag{2.5a}$$

$$U^{(2)}(z,s) = \int_0^\infty u^{(2)}(z,t) \mathrm{e}^{-st}\, dt \tag{2.5b}$$

Before we can proceed, we must decide on the initial flow and on the behavior in the far-field.

2.1 Uniform Flow Initially Throughout

The simplest case is to assume

$$u^{(1)}(z,0) = U_\infty\,, \qquad \text{for } 0 < z < \infty \tag{2.6a}$$

$$u^{(2)}(z,0) = 0\,, \qquad \text{for } -\infty < z < 0 \tag{2.6b}$$

Then, after introducing the kinematic viscosity $\nu = \mu/\rho$,

$$sU^{(1)} - U_\infty = \nu_1 U^{(1)}_{zz} \tag{2.7a}$$

$$sU^{(2)} = \nu_2 U^{(2)}_{zz} \tag{2.7b}$$

which have the solutions,

$$U^{(1)} = \frac{U_\infty}{s} + A\,\mathrm{e}^{-\sqrt{s/\nu_1}\,z} \tag{2.8a}$$

$$U^{(2)} = B\,\mathrm{e}^{\sqrt{s/\nu_2}\,z} \tag{2.8b}$$

where we have chosen the solutions that decay away from the interface.

To determine the constants A and B, we apply the Laplace transform to the interface conditions (2.3a) and (2.3b). At $z = 0$,

$$U^{(1)} = U^{(2)} \tag{2.9a}$$

$$\mu_1 U^{(1)}_z = \mu_2 U^{(2)}_z \tag{2.9b}$$

which leads to the equations,

$$\frac{U_\infty}{s} + A = B$$

$$-\mu_1\sqrt{\frac{s}{\nu_1}}\,A = \mu_2\sqrt{\frac{s}{\nu_2}}\,B$$

and their solutions,

$$A = -\frac{U_\infty}{(1+R)s}$$

$$B = \frac{RU_\infty}{(1+R)s}$$

where

$$R = \left(\frac{\rho_1\mu_1}{\rho_2\mu_2}\right)^{\frac{1}{2}} \tag{2.10}$$

Finally, the inverse Laplace transform gives

$$u^{(1)} = U_\infty - \frac{U_\infty}{1+R}\,\mathrm{erfc}\left(\frac{z}{2\sqrt{\nu_1 t}}\right) \tag{2.11a}$$

$$u^{(2)} = \frac{RU_\infty}{1+R}\,\mathrm{erfc}\left(-\frac{z}{2\sqrt{\nu_2 t}}\right) \tag{2.11b}$$

From Batchelor's book, the values of the physical parameters are given in Table 1. As a consequence, $R = 0.0044$, which means the velocity of the water at the interface is very small. Also, the length of the boundary layer in the air is longer than that in the water. As $t \to \infty$, the flow becomes uniform near the interface of speed

$$\frac{RU_\infty}{1+R} \tag{2.12}$$

	water	air
ρ	1.0 gm/cm^3	1.2×10^{-3} gm/cm^3
μ	1.1×10^{-2} gm/(cm sec)	1.8×10^{-4} gm/(cm sec)
ν	1.1×10^{-2} cm^2/sec	0.15 cm^2/sec
T	74 gm/sec^2	

Table 1: Physical parameters

2.2 Initially at Rest with Accelerated Flow in the Far-Field

For numerical studies, we may wish to turn the wind on slowly. So now assume that the wind speed far from the water surface is acclerated to a constant speed as

$$u^{(1)}(\infty) = U_\infty \left[1 - \exp\left(-\alpha t\right)\right] \tag{2.13}$$

and that the air is at rest initially. There must be a horizontal pressure gradient to generate the rate of change of the air speed at $z = \infty$. Thus, (2.1a) must be replaced by

$$\rho_1 u_t^{(1)} = -p_x^{(1)} + \mu_1 u_{zz}^{(1)} \tag{2.14}$$

Equation (2.1b) is still valid, so

$$p^{(1)} = -\rho_1 g z + \rho_1 \hat{p}(x,t)$$

Substitute into (2.14);

$$u_t^{(1)} = -\hat{p}_x + \nu_1 u_{zz}^{(1)}$$

Since $u^{(1)}$ is only a function of z, t, $\hat{p}_x = f(t)$, or

$$p^{(1)} = -\rho_1 g z + \rho_1 f(t) x \tag{2.15}$$

The Laplace transform applied to (2.14)) gives

$$sU^{(1)} = -F + \nu_1 U_{zz}^{(1)}$$

where F is the Laplace transform of f. The solution with bounded behaviour for large z is

$$U^{(1)} = -\frac{F}{s} + A \exp\left(-\sqrt{\frac{s}{\nu_1}}\, z\right) \tag{2.16}$$

The Laplace Transform of (2.13) implies

$$U^{(1)} \to U_\infty \left(\frac{1}{s} - \frac{1}{s+\alpha}\right) = \frac{U_\infty \alpha}{s(s+\alpha)}$$

as $z \to \infty$. Consequently,

$$F = -\frac{U_\infty \alpha}{s+\alpha} \quad \text{or} \quad f = -\alpha U_\infty \exp\left(-\alpha t\right) \tag{2.17}$$

Since there is a horizontal pressure gradient in the air at the interface, the pressure in the water must also have a horizontal pressure gradient. By solving (2.2b), and ensuring continuity of the pressure at the interface, we obtain

$$p^{(2)} = -\rho_2 g z + \rho_1 f(t) x \tag{2.18}$$

which means (2.2a) must be replaced by

$$\rho_2 u_t^{(2)} = -p_x^{(2)} + \mu_2 u_{zz}^{(2)} \tag{2.19}$$

Applying the Laplace transform leads to the solution, bounded at $z = -\infty$,

$$U^{(2)} = -\frac{\rho_1}{\rho_2}\frac{F}{s} + B\exp\left(\sqrt{\frac{s}{\nu_2}}\,z\right) \tag{2.20}$$

The matching conditions, (2.9), are the same, so

$$\begin{aligned} \frac{U_\infty \alpha}{s(s+\alpha)} + A &= \frac{\rho_1}{\rho_2}\frac{U_\infty \alpha}{s(s+\alpha)} + B \\ -\mu_1\sqrt{\frac{s}{\nu_1}}\,A &= \mu_2\sqrt{\frac{s}{\nu_2}}\,B \end{aligned}$$

and their solutions are

$$\begin{aligned} A &= -\frac{1}{1+R}\left(1-\frac{\rho_1}{\rho_2}\right)\frac{U_\infty \alpha}{s(s+\alpha)} \\ B &= \frac{R}{1+R}\left(1-\frac{\rho_1}{\rho_2}\right)\frac{U_\infty \alpha}{s(s+\alpha)} \end{aligned}$$

Thus,

$$U^{(1)} = \frac{U_\infty \alpha}{s(s+\alpha)}\left[1 - \frac{1}{1+R}\left(1-\frac{\rho_1}{\rho_2}\right)\exp\left(-\sqrt{\frac{s}{\nu_1}}\,z\right)\right] \tag{2.21a}$$

$$U^{(2)} = \frac{U_\infty \alpha}{s(s+\alpha)}\left[\frac{\rho_1}{\rho_2} + \frac{R}{1+R}\left(1-\frac{\rho_1}{\rho_2}\right)\exp\left(\sqrt{\frac{s}{\nu_2}}\,z\right)\right] \tag{2.21b}$$

A result from Appendix V in Carslaw and Jaeger proves useful: the function

$$F(s) = \frac{1}{s-a}\exp\left(-\sqrt{\frac{s}{\kappa}}\,x\right) \tag{2.22a}$$

has the inversion

$$f(x) = \frac{1}{2}e^{at}\left\{e^{-x\sqrt{a/\kappa}}\operatorname{erfc}\left[\frac{x}{2\sqrt{\kappa t}} - \sqrt{at}\right] + e^{x\sqrt{a/\kappa}}\operatorname{erfc}\left[\frac{x}{2\sqrt{\kappa t}} + \sqrt{at}\right]\right\} \tag{2.22b}$$

Thus, the inversion of (2.21) is

$$u^{(1)} = U_\infty\,[1-\exp(-\alpha t)] - \frac{U_\infty}{1+R}\left(1-\frac{\rho_1}{\rho_2}\right)\left[\operatorname{erfc}\left(\frac{z}{2\sqrt{\nu_1 t}}\right) - \exp(-\alpha t)\,\Re\left\{\exp(\mathrm{i}\sqrt{\alpha/\nu_1}\,z)\operatorname{erfc}\left(\frac{z}{2\sqrt{\nu_1 t}}+\mathrm{i}\sqrt{\alpha t}\right)\right\}\right] \tag{2.23a}$$

$$u^{(2)} = \frac{\rho_1}{\rho_2}\,U_\infty\,[1-\exp(-\alpha t)] + \frac{RU_\infty}{1+R}\left(1-\frac{\rho_1}{\rho_2}\right)\left[\operatorname{erfc}\left(-\frac{z}{2\sqrt{\nu_2}\,t}\right) - \exp(-\alpha t)\,\Re\left\{\exp(\mathrm{i}\sqrt{\alpha/\nu_2}\,z)\operatorname{erfc}\left(-\frac{z}{2\sqrt{\nu_2 t}}-\mathrm{i}\sqrt{\alpha t}\right)\right\}\right] \tag{2.23b}$$

The complementary error function with complex arguments may be evaluated by splitting the integral in its definition into two parts, along the real axis and up in the direction of the imaginary axis. Thus,

$$\operatorname{erfc}(x+\mathrm{i}y) = \frac{2}{\sqrt{\pi}}\int_x^\infty \mathrm{e}^{-p^2}\,dp - \frac{2\mathrm{i}}{\sqrt{\pi}}\mathrm{e}^{-x^2}\int_0^y \mathrm{e}^{p^2}\left[\cos(2xp)-\mathrm{i}\sin(2xp)\right]\,dp \tag{2.24}$$

Curiously, the speed at the interface settles to a constant value,

$$\frac{RU_\infty}{1+R} + \frac{U_\infty}{1+R}\frac{\rho_1}{\rho_2} \tag{2.25}$$

different from the previous case (2.12). While a pressure gradient is acting, the water is accelerated up to a constant speed far from the interface and the result above reflects the additional conribution.

The bad news is that the initial motion of the fluids is still non-uniform in z: A boundary layer still forms instantaneously, which complicates the choice of initial conditions for any numerical method.

2.3 Bounded flow in the Air, Initially at Rest

Now imagine a flat plate above the interface ($z = L$) which is set into motion spontaneously with speed U_L. The equations are the same as (2.1 – 2.3) but the flow in the air is now subject to the boundary condition, $u(z = L) = U_L$, and there is no initial motion. Thus, (2.8a) is replaced by

$$U^{(1)} = A\mathrm{e}^{-\sqrt{s/\nu_1}\,z} + C\mathrm{e}^{\sqrt{s/\nu_1}\,z} \tag{2.26}$$

At the interface, we must have

$$A + C = B$$
$$-RA + RC = B$$

which means

$$A = -\Omega C$$

where

$$\Omega = \frac{1-R}{1+R} \tag{2.27}$$

At the upper boundary,

$$A\mathrm{e}^{-\sqrt{s/\nu_1}\,L} + C\mathrm{e}^{\sqrt{s/\nu_1}\,L} = \frac{U_L}{s}$$

which gives the solution,

$$\begin{aligned}
A &= \frac{U_L}{s}\frac{\Omega}{\Omega\exp(-\sqrt{s/\nu_1}\,L) - \exp(\sqrt{s/\nu_1}\,L)} \\
C &= -\frac{U_L}{s}\frac{1}{\Omega\exp(-\sqrt{s/\nu_1}\,L) - \exp(\sqrt{s/\nu_1}\,L)} \\
B &= \frac{U_L}{s}\frac{\Omega - 1}{\Omega\exp(-\sqrt{s/\nu_1}\,L) - \exp(\sqrt{s/\nu_1}\,L)}
\end{aligned}$$

Finally, we may write the solution in the form,

$$U^{(1)} = \frac{U_L}{s}\frac{\exp\left[\sqrt{s/\nu_1}\,(z-L)\right] - \Omega\exp\left[-\sqrt{s/\nu_1}\,(z+L)\right]}{1 - \Omega\exp(-2\sqrt{s/\nu_1}\,L)} \tag{2.28a}$$

$$U^{(2)} = \frac{U_L}{s}\frac{(1-\Omega)\exp\left[\sqrt{s/\nu_2}\,z - \sqrt{s/\nu_1}\,L\right]}{1 - \Omega\exp(-2\sqrt{s/\nu_1}\,L)} \tag{2.28b}$$

An analytic inversion is not known, but we can approximate the solution for small times, specifically, for $\nu_1 t \ll L^2$.

$$\begin{aligned}
U^{(1)} &= \frac{U_L}{s}\left[\mathrm{e}^{\sqrt{s/\nu_1}\,(z-L)} - \Omega\mathrm{e}^{-\sqrt{s/\nu_1}\,(z+L)}\right]\sum_{n=0}^{\infty}\Omega^n\mathrm{e}^{-2n\sqrt{s/\nu_1}\,L} \\
&= U_L\sum_{n=0}^{\infty}\frac{\Omega^n}{s}\mathrm{e}^{\sqrt{s/\nu_1}\,[z-(2n+1)L]} - U_L\sum_{n=0}^{\infty}\frac{\Omega^{n+1}}{s}\mathrm{e}^{-\sqrt{s/\nu_1}\,[z+(2n+1)L]} \\
u^{(1)} &= U_L\sum_{n=0}^{\infty}\Omega^n\mathrm{erfc}\left[\frac{(2n+1)\,L - z}{2\sqrt{\nu_1 t}}\right] - U_L\Omega\sum_{n=0}^{\infty}\Omega^n\mathrm{erfc}\left[\frac{z + (2n+1)\,L}{2\sqrt{\nu_1 t}}\right] && (2.29a) \\
&\approx U_L\left[\mathrm{erfc}\left(\frac{L-z}{2\sqrt{\nu_1 t}}\right) - \Omega\,\mathrm{erfc}\left(\frac{z+L}{2\sqrt{\nu_1 t}}\right)\right] && (2.29b)
\end{aligned}$$

Similarly,

$$\begin{aligned}
U^{(2)} &= U_L(1-\Omega)\sum_{n=0}^{\infty}\frac{\Omega^n}{s}\mathrm{e}^{\sqrt{s}\,[z/\sqrt{\nu_2} - (2n+1)L/\sqrt{\nu_1}]} \\
u^{(2)} &= U_L(1-\Omega)\sum_{n=0}^{\infty}\Omega^n\mathrm{erfc}\left[\frac{(2n+1)L}{2\sqrt{\nu_1 t}} - \frac{z}{2\sqrt{\nu_2 t}}\right] && (2.29c) \\
&\approx U_L(1-\Omega)\mathrm{erfc}\left(\frac{L}{2\sqrt{\nu_1 t}} - \frac{z}{2\sqrt{\nu_2 t}}\right) && (2.29d)
\end{aligned}$$

Curiously, there is no boundary layer near the interface.

2.4 Bounded Flow in the Air, Uniform Flow Initially

This case differs form the previous one in that the air below the plate (at $z = L$) is also set into motion initially. The solution may be written as

$$U^{(1)} = \frac{U_L}{s} + A\mathrm{e}^{-\sqrt{s/\nu_1}\,z} + C\mathrm{e}^{\sqrt{s/\nu_1}\,z}$$
$$U^{(2)} = B\mathrm{e}^{\sqrt{s/\nu_2}\,z}$$

After applying the boundary and interface conditions,

$$U^{(1)} = \frac{U_L}{s}\left\{1 - \frac{1}{1+R}\frac{\exp\left(-\sqrt{s/\nu_1}\,z\right) - \exp\left[\sqrt{s/\nu_1}\,(z-L)\right]}{1-\Omega\exp\left(-2\sqrt{s/\nu_1}\,L\right)}\right\} \tag{2.30a}$$

$$U^{(2)} = \frac{U_L}{s}\frac{R}{1+R}\frac{\left[1+\exp\left(-2\sqrt{s/\nu_1}\,L\right)\right]\exp\left(\sqrt{s/\nu_2}\,z\right)}{1-\Omega\exp\left(-2\sqrt{s/\nu_1}\,L\right)} \tag{2.30b}$$

Making the same approximation as above, we obtain

$$u^{(1)} = U_L - \frac{U_L}{1+R}\sum_{n=0}^{\infty}\Omega^n\left[\mathrm{erfc}\left(\frac{z+2nL}{2\sqrt{\nu_1 t}}\right) + \mathrm{erfc}\left(\frac{2(n+1)L-z}{2\sqrt{\nu_1 t}}\right)\right] \tag{2.31a}$$

$$\approx U_L - \frac{U_L}{1+R}\mathrm{erfc}\left(\frac{z}{2\sqrt{\nu_1 t}}\right) \tag{2.31b}$$

$$u^{(2)} = \frac{U_L R}{1+R}\sum_{n=0}^{\infty}\Omega^n\left[\mathrm{erfc}\left(\frac{nL}{\sqrt{\nu_1 t}} - \frac{z}{2\sqrt{\nu_2 t}}\right) + \mathrm{erfc}\left(\frac{(n+1)L}{\sqrt{\nu_1 t}} - \frac{z}{2\sqrt{\nu_2 t}}\right)\right] \tag{2.31c}$$

$$\approx \frac{U_L R}{1+R}\mathrm{erfc}\left(-\frac{z}{2\sqrt{\nu_2 t}}\right) \tag{2.31d}$$

which is the same as (2.11).

3 Stability of Parallel Flow

Since the boundary layer is growing in time, the notion of stability depends on what is sought. We shall ask what happens to a small perturbation, and develop the linear equations. We write the perturbed flow as $\mathbf{u} = (u(z,t) + \hat{u}, \hat{v}, \hat{w})$, and the perturbed pressure as $p_0(x,t) - \rho g z + \hat{p}$. The presence of p_0 is to accommodate any acceleration in the motion far from the interface. The incompressibility condition is simply

$$\hat{u}_x + \hat{v}_y + \hat{w}_z = 0 \tag{3.1a}$$

while the momentum equations become

$$\rho\left(\hat{u}_t + u\hat{u}_x + \hat{w}u_z\right) = -\hat{p}_x + \mu\left(\hat{u}_{xx} + \hat{u}_{yy} + \hat{u}_{zz}\right) \tag{3.1b}$$
$$\rho\left(\hat{v}_t + u\hat{v}_x\right) = -\hat{p}_y + \mu\left(\hat{v}_{xx} + \hat{v}_{yy} + \hat{v}_{zz}\right) \tag{3.1c}$$
$$\rho\left(\hat{w}_t + u\hat{w}_x\right) = -\hat{p}_z + \mu\left(\hat{w}_{xx} + \hat{w}_{yy} + \hat{w}_{zz}\right) \tag{3.1d}$$

By introducing the representation,

$$\hat{u} = U(k,l,z,t)\mathrm{e}^{\mathrm{i}(kx+ly)} \tag{3.2a}$$

$$\hat{v} = V(k,l,z,t)\mathrm{e}^{\mathrm{i}(kx+ly)} \tag{3.2b}$$

$$\hat{w} = W(k,l,z,t)\mathrm{e}^{\mathrm{i}(kx+ly)} \tag{3.2c}$$

$$\hat{p} = P(k,l,z,t)\mathrm{e}^{\mathrm{i}(kx+ly)} \tag{3.2d}$$

the stability of a general perturbation may be constructed through the Fourier transform. After substitution into (3.1), the following system is obtained for the Fourier coefficients of fixed mode numbers k, l.

$$\rho\left(U_t + \mathrm{i}kuU + Wu_z\right) = -\mathrm{i}kP + \mu\left[U_{zz} - \left(k^2+l^2\right)U\right] \tag{3.3a}$$

$$\rho\left(V_t + \mathrm{i}kuV\right) = -\mathrm{i}lP + \mu\left[V_{zz} - \left(k^2+l^2\right)V\right] \tag{3.3b}$$

$$\rho\left(W_t + \mathrm{i}kuW\right) = -P_z + \mu\left[W_{zz} - \left(k^2+l^2\right)W\right] \tag{3.3c}$$

$$\mathrm{i}kU + \mathrm{i}lV + W_z = 0 \tag{3.3d}$$

In general, we must solve this system by numerical means. It contains all the difficulties associated with solving the nonlinear system, in particular, the need to treat the pressure appropriately.

3.1 Standard Analysis

The standard approach is to invoke a separation of time scales. Since the diffusive time scale is very slow for very small viscosity, the profile is imagined to be frozen at some time τ. For the linearization to be valid, we require $\nu\tau >> \lambda$, where λ is a typical length scale of the perturbation. With this assumption, we apply the Laplace Transform to the system (3.3). Let $\mathcal{U} = \mathcal{L}\{U\}$, etc. Thus,

$$\rho\left(s\mathcal{U} - F + \mathrm{i}ku\mathcal{U} + \mathcal{W}u_z\right) = -\mathrm{i}k\mathcal{P} + \mu\left[\mathcal{U}_{zz} - \left(k^2+l^2\right)\mathcal{U}\right] \tag{3.4a}$$

$$\rho\left(s\mathcal{V} - G + \mathrm{i}ku\mathcal{V}\right) = -\mathrm{i}l\mathcal{P} + \mu\left[\mathcal{V}_{zz} - \left(k^2+l^2\right)\mathcal{V}\right] \tag{3.4b}$$

$$\rho\left(s\mathcal{W} - K + \mathrm{i}ku\mathcal{W}\right) = -\mathcal{P}_z + \mu\left[\mathcal{W}_{zz} - \left(k^2+l^2\right)\mathcal{W}\right] \tag{3.4c}$$

$$\mathrm{i}k\mathcal{U} + \mathrm{i}l\mathcal{V} + \mathcal{W}_z = 0 \tag{3.4d}$$

where F, G, K are the initial profiles for U, V, W. There are assumed to satisfy the incompressibility constraint,

$$\mathrm{i}kF + \mathrm{i}lG + K_z = 0 \tag{3.4e}$$

This system (3.4) constitutes a boundary value problem in the variable z. Standard analysis follows the approach of reducing the system to a single high-order pde for a single unknown variable. One way to achieve this is to use (3.4a), (3.4b) and (3.4d) to derive a relationship between $\mathcal{P}$ and $\mathcal{W}$, which is subsequently substituted into (3.4c). Multiply (3.4a) by ik and add it to (3.4b) multiplied by il with the result,

$$\rho(-s\mathcal{W}_z + K_z - \mathrm{i}ku\mathcal{W}_z + \mathrm{i}ku_z\mathcal{W}) = (k^2+l^2)\mathcal{P} - \mu\left[\mathcal{W}_{zzz} - \left(k^2+l^2\right)\mathcal{W}_z\right] \tag{3.5}$$

Differentiate (3.5) with respect to z and substitute into (3.4c) multiplied by $k^2 + l^2$ with the result,

$$-\rho\,(s + \mathrm{i}ku)\,\mathcal{D}_z\mathcal{W} + \rho\mathrm{i}ku_{zz}\mathcal{W} + \mu\mathcal{D}_z^2\mathcal{W} = -\rho\mathcal{D}_z K \tag{3.6a}$$

where

$$\mathcal{D}_z = \frac{d^2}{dz^2} - \left(k^2 + l^2\right) \tag{3.6b}$$

This is a fourth-order differential equation, unfortunately with variable coefficients. We need four boundary conditions.

3.2 Alternate Approach – Boundary Value Problem

In contrast to the standard approach, let's write (3.4) as a first-order system. Define

$$y_1 = \mathcal{U}\,, \quad y_2 = \mathcal{U}_z\,, \quad y_3 = \mathcal{V}\,, \quad y_4 = \mathcal{V}_z\,, \quad y_5 = \mathcal{W}\,, \quad y_6 = \mathcal{W}_z\,, \quad y_7 = \mathcal{P} \tag{3.7}$$

Then (3.4) may be written as

$$A\frac{d\mathbf{y}}{dz} = B\mathbf{y} + \mathbf{r} \tag{3.8a}$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \nu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \nu & -1/\rho \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.8b}$$

and

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ S & 0 & 0 & 0 & u_z & 0 & \mathrm{i}k/\rho \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & S & 0 & 0 & 0 & \mathrm{i}l/\rho \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & S & 0 & 0 \\ \mathrm{i}k & 0 & \mathrm{i}l & 0 & 0 & 1 & 0 \end{bmatrix} \tag{3.8c}$$

We have introduced

$$S = s + \mathrm{i}ku + \nu(k^2 + l^2) \tag{3.8d}$$

and

$$r = \begin{pmatrix} 0 \\ -F \\ 0 \\ -G \\ 0 \\ -K \\ 0 \end{pmatrix} \tag{3.8e}$$

The last equation could be written differently, but the conclusions are the same – the matrix A is singular.

To overcome this difficulty, we will drop y_6, and manipulate the last three equations. Differentiate the last equation to obtain

$$\frac{dy_6}{dz} = -\mathrm{i}k\frac{dy_1}{dz} - \mathrm{i}l\frac{dy_3}{dz} = -\mathrm{i}ky_2 - \mathrm{i}ly_4 \tag{3.9a}$$

Substitute into the second to last equation to obtain

$$\frac{dy_7}{dz} = \mu\frac{dy_6}{dz} - \rho S y_5 + \rho K = -\mathrm{i}\mu k y_2 - \mathrm{i}\mu l y_4 - \rho S y_5 + \rho K \tag{3.9b}$$

Redefine $y_6 = y_7$, then (3.8a) still holds but with

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \nu & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/\rho \end{bmatrix} \tag{3.10a}$$

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ S & 0 & 0 & 0 & u_z & \mathrm{i}k/\rho \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & S & 0 & 0 & \mathrm{i}l/\rho \\ -\mathrm{i}k & 0 & -\mathrm{i}l & 0 & 0 & 0 \\ 0 & -\mathrm{i}\nu k & 0 & -\mathrm{i}\nu l & -S & 0 \end{bmatrix} \tag{3.10b}$$

and

$$r = \begin{pmatrix} 0 \\ -F \\ 0 \\ -G \\ 0 \\ K \end{pmatrix} \tag{3.10c}$$

Boundary conditions must be added to determine the solution.

3.2.1 Two-Dimensional Flow

The simpler case of two-dimensional flow carries all the important behavior of the system. Drazin and Reed give a transformation that reduces the three-dimensional flow to a two-dimensional flow. The easiest for us is to simply set $\mathcal{V} = l = 0$. Then, (3.6a) becomes

$$-\left(s + \mathrm{i}ku\right)\mathcal{D}_z\mathcal{W} + \mathrm{i}ku_{zz}\mathcal{W} + \nu\mathcal{D}_z^2\mathcal{W} = -\mathcal{D}_zK \tag{3.11a}$$

with

$$\mathcal{D}_z = \frac{d^2}{dz^2} - k^2 \tag{3.11b}$$

The system (3.10) becomes

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \nu & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/\rho \end{bmatrix} \tag{3.12a}$$

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ S & 0 & u_z & \mathrm{i}k/\rho \\ -\mathrm{i}k & 0 & 0 & 0 \\ 0 & -\mathrm{i}\nu k & -S & 0 \end{bmatrix} \tag{3.12b}$$

and

$$r = \begin{pmatrix} 0 \\ -F \\ 0 \\ K \end{pmatrix} \tag{3.12c}$$

3.3 Solid Boundary

There is one case we can solve analytically, namely, $u = 0$. Then, (3.11a) becomes

$$-s\mathcal{D}_z\mathcal{W} + \nu\mathcal{D}_z^2\mathcal{W} = -\mathcal{D}_zK \tag{3.13}$$

A particular solution, $\mathcal{W}^{(p)}$, satisfies,

$$\left(\nu\frac{d^2}{dz^2} - \left(\nu k^2 + s\right)\right)\mathcal{W}^{(p)} = -K \tag{3.14}$$

As a specific example, let

$$K = -Cz\mathrm{e}^{-mz} \tag{3.15}$$

where C and m will depend on k in order for the Fourier series of the initial vertical velocity to converge. Also, the choice (3.15) ensures $\mathcal{D}_zK \neq 0$. A particular solution to (3.14) is

$$\mathcal{W}^{(p)} = \left(\frac{z}{\nu m^2 - \nu k^2 - s} + \frac{2m\nu}{\left(\nu m^2 - \nu k^2 - s\right)^2}\right)C\mathrm{e}^{-mz} \tag{3.16a}$$

To this, we must add homogeneous solutions, but only those with negative exponents:

$$\mathcal{W}^{(h)} = A\mathrm{e}^{-kz} + B\mathrm{e}^{-\sqrt{\nu k^2+s}\,(z/\sqrt{\nu})} \tag{3.16b}$$

where A and B are determined by the requirement that $\mathcal{W}(0) = \mathcal{W}_z(0) = 0$. In particular,

$$\frac{2m\nu C}{(\nu m^2 - \nu k^2 - s)^2} + A + B = 0$$
$$\frac{C}{\nu m^2 - \nu k^2 - s} - \frac{2\nu m^2 C}{(\nu m^2 - \nu k^2 - s)^2} - kA - \frac{\sqrt{\nu k^2 + s}}{\sqrt{\nu}}\, B = 0$$

which leads to

$$A = \frac{\sqrt{\nu}}{\left(\sqrt{\nu k^2 + s} - \sqrt{\nu}\,k\right)} \left[\frac{1}{\sqrt{\nu k^2 + s} + \sqrt{\nu}\,m}\right]^2 C \tag{3.17a}$$
$$B = \frac{\nu(k-m)^2 + s}{\left(\sqrt{\nu k^2 + s} - \sqrt{\nu} k\right)\, (\nu m^2 - \nu k^2 - s)^2}\,\sqrt{\nu}\,C \tag{3.17b}$$

These results are valid provided $k \neq 0$, $m \neq k$ and $s \neq 0$.

The inversion formula involves the Bromwich contour. We may shift the contour to the left provided we pick up the contributions from any poles. By looking at (3.17), we find poles at

$$s = 0 \quad \text{and} \quad s = \nu(m^2 - k^2) \tag{3.18}$$

The first pole is associated with non-trivial homogeneous solutions, while the second is associated with the nature of the initial conditions. The location of the poles, s_j say, give the growth rates for exponential grow of the solutions, $\exp(s_j t)$. Unfortunately, the solution is not valid for $s = 0$; instead the homogeneous solution must be written in the form $(Az + B)\exp(-kz)$. The point, though, is that we look for non-trivial homogeneous solutions, which will require certain choices for s – actually, eigenvalues for the linear boundary value problem. This step is equivalent to seeking solutions of the form,

$$\hat{w} = W\mathrm{e}^{st+\mathrm{i}kx} \tag{3.19}$$

instead of using (3.2c) with the Laplace Transform. As a result, we establish necessary conditions for instability. The form (3.19) is the standard form for a normal mode analysis (with s usually written as σ, the growth rate). The procedure, then, is straightforward; we seek non-trivial homogeneous solutions to (3.11), even when $u \neq 0$, a calculation that must be done numerically in general.

3.4 Interface Between Immiscible Fluids

The linear equations (3.1) must hold in both the air and the water. Their solutions are connected through the linear version of the interface conditions. The height of the interface

h is now the small parameter. Quantities at the interface have the expansions;

$$u^{(1)}(h) = u(0) + u_z^{(1)}(0)h + \hat{u}^{(1)}(0) \tag{3.20a}$$
$$w^{(1)}(h) = \hat{w}^{(1)}(0) \tag{3.20b}$$
$$u^{(2)}(h) = u(0) + u_z^{(2)}(0)h + \hat{u}^{(2)}(0) \tag{3.20c}$$
$$w^{(2)}(h) = \hat{w}^{(2)}(0) \tag{3.20d}$$
$$u_x^{(1)}(h) = u_x(0) + u_{xz}^{(1)}(0)h + \hat{u}_x^{(1)}(0) \tag{3.20e}$$
$$u_z^{(1)}(h) = u_z^{(1)}(0) + u_{zz}^{(1)}(0)h + \hat{u}_z^{(1)}(0) \tag{3.20f}$$
$$w_x^{(1)}(h) = \hat{w}_x^{(1)}(0) \tag{3.20g}$$
$$w_z^{(1)}(h) = \hat{w}_z^{(1)}(0) \tag{3.20h}$$
$$u_x^{(2)}(h) = u_x(0) + u_{xz}^{(2)}(0)h + \hat{u}_x^{(2)}(0) \tag{3.20i}$$
$$u_z^{(2)}(h) = u_z^{(2)}(0) + u_{zz}^{(2)}(0)h + \hat{u}_z^{(2)}(0) \tag{3.20j}$$
$$w_x^{(2)}(h) = \hat{w}_x^{(2)}(0) \tag{3.20k}$$
$$w_z^{(2)}(h) = \hat{w}_z^{(2)}(0) \tag{3.20l}$$
$$p^{(1)}(h) = p_0 - \rho_1 g h + \hat{p}^{(1)}(0) \tag{3.20m}$$
$$p^{(2)}(h) = p_0 - \rho_2 g h + \hat{p}^{(2)}(0) \tag{3.20n}$$

The continuity of the velocity imposes the equations,

$$u_z^{(1)}(0)h + \hat{u}^{(1)}(0) = u_z^{(2)}(0)h + \hat{u}^{(2)}(0)\,, \quad \hat{w}^{(1)}(0) = \hat{w}^{(2)}(0) \equiv \hat{w}^{(I)} \tag{3.21a}$$

The motion of the interface is governed by

$$h_t + u(0)h_x = \hat{w}^{(I)} \tag{3.21b}$$

The linearized versions of the balance of stresses at the interface are is

$$\mu_1\left(u_{zz}^{(1)}(0)h + \hat{u}_z^{(1)}(0) + \hat{w}_x^{(1)}(0)\right) = \mu_2\left(u_{zz}^{(2)}(0)h + \hat{u}_z^{(2)}(0) + \hat{w}_x^{(2)}(0)\right) \tag{3.21c}$$

$$\begin{aligned}(\rho_2 - \rho_1)\,gh + &\hat{p}^{(1)}(0) - \hat{p}^{(2)}(0) + h_x\left(\mu_1 u_z^{(1)}(0) - \mu_2 u_z^{(2)}(0)\right)\\ &- 2\left(\mu_1 \hat{w}_z^{(1)}(0) - \mu_2 \hat{w}_z^{(2)}(0)\right) = T h_{xx}\end{aligned} \tag{3.21d}$$

By invoking the assumption (3.2) (with $l = 0$) and including

$$h = H(k,t)\,\mathrm{e}^{\mathrm{i}kx}\,, \tag{3.22}$$

we obtain upon substitution into (3.21)

$$u_z^{(1)}(0)H + U^{(1)}(0) = u_z^{(2)}(0)H + U^{(2)}(0) \tag{3.23a}$$
$$H_t + \mathrm{i}ku(0)H = W^{(1)}(0) = W^{(2)}(0) \tag{3.23b}$$
$$\mu_1\left(u_{zz}^{(1)}(0)H + U_z^{(1)}(0) + \mathrm{i}kW^{(1)}(0)\right) = \mu_2\left(u_{zz}^{(2)}(0)H + U_z^{(2)}(0) + \mathrm{i}kW^{(2)}(0)\right) \tag{3.23c}$$

$$\begin{aligned}(\rho_2 - \rho_1)\,gH + P^{(1)}(0) - P^{(2)}(0) + \mathrm{i}kH\left(\mu_1 u_z^{(1)}(0) - \mu_2 u_z^{(2)}(0)\right)&\\ - 2\left(\mu_1 W_z^{(1)}(0) - \mu_2 W_z^{(2)}(0)\right) = -k^2 TH&\end{aligned} \tag{3.23d}$$

Altogether, we must solve (3.3) in both the air and water, subject to the interface conditions (3.23).

4 Stability with No Base Flow

The only case we can solve in closed form is when the base flow vanishes, $u(z,t) \equiv 0$. Then, we may assume solutions have the form,

$$\begin{pmatrix} U \\ W \\ P \\ H \end{pmatrix} = \begin{pmatrix} \mathcal{U} \\ \mathcal{W} \\ \mathcal{P} \\ \mathcal{H} \end{pmatrix} \mathrm{e}^{\sigma t} \tag{4.1}$$

The equations of motion (3.3) in the air become

$$\rho_1 \sigma \mathcal{U}^{(1)} = -\mathrm{i}k\mathcal{P}^{(1)} + \mu_1 \left(\mathcal{U}^{(1)}_{zz} - k^2 \mathcal{U}^{(1)} \right) \tag{4.2a}$$

$$\rho_1 \sigma \mathcal{W}^{(1)} = -\mathcal{P}^{(1)}_z + \mu_1 \left(\mathcal{W}^{(1)}_{zz} - k^2 \mathcal{W}^{(1)} \right) \tag{4.2b}$$

$$\mathrm{i}k\mathcal{U}^{(1)} + \mathcal{W}^{(1)}_z = 0 \tag{4.2c}$$

while in the water, they are

$$\rho_2 \sigma \mathcal{U}^{(2)} = -\mathrm{i}k\mathcal{P}^{(2)} + \mu_2 \left(\mathcal{U}^{(2)}_{zz} - k^2 \mathcal{U}^{(2)} \right) \tag{4.2d}$$

$$\rho_2 \sigma \mathcal{W}^{(2)} = -\mathcal{P}^{(2)}_z + \mu_2 \left(\mathcal{W}^{(2)}_{zz} - k^2 \mathcal{W}^{(2)} \right) \tag{4.2e}$$

$$\mathrm{i}k\mathcal{U}^{(2)} + \mathcal{W}^{(2)}_z = 0 \tag{4.2f}$$

The interface conditions (3.23) become

$$\mathcal{U}^{(1)}(0) = \mathcal{U}^{(2)}(0) \tag{4.3a}$$

$$\sigma \mathcal{H} = \mathcal{W}^{(1)}(0) = \mathcal{W}^{(2)}(0) \tag{4.3b}$$

$$\mu_1 \left(\mathcal{U}^{(1)}_z(0) + \mathrm{i}k\mathcal{W}^{(1)}(0) \right) = \mu_2 \left(\mathcal{U}^{(2)}_z(0) + \mathrm{i}k\mathcal{W}^{(2)}(0) \right) \tag{4.3c}$$

$$(\rho_2 - \rho_1)\, g\mathcal{H} + \mathcal{P}^{(1)}(0) - \mathcal{P}^{(2)}(0) - 2\left(\mu_1 \mathcal{W}^{(1)}_z(0) - \mu_2 \mathcal{W}^{(2)}_z(0) \right) = -k^2 T \mathcal{H} \tag{4.3d}$$

4.1 Standard Approach

We may solve these equations be following the standard approach. First, seek $\mathcal{P}$ in terms of $\mathcal{W}$ (we drop the superscripts for this calculation and simply return them when the results is known). Multiply (4.2a) by $\mathrm{i}k$ and use (4.2c) to replace $-\mathrm{i}k\mathcal{U}$ by $\mathcal{W}_z$. Thus,

$$k^2 \mathcal{P} = -\rho\sigma \mathcal{W}_z + \mu \left(\mathcal{W}_{zzz} - k^2 \mathcal{W}_z \right) \tag{4.4}$$

Differentiate (4.4) and substitute into (4.2b) to obtain

$$\left[\sigma - \nu \left(\frac{d^2}{dz^2} - k^2 \right) \right] \left(\frac{d^2}{dz^2} - k^2 \right) \mathcal{W} = 0 \tag{4.5}$$

The solutions we require decay away from the interface.

$$\mathcal{W}^{(1)} = A \exp(-kz) + B \exp\left(-\sqrt{\sigma + \nu_1 k^2}\, \frac{z}{\sqrt{\nu_1}} \right) \tag{4.6a}$$

$$\mathcal{W}^{(2)} = C \exp(kz) + D \exp\left(\sqrt{\sigma + \nu_2 k^2}\, \frac{z}{\sqrt{\nu_2}} \right) \tag{4.6b}$$

Here the square roots must be taken with positive real parts. (σ may be complex).

The constants A, B, C and D are determined by (4.3). From (4.3b), we have

$$A + B = C + D \tag{4.7a}$$

By using (4.3a) with (4.2c) and (4.2f), we conclude $\mathcal{W}_z^{(1)}(0) = \mathcal{W}_z^{(2)}(0)$. So

$$-kA - \sqrt{\sigma + \nu_1 k^2}\,\frac{B}{\sqrt{\nu_1}} = kC + \sqrt{\sigma + \nu_2 k^2}\,\frac{D}{\sqrt{\nu_2}} \tag{4.7b}$$

After substituting (4.2c) and (4.2f) into (4.3c), we find

$$\mu_1 \left(\mathcal{W}_{zz}^{(1)}(0) + k^2 \mathcal{W}^{(1)}(0)\right) = \mu_2 \left(\mathcal{W}_{zz}^{(2)}(0) + k^2 \mathcal{W}^{(2)}(0)\right)$$

Thus,

$$2\mu_1 k^2 A + \rho_1 \left(\sigma + 2\nu_1 k^2\right) B = 2\mu_2 k^2 C + \rho_2 \left(\sigma + 2\nu_2 k^2\right) D \tag{4.7c}$$

We proceed in several steps to evaluate (4.3d). Consider

$$k^2 \mathcal{P} - 2\mu k^2 \mathcal{W}_z = -\rho\sigma \mathcal{W}_z + \mu \mathcal{W}_{zzz} - 3\mu k^2 \mathcal{W}_z$$

where we have used (4.4). Then, using this result in (4.3d),

$$\left(\mu_1 \mathcal{W}_{zzz}^{(1)}(0) - \mu_2 \mathcal{W}_{zzz}^{(2)}(0)\right) + \left[(\rho_2 - \rho_1)\,\sigma + 3\,(\mu_2 - \mu_1)\,k^2\right] \mathcal{W}_z^{(1)}(0) = \\ - \left[(\rho_2 - \rho_1)\,g k^2 + k^4 T\right] \mathcal{H}$$

where we have used $\mathcal{W}_z^{(2)}(0) = \mathcal{W}_z^{(1)}(0)$. Finally, after multiplying by σ and using (4.3b), we have

$$\begin{aligned} &\left[(\rho_2 - \rho_1)\,g k^2 + k^4 T\right]\,(A + B) - \sigma\mu_1 \left[k^3 A + \left(\sigma + \nu_1 k^2\right)^{3/2} \frac{B}{(\nu_1)^{3/2}}\right] \\ &- \sigma \left[(\rho_2 - \rho_1)\,\sigma + 3\,(\mu_2 - \mu_1)\,k^2\right] \left[kA + \sqrt{\sigma + \nu_1 k^2}\,\frac{B}{\sqrt{\nu_1}}\right] = \\ &\sigma\mu_2 \left[k^3 C + \left(\sigma + \nu_2 k^2\right)^{3/2} \frac{D}{(\nu_2)^{3/2}}\right] \end{aligned} \tag{4.7d}$$

Before solving (4.7), it is convenient to define

$$\Omega_1 = \sqrt{\sigma + \nu_1 k^2} \tag{4.8a}$$

$$\Omega_2 = \sqrt{\sigma + \nu_2 k^2} \tag{4.8b}$$

From (4.7a), $D = A + B - C$. Substitute into (4.7b).

$$C = -\frac{\sqrt{\nu_2}\,k + \Omega_2}{\sqrt{\nu_2}\,k - \Omega_2} A - \frac{\sqrt{\nu_2}\,\Omega_1 + \sqrt{\nu_1}\,\Omega_2}{\sqrt{\nu_2}\,k - \Omega_2}\,\frac{B}{\sqrt{\nu_1}} \tag{4.9a}$$

which also means,

$$D = \frac{2\sqrt{\nu_2}\,k}{\sqrt{\nu_2}\,k - \Omega_2}\,A + \frac{\sqrt{\nu_1}\,k + \Omega_1}{\sqrt{\nu_2}\,k - \Omega_2}\,\frac{\sqrt{\nu_2}}{\sqrt{\nu_1}}\,B \tag{4.9b}$$

Now substitute (4.9) into (4.7c).

$$\rho_1\left[2\nu_1 k^2 A+\left(\nu_1 k^2+\Omega_1^2\right)B\right]=\rho_2\left[2\nu_2 k^2 C+\left(\nu_2 k^2+\Omega_2^2\right)D\right]$$
$$=PA+QB$$

where

$$P=\frac{\rho_2}{\sqrt{\nu_2}\,k-\Omega_2}\left[-2\nu_2 k^2\left(\sqrt{\nu_2}\,k+\Omega_2\right)+2\sqrt{\nu_2}\,k\left(\nu_2 k^2+\Omega_2^2\right)\right]$$
$$=-2\rho_2\sqrt{\nu_2}\,k\Omega_2$$

and

$$Q=\frac{\rho_2}{\sqrt{\nu_1}\left(\sqrt{\nu_2}\,k-\Omega_2\right)}\left[-2\nu_2 k^2\left(\sqrt{\nu_2}\,\Omega_1+\sqrt{\nu_1}\,\Omega_2\right)\right.$$
$$\left.+\sqrt{\nu_2}\left(\nu_2 k^2+\Omega_2^2\right)\left(\sqrt{\nu_1}\,k+\Omega_1\right)\right]$$
$$=\rho_2\sqrt{\frac{\nu_2}{\nu_1}}\left[\sqrt{\nu_1}\,k\left(\sqrt{\nu_2}\,k-\Omega_2\right)-\Omega_1\left(\sqrt{\nu_2}\,k+\Omega_2\right)\right]$$

Finally, we write (4.7c) in the form

$$RA+SB=0 \tag{4.10a}$$

where

$$R=2\rho_1\nu_1 k^2-P$$
$$=2\rho_1\nu_1 k^2+2\rho_2\sqrt{\nu_2}\,k\Omega_2 \tag{4.10b}$$
$$S=\rho_1\sigma+2\rho_1\nu_1 k^2-Q$$
$$=\rho_1\left(\sigma+2\nu_1 k^2\right)+\rho_2\Omega_1\sqrt{\frac{\nu_2}{\nu_1}}\left(\sqrt{\nu_2}\,k+\Omega_2\right)-\rho_2\sqrt{\nu_2}\,k\left(\sqrt{\nu_2}\,k-\Omega_2\right) \tag{4.10c}$$

The last equation (4.7d) may be written in the form

$$UA+VB=0 \tag{4.11a}$$

where

$$U=\left(\rho_2-\rho_1\right)gk^2+Tk^4-\rho_1\sigma\nu_1 k^3-\sigma k\left[\left(\rho_2-\rho_1\right)\sigma+3\left(\rho_2\nu_2-\rho_1\nu_1\right)k^2\right]$$
$$+\frac{\rho_2\sigma k}{\sqrt{\nu_2}\,k-\Omega_2}\left[\nu_2 k^2\left(\sqrt{\nu_2}\,k+\Omega_2\right)-2\Omega_2^3\right]$$
$$=\left(\rho_2-\rho_1\right)gk^2+Tk^4+\sigma k\left[\sigma\left(\rho_1+\rho_2\right)+2\rho_1\nu_1 k^2+2\rho_2\sqrt{\nu_2}\,k\Omega_2\right] \tag{4.11b}$$
$$V=\left(\rho_2-\rho_1\right)gk^2+Tk^4-\frac{\rho_1\sigma}{\sqrt{\nu_1}}\Omega_1^3$$
$$-\frac{\sigma\Omega_1}{\sqrt{\nu_1}}\left[\left(\rho_2-\rho_1\right)\sigma+3\left(\rho_2\nu_2-\rho_1\nu_1\right)k^2\right]$$
$$-\frac{\rho_2\sigma\Omega_2^3}{\sqrt{\nu_1}}\frac{\sqrt{\nu_1}\,k+\Omega_1}{\sqrt{\nu_2}\,k-\Omega_2}+\frac{\rho_2\sigma\nu_2 k^3}{\sqrt{\nu_1}}\frac{\sqrt{\nu_2}\,\Omega_1+\sqrt{\nu_1}\,\Omega_2}{\sqrt{\nu_2}\,k-\Omega_2}$$
$$=\left(\rho_2-\rho_1\right)gk^2+Tk^4+\frac{\sigma}{\sqrt{\nu_1}}\left[\Omega_1\left(2\rho_1\nu_1 k^2-\rho_2\nu_2 k^2\right)\right.$$
$$\left.+\rho_2\sqrt{\nu_2}\,k\Omega_1\Omega_2+\rho_2\sqrt{\nu_1}\,k\left(\sigma+\nu_2 k^2+\sqrt{\nu_2}\,k\Omega_2\right)\right] \tag{4.11c}$$

A nontrivial solution occurs for (4.10a) and (4.11a) when

$$US - VR = 0 \tag{4.12}$$

Unfortunately, the algebra is very tedious, and is best broken up into several parts. First, let's write the result in the form,

$$US - VR = \left[(\rho_2 - \rho_1)\, gk^2 + Tk^4\right] X + Y \tag{4.13a}$$

where

$$\begin{aligned} X &= \rho_1(\sigma + 2\nu_1 k^2) + \rho_2\sqrt{\frac{\nu_2}{\nu_1}}\,\Omega_1\left(\Omega_2 + \sqrt{\nu_2}\,k\right) - \rho_2\sqrt{\nu_2}\,k\left(\sqrt{\nu_2}\,k - \Omega_2\right) \\ &\qquad - 2\rho_1\nu_1 k^2 - 2\rho_2\sqrt{\nu_2}\,k\Omega_2 \\ &= \rho_1\sigma + \rho_2\sqrt{\frac{\nu_2}{\nu_1}}\left(\Omega_2 + \sqrt{\nu_2}\,k\right)\left(\Omega_1 - \sqrt{\nu_1}\,k\right) \\ &= \frac{\rho_1\sqrt{\nu_1}\left(\Omega_1 + \sqrt{\nu_1}\,k\right) + \rho_2\sqrt{\nu_2}\left(\Omega_2 + \sqrt{\nu_2}\,k\right)}{\sqrt{\nu_1}\left(\Omega_1 + \sqrt{\nu_1}\,k\right)}\,\sigma \end{aligned} \tag{4.13b}$$

The expression for Y can be further written as

$$Y = \frac{\sigma}{\sqrt{\nu_1}}\left(\alpha\sigma^2 + \beta\sigma + \gamma\right) \tag{4.13c}$$

where

$$\alpha = \rho_1\left(\rho_1 + \rho_2\right)\sqrt{\nu_1}\,k \tag{4.13d}$$

$$\begin{aligned} \beta &= \rho_2\left(\rho_1 + \rho_2\right)\nu_2 k^2\Omega_1 + \left(3\rho_1\rho_2 - \rho_2^2\right)\sqrt{\nu_1}\,\sqrt{\nu_2}\,k^2\Omega_2 \\ &\qquad + \rho_2\left(\rho_1 + \rho_2\right)\sqrt{\nu_2}\,k\Omega_1\Omega_2 + 4\rho_1^2\nu_1\sqrt{\nu_1}\,k^3 \\ &\qquad - \rho_2\left(\rho_1 + \rho_2\right)\sqrt{\nu_1}\,\nu_2 k^3 \end{aligned} \tag{4.13e}$$

$$\begin{aligned} \gamma &= 4\left(\rho_1\rho_2\nu_1\nu_2 - \rho_1^2\left(\nu_1\right)^2\right)k^4\Omega_1 \\ &\qquad + 4\left(\rho_1\rho_2\nu_1\sqrt{\nu_1}\,\sqrt{\nu_2} - \rho_2^2\sqrt{\nu_1}\,\nu_2\sqrt{\nu_2}\right)k^4\Omega_2 \\ &\qquad + 4\left(\left(\rho_2\right)^2\nu_2\sqrt{\nu_2} - \rho_1\rho_2\nu_1\sqrt{\nu_2}\right)k^3\Omega_1\Omega_2 \\ &\qquad + 4\left(\left(\rho_1\nu_1\right)^2\sqrt{\nu_1} - \rho_1\rho_2\nu_1\sqrt{\nu_1}\,\nu_2\right)k^5 \end{aligned} \tag{4.13f}$$

Now let's simplify γ.

$$\begin{aligned} \gamma &= -4\left(\rho_1\nu_1 - \rho_2\nu_2\right)\left(\Omega_1 - \sqrt{\nu_1}\,k\right)\left(\rho_2\sqrt{\nu_2}\,k^3\Omega_2 + \rho_1\nu_1 k^4\right) \\ &= -\frac{4\left(\rho_1\nu_1 - \rho_2\nu_2\right)}{\Omega_1 + \sqrt{\nu_1}\,k}\left(\rho_2\sqrt{\nu_2}\,k^3\Omega_2 + \rho_1\nu_1 k^4\right)\sigma \end{aligned}$$

Thus, γ should be absorbed with the term containing β. Combining those terms,

$$\begin{aligned}
\beta\sigma + \gamma = &\Big\{\big[\rho_2\,(\rho_1+\rho_2)\,\nu_2 k^2 + \rho_2^2\sqrt{\nu_2}\,k\Omega_2\big]\,(\Omega_1 - \sqrt{\nu_1}\,k) \\
&\quad + \rho_1\rho_2\sqrt{\nu_2}\,k\Omega_1\Omega_2 + 3\rho_1\rho_2\sqrt{\nu_1}\,\sqrt{\nu_2}\,k^2\Omega_2 + 4\rho_1^2\nu_1\sqrt{\nu_1}\,k^3 \\
&\quad - \frac{4\,(\rho_1\nu_1 - \rho_2\nu_2)}{\Omega_1 + \sqrt{\nu_1}\,k}\,\big(\rho_2\sqrt{\nu_2}\,k^3\Omega_2 + \rho_1\nu_1 k^4\big)\Big\}\,\sigma \\
= &\frac{(\rho_1+\rho_2)\rho_2\nu_2 k^2 + \rho_2^2\sqrt{\nu_2}\,k\Omega_2}{\Omega_1 + \sqrt{\nu_1}\,k}\,\sigma^2 \\
&+ \frac{\rho_1\rho_2\,\big(\sigma + \nu_1 k^2 + \sqrt{\nu_1}\,k\Omega_1\big)\,\sqrt{\nu_2}\,k\Omega_2}{\Omega_1 + \sqrt{\nu_1}\,k}\,\sigma \\
&\quad + \big[3\rho_1\rho_2\sqrt{\nu_1}\,\sqrt{\nu_2}\,k^2\Omega_2 + 4\rho_1^2\nu_1\sqrt{\nu_1}\,k^3\big]\,\sigma \\
&\quad - \frac{4\,(\rho_1\nu_1 - \rho_2\nu_2)}{\Omega_1 + \sqrt{\nu_1}\,k}\,\big(\rho_2\sqrt{\nu_2}\,k^3\Omega_2 + \rho_1\nu_1 k^4\big)\,\sigma
\end{aligned}$$

This time terms with σ^2 have appeared. Finally, combining all the terms,

$$\begin{aligned}
\alpha\sigma^2 + \beta\sigma + \gamma = &\frac{1}{\Omega_1 + \sqrt{\nu_1}\,k}\,\{[(\rho_1+\rho_2)\rho_1\sqrt{\nu_1}\,k\,(\Omega_1 + \sqrt{\nu_1}\,k) \\
&(\rho_1+\rho_2)\rho_2\nu_2 k^2 + \rho_2^2\sqrt{\nu_2}\,k\Omega_2 + \rho_1\rho_2\sqrt{\nu_2}\,k\Omega_2\big]\,\sigma^2 \\
&+ 4\,\big[\rho_1\rho_2\nu_1\sqrt{\nu_2}\,k^3\Omega_2 + \rho_1\rho_2\sqrt{\nu_1}\,\sqrt{\nu_2}\,k^2\Omega_1\Omega_2 \\
&+ \rho_1^2\nu_1\sqrt{\nu_1}\,k^3\Omega_1 + \rho_1^2\nu_1^2 k^4 \\
&- (\rho_1\nu_1 - \rho_2\nu_2)\,\rho_2\sqrt{\nu_2}\,k^3\Omega_2 - (\rho_1\nu_1 - \rho_2\nu_2)\,\rho_1\nu_1 k^4\big]\,\sigma\big\} \\
= &\frac{1}{\Omega_1 + \sqrt{\nu_1}\,k}\,\{(\rho_1+\rho_2)\,[\rho_1\sqrt{\nu_1}\,(\Omega_1 + \sqrt{\nu_1}\,k) \\
&\qquad + \rho_2\sqrt{\nu_2}\,(\Omega_2 + \sqrt{\nu_2}\,k)]\,\sigma^2 k \\
&\quad + 4\,[\rho_1\sqrt{\nu_1}\,\Omega_1 + \rho_2\nu_2 k]\,[\rho_2\sqrt{\nu_2}\,\Omega_2 + \rho_1\nu_1 k]\,\sigma k^2\big\}
\end{aligned}$$

Finally, then, σ must satisfy the equation,

$$\begin{aligned}
&[\rho_1\sqrt{\nu_1}\,(\Omega_1 + \sqrt{\nu_1}\,k) + \rho_2\sqrt{\nu_2}\,(\Omega_2 + \sqrt{\nu_2}\,k)] \\
&\qquad \times \big[(\rho_2 - \rho_1)\,gk + Tk^3 + (\rho_1+\rho_2)\,\sigma^2\big] \\
&\qquad\quad + 4\,(\rho_1\sqrt{\nu_1}\,\Omega_1 + \rho_2\nu_2 k)\,(\rho_2\sqrt{\nu_2}\,\Omega_2 + \rho_1\nu_1 k)\,\sigma k = 0 \qquad (4.14)
\end{aligned}$$

4.2 Alternate Analysis

A different approach to the analysis in the previous section is to take advantage of (3.12) which may now be written as

$$\frac{d\mathbf{y}}{dz} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \Omega^2/\nu & 0 & 0 & \mathrm{i}k/(\rho\nu) \\ -\mathrm{i}k & 0 & 0 & 0 \\ 0 & -\mathrm{i}\rho\nu k & -\rho\Omega^2 & 0 \end{bmatrix}\mathbf{y} \equiv A\mathbf{y} \qquad (4.15)$$

where $\Omega^2 = \sigma + \nu k^2$. This system must be solved in the air and water with decaying solutions away from the interface and subject to the interface conditions (3.23).

The solutions to (4.15) take the form $\mathbf{y} = \mathcal{Y}\exp(\lambda z)$, where λ is an eigenvalues of the matrix in (4.15). The eigenvalues are determined through the vanishing of the determinant of $A - \lambda I$. After some tedious algebra,

$$\det(A - \lambda I) = \left(\lambda^2 - \frac{\Omega^2}{\nu}\right)\left(\lambda^2 - k^2\right) \tag{4.16}$$

For decaying solutions in air, we pick $\lambda = -k,\ -\Omega/\sqrt{\nu}$, while in water, we pick $\lambda = k,\ \Omega/\sqrt{\nu}$.[1]

The eigenvectors may be determined by standard row operations on the matrix $A - \lambda I$. The most useful intermediate row reduced form is

$$\begin{bmatrix} \lambda k\mathrm{i} & 0 & \lambda^2 & 0 \\ 0 & k\mathrm{i} & \lambda^2 & 0 \\ 0 & 0 & \rho\lambda\left(\Omega^2 - \nu\lambda^2\right) & k^2 \\ 0 & 0 & \rho\left(\Omega^2 - \nu\lambda^2\right) & \lambda \end{bmatrix} \tag{4.17}$$

For $\lambda = -\Omega/\sqrt{\nu}$ and $\Omega/\sqrt{\nu}$, the eigenvectors are

$$\mathbf{a}_1 = \begin{pmatrix} \Omega/\sqrt{\nu} \\ -\Omega^2/\nu \\ k\mathrm{i} \\ 0 \end{pmatrix} \qquad \mathbf{a}_2 = \begin{pmatrix} -\Omega/\sqrt{\nu} \\ -\Omega^2/\nu \\ k\mathrm{i} \\ 0 \end{pmatrix} \tag{4.18a}$$

respectively. For $\lambda = -k$ and k, they are

$$\mathbf{b}_1 = \begin{pmatrix} -k\mathrm{i} \\ k^2\mathrm{i} \\ k \\ \rho\sigma \end{pmatrix} \qquad \mathbf{b}_2 = \begin{pmatrix} -k\mathrm{i} \\ -k^2\mathrm{i} \\ -k \\ \rho\sigma \end{pmatrix} \tag{4.18b}$$

The solutions, then, may be written as

$$\mathbf{y}_1 = A\mathbf{a}_1\mathrm{e}^{-\Omega_1 z/\sqrt{\nu_1}} + B\mathbf{b}_1\mathrm{e}^{-kz} \tag{4.19a}$$

$$\mathbf{y}_2 = C\mathbf{a}_2\mathrm{e}^{\Omega_2 z/\sqrt{\nu_2}} + D\mathbf{b}_2\mathrm{e}^{kz} \tag{4.19b}$$

where the constants A, B, C, D are determined by enforcing the interface conditions (4.3). First, (4.3a) implies

$$\frac{\Omega_1}{\sqrt{\nu_1}}\,A - \mathrm{i}kB = -\frac{\Omega_2}{\sqrt{\nu_2}}\,C - \mathrm{i}kD \tag{4.20a}$$

Second,

$$\mathrm{i}kA + kB = \mathrm{i}kC - kD = \sigma\mathcal{H} \tag{4.20b}$$

[1] As before, we consider $k > 0$.

Third,

$$\rho_1\nu_1\left[-\frac{\Omega_1^2}{\nu_1}A+\mathrm{i}k^2B+\mathrm{i}k\left(\mathrm{i}kA+kB\right)\right]=\rho_2\nu_2\left[-\frac{\Omega_2^2}{\nu_2}C-\mathrm{i}k^2D+\mathrm{i}k\left(\mathrm{i}kC-kD\right)\right]\quad(4.20\mathrm{c})$$

Finally,

$$\begin{aligned}&\left[(\rho_2-\rho_1)\,g+Tk^2\right]\mathcal{H}+\rho_1\sigma B-\rho_2\sigma D\\&\qquad+2\mathrm{i}k\left[\rho_1\nu_1\left(\frac{\Omega_1}{\sqrt{\nu_1}}A-\mathrm{i}kB\right)-\rho_2\nu_2\left(-\frac{\Omega_2}{\sqrt{\nu_2}}C-\mathrm{i}kD\right)\right]=0\quad(4.20\mathrm{d})\end{aligned}$$

Equations (4.20) constitute a linear system for $A,B,C,D,\mathcal{H}$ with a coefficient matrix

$$\begin{bmatrix}\Omega_1/\sqrt{\nu_1} & -\mathrm{i}k & \Omega_2/\sqrt{\nu_2} & \mathrm{i}k & 0\\ \mathrm{i} & 1 & -\mathrm{i} & 1 & 0\\ 0 & 0 & \mathrm{i}k & -k & -\sigma\\ -\rho_1\left(\sigma+2\nu_1k^2\right) & 2\rho_1\nu_1\mathrm{i}k^2 & \rho_2\left(\sigma+2\nu_2k^2\right) & 2\rho_2\nu_2\mathrm{i}k^2 & 0\\ 2\rho_1\sqrt{\nu_1}\,\mathrm{i}k\Omega_1 & \rho_1\left(\sigma+2\nu_1k^2\right) & 2\rho_2\sqrt{\nu_2}\,\mathrm{i}k\Omega_2 & -\rho_2\left(\sigma+2\nu_2k^2\right) & \left[(\rho_2-\rho_1)\,g+Tk^2\right]\end{bmatrix}\quad(4.21)$$

Interchange the first two rows and perform the first step of Gaussian elimination.

$$\begin{bmatrix}\mathrm{i} & 1 & -\mathrm{i} & 1 & 0\\ 0 & -(\Omega_1/\sqrt{\nu_1}-k) & \mathrm{i}\left(\Omega_1/\sqrt{\nu_1}+\Omega_2/\sqrt{\nu_2}\right) & -\left(\Omega_1/\sqrt{\nu_1}+k\right) & 0\\ 0 & 0 & \mathrm{i}k & -k & -\sigma\\ 0 & \rho_1\sigma & a_{43} & a_{44} & 0\\ 0 & a_{52} & a_{53} & a_{54} & a_{55}\end{bmatrix}$$

where

$$\begin{aligned}a_{43}&=(\rho_2-\rho_1)\,\mathrm{i}\sigma-2\left(\rho_1\nu_1-\rho_2\nu_2\right)\mathrm{i}k^2\\ a_{44}&=\rho_1\sigma+2\left(\rho_1\nu_1-\rho_2\nu_2\right)k^2\\ a_{52}&=\mathrm{i}\rho_1\left(\sigma+2\nu_1k^2\right)-2\rho_1\sqrt{\nu_1}\,\mathrm{i}k\Omega_1\\ a_{53}&=-2\rho_2\sqrt{\nu_2}\,k\Omega_2-2\rho_1\sqrt{\nu_1}\,k\Omega_1\\ a_{54}&=-\rho_2\mathrm{i}\left(\sigma+2\nu_2k^2\right)-2\rho_1\sqrt{\nu_1}\,\mathrm{i}k\Omega_1\\ a_{55}&=\mathrm{i}\left[(\rho_2-\rho_1)\,g+Tk^2\right]\end{aligned}$$

Next, use the second row to eliminate the variable B.

$$\begin{bmatrix}\mathrm{i} & 1 & -\mathrm{i} & 1 & 0\\ 0 & -(\Omega_1/\sqrt{\nu_1}-k) & \mathrm{i}\left(\Omega_1/\sqrt{\nu_1}+\Omega_2/\sqrt{\nu_2}\right) & -\left(\Omega_1/\sqrt{\nu_1}+k\right) & 0\\ 0 & 0 & \mathrm{i}k & -k & -\sigma\\ 0 & 0 & b_{43} & b_{44} & 0\\ 0 & 0 & b_{53} & b_{54} & b_{55}\end{bmatrix}$$

where

$$
\begin{aligned}
b_{43} &= -a_{43}\left(\frac{\Omega_1}{\sqrt{\nu_1}} - k\right) - \rho_1 \mathrm{i}\sigma \left(\frac{\Omega_1}{\sqrt{\nu_1}} + \frac{\Omega_2}{\sqrt{\nu_2}}\right) \\
&= -\mathrm{i}\sigma \left[\rho_2 \left(\frac{\Omega_1}{\sqrt{\nu_1}} - k\right) + \rho_1 \left(\frac{\Omega_2}{\sqrt{\nu_2}} + k\right)\right] - 2\left(\frac{\Omega_1}{\sqrt{\nu_1}} - k\right)\left(\rho_2\nu_2 - \rho_1\nu_1\right)\mathrm{i}k^2 \\
b_{44} &= -a_{44}\left(\frac{\Omega_1}{\sqrt{\nu_1}} - k\right) + \rho_1\sigma\left(\frac{\Omega_1}{\sqrt{\nu_1}} + k\right) \\
&= 2\rho_1\sigma k + 2\left(\frac{\Omega_1}{\sqrt{\nu_1}} - k\right)\left(\rho_2\nu_2 - \rho_1\nu_1\right)k^2 \\
b_{53} &= -a_{53}\left(\frac{\Omega_1}{\sqrt{\nu_1}} - k\right) - \mathrm{i}a_{52}\left(\frac{\Omega_1}{\sqrt{\nu_1}} + \frac{\Omega_2}{\sqrt{\nu_2}}\right) \\
&= 2\left(\frac{\Omega_1}{\sqrt{\nu_1}} - k\right)\left(\rho_2\nu_2 - \rho_1\nu_1\right)k\frac{\Omega_2}{\sqrt{\nu_2}} + \rho_1\sigma\left(\frac{\Omega_1}{\sqrt{\nu_1}} + \frac{\Omega_2}{\sqrt{\nu_2}}\right) \\
b_{54} &= -a_{54}\left(\frac{\Omega_1}{\sqrt{\nu_1}} - k\right) + a_{52}\left(\frac{\Omega_1}{\sqrt{\nu_1}} + k\right) \\
&= \mathrm{i}\sigma\left[\rho_1\left(\frac{\Omega_1}{\sqrt{\nu_1}} + k\right) + \rho_2\left(\frac{\Omega_1}{\sqrt{\nu_1}} - k\right)\right] + 2\mathrm{i}k^2\left(\frac{\Omega_1}{\sqrt{\nu_1}} - k\right)\left(\rho_2\nu_2 - \rho_1\nu_1\right) \\
b_{55} &= -\mathrm{i}\left(\frac{\Omega_1}{\sqrt{\nu_1}} - k\right)\left[\left(\rho_2 - \rho_1\right)g + Tk^2\right]
\end{aligned}
$$

Next, use the third equation to eliminate C.

$$
\begin{bmatrix}
\mathrm{i} & 1 & -\mathrm{i} & 1 & 0 \\
0 & -(\Omega_1/\sqrt{\nu_1} - k) & \mathrm{i}\left(\Omega_1/\sqrt{\nu_1} + \Omega_2/\sqrt{\nu_2}\right) & -\left(\Omega_1/\sqrt{\nu_1} + k\right) & 0 \\
0 & 0 & \mathrm{i}k & -k & -\sigma \\
0 & 0 & 0 & c_{44} & c_{45} \\
0 & 0 & 0 & c_{54} & c_{55}
\end{bmatrix}
$$

where

$$
\begin{aligned}
c_{44} &= \mathrm{i}kb_{44} + kb_{43} \\
&= -\mathrm{i}\sigma k\left[\rho_2\left(\frac{\Omega_1}{\sqrt{\nu_1}} - k\right) + \rho_1\left(\frac{\Omega_2}{\sqrt{\nu_2}} - k\right)\right] \\
c_{45} &= \sigma b_{43} \\
c_{54} &= \mathrm{i}kb_{54} + kb_{53} \\
&= 2k^2\left(\rho_2\nu_2 - \rho_1\nu_1\right)\left(\frac{\Omega_1}{\sqrt{\nu_1}} - k\right)\left(\frac{\Omega_2}{\sqrt{\nu_2}} - k\right) + \sigma k\left[\rho_1\left(\frac{\Omega_2}{\sqrt{\nu_2}} - k\right) - \rho_2\left(\frac{\Omega_1}{\sqrt{\nu_1}} - k\right)\right] \\
c_{55} &= \left(\frac{\Omega_1}{\sqrt{\nu_1}} - k\right)\left[\left(\rho_1 - \rho_2\right)gk + Tk^3\right] + \rho_1\sigma^2\left(\frac{\Omega_1}{\sqrt{\nu_1}} + \frac{\Omega_2}{\sqrt{\nu_2}}\right) \\
&\qquad + 2\sigma k\left(\frac{\Omega_1}{\sqrt{\nu_1}} - k\right)\left(\rho_2\nu_2 - \rho_1\nu_1\right)\frac{\Omega_2}{\sqrt{\nu_2}}
\end{aligned}
$$

The final step creates an upper triangular matrix,

$$\begin{bmatrix} \mathrm{i} & 1 & -\mathrm{i} & 1 & 0 \\ 0 & -(\Omega_1/\sqrt{\nu_1}-k) & \mathrm{i}\left(\Omega_1/\sqrt{\nu_1}+\Omega_2/\sqrt{\nu_2}\right) & -\left(\Omega_1/\sqrt{\nu_1}+k\right) & 0 \\ 0 & 0 & \mathrm{i}k & -k & -\sigma \\ 0 & 0 & 0 & c_{44} & c_{45} \\ 0 & 0 & 0 & 0 & d_{55} \end{bmatrix} \tag{4.22}$$

where

$$\begin{aligned} d_{55} &= c_{44}c_{55} - c_{54}c_{45} \\ &= -\mathrm{i}\sigma k\left[\rho_1\left(\frac{\Omega_2}{\sqrt{\nu_2}}-k\right)+\rho_2\left(\frac{\Omega_1}{\sqrt{\nu_1}}-k\right)\right]\left\{\mathcal{G}\left(\frac{\Omega_1}{\sqrt{\nu_1}}-k\right)\right. \\ &\qquad \left. + 2\sigma k\left(\frac{\Omega_1}{\sqrt{\nu_1}}-k\right)\left(\rho_2\nu_2-\rho_1\nu_1\right)\frac{\Omega_2}{\sqrt{\nu_2}} + \rho_1\sigma^2\left(\frac{\Omega_1}{\sqrt{\nu_1}}+\frac{\Omega_2}{\sqrt{\nu_2}}\right)\right\} + \\ &\left\{\sigma k\left[\rho_1\left(\frac{\Omega_2}{\sqrt{\nu_2}}-k\right)-\rho_2\left(\frac{\Omega_1}{\sqrt{\nu_1}}-k\right)\right] + 2k^2\left(\rho_2\nu_2-\rho_1\nu_1\right)\left(\frac{\Omega_1}{\sqrt{\nu_1}}-k\right)\left(\frac{\Omega_2}{\sqrt{\nu_2}}-k\right)\right\} \\ &\times\left\{\mathrm{i}\sigma^2\left[\rho_2\left(\frac{\Omega_1}{\sqrt{\nu_1}}-k\right)+\rho_1\left(\frac{\Omega_2}{\sqrt{\nu_2}}+k\right)\right] + 2\mathrm{i}\sigma k^2\left(\rho_2\nu_2-\rho_1\nu_1\right)\left(\frac{\Omega_1}{\sqrt{\nu_1}}-k\right)\right\} \end{aligned}$$

and

$$\mathcal{G} = (\rho_2-\rho_1)\,gk + Tk^3$$

Multiply out the last expression. The result $\mathcal{R}$ is

$$\begin{aligned} \mathcal{R} = \mathrm{i}\sigma^3 k &\left\{\rho_1^2\left(\frac{\Omega_2^2}{\nu_2}-k^2\right) - 2k\rho_1\rho_2\left(\frac{\Omega_1}{\sqrt{\nu_1}}-k\right) - \rho_2^2\left(\frac{\Omega_1}{\sqrt{\nu_1}}-k\right)^2\right\} \\ &+ 2\mathrm{i}\sigma^2 k^2\left(\rho_2\nu_2-\rho_1\nu_1\right)\left(\frac{\Omega_1}{\sqrt{\nu_1}}-k\right)\mathcal{S} \\ &+ 4\mathrm{i}\sigma k^4\left(\rho_2\nu_2-\rho_1\nu_1\right)^2\left(\frac{\Omega_1}{\sqrt{\nu_1}}-k\right)^2\left(\frac{\Omega_2}{\sqrt{\nu_2}}-k\right) \end{aligned}$$

where

$$\mathcal{S} = \rho_1\left(\frac{\Omega_2^2}{\nu_2}-k^2\right) + \rho_2\left(\frac{\Omega_1}{\sqrt{\nu_1}}-k\right)\left(\frac{\Omega_2}{\sqrt{\nu_2}}-k\right) + \rho_1 k\left(\frac{\Omega_2}{\sqrt{\nu_2}}-k\right) - \rho_2 k\left(\frac{\Omega_1}{\sqrt{\nu_1}}-k\right)$$

By collecting expressions together, we obtain

$$\begin{aligned} d_{55} = &-\mathrm{i}\sigma k\left[\rho_1\left(\frac{\Omega_2}{\sqrt{\nu_2}}-k\right)+\rho_2\left(\frac{\Omega_1}{\sqrt{\nu_1}}-k\right)\right]\left(\frac{\Omega_1}{\sqrt{\nu_1}}-k\right)\mathcal{G} + \mathrm{i}\sigma^3 k\mathcal{U} \\ &+ 2\mathrm{i}\sigma^2 k^2\left(\rho_2\nu_2-\rho_1\nu_1\right)\left(\frac{\Omega_1}{\sqrt{\nu_1}}-k\right)\mathcal{V} + 4\mathrm{i}\sigma k^4\left(\rho_2\nu_2-\rho_1\nu_1\right)^2\left(\frac{\Omega_1}{\sqrt{\nu_1}}-k\right)^2\left(\frac{\Omega_2}{\sqrt{\nu_2}}-k\right) \end{aligned}$$

where

$$\mathcal{U} = -\left(\frac{\Omega_1}{\sqrt{\nu_1}} - k\right)\left[\rho_1^2\left(\frac{\Omega_2}{\sqrt{\nu_2}} - k\right) + \rho_1\rho_2\left(\frac{\Omega_1}{\sqrt{\nu_1}} + \frac{\Omega_2}{\sqrt{\nu_2}} + 2k\right) + \rho_2^2\left(\frac{\Omega_1}{\sqrt{\nu_1}} - k\right)\right]$$

$$\mathcal{V} = 2\rho_1 k\left(\frac{\Omega_2}{\sqrt{\nu_2}} - k\right) - 2\rho_2 k\left(\frac{\Omega_1}{\sqrt{\nu_1}} - k\right)$$

Now cancel the common term,

$$-\mathrm{i}\sigma k\left(\frac{\Omega_1}{\sqrt{\nu_1}} - k\right)$$

with the result,

$$d_{55} = \left[\rho_1\left(\frac{\Omega_2}{\sqrt{\nu_2}} - k\right) + \rho_2\left(\frac{\Omega_1}{\sqrt{\nu_1}} - k\right)\right]\mathcal{G} + \sigma^2\mathcal{W}$$
$$- 2\sigma k\big(\rho_2\nu_2 - \rho_1\nu_1\big)\mathcal{V} - 4k^3\big(\rho_2\nu_2 - \rho_1\nu_1\big)^2\left(\frac{\Omega_1}{\sqrt{\nu_1}} - k\right)\left(\frac{\Omega_2}{\sqrt{\nu_2}} - k\right)$$

where

$$\mathcal{W} = \rho_1^2\left(\frac{\Omega_2}{\sqrt{\nu_2}} - k\right) + \rho_1\rho_2\left(\frac{\Omega_1}{\sqrt{\nu_1}} + \frac{\Omega_2}{\sqrt{\nu_2}} + 2k\right) + \rho_2^2\left(\frac{\Omega_1}{\sqrt{\nu_1}} - k\right)$$

Next, multiply by

$$\nu_1\nu_2\left(\frac{\Omega_1}{\sqrt{\nu_1}} + k\right)\left(\frac{\Omega_2}{\sqrt{\nu_2}} + k\right)$$

and use the result

$$\nu\left(\frac{\Omega^2}{\sqrt{\nu}} - k^2\right) = \sigma$$

to obtain

$$d_{55} = \sigma\left[\rho_1\nu_1\left(\frac{\Omega_1}{\sqrt{\nu_1}} + k\right) + \rho_2\nu_2\left(\frac{\Omega_2}{\sqrt{\nu_2}} + k\right)\right]\mathcal{G} + \sigma^2\mathcal{A}$$
$$- 2\sigma^2 k\big(\rho_2\nu_2 - \rho_1\nu_1\big)\mathcal{B} - 4\sigma^2 k^3\big(\rho_2\nu_2 - \rho_1\nu_1\big)^2$$

where

$$\mathcal{A} = \sigma\rho_1^2\nu_1\left(\frac{\Omega_1}{\sqrt{\nu_1}} + k\right) + \rho_1\nu_1\rho_2\nu_2\left(\frac{\Omega_1}{\sqrt{\nu_1}} + k\right)\left(\frac{\Omega_2}{\sqrt{\nu_2}} + k\right)\left(\frac{\Omega_1}{\sqrt{\nu_1}} + \frac{\Omega_2}{\sqrt{\nu_2}} + 2k\right)$$
$$+ \sigma\rho_2^2\nu_2\left(\frac{\Omega_2}{\sqrt{\nu_2}} + k\right)$$

$$\mathcal{B} = 2\sigma\rho_1\nu_1 k\left(\frac{\Omega_1}{\sqrt{\nu_1}} + k\right) - 2\sigma\rho_2\nu_2 k\left(\frac{\Omega_2}{\sqrt{\nu_2}} + k\right)$$

Since

$$\left(\frac{\Omega_1}{\sqrt{\nu_1}} + k\right)\left(\frac{\Omega_2}{\sqrt{\nu_2}} + k\right)\left(\frac{\Omega_1}{\sqrt{\nu_1}} + \frac{\Omega_2}{\sqrt{\nu_2}} + 2k\right) =$$
$$\frac{\sigma}{\nu_1}\left(\frac{\Omega_1}{\sqrt{\nu_1}} + k\right) + \frac{\sigma}{\nu_2}\left(\frac{\Omega_2}{\sqrt{\nu_2}} + k\right) + 4k\left(\frac{\Omega_1}{\sqrt{\nu_1}} + k\right)\left(\frac{\Omega_2}{\sqrt{\nu_2}} + k\right)$$

we find

$$\mathcal{A} = \sigma\big(\rho_1+\rho_2\big)\left[\rho_1\nu_1\left(\frac{\Omega_1}{\sqrt{\nu_1}}+k\right)+\rho_2\nu_2\left(\frac{\Omega_2}{\sqrt{\nu_2}}+k\right)\right] \\ + 4\rho_1\nu_1\rho_2\nu_2 k\left(\frac{\Omega_1}{\sqrt{\nu_1}}+k\right)\left(\frac{\Omega_2}{\sqrt{\nu_2}}+k\right)$$

By dropping σ and absorbing the first term of $\mathcal{A}$ with $\mathcal{W}$, we obtain

$$d_{55} = \left[\rho_1\nu_1\left(\frac{\Omega_1}{\sqrt{\nu_1}}+k\right)+\rho_2\nu_2\left(\frac{\Omega_2}{\sqrt{\nu_2}}+k\right)\right]\big(\mathcal{G}+\big(\rho_1+\rho_2\big)\sigma^2\big)+4\sigma k\mathcal{C}$$

where

$$\begin{aligned}\mathcal{C} &= \rho_1\nu_1\rho_2\nu_2 k\left(\frac{\Omega_1}{\sqrt{\nu_1}}+k\right)\left(\frac{\Omega_2}{\sqrt{\nu_2}}+k\right)-\big(\rho_2\nu_2-\rho_1\nu_1\big)\left[\rho_1\nu_1\left(\frac{\Omega_1}{\sqrt{\nu_1}}+k\right)-\rho_2\nu_2\left(\frac{\Omega_2}{\sqrt{\nu_2}}+k\right)\right] \\ &\qquad - k^2\big(\rho_2\nu_2-\rho_1\nu_1\big)^2 \\ &= \left(\rho_1\nu_1\frac{\Omega_1}{\sqrt{\nu_1}}+\rho_2\nu_2 k\right)\left(\rho_2\nu_2\frac{\Omega_2}{\sqrt{\nu_2}}+\rho_1\nu_1 k\right)\end{aligned}$$

This result agrees with (4.14).

The case $k=0$ must be treated separately. However, it produces no interesting results. The system (4.5) becomes

$$\frac{d\mathbf{y}}{dz} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \sigma/\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\rho\sigma & 0 \end{bmatrix}\mathbf{y} \equiv A\mathbf{y}$$

The third equation implies $\mathcal{W}^{(1)} = \mathcal{W}^{(2)} = 0$ (applying the far-field boundary conditions), which means $\sigma = 0$ by (4.3b). Thus, the first two equations become $\mathcal{U}_{zz} = 0$ and $\mathcal{U}^{(1)} = \mathcal{U}^{(2)} = 0$ again by the far-field boundary conditions. Finally, the last equation gives $\mathcal{P}_z = 0$ which is satisfied by any choice of constant for the pressure in the air and water.

4.3 Analysis of the Dispersion Relation

The dispersion relation (4.14) involves several physical parameters and may contain many solutions. Even with the material parameters held fixed, there may be many branches of solutions as the wavenumber k is varied. It is best to build up an understanding of the various solutions by invoking some simplifying assumptions. The first obvious one is to remove the influence of the air by setting $\rho_1 = \nu_1 = 0$.

4.3.1 No Air

With $\rho_1 = \nu_1 = 0$, (4.14) becomes

$$\sigma^2 + gk + \frac{T}{\rho_2}k^3 + 4\frac{\nu_2 k^2\Omega_2}{\Omega_2+\sqrt{\nu_2}\,k}\,\sigma = 0 \tag{4.23}$$

which may be simplified by multiplying the numerator and denominator of the last term by $\Omega_2 - \sqrt{\nu_2}\, k$.

$$\sigma^2 + gk + \frac{T}{\rho_2} k^3 + 4\nu_2 k^2 \Omega_2 (\Omega_2 - \sqrt{\nu_2}\, k) = 0 \tag{4.24}$$

Since $\sigma = \Omega_2^2 - \nu_2 k^2$, we obtain a quartic in Ω_2.

$$\Omega_2^4 + 2\nu_2 k^2 \Omega_2^2 - 4\nu_2^{3/2} k^3 \Omega_2 + \nu_2^2 k^4 + gk + \frac{T}{\rho_2} k^3 = 0 \tag{4.25}$$

It is time to introduce dimensionless variables. For water ν_2 is quite small, so we expect the dominant contribution to come from the terms containing the gravity and surface tension. Thus, define

$$G = \left(gk + \frac{T}{\rho_2} k^3 \right)^{1/4} \tag{4.26a}$$

and set

$$\Omega_2 = GY \tag{4.26b}$$

Then, (4.25) becomes

$$Y^4 + 2\varepsilon^2 Y^2 - 4\varepsilon^3 Y + \varepsilon^4 + 1 = 0 \tag{4.26c}$$

where

$$\varepsilon = \frac{\sqrt{\nu_2}\, k}{G} \tag{4.26d}$$

The value of these definitions becomes clear when we examine the value of ε for all positive k, shown in Figure 1. We have chosen the values for water to evaluate ε. Clearly ε is very small even for large values of k, and only becomes sizeable for $k >> 1$: we may solve (4.26c) as a regular perturbation. Let

$$Y = \mathrm{e}^{\mathrm{i}n\pi/4} + \varepsilon^2 \alpha_2 + \varepsilon^3 \alpha_3 + \cdots, \quad \text{where } n = -1, 1 \tag{4.27a}$$

and substitute into (4.26b). (Note that we pick the two roots for Y when $\varepsilon = 0$ that have positive real parts as assumed in the derivation of (4.15). By balancing terms of equal powers in ε, we find

$$\alpha_2 = -\frac{1}{2}\, \mathrm{e}^{-\mathrm{i}n\pi/4} \tag{4.27b}$$

$$\alpha_3 = \mathrm{e}^{-\mathrm{i}n\pi/2} \tag{4.27c}$$

Finally,

$$\begin{aligned} \sigma &= (GY)^2 - \nu_2 k^2 \\ &= G^2 \mathrm{e}^{\mathrm{i}n\pi/2} - G^2\varepsilon^2 + 2G^2\varepsilon^3 \mathrm{e}^{-\mathrm{i}n\pi/4} - \nu_2 k^2 \\ &= \sqrt{gk + \frac{Tk^3}{\rho_2}}\, \mathrm{e}^{\mathrm{i}n\pi/2} - 2\nu_2 k^2 + \frac{2\nu_2^{3/2} k^3}{(gk + Tk^3/\rho_2)^{1/4}}\, \mathrm{e}^{-\mathrm{i}n\pi/4} \end{aligned} \tag{4.27d}$$

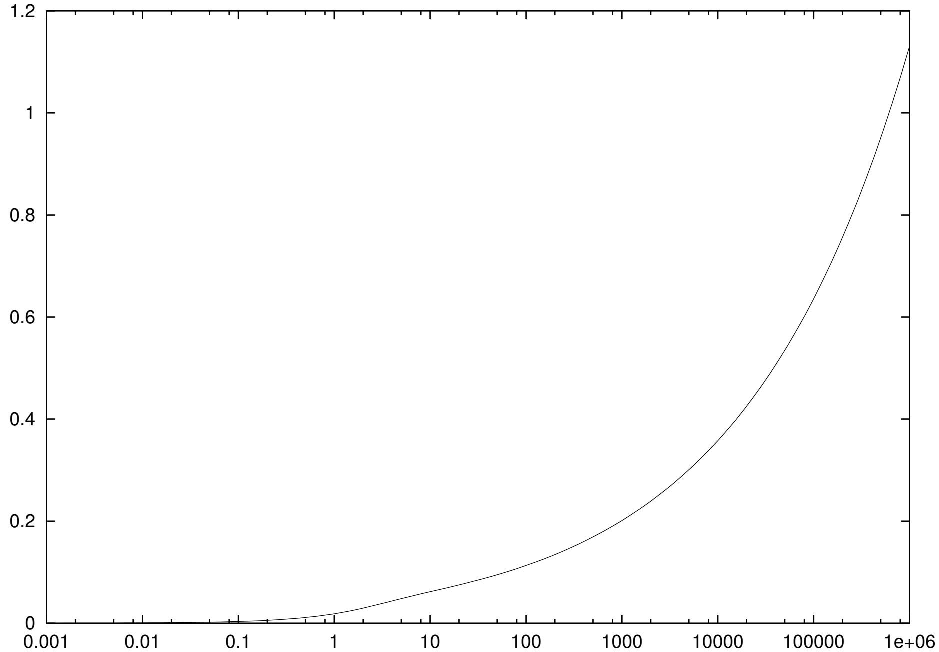

Figure 1: Variation of ε with k

Writing the result in real and imaginary parts,

$$\sigma = -2\nu_2 k^2 + \frac{\sqrt{2}\,\nu_2^{3/2} k^3}{(gk + Tk^3/\rho_2)^{1/4}} \pm \mathrm{i}\sqrt{gk + \frac{Tk^3}{\rho_2}}\left[1 - \frac{\sqrt{2}\,\nu_2^{3/2} k^3}{(gk + Tk^3/\rho_2)^{3/4}}\right] \tag{4.28}$$

Now consider the nature of the solution when ε is large. Set $Y = \varepsilon Z$ and write (4.26c) as

$$(Z-1)(Z^3 + Z^2 + 3Z - 1) + \frac{1}{\varepsilon^4} = 0 \tag{4.29}$$

There are three roots for the cubic

$$\begin{aligned} Z_1 &= 0.2956\,, \\ Z_{\pm} &= -0.6478 \pm 1.7214\mathrm{i}\,, \end{aligned} \tag{4.30}$$

but only Z_1 is relevant. Thus,

$$\begin{aligned} \sigma &= G^2\varepsilon^2 Z_1^2 - \nu_2 k^2 \\ &= -0.9126\nu_2 k^2\,. \end{aligned} \tag{4.31a}$$

There is another solution to (4.29) given approximately by $1 - 1/(4\varepsilon^4)$ and then

$$\begin{aligned} \sigma &= -\frac{G^2}{2\varepsilon^2} \\ &= -\frac{1}{2\nu_2 k^2}\left(gk + \frac{Tk^3}{\rho_2}\right) \end{aligned} \tag{4.31b}$$

It is perhaps useful to reconsider the results from a more physical view. Then, it is more appropriate to use gravity waves as the base state. In other words, scale σ by $\sqrt{gk}$, and write

$$\varepsilon^2 = \frac{\beta}{\sqrt{1+\alpha}} \tag{4.32a}$$

where

$$\alpha = \frac{Tk^2}{\rho g} \tag{4.32b}$$

$$\beta = \nu\left(\frac{k^3}{g}\right)^{\frac{1}{2}} \tag{4.32c}$$

Clearly, when $\beta \ll 1$ or $\beta \ll \sqrt{\alpha}$, then (4.28) applies, and when $\beta \gg 1$ and $\beta \gg \sqrt{\alpha}$, then (4.31) applies. For water, gravity dominates for wavelengths much greater than 1.7 cm; surface tension dominates for wavelengths much smaller than 1.7cm ($\alpha = 1$), but much greater than 1.0×10^{-2} cm ($\beta = \sqrt{\alpha}$); viscosity dominates for wavelengths much smaller than 1.0×10^{-2} cm.

4.3.2 Full Relation

Guided by the results in the previous section, let's use

$$F = \left[\left(\frac{\rho_2-\rho_1}{\rho_2+\rho_1}\right) gk + \frac{Tk^3}{\rho_2+\rho_1}\right]^{\frac{1}{2}} \tag{4.33a}$$

to define the dimensionless variable X by,

$$\sigma = FX \tag{4.33b}$$

Then, (4.14) becomes

$$(1+r)\left[r\varepsilon_1\left(\sqrt{X+\varepsilon_1^2}+\varepsilon_1\right)+\varepsilon_2\left(\sqrt{X+\varepsilon_2^2}+\varepsilon_2\right)\right]\left(X^2+1\right) + 4\left(r\varepsilon_1\sqrt{X+\varepsilon_1^2}+\varepsilon_2^2\right)\left(\varepsilon_2\sqrt{X+\varepsilon_2^2}+r\varepsilon_1^2\right)X = 0 \tag{4.33c}$$

with the dimensionless parameters given by

$$r = \frac{\rho_1}{\rho_2} \qquad \varepsilon_1 = \sqrt{\frac{\nu_1}{F}}\,k \qquad \varepsilon_2 = \sqrt{\frac{\nu_2}{F}}\,k \tag{4.33d}$$

When viscous effects are small, we expect the solution to be close to $X = \pm\mathrm{i}$. It is not clear at first sight which of the small terms should be included to calculate the correction to this solution. However, we do notice that all square roots may be approximated as $\sqrt{\mathrm{i}}$. To better appreciate which terms may dominate, let's write

$$X = \mathrm{i} + \hat{X} \tag{4.34a}$$

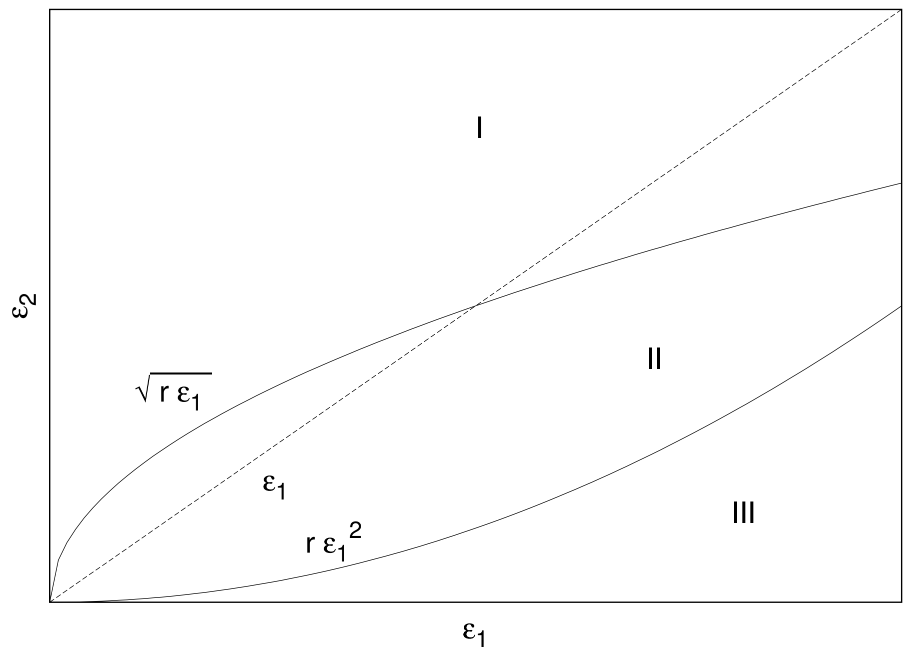

Figure 2: Regions of Different Asymptotic Solutions

and obtain an approximate solution

$$\hat{X} = -\frac{2\left(r\varepsilon_1\sqrt{\mathrm{i}} + \varepsilon_2^2\right)\left(\varepsilon_2\sqrt{\mathrm{i}} + r\varepsilon_1^2\right)}{(1+r)(r\varepsilon_1+\varepsilon_2)\sqrt{\mathrm{i}}}, \tag{4.34b}$$

which may be further simplied as

$$\Re\left\{\hat{X}\right\} = -\frac{\sqrt{2}\,\left(r\varepsilon_1\varepsilon_2 + \sqrt{2}\,(r^2\varepsilon_1^3+\varepsilon_2^3)\right)}{(1+r)\,(r\varepsilon_1+\varepsilon_2)} \tag{4.34c}$$

$$\Im\left\{\hat{X}\right\} = -\frac{\sqrt{2}\,r\varepsilon_1\varepsilon_2}{(1+r)\,(r\varepsilon_1+\varepsilon_2)} \tag{4.34d}$$

There are three special cases worth noting, and they can be identified by three regions in the diagram given in Figure 2.

Region I. Since $\varepsilon_2^2 >> r\varepsilon_1/\sqrt{2}$,

$$\hat{X} \approx -\frac{2\varepsilon_2^2}{1+r} \tag{4.35a}$$

Region II. Here $\varepsilon_2^2 << r\varepsilon_1/\sqrt{2}$, and $\varepsilon_2 >> \sqrt{2}\,r\varepsilon_1^2$, so

$$\hat{X} \approx -\frac{\sqrt{2}\,r\varepsilon_1\varepsilon_2}{(1+r)(r\varepsilon_1+\varepsilon_2)}\,(1+\mathrm{i}) \tag{4.35b}$$

Region III. Now $\varepsilon_2 << \sqrt{2}\,r\varepsilon_1^2$, and

$$\hat{X} \approx -\frac{2r\varepsilon_1^2}{1+r} \tag{4.35c}$$

On physical grounds, it is better to set $\varepsilon_2 = R\varepsilon_1$ where $R = \sqrt{\nu_2/\nu_1}$ is a parameter that depends only on material properties. The variable ε_1 can then be used to measure the wavenumber. For small enough k, the results form Region II are the appropriate choice.

$$\hat{X} \approx -\frac{\sqrt{2}\, rR\varepsilon_1}{(1+r)(r+R)}\,(1+\mathrm{i}) \tag{4.36a}$$

provided

$$\varepsilon_1 << \frac{r}{\sqrt{2}\, R^2} \qquad \text{and} \qquad \varepsilon_1 << \frac{R}{\sqrt{2}\, r} \tag{4.36b}$$

For water and air, $r = 1.2\times 10^{-3}$ and $\varepsilon_2/\varepsilon_1 = R = 0.271$. It is only the first inequality in (4.36b) that is of consequence: $\varepsilon_1 << 1.2\times 10^{-2}$ or $k << 0.0046/\mathrm{cm}$. The viscosity of the air is dominant for wavelengths well above 10m. For $k >> 0.0046/\mathrm{cm}$, the viscosity of the water dominates and the solution in Region I is the appropriate choice. It agress with (4.28). For Region III to be appropriate, $R/r < \varepsilon_1 < 1$. However, for air and water $R/r = 2.26\times 10^2$ and Region III is never relevant.

For R fixed, there are two real solutions to (4.33c) when ε_1 is very large. One solution is very small,

$$X \approx -\frac{(1+r)}{2(r+R^2)\varepsilon_1^2} \tag{4.37a}$$

and the other is very large,

$$X \approx a\varepsilon_1^2 \tag{4.37b}$$

where a satisfies

$$(1+r)\left[r\left(\sqrt{a+1}+1\right) + R\left(\sqrt{a+R^2}+R\right)\right]a + 4\left(r\sqrt{a+1}+R^2\right)\left(R\sqrt{a+R^2}+r\right) = 0\,. \tag{4.37c}$$

For the air/water interface, $a = -0.0676$.

The results for the air/water interface are displayed in Figure 3 which shows the $\Re\{X\}$ as a function of ε_1. The asymptotic results in (4.34c) for small ε_1 and the two asymptotic results in (4.37a) and (4.37b) are drawn over limited ranges as dashed lines. For large ε_1, the asymptotic results are already very good when ε_1 reaches 5. For small ε_1, the results agree well with (4.35b) for a value of ε_1 as large as 0.3. For $0.05 < \varepsilon_1 < 0.3$, the dominant part of (4.34c) is given by (4.35a) (Region I), while below 0.001 the dominant contribution is given by (4.35b) (Region II). The transition occurs around $\varepsilon_1 \approx 0.01$ and is shown in Figure 4. The $\Im\{X\}$ is shown in Figure 5. Clearly, the two real roots for X occur when $\Im\{X\} = 0$.

4.4 The Physical States

It is clear that the roots to (4.14), or its dimensionless form (4.33c), occur as complex conjugate pairs. Let $\sigma = \sigma_{\mathrm{r}} + \mathrm{i}\sigma_{\mathrm{i}}$ and from now on subscripts r and i will refer to real and

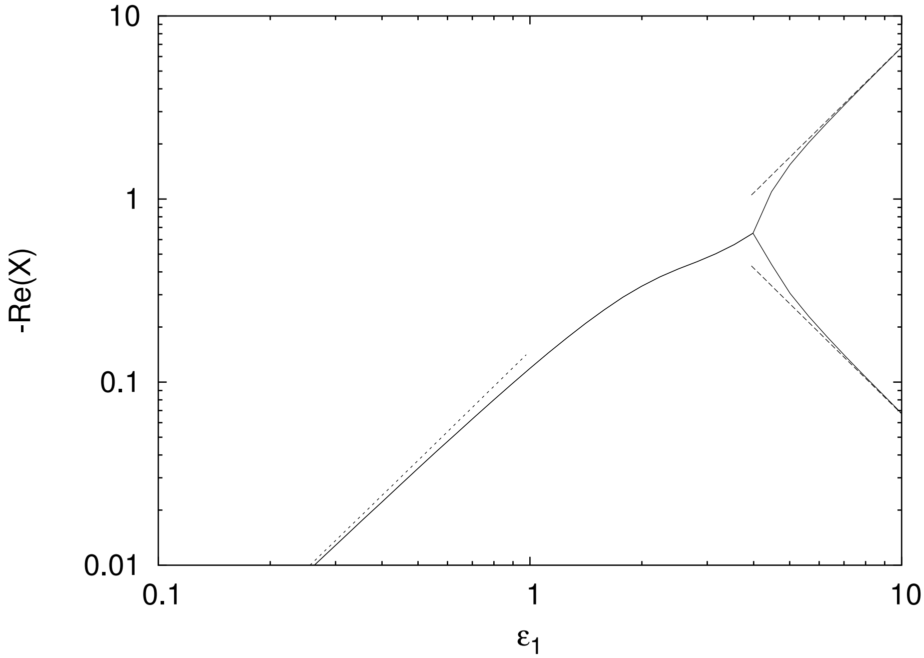

Figure 3: The $\Re\{X\}$ for air and water.

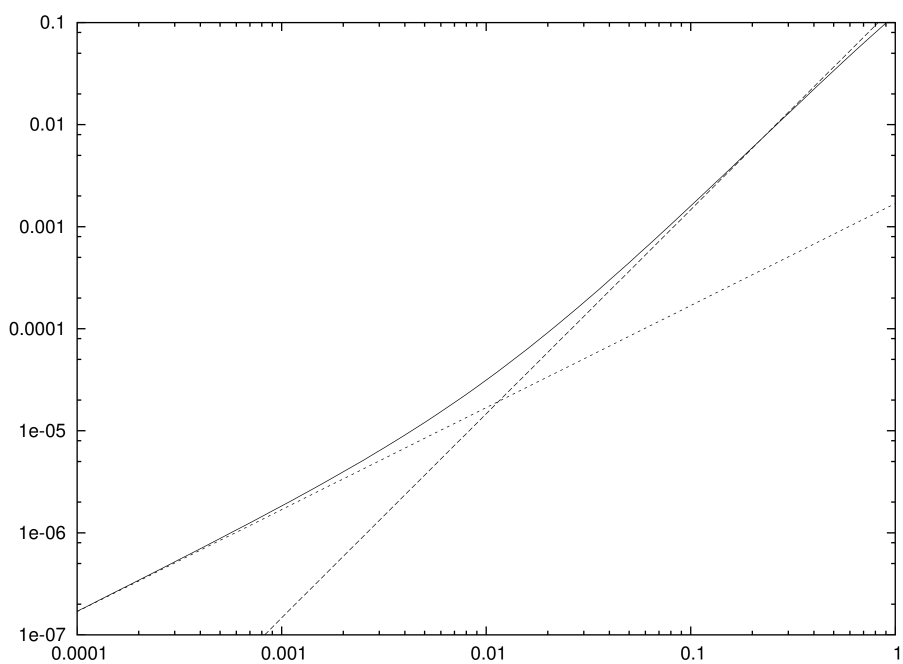

Figure 4: The $\Re\{X\}$ for air and water.

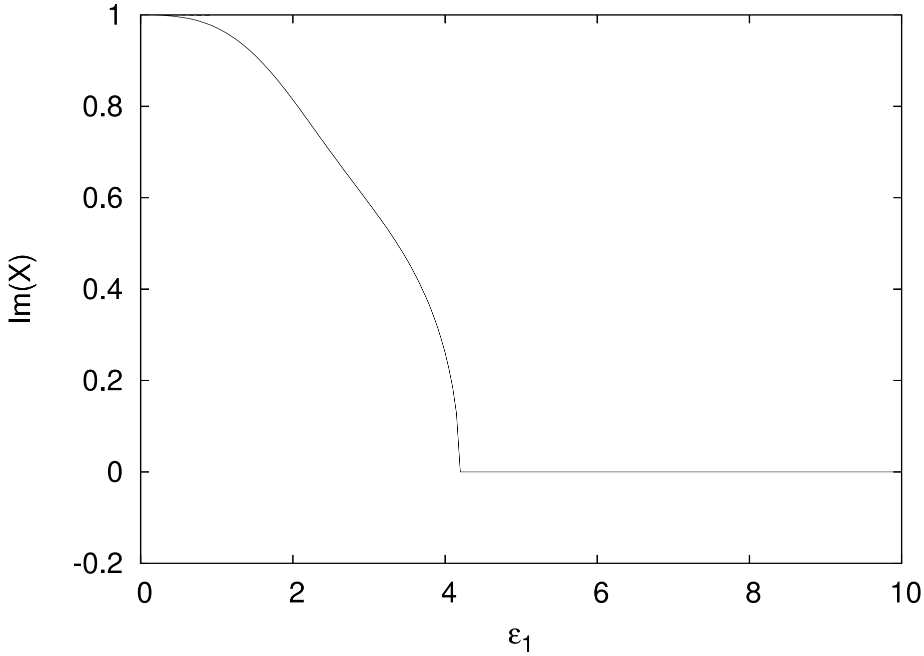

Figure 5: The $\Im\{X\}$ for air and water.

imaginary parts. Let us pick $\mathcal{H} = a$ as a measure of the amplitude of the surface elevation. Then,

$$h(x,t) = a\,\mathrm{e}^{(\sigma_r + \mathrm{i}\sigma_i)t}\,\mathrm{e}^{\mathrm{i}kx} = a\,\mathrm{e}^{F(X_r + \mathrm{i}X_i)t}\,\mathrm{e}^{\mathrm{i}kx} \tag{4.38}$$

where we have used (4.33b).

Next, we solve for D in dimensionless form using (4.33d), but after introducing,

$$\Omega_1 = \sqrt{\sigma + \nu_1 k^2} = \sqrt{F}\,\sqrt{X + \varepsilon_1^2} \tag{4.39a}$$

$$\Omega_2 = \sqrt{\sigma + \nu_2 k^2} = \sqrt{F}\,\sqrt{X + \varepsilon_2^2} \tag{4.39b}$$

$$\Omega_1 - \sqrt{\nu_1}\,k = \sqrt{F}\,\left(\sqrt{X + \varepsilon_1^2} - \varepsilon_1\right) \equiv \sqrt{F}\,X_1 \tag{4.39c}$$

$$\Omega_2 - \sqrt{\nu_2}\,k = \sqrt{F}\,\left(\sqrt{X + \varepsilon_2^2} - \varepsilon_2\right) \equiv \sqrt{F}\,X_2 \tag{4.39d}$$

Now (4.22) is in upper triangular form, and σ ensures $d_{55} = 0$. So we use the fourth equation to relate D to $\mathcal{H} = a$.

$$D = \frac{F}{k}\,\mathcal{D}a \tag{4.40a}$$

where

$$\begin{aligned}\mathcal{D} &= -\frac{RX\left[RX_1 + r\left(X_2 + 2\varepsilon_2\right)\right] + 2\varepsilon_2^2 X_1\left(R^2 - r\right)}{R\left(RX_1 + rX_2\right)} \\ &= -X - \varepsilon_1\Phi\end{aligned} \tag{4.40b}$$

where

$$\Phi = 2\,\frac{rRX + \varepsilon_2 X_1(R^2 - r)}{RX_1 + rX_2} \tag{4.40c}$$

Next, from the third equation of (4.22), we find C in the form

$$C = \mathrm{i}\frac{F}{k}\,\mathcal{C}a \tag{4.41a}$$

where

$$\mathcal{C} = -\mathcal{D} - X = \varepsilon_1 \Phi \tag{4.41b}$$

Also, from the second equation of (4.22), we find B in the form

$$B = \frac{F}{k}\,\mathcal{B}a \tag{4.42a}$$

where

$$\begin{aligned}\mathcal{B} &= -\frac{R\,(X_1 + 2\varepsilon_1)\,\mathcal{D} + \left(R\sqrt{X + \varepsilon_1^2} + \sqrt{X + \varepsilon_2^2}\right)\mathcal{C}}{RX_1} \\ &= X + \varepsilon_1 \Psi \end{aligned} \tag{4.42b}$$

and

$$\Psi = \frac{2RX - X_2\Phi}{RX_1} \tag{4.42c}$$

Finally, from the first equation of (4.22), we write A in the form

$$A = \mathrm{i}\frac{F}{k}\,\mathcal{A}a \tag{4.43a}$$

where

$$\mathcal{A} = \mathcal{B} + \mathcal{C} + \mathcal{D} = \varepsilon_1 \Psi \tag{4.43b}$$

The consequence of this choice of dimensionless variables is that they satisfy a dimensionless version of (4.21)

$$\begin{bmatrix} R\sqrt{X+\varepsilon_1^2} & -\varepsilon_2 & \sqrt{X+\varepsilon_2^2} & \varepsilon_2 & 0 \\ -1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & X \\ -r(X+2\varepsilon_1^2) & 2r\varepsilon_1^2 & X+2\varepsilon_2^2 & 2\varepsilon_2^2 & 0 \\ -2r\varepsilon_1\sqrt{X+\varepsilon_1^2} & r(X+2\varepsilon_1^2) & -2\varepsilon_2\sqrt{X+\varepsilon_2^2} & -X-2\varepsilon_2^2 & 1+r \end{bmatrix} \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \\ \mathcal{C} \\ \mathcal{D} \\ 1 \end{pmatrix} = 0 \tag{4.44}$$

With these results, we may write down the linear solutions from (4.19). In air,

$$\begin{pmatrix} \hat{u} \\ \hat{u}_z \\ \hat{w} \\ \hat{p} \end{pmatrix} = \left[A \begin{pmatrix} \frac{\Omega_1}{\sqrt{\nu_1}} \\ -\frac{\Omega_1^2}{\nu_1} \\ \mathrm{i}k \\ 0 \end{pmatrix} \mathrm{e}^{-\Omega_1 z/\sqrt{\nu_1}} + B \begin{pmatrix} -\mathrm{i}k \\ \mathrm{i}k^2 \\ k \\ \rho_1\sigma \end{pmatrix} \mathrm{e}^{-kz} \right] \mathrm{e}^{\sigma t}\,\mathrm{e}^{\mathrm{i}kx} \tag{4.45}$$

or individually,

$$\hat{u} = \mathrm{i}Fa\left[\sqrt{X+\varepsilon_1^2}\,\Psi\mathrm{e}^{-\sqrt{X+\varepsilon_1^2}\,kz/\varepsilon_1} - (X+\varepsilon_1\Psi)\mathrm{e}^{-kz}\right]\mathrm{e}^{FXt}\,\mathrm{e}^{\mathrm{i}kx} \tag{4.46a}$$

$$\hat{w} = Fa\left[-\varepsilon_1\Psi\mathrm{e}^{-\sqrt{X+\varepsilon_1^2}\,kz/\varepsilon_1} + (X+\varepsilon_1\Psi)\mathrm{e}^{-kz}\right]\mathrm{e}^{FXt}\,\mathrm{e}^{\mathrm{i}kx} \tag{4.46b}$$

$$\hat{p} = \frac{\rho_1 F^2 a}{k}X(X+\varepsilon_1\Psi)\mathrm{e}^{-kz}\mathrm{e}^{FXt}\,\mathrm{e}^{\mathrm{i}kx} \tag{4.46c}$$

In water,

$$\begin{pmatrix}\hat{u}\\ \hat{u}_z\\ \hat{w}\\ \hat{p}\end{pmatrix} = \left[C\begin{pmatrix}-\frac{\Omega_2}{\sqrt{\nu_2}}\\ -\frac{\Omega_2^2}{\nu_2}\\ \mathrm{i}k\\ 0\end{pmatrix}\mathrm{e}^{\Omega_2 z/\sqrt{\nu_2}} + D\begin{pmatrix}-\mathrm{i}k\\ -\mathrm{i}k^2\\ -k\\ \rho_2\sigma\end{pmatrix}\mathrm{e}^{kz}\right]\mathrm{e}^{\sigma t}\,\mathrm{e}^{\mathrm{i}kx} \tag{4.47}$$

or individually,

$$\hat{u} = -\mathrm{i}Fa\left[\sqrt{X+\varepsilon_2^2}\,\frac{\Phi}{R}\mathrm{e}^{\sqrt{X+\varepsilon_2^2}\,kz/\varepsilon_2} - (X+\varepsilon_1\Phi)\mathrm{e}^{kz}\right]\mathrm{e}^{FXt}\,\mathrm{e}^{\mathrm{i}kx} \tag{4.48a}$$

$$\hat{w} = Fa\left[-\varepsilon_1\Phi\mathrm{e}^{\sqrt{X+\varepsilon_2^2}\,kz/\varepsilon_2} + (X+\varepsilon_1\Phi)\mathrm{e}^{kz}\right]\mathrm{e}^{FXt}\,\mathrm{e}^{\mathrm{i}kx} \tag{4.48b}$$

$$\hat{p} = -\frac{\rho_2 F^2}{k}aX(X+\varepsilon_1\Phi)\mathrm{e}^{kz}\mathrm{e}^{FXt}\,\mathrm{e}^{\mathrm{i}kx} \tag{4.48c}$$

Real solutions are generated by considering the real and imaginary parts of (4.46) and (4.48). First, the real part of (4.38) gives

$$h = a\,\mathrm{e}^{FX_r t}\cos(FX_i t + kx) \tag{4.49}$$

which represents a wave travelling to the left. To simplify the notation, let us define

$$\sqrt{X+\varepsilon_1^2} = \sqrt{X_r+\varepsilon_1^2+\mathrm{i}X_i} \equiv Y_{1r} + \mathrm{i}Y_{1i} \tag{4.50a}$$

$$\sqrt{X+\varepsilon_2^2} = \sqrt{X_r+\varepsilon_2^2+\mathrm{i}X_i} \equiv Y_{2r} + \mathrm{i}Y_{2i} \tag{4.50b}$$

Also, let

$$\left[\sqrt{X+\varepsilon_1^2}\,\Psi\mathrm{e}^{-\sqrt{X+\varepsilon_1^2}\,kz/\varepsilon_1} - (X+\varepsilon_1\Psi)\mathrm{e}^{-kz}\right] = P_{ur} + \mathrm{i}P_{ui} \tag{4.51a}$$

$$\left[-\varepsilon_1\Psi\mathrm{e}^{-\sqrt{X+\varepsilon_1^2}\,kz/\varepsilon_1} + (X+\varepsilon_1\Psi)\mathrm{e}^{-kz}\right] = P_{wr} + \mathrm{i}P_{wi} \tag{4.51b}$$

$$X(X+\varepsilon_1\Psi)\mathrm{e}^{-kz} = P_{pr} + \mathrm{i}P_{pi} \tag{4.51c}$$

where

$$P_{ur} = \left[(Y_{1r}\Psi_r - Y_{1i}\Psi_i)\ \cos(\psi_1) + (Y_{1i}\Psi_r + Y_{1r}\Psi_i)\ \sin(\psi_1)\right]\ \mathrm{e}^{-Y_{1r}kz/\varepsilon_1} - (X_r + \varepsilon_1\Psi_r)\mathrm{e}^{-kz} \tag{4.52a}$$

$$P_{ui} = \left[(Y_{1i}\Psi_r + Y_{1r}\Psi_i)\ \cos(\psi_1) - (Y_{1r}\Psi_r - Y_{1i}\Psi_i)\ \sin(\psi_1)\right]\ \mathrm{e}^{-Y_{1r}kz/\varepsilon_1} - (X_i + \varepsilon_1\Psi_i)\mathrm{e}^{-kz} \tag{4.52b}$$

$$P_{wr} = \left[-\varepsilon_1\Psi_r\ \cos(\psi_1) - \varepsilon_1\Psi_i\ \sin(\psi_1)\right]\ \mathrm{e}^{-Y_{1r}kz/\varepsilon_1} + (X_r + \varepsilon_1\Psi_r)\mathrm{e}^{-kz} \tag{4.52c}$$

$$P_{wi} = \left[-\varepsilon_1\Psi_i\ \cos(\psi_1) + \varepsilon_1\Psi_r\ \sin(\psi_1)\right]\ \mathrm{e}^{-Y_{1r}kz/\varepsilon_1} + (X_i + \varepsilon_1\Psi_i)\mathrm{e}^{-kz} \tag{4.52d}$$

$$P_{pr} = \left[X_r(X_r + \varepsilon_1\Psi_r) - X_i(X_i + \varepsilon_1\Psi_i)\right]\ \mathrm{e}^{-kz} \tag{4.52e}$$

$$P_{pi} = \left[X_r(X_i + \varepsilon_1\Psi_i) + X_i(X_r + \varepsilon_1\Psi_r)\right]\ \mathrm{e}^{-kz} \tag{4.52f}$$

and

$$\psi_1 = \frac{Y_{1i}kz}{\varepsilon_1} \tag{4.52g}$$

Using these definitions, the real parts of (4.46) become

$$\hat{u} = -Fa\left[P_{ui}\cos\left(FX_it + kx\right) + P_{ur}\sin\left(FX_it + kx\right)\right]\ \mathrm{e}^{FX_rt} \tag{4.53a}$$

$$\hat{w} = Fa\left[P_{wr}\cos\left(FX_it + kx\right) - P_{wi}\sin\left(FX_it + kx\right)\right]\ \mathrm{e}^{FX_rt} \tag{4.53b}$$

$$\hat{p} = \rho_1\frac{F^2a}{k}\left[P_{pr}\cos\left(FX_it + kx\right) - P_{pi}\sin\left(FX_it + kx\right)\right]\ \mathrm{e}^{FX_rt} \tag{4.53c}$$

Similarly for the flow in the water, we introduce

$$\left[\sqrt{X + \varepsilon_2^2}\,\frac{\Phi}{R}\mathrm{e}^{\sqrt{X+\varepsilon_2^2}\,kz/\varepsilon_2} - (X + \varepsilon_1\Phi)\mathrm{e}^{kz}\right] = Q_{ur} + \mathrm{i}Q_{ui} \tag{4.54a}$$

$$\left[-\varepsilon_1\Phi\mathrm{e}^{\sqrt{X+\varepsilon_2^2}\,kz/\varepsilon_2} + (X + \varepsilon_1\Phi)\mathrm{e}^{kz}\right] = Q_{wr} + \mathrm{i}Q_{wi} \tag{4.54b}$$

$$X(X + \varepsilon_1\Phi)\mathrm{e}^{kz} = Q_{pr} + \mathrm{i}Q_{pi} \tag{4.54c}$$

where

$$Q_{ur} = \frac{1}{R}\left[(Y_{2r}\Phi_r - Y_{2i}\Phi_i)\ \cos(\psi_2) - (Y_{2i}\Phi_r + Y_{2r}\Phi_i)\ \sin(\psi_2)\right]\ \mathrm{e}^{Y_{2r}kz/\varepsilon_2} - (X_r + \varepsilon_1\Phi_r)\mathrm{e}^{kz} \tag{4.55a}$$

$$Q_{ui} = \frac{1}{R}\left[(Y_{2i}\Phi_r + Y_{2r}\Phi_i)\ \cos(\psi_2) + (Y_{2r}\Phi_r - Y_{2i}\Phi_i)\ \sin(\psi_2)\right]\ \mathrm{e}^{Y_{2r}kz/\varepsilon_2} - (X_i + \varepsilon_1\Phi_i)\mathrm{e}^{kz} \tag{4.55b}$$

$$Q_{wr} = \left[-\varepsilon_1\Phi_r\ \cos(\psi_2) + \varepsilon_1\Phi_i\ \sin(\psi_2)\right]\ \mathrm{e}^{Y_{2r}kz/\varepsilon_2} + (X_r + \varepsilon_1\Phi_r)\mathrm{e}^{kz} \tag{4.55c}$$

$$Q_{wi} = \left[-\varepsilon_1\Phi_i\ \cos(\psi_2) - \varepsilon_1\Phi_r\ \sin(\psi_2)\right]\ \mathrm{e}^{Y_{2r}kz/\varepsilon_2} + (X_i + \varepsilon_1\Phi_i)\mathrm{e}^{kz} \tag{4.55d}$$

$$Q_{pr} = \left[X_r(X_r + \varepsilon_1\Phi_r) - X_i(X_i + \varepsilon_1\Phi_i)\right]\ \mathrm{e}^{kz} \tag{4.55e}$$

$$Q_{pi} = \left[X_r(X_i + \varepsilon_1\Phi_i) + X_i(X_r + \varepsilon_1\Phi_r)\right]\ \mathrm{e}^{kz} \tag{4.55f}$$

and

$$\psi_2 = \frac{Y_{2i}kz}{\varepsilon_2} \tag{4.55g}$$

Using these definitions, the real parts of (4.48) become

$$\hat{u} = Fa\left[Q_{ui}\cos\left(FX_it + kx\right) + Q_{ur}\sin\left(FX_it + kx\right)\right]\, \mathrm{e}^{FX_rt} \tag{4.56a}$$

$$\hat{w} = Fa\left[Q_{wr}\cos\left(FX_it + kx\right) - Q_{wi}\sin\left(FX_it + kx\right)\right]\, \mathrm{e}^{FX_rt} \tag{4.56b}$$

$$\hat{p} = -\rho_2\frac{F^2a}{k}\left[Q_{pr}\cos\left(FX_it + kx\right) - Q_{pi}\sin\left(FX_it + kx\right)\right]\, \mathrm{e}^{FX_rt} \tag{4.56c}$$

To obtain waves travelling to the right, we seek the solutions that are conjugate to the previous ones. In other words, $X = X_r - \mathrm{i}X_i$, and

$$h = a\,\mathrm{e}^{FX_rt}\,\cos(FX_it - kx) \tag{4.57}$$

Notice that the conjugate growth rate is a solution because we may take the conjugate of (4.44). The conjugates of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ will be the appropriate ones to determine the flow variables. As a consequence of these results, we obtain another solution by replacing all imaginary parts in (4.53) and (4.56) by their negatives. Thus, in air,

$$\hat{u} = Fa\left[P_{ui}\cos\left(FX_it - kx\right) + P_{ur}\sin\left(FX_it - kx\right)\right]\, \mathrm{e}^{FX_rt} \tag{4.58a}$$

$$\hat{w} = Fa\left[P_{wr}\cos\left(FX_it - kx\right) - P_{wi}\sin\left(FX_it - kx\right)\right]\, \mathrm{e}^{FX_rt} \tag{4.58b}$$

$$\hat{p} = \rho_1\frac{F^2a}{k}\left[P_{pr}\cos\left(FX_it - kx\right) - P_{pi}\sin\left(FX_it - kx\right)\right]\, \mathrm{e}^{FX_rt} \tag{4.58c}$$

and in water,

$$\hat{u} = -Fa\left[Q_{ui}\cos\left(FX_it - kx\right) + Q_{ur}\sin\left(FX_it - kx\right)\right]\, \mathrm{e}^{FX_rt} \tag{4.59a}$$

$$\hat{w} = Fa\left[Q_{wr}\cos\left(FX_it - kx\right) - Q_{wi}\sin\left(FX_it - kx\right)\right]\, \mathrm{e}^{FX_rt} \tag{4.59b}$$

$$\hat{p} = -\rho_2\frac{F^2a}{k}\left[Q_{pr}\cos\left(FX_it - kx\right) - Q_{pi}\sin\left(FX_it - kx\right)\right]\, \mathrm{e}^{FX_rt} \tag{4.59c}$$

Finally, we may obtain standing waves by superposing two travelling waves in opposite directions. In particular, adding (4.49) and (4.57) and halving the result leads to

$$h = a\,\mathrm{e}^{FX_rt}\,\cos(FX_it)\cos(kx) \tag{4.60}$$

Repeat the procedure for the motion in the air;

$$\hat{u} = Fa\left[P_{ui}\sin\left(FX_it\right) - P_{ur}\cos\left(FX_it\right)\right]\, \mathrm{e}^{FX_rt}\,\sin\left(kx\right) \tag{4.61a}$$

$$\hat{w} = Fa\left[P_{wr}\cos\left(FX_it\right) - P_{wi}\sin\left(FX_it\right)\right]\, \mathrm{e}^{FX_rt}\,\cos\left(kx\right) \tag{4.61b}$$

$$\hat{p} = \rho_1\frac{F^2a}{k}\left[P_{pr}\cos\left(FX_it\right) - P_{pi}\sin\left(FX_it\right)\right]\, \mathrm{e}^{FX_rt}\,\cos\left(kx\right) \tag{4.61c}$$

and in water,

$$\hat{u} = -Fa\left[Q_{ui}\sin\left(FX_it\right) - Q_{ur}\cos\left(FX_it\right)\right]\, \mathrm{e}^{FX_rt}\,\sin\left(kx\right) \tag{4.62a}$$

$$\hat{w} = Fa\left[Q_{wr}\cos\left(FX_it\right) - Q_{wi}\sin\left(FX_it\right)\right]\, \mathrm{e}^{FX_rt}\,\cos\left(kx\right) \tag{4.62b}$$

$$\hat{p} = -\rho_2\frac{F^2a}{k}\left[Q_{pr}\cos\left(FX_it\right) - Q_{pi}\sin\left(FX_it\right)\right]\, \mathrm{e}^{FX_rt}\,\cos\left(kx\right) \tag{4.62c}$$

There are several other physical quantities of interest, the vorticity and the stress components at the surface. To calculate these quantities we need the following expressions: in air

$$\begin{aligned}\frac{\partial P_{ur}}{\partial z} = -\frac{k}{\varepsilon_1}\Big\{ &\Big[\big(Y_{1r}^2 - Y_{1i}^2\big)\Psi_r - 2Y_{1r}Y_{1i}\Psi_i\Big]\,\cos(\psi_1) \\ &+ \Big[\big(Y_{1r}^2 - Y_{1i}^2\big)\Psi_i + 2Y_{1r}Y_{1i}\Psi_r\Big]\,\sin(\psi_1)\Big\}\,\mathrm{e}^{-Y_{1r}kz/\varepsilon_1} \\ &+ k\big(X_r + \varepsilon_1\Psi_r\big)\mathrm{e}^{-kz}\end{aligned} \tag{4.63a}$$

$$\begin{aligned}\frac{\partial P_{ui}}{\partial z} = -\frac{k}{\varepsilon_1}\Big\{ &\Big[\big(Y_{1r}^2 - Y_{1i}^2\big)\Psi_i + 2Y_{1r}Y_{1i}\Psi_r\Big]\,\cos(\psi_1) \\ &- \Big[\big(Y_{1r}^2 - Y_{1i}^2\big)\Psi_r - 2Y_{1r}Y_{1i}\Psi_i\Big]\,\sin(\psi_1)\Big\}\,\mathrm{e}^{-Y_{1r}kz/\varepsilon_1} \\ &+ k\big(X_i + \varepsilon_1\Psi_i\big)\mathrm{e}^{-kz}\end{aligned} \tag{4.63b}$$

$$\begin{aligned}\frac{\partial P_{wr}}{\partial z} = k\Big[&\big(Y_{1r}\Psi_r - Y_{1i}\Psi_i\big)\,\cos(\psi_1) + \big(Y_{1r}\Psi_i + Y_{1i}\Psi_r\big)\,\sin(\psi_1)\Big]\,\mathrm{e}^{-Y_{1r}kz/\varepsilon_1} \\ &- k(X_r + \varepsilon_1\Psi_r)\mathrm{e}^{-kz}\end{aligned} \tag{4.63c}$$

$$\begin{aligned}\frac{\partial P_{wi}}{\partial z} = k\Big[&\big(Y_{1i}\Psi_r + Y_{1r}\Psi_i\big)\,\cos(\psi_1) + \big(Y_{1i}\Psi_i - Y_{1r}\Psi_r\big)\,\sin(\psi_1)\Big]\,\mathrm{e}^{-Y_{1r}kz/\varepsilon_1} \\ &- k(X_i + \varepsilon_1\Psi_i)\mathrm{e}^{-kz}\end{aligned} \tag{4.63d}$$

and in water

$$\begin{aligned}\frac{\partial Q_{ur}}{\partial z} = \frac{k}{R\varepsilon_2}\Big\{ &\Big[\big(Y_{2r}^2 - Y_{2i}^2\big)\Phi_r - 2Y_{2r}Y_{2i}\Phi_i\Big]\,\cos(\psi_2) \\ &- \Big[\big(Y_{2r}^2 - Y_{2i}^2\big)\Phi_i + 2Y_{2r}Y_{2i}\Phi_r\Big]\,\sin(\psi_2)\Big\}\,\mathrm{e}^{Y_{2r}kz/\varepsilon_2} \\ &- k\big(X_r + \varepsilon_1\Phi_r\big)\mathrm{e}^{kz}\end{aligned} \tag{4.64a}$$

$$\begin{aligned}\frac{\partial Q_{ui}}{\partial z} = \frac{k}{R\varepsilon_2}\Big\{ &\Big[\big(Y_{2r}^2 - Y_{2i}^2\big)\Phi_i + 2Y_{2r}Y_{2i}\Phi_r\Big]\,\cos(\psi_2) \\ &+ \Big[\big(Y_{2r}^2 - Y_{2i}^2\big)\Phi_r - 2Y_{2r}Y_{2i}\Phi_i\Big]\,\sin(\psi_2)\Big\}\,\mathrm{e}^{Y_{2r}kz/\varepsilon_2} \\ &- k\big(X_i + \varepsilon_1\Phi_i\big)\mathrm{e}^{kz}\end{aligned} \tag{4.64b}$$

$$\begin{aligned}\frac{\partial Q_{wr}}{\partial z} = \frac{k}{R}\Big[&-\big(Y_{2r}\Phi_r - Y_{2i}\Phi_i\big)\,\cos(\psi_2) + \big(Y_{2r}\Phi_i + Y_{2i}\Phi_r\big)\,\sin(\psi_2)\Big]\,\mathrm{e}^{Y_{2r}kz/\varepsilon_2} \\ &+ k(X_r + \varepsilon_1\Phi_r)\mathrm{e}^{kz}\end{aligned} \tag{4.64c}$$

$$\begin{aligned}\frac{\partial Q_{wi}}{\partial z} = -\frac{k}{R}\Big[&\big(Y_{2i}\Phi_r + Y_{2r}\Phi_i\big)\,\cos(\psi_2) + \big(Y_{2r}\Phi_r - Y_{2i}\Phi_i\big)\,\sin(\psi_2)\Big]\,\mathrm{e}^{Y_{2r}kz/\varepsilon_2} \\ &+ k(X_i + \varepsilon_1\Phi_i)\mathrm{e}^{kz}\end{aligned} \tag{4.64d}$$

For waves travelling to the right, we may express the initial vorticity

$$\omega = \frac{\partial \hat{w}}{\partial x} - \frac{\partial \hat{u}}{\partial z}$$

in the air as

$$\omega^{(1)} = -Fka\Big[\big(\omega_1^{(1)}\cos(\psi_1) - \omega_2^{(1)}\sin(\psi_1)\big)\cos(kx) - \big(\omega_2^{(1)}\cos(\psi_1) + \omega_1^{(1)}\sin(\psi_1)\big)\sin(kx)\Big]\exp\left(\frac{-Y_{1r}kz}{\varepsilon_1}\right) \tag{4.65a}$$

where

$$\omega_1^{(1)} = \frac{(Y_{1r}^2 - Y_{1i}^2)\Psi_i + 2Y_{1r}Y_{1i}\Psi_r}{\varepsilon_1} - \varepsilon_1\Psi_i$$
$$\omega_2^{(1)} = \frac{(Y_{1r}^2 - Y_{1i}^2)\Psi_r - 2Y_{1r}Y_{1i}\Psi_i}{\varepsilon_1} - \varepsilon_1\Psi_r$$

and in water as

$$\omega^{(2)} = -Fka\Big[\big(\omega_1^{(2)}\cos(\psi_2) + \omega_2^{(2)}\sin(\psi_2)\big)\cos(kx) - \big(\omega_2^{(2)}\cos(\psi_2) - \omega_1^{(2)}\sin(\psi_2)\big)\sin(kx)\Big]\exp\left(\frac{Y_{2r}kz}{R\varepsilon_1}\right) \tag{4.65b}$$

where

$$\omega_1^{(2)} = \frac{(Y_{2r}^2 - Y_{2i}^2)\Phi_i + 2Y_{2r}Y_{2i}\Phi_r}{R^2\varepsilon_1} - \varepsilon_1\Phi_i$$
$$\omega_2^{(2)} = \frac{(Y_{2r}^2 - Y_{2i}^2)\Phi_r - 2Y_{2r}Y_{2i}\Phi_i}{R^2\varepsilon_1} - \varepsilon_1\Phi_r$$

At the surface ($z = 0$), the pressure in the air for a wave moving to the right is initially

$$p_{\text{surface}}^{(1)} = (\rho_1 + \rho_2)\frac{F^2 a}{k}\Big[\frac{r}{1+r}\big(P_{pr}\cos(kx) + P_{pi}\sin(kx)\big)\Big] \tag{4.66a}$$

and the normal stress is

$$-2\mu_1\frac{\partial\hat{w}}{\partial z}^{(1)}(0) = (\rho_1 + \rho_2)\frac{F^2 a}{k}\mathcal{N}^{(1)} \tag{4.66b}$$

where

$$\mathcal{N}^{(1)} = -\frac{2r\varepsilon_1^2}{(1+r)}\big[(Y_{1r}\Psi_r - Y_{1i}\Psi_i - X_r - \varepsilon_1\Psi_r)\cos(kx) + (Y_{1i}\Psi_r + Y_{1r}\Psi_i - X_i - \varepsilon_1\Psi_i)\sin(kx)\big] \tag{4.66c}$$

Similarly, the surface pressure in the water is

$$p_{\text{surface}}^{(2)} = -(\rho_1 + \rho_2)\frac{F^2 a}{k}\Big[\frac{1}{1+r}\big(Q_{pr}\cos(kx) + Q_{pi}\sin(kx)\big)\Big] \tag{4.67a}$$

and the normal stress is

$$-2\mu_2\frac{\partial\hat{w}}{\partial z}^{(2)}(0) = (\rho_1 + \rho_2)\frac{F^2 a}{k}\mathcal{N}^{(2)} \tag{4.67b}$$

where

$$\mathcal{N}^{(2)} = -\frac{2R\varepsilon_1^2}{(1+r)}\Big[\big(-Y_{2r}\Phi_r + Y_{2i}\Phi_i + R(X_r + \varepsilon_1\Phi_r)\big)\cos(kx) + \big(-Y_{2i}\Phi_r - Y_{2r}\Phi_i + R(X_i + \varepsilon_1\Phi_i)\big)\sin(kx)\Big] \quad (4.67c)$$

The tangential stress at the surface is given by

$$\mu_1\left(\frac{\partial\hat{u}}{\partial z}^{(1)} + \frac{\partial\hat{w}}{\partial x}^{(1)}\right) = (\rho_1+\rho_2)\frac{F^2 a}{k}\,\mathcal{T}^{(1)} \quad (4.68a)$$

where

$$\mathcal{T}^{(1)} = \frac{r\varepsilon_1^2}{(1+r)}\Big\{\Big[2X_i + \varepsilon_1\Psi_i - \frac{1}{\varepsilon_1}\big(Y_{1r}^2 - Y_{1i}^2\big)\Psi_i - \frac{2}{\varepsilon_1}\,Y_{1r}Y_{1i}\Psi_r\Big]\cos(kx) + \Big[-2X_r - \varepsilon_1\Psi_r + \frac{1}{\varepsilon_1}\big(Y_{1r}^2 - Y_{1i}^2\big)\Psi_r - \frac{2}{\varepsilon_1}Y_{1r}Y_{1i}\Psi_i\Big]\sin(kx)\Big\} \quad (4.68b)$$

In water,

$$\mu_1\left(\frac{\partial\hat{u}}{\partial z}^{(2)} + \frac{\partial\hat{w}}{\partial x}^{(2)}\right) = (\rho_1+\rho_2)\frac{F^2 a}{k}\,\mathcal{T}^{(2)} \quad (4.69a)$$

where

$$\mathcal{T}^{(2)} = \frac{R^2\varepsilon_1^2}{(1+r)}\Big\{\Big[2X_i + \varepsilon_1\Phi_i - \frac{1}{R^2\varepsilon_1}\big(Y_{2r}^2 - Y_{2i}^2\big)\Phi_i - \frac{2}{R^2\varepsilon_1}\,Y_{2r}Y_{2i}\Phi_r\Big]\cos(kx) + \Big[-2X_r - \varepsilon_1\Phi_r + \frac{1}{R^2\varepsilon_1}\big(Y_{2r}^2 - Y_{2i}^2\big)\Phi_r - \frac{2}{R^2\varepsilon_1}Y_{2r}Y_{2i}\Phi_i\Big]\sin(kx)\Big\} \quad (4.69b)$$

4.5 Asymptotic Approximations

As the approximate solutions (4.35) show, it is necessary to expand up to second order in the effects of viscosity to capture both regions of interest. We may proceed formally by setting $\varepsilon_2 = R\varepsilon_1$ and expanding all quantities to second order. The starting point is the dispersion relation (4.33c). Assume

$$X = \mathrm{i} + \hat{X}_1\varepsilon_1 + \hat{X}_2\varepsilon_1^2\dots \quad (4.70)$$

and expand the following quantities:

$$\sqrt{X+\varepsilon_1^2} \approx \sqrt{\mathrm{i}} - \frac{\mathrm{i}\sqrt{\mathrm{i}}}{2}\hat{X}_1\varepsilon_1 + \left(\frac{\sqrt{\mathrm{i}}}{8}\hat{X}_1^2 - \frac{\mathrm{i}\sqrt{\mathrm{i}}}{2}(1+\hat{X}_2)\right)\varepsilon_1^2 \tag{4.71a}$$

$$\sqrt{X+R^2\varepsilon_1^2} \approx \sqrt{\mathrm{i}} - \frac{\mathrm{i}\sqrt{\mathrm{i}}}{2}\hat{X}_1\varepsilon_1 + \left(\frac{\sqrt{\mathrm{i}}}{8}\hat{X}_1^2 - \frac{\mathrm{i}\sqrt{\mathrm{i}}}{2}(R^2+\hat{X}_2)\right)\varepsilon_1^2 \tag{4.71b}$$

$$r\varepsilon_1\sqrt{X+\varepsilon_1^2} + R^2\varepsilon_1^2 \approx \sqrt{\mathrm{i}}\, r\varepsilon_1 + \left(R^2 - \frac{\mathrm{i}\sqrt{\mathrm{i}}}{2}r\hat{X}_1\right)\varepsilon_1^2 \tag{4.71c}$$

$$R\varepsilon_1\sqrt{X+R^2\varepsilon_1^2} + r\varepsilon_1^2 \approx \sqrt{\mathrm{i}}\, R\varepsilon_1 + \left(r - \frac{\mathrm{i}\sqrt{\mathrm{i}}}{2}R\hat{X}_1\right)\varepsilon_1^2 \tag{4.71d}$$

$$\begin{aligned} r\varepsilon_1\left(\sqrt{X+\varepsilon_1^2} + \varepsilon_1\right) + R\varepsilon_1\left(\sqrt{X+R^2\varepsilon_1^2} + R\varepsilon_1\right) \\ \approx \sqrt{\mathrm{i}}(R+r)\varepsilon_1 + \left((R^2+r) - \frac{\mathrm{i}\sqrt{\mathrm{i}}}{2}(R+r)\hat{X}_1\right)\varepsilon_1^2 \end{aligned} \tag{4.71e}$$

$$\begin{aligned} \left(r\varepsilon_1\sqrt{X+\varepsilon_1^2} + R^2\varepsilon_1^2\right)\left(R\varepsilon_1\sqrt{X+R^2\varepsilon_1^2} + r\varepsilon_1^2\right) \\ \approx \mathrm{i}rR\varepsilon_1^2 + rR\hat{X}_1\varepsilon_1^3 + \sqrt{\mathrm{i}}\left(R^3+r^2\right)\varepsilon_1^3 \end{aligned} \tag{4.71f}$$

By substituting these expressions into (4.33c) and balancing powers of ε_1, we obtain expressions for $\hat{X}_1$ and $\hat{X}_2$. The lowest power is ε_1^2 and the terms are

$$2\mathrm{i}\sqrt{\mathrm{i}}\,(1+r)(R+r)\hat{X}_1 - 4rR = 0\,,$$

which leads to

$$\hat{X}_1 = -\sqrt{\mathrm{i}}\frac{2rR}{(1+r)(R+r)}\,. \tag{4.72}$$

At the next order,

$$\begin{aligned} \sqrt{\mathrm{i}}\,(1+r)(R+r)(2\mathrm{i}\hat{X}_2 + \hat{X}_1^2) + 2\mathrm{i}\hat{X}_1(1+r)\left(R^2 + r - \frac{\mathrm{i}\sqrt{\mathrm{i}}}{2}(R+r)\hat{X}_1\right) \\ + 4\mathrm{i}\left(rR\hat{X}_1 + \sqrt{\mathrm{i}}(R^3+r^2)\right) + 4\mathrm{i}rR\hat{X}_1 = 0\,, \end{aligned}$$

which leads to

$$\hat{X}_2 = -2\,\frac{R^4 + rR^4 - 2r^2R^2 + r^3 + r^4}{(1+r)^2(R+r)^2} \tag{4.73}$$

When we apply the assumption $r << 1$ as in the case for air/water, the dominant contibutions give

$$X \approx \mathrm{i} - \sqrt{\mathrm{i}}\,2r\varepsilon_1 - 2R^2\varepsilon_1^2 \tag{4.74}$$

which agrees with the previous results (4.35). The important observation is that the first-order term is proportional to r and is in effect only for very small values of ε_1 when the

ε_1	numerical	asymptotic
0.1	-1.596 $\times 10^{-3}$	-1.639 $\times 10^{-3}$
0.001	-1.833 $\times 10^{-6}$	-1.845 $\times 10^{-6}$

Table 2: The real part of X

second-order term is smaller then the first. Specifically, $\sqrt{2}\, r = 1.697 \times 10^{-3}$ while $2R^2 = 1.469 \times 10^{-1}$. In Table 2, we give two representative results that illustrate how one term may dominate the other.

To proceed, we need the expansions for Φ and Ψ. Since

$$RX_1 + rX_2 \approx \sqrt{\mathrm{i}}\,(R+r)\left[1 - \mathrm{i}\left(\frac{\hat{X}_1}{2} - \sqrt{\mathrm{i}}\,\frac{R(1+r)}{R+r}\right)\varepsilon_1\right]$$

and

$$rX + (R^2 - r)X_1\varepsilon_1 \approx \mathrm{i}r + \left(\sqrt{\mathrm{i}}\,(R^2 - r) + r\hat{X}_1\right)\varepsilon_1$$

the expansion for

$$\Phi = \Phi_0 + \Phi_1\varepsilon_1 \ldots \tag{4.75a}$$

has

$$\Phi_0 = \sqrt{\mathrm{i}}\,\frac{2rR}{R+r}\,, \tag{4.75b}$$

$$\Phi_1 = 2R\,\frac{R^3 + rR^3 + rR^2 + r^2R^2 + r^3R - r^2 - r^3}{(1+r)(R+r)^2} \tag{4.75c}$$

For Ψ we need the expansion,

$$\begin{aligned} 2RX - X_2\Phi &\approx 2\mathrm{i}R - \sqrt{\mathrm{i}}\,\Phi_0 + \left(2R\hat{X}_1 - \sqrt{\mathrm{i}}\,\Phi_1 + R\Phi_0 + \frac{\mathrm{i}\sqrt{\mathrm{i}}}{2}\hat{X}_1\Phi_0\right)\varepsilon_1 \\ &\approx \mathrm{i}\frac{2R^2}{R+r} - 2\sqrt{\mathrm{i}}\,R\frac{R^3 + 2rR^2 + rR^3 - r^2 - r^3}{(1+r)(R+r)^2}\,\varepsilon_1\,. \end{aligned}$$

Then the expansion for

$$\Psi = \Psi_0 + \Psi_1\varepsilon_1 \ldots \tag{4.76a}$$

has

$$\Psi_0 = \sqrt{\mathrm{i}}\,\frac{2R}{R+r} \tag{4.76b}$$

$$\Psi_1 = 2\,\frac{R^2 - R^3 - rR^3 + rR + r^2R + r^2 + r^3}{(1+r)(R+r)^2} \tag{4.76c}$$

The horizontal velocity profiles require expansions for

$$P_u \approx \mathrm{i}\frac{2R}{R+r}\exp\left(-\frac{\sqrt{\mathrm{i}}\,kz}{\varepsilon_1}\right) - \mathrm{i}\exp(-kz)\,, \tag{4.77a}$$

$$Q_u \approx \left[\mathrm{i}\frac{2r}{R+r} + \left(\frac{\Phi_1}{R} - \frac{2r^2R}{(1+r)(R+r)^2}\right)\sqrt{\mathrm{i}}\,\varepsilon_1\right]\exp\left(\frac{\sqrt{\mathrm{i}}\,kz}{R\varepsilon_1}\right) - \mathrm{i}\exp(kz)\,, \tag{4.77b}$$

where we must include the first two terms for Q_u since the first term is proportional to r. Assuming $r << 1$ and $r << R$, we obtain the initial profiles for waves travelling to the right,

$$\hat{u}^{(1)} = 2Fa\Big[\cos\Big(\frac{kz}{\sqrt{2}\,\varepsilon_1}\Big)\cos(kx) - \sin\Big(\frac{kz}{\sqrt{2}\,\varepsilon_1}\Big)\sin(kx)\Big]\exp\Big(-\frac{kz}{\sqrt{2}\,\varepsilon_1}\Big) - Fa\cos(kx)\exp(-kz)\,, \quad (4.78a)$$

in air and

$$\hat{u}^{(2)} = -Fa\Bigg\{\Big[\Big(\frac{2r}{R} + \sqrt{2}\,R\varepsilon_1\Big)\cos\Big(\frac{kz}{\sqrt{2}\,R\varepsilon_1}\Big) + \sqrt{2}\,R\varepsilon_1\sin\Big(\frac{kz}{\sqrt{2}\,R\varepsilon_1}\Big)\Big]\cos(kx) - \Big[\sqrt{2}\,R\varepsilon_1\cos\Big(\frac{kz}{\sqrt{2}\,R\varepsilon_1}\Big) - \Big(\frac{2r}{R} + \sqrt{2}\,R\varepsilon_1\Big)\sin\Big(\frac{kz}{\sqrt{2}\,R\varepsilon_1}\Big)\Big]\sin(kx)\Bigg\}\exp\Big(\frac{kz}{\sqrt{2}\,R\varepsilon_1}\Big) + Fa\cos(kx)\exp(kz)\,. \quad (4.78b)$$

in water. It is interesting to compare these results with those from inviscid theory:

$$\hat{u}^{(1)} = -Fa\exp(-kz)\cos(kx)\,,$$
$$\hat{u}^{(2)} = Fa\exp(kz)\cos(kx)\,.$$

There is a correction in the boundary layer in the air that is comparable in magnitude to the far-field flow. This is not the case for the vertical velocity which is the same as the inviscid theory.

$$\hat{w}^{(1)} = Fa\exp(-kz)\sin(kx) \quad (4.79a)$$
$$\hat{w}^{(2)} = Fa\exp(kz)\sin(kx) \quad (4.79b)$$

The easiest way to identify the boundary layer contribution is through the vorticity. By using (4.78) and (4.79), the vorticity has no leading order contribution from the far-field flow and only has non-zero values in the boundary layer. In air,

$$\omega^{(1)} = -\sqrt{2}\,\frac{Fak}{\varepsilon_1}\exp\Big(-\frac{kz}{\sqrt{2}\,\varepsilon_1}\Big)\Bigg\{\Big[\cos\Big(\frac{kz}{\sqrt{2}\,\varepsilon_1}\Big) + \sin\Big(\frac{kz}{\sqrt{2}\,\varepsilon_1}\Big)\Big]\cos(kx) + \Big[\cos\Big(\frac{kz}{\sqrt{2}\,\varepsilon_1}\Big) - \sin\Big(\frac{kz}{\sqrt{2}\,\varepsilon_1}\Big)\Big]\sin(kx)\Bigg\}\,. \quad (4.80a)$$

In water,

$$\omega^{(2)} = -Fak\exp\Big(\frac{kz}{\sqrt{2}\,R\varepsilon_1}\Big)\Bigg\{\Big[\Big(\frac{\sqrt{2}\,r}{R^2\varepsilon_1} + 2\Big)\cos\Big(\frac{kz}{\sqrt{2}\,R\varepsilon_1}\Big) - \frac{\sqrt{2}\,r}{R^2\varepsilon_1}\sin\Big(\frac{kz}{\sqrt{2}\,R\varepsilon_1}\Big)\Big]\cos(kx) + \Big[\frac{\sqrt{2}\,r}{R^2\varepsilon_1}\cos\Big(\frac{kz}{\sqrt{2}\,R\varepsilon_1}\Big) + \Big(\frac{\sqrt{2}\,r}{R^2\varepsilon_1} + 2\Big)\sin\Big(\frac{kz}{\sqrt{2}\,R\varepsilon_1}\Big)\Big]\sin(kx)\Bigg\}\,. \quad (4.80b)$$

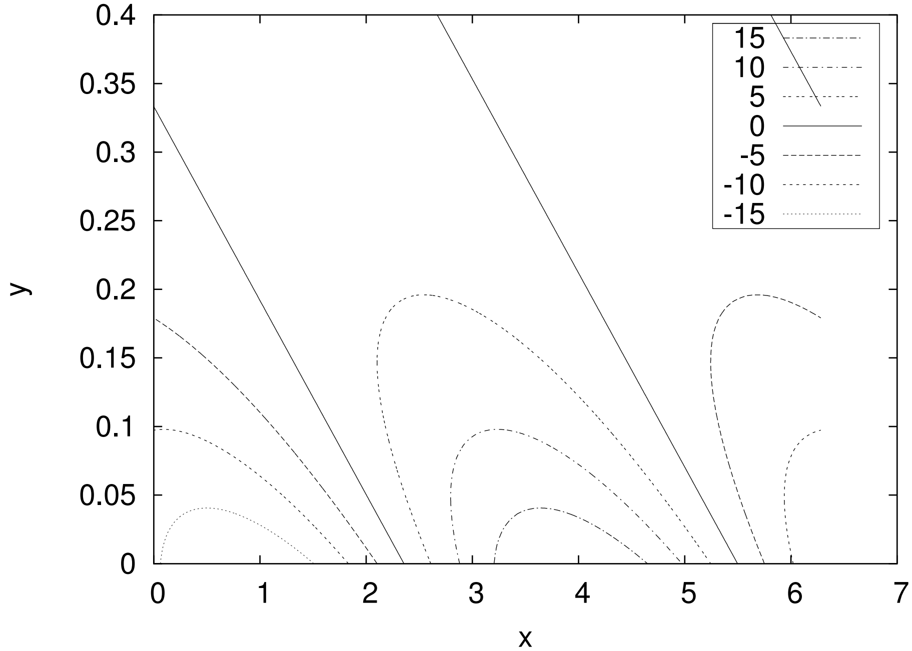

Figure 6: Contour lines of the vorticity in air when $\varepsilon_1 = 0.1$: asymptotic theory.

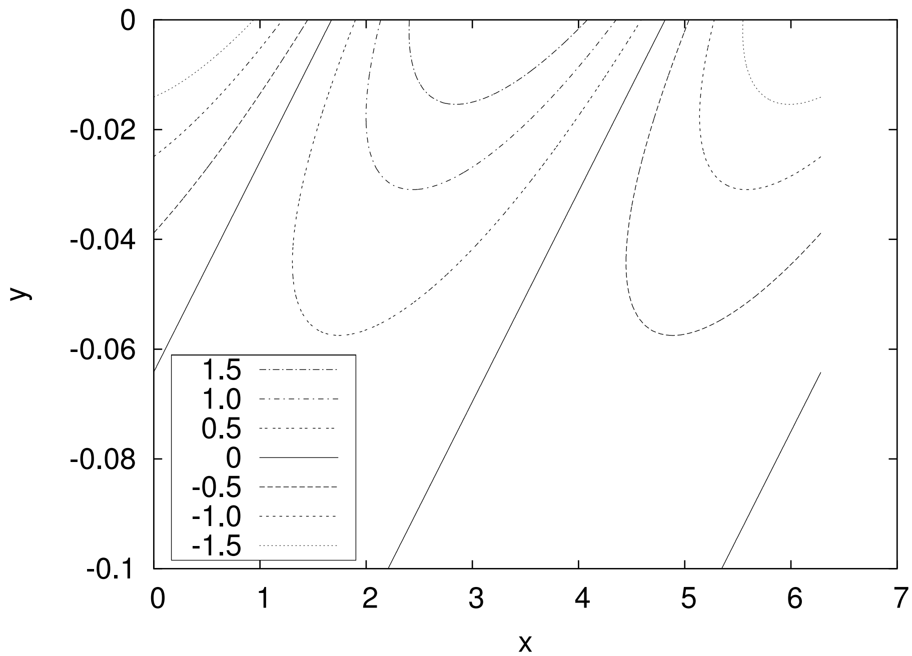

Figure 7: Contour lines of the vorticity in water when $\varepsilon_1 = 0.1$: asymptotic theory.

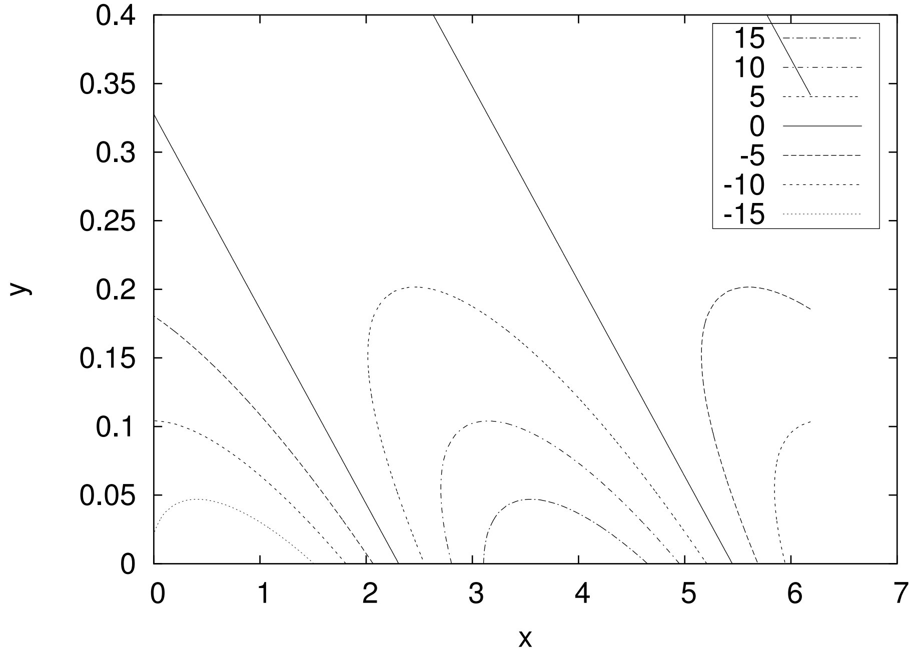

Figure 8: Contour lines of the vorticity in air when $\varepsilon_1 = 0.1$: exact results .

In Figures 6 and 7, we show the asymptotic predictions for the contour lines of the vorticity in air and water for the case of air and water with $\varepsilon_1 = 0.1$. The vorticity contour lines in water are essentially those of (4.80b) when $r = 0$. The exact contour lines are given in Figures 8 and 9.

The close agreement between asymptotic contour lines and the exact contour lines is also true when $\varepsilon_1 = 0.001$. Except for scale, the pattern of the vorticity contour lines is very similar to those in Figures 6,7,8 and 9. The only difference is that $\sqrt{2}\,r/(R^2\varepsilon_1) = 23$ which is much larger than the other term. The implication is that the presence of the air is now dominating the creation of vorticity at the interface. Consequently, it is worth studying the balance of forces at the interface.

To obtain the pressure, we expand $P_p = X(X + \varepsilon_1\Psi)$ and $Q_p = X(X + \varepsilon\Phi)$ to obtain

$$
\begin{aligned}
P_p &\approx -1 + \left(2\hat{X}_1 + \Psi_0\right) \mathrm{i}\varepsilon_1 + \left(2\hat{X}_2\mathrm{i} + \hat{X}_1^2 + \hat{X}_1\Psi_0 + \Psi_1\mathrm{i}\right)\varepsilon_1^2 \dots \\
&\approx -1 + \frac{2R(1-r)}{(1+r)(R+r)}\,\mathrm{i}\sqrt{\mathrm{i}}\,\varepsilon_1 + \frac{N_p}{(1+r)^2(R+r)^2}\mathrm{i}\varepsilon_1^2 \qquad (4.81a)
\end{aligned}
$$

$$
\begin{aligned}
Q_p &\approx -1 + \left(2\hat{X}_1 + \Phi_0\right) \mathrm{i}\varepsilon_1 + \left(2\hat{X}_2\mathrm{i} + \hat{X}_1^2 + \hat{X}_1\Phi_0 + \Phi_1\mathrm{i}\right)\varepsilon_1^2 \dots \\
&\approx -1 - \frac{2rR(1-r)}{(1+r)(R+r)}\mathrm{i}\sqrt{\mathrm{i}}\,\varepsilon_1 + \frac{N_q}{(1+r)^2(R+r)^2}\,\mathrm{i}\varepsilon_1^2 \qquad (4.81b)
\end{aligned}
$$

where

$$
N_p = 2R^2 - 2R^3 - 4R^4 + 2rR - 2rR^2 - 4rR^3 - 4rR^4 + 2r^2
$$

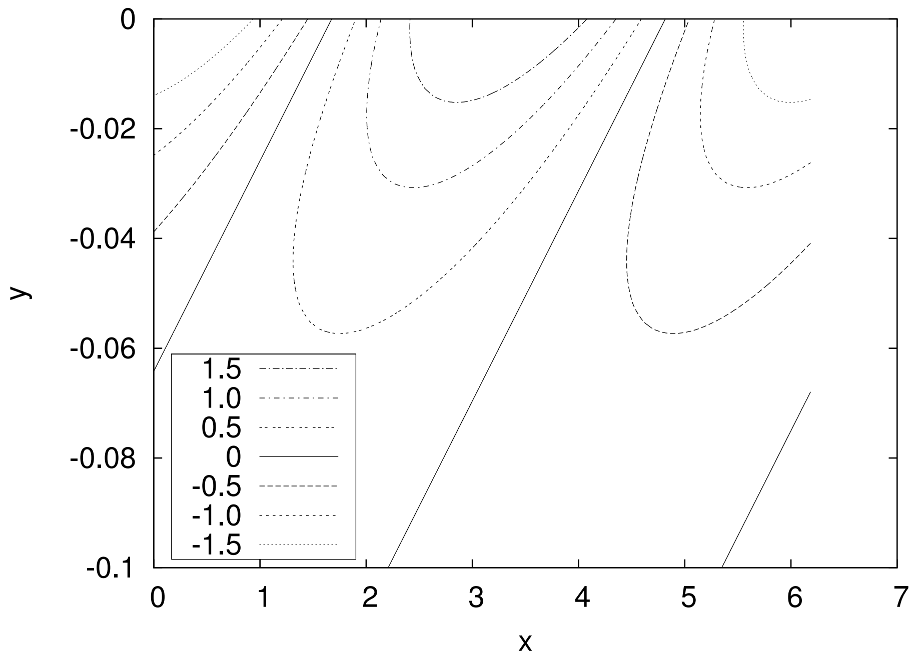

Figure 9: Contour lines of the vorticity in water when $\varepsilon_1 = 0.1$: exact results .

$$
\begin{aligned}
&+ 4r^2R + 8r^2R^2 - 2r^2R^3 + 2r^3R - 2r^4 \\
N_q = &-2R^4 + 2rR^3 - 2r^2R + 8r^2R^2 + 4r^2R^3 + 2r^2R^4 \\
&- 4r^3 - 4r^3R - 2r^3R^2 + 2r^3R^3 - 4r^4 - 2r^4R + 2r^4R^2
\end{aligned}
$$

These results allow the pressure at the surface to be determined. From (4.66a) and (4.67a),

$$
\begin{aligned}
p^{(1)}_{\text{surface}} \approx \frac{r}{1+r}\Bigg[&\left(-1 - \frac{\sqrt{2}\,R(1-r)}{(1+r)(R+r)}\varepsilon_1\right)\cos(kx) \\
&+ \left(\frac{\sqrt{2}\,R(1-r)}{(1+r)(R+r)}\varepsilon_1 + \frac{N_p}{(1+r)^2(R+r)^2}\varepsilon_1^2\right)\sin(kx)\Bigg]
\end{aligned} \tag{4.82a}
$$

$$
\begin{aligned}
p^{(2)}_{\text{surface}} \approx -\frac{1}{1+r}\Bigg[&\left(-1 + \frac{\sqrt{2}\,rR(1-r)}{(1+r)(R+r)}\varepsilon_1\right)\cos(kx) \\
&+ \left(-\frac{\sqrt{2}\,rR(1-r)}{(1+r)(R+r)}\varepsilon_1 + \frac{N_q}{(1+r)^2(R+r)^2}\varepsilon_1^2\right)\sin(kx)\Bigg]
\end{aligned} \tag{4.82b}
$$

The normal stresses at the surface only start with second-order contributions. From (4.66c) and (4.67c),

$$
\mathcal{N}^{(1)} \approx -\frac{2r(R-r)}{(1+r)(R+r)}\sin(kx)\varepsilon_1^2 \tag{4.83a}
$$

$$\mathcal{N}^{(2)} \approx -\frac{2R^2(R-r)}{(1+r)(R+r)} \sin(kx)\varepsilon_1^2 \tag{4.83b}$$

Under the assumption $r << 1$ and $r << R$, we keep only the dominant contributions,

$$p^{(1)}_{\text{surface}} \approx -r\big(1+\sqrt{2}\,\varepsilon_1\big)\cos(kx) + \sqrt{2}\,r\varepsilon_1 \sin(kx) \tag{4.84a}$$

$$p^{(2)}_{\text{surface}} \approx \big(1-r-\sqrt{2}\,r\varepsilon_1\big)\cos(kx) + \big(\sqrt{2}\,r\varepsilon_1 + 2R^2\varepsilon_1^2\big)\sin(kx) \tag{4.84b}$$

$$\mathcal{N}^{(1)} \approx -2r\varepsilon_1^2 \sin(kx) \tag{4.84c}$$

$$\mathcal{N}^{(2)} \approx -2R^2\varepsilon_1^2 \sin(kx) \tag{4.84d}$$

For the tangential stresses at the surface, there are quantities of only first order. Thus we need the expansions to first order of several expressions. The quantities of interest are:

$$Y_{1r}^2 - Y_{1i}^2 \approx -\frac{\sqrt{2}\,rR}{(1+r)(R+r)}\,\varepsilon_1\,,$$

$$2Y_{1r}Y_{1i} \approx 1 - \frac{\sqrt{2}\,rR}{(1+r)(R+r)}\,\varepsilon_1\,,$$

$$Y_{2r}^2 - Y_{2i}^2 \approx -\frac{\sqrt{2}\,rR}{(1+r)(R+r)}\,\varepsilon_1\,,$$

$$2Y_{2r}Y_{2i} \approx 1 - \frac{\sqrt{2}\,rR}{(1+r)(R+r)}\,\varepsilon_1\,,$$

which lead to

$$\mathcal{T}^{(1)} \approx \frac{r\varepsilon_1}{1+r}\left\{\left[-\frac{\sqrt{2}\,R}{R+r} + \left(2 + \frac{4rR^2}{(1+r)(R+r)^2} - \Psi_1\right)\varepsilon_1\right]\cos(kx) - \frac{\sqrt{2}\,R}{R+r}\sin(kx)\right\} \tag{4.85a}$$

$$\mathcal{T}^{(2)} \approx \frac{\varepsilon_1}{1+r}\left\{\left[-\frac{\sqrt{2}\,rR}{R+r} + \left(2R^2 + \frac{4r^2R^2}{(1+r)(R+r)^2} - \Phi_1\right)\varepsilon_1\right]\cos(kx) - \frac{\sqrt{2}\,rR}{R+r}\sin(kx)\right\} \tag{4.85b}$$

Under the assumption $r << 1$ and R, the tangential stresses become

$$\mathcal{T}^{(1)} \approx -\sqrt{2}\,r\varepsilon_1\big(\cos(kx)+\sin(kx)\big) \tag{4.86a}$$

$$\mathcal{T}^{(2)} \approx -\sqrt{2}\,r\varepsilon_1\big(\cos(kx)+\sin(kx)\big) \tag{4.86b}$$

As a consequence of the choice of dimensionless variables, the balance of normal stresses at the interface must satisfy

$$p^{(2)}_{\text{surface}} + \mathcal{N}^{(2)} - p^{(1)}_{\text{surface}} - \mathcal{N}^{(1)} = \cos(kx)\,. \tag{4.87}$$

Since inviscid theory predicts

$$p^{(1)}_{\text{surface}} = -\frac{r}{1+r}\cos(kx)\,, \tag{4.88a}$$

$$p^{(2)}_{\text{surface}} = \frac{1}{1+r}\cos(kx)\,, \tag{4.88b}$$

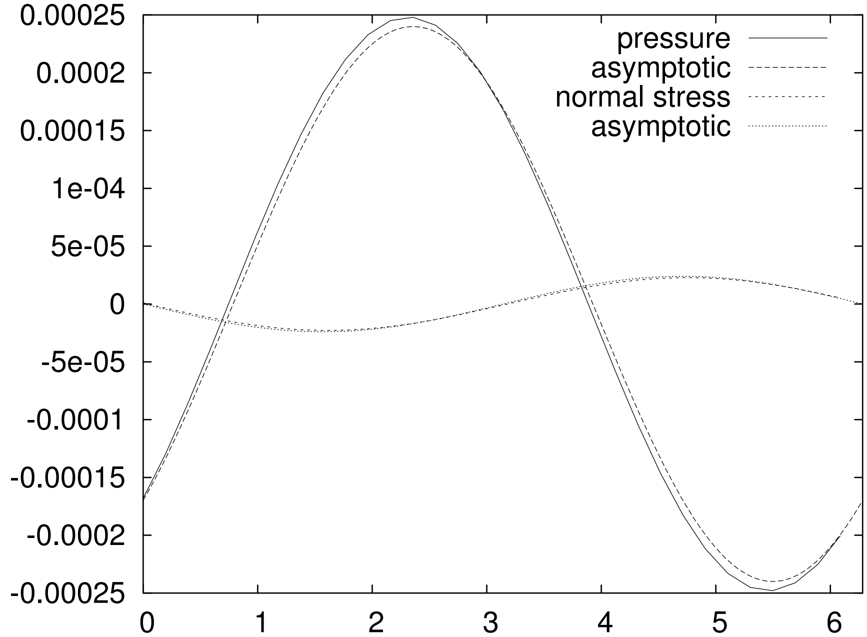

Figure 10: Profiles of the modified pressure and viscous contribution to the normal stresses in air: $\varepsilon_1 = 0.1$.

the modified pressures

$$\tilde{p}^{(1)}_{\text{surface}} = p^{(1)}_{\text{surface}} + \frac{r}{1+r}\,\cos(kx)\,, \tag{4.89a}$$

$$\tilde{p}^{(2)}_{\text{surface}} = p^{(2)}_{\text{surface}} - \frac{1}{1+r}\,\cos(kx)\,, \tag{4.89b}$$

will show the influence of viscous effects.

In Figure 10 and 11, we show the modified pressure and the viscous contribution to the normal stress along the interface for the air/water case in air and water respectively. The choice $\varepsilon_1 = 0.1$ corresponds to Figures 6,7,8 and 9. What is clear from these profiles is that the pressure and viscous contributions to the normal stress are in balance in the water and that there is little influence from the air. Furthermore, by looking at the profiles of the tangential stress shown in Figure 12, we see that the tangential stresses are much smaller than the normal stress. We show the tangential stress only in water since it is exactly the same as that in air.

The situation is different for $\varepsilon_1 = 0.001$. In Figures 13 and 14, the profiles of the modified surface pressure are in balance and the viscous contributions to the normal stress are insignificant. The tangential stress is shown in Figure 15 and its magnitude is comparable to the pressure. The magnitude of all the forces is considerably smaller than those when $\varepsilon_1 = 0.1$.

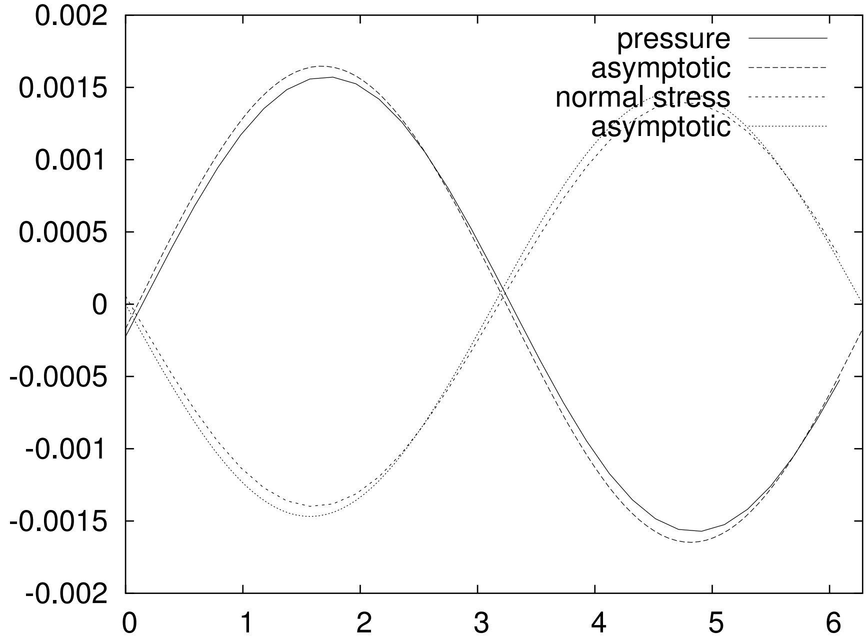

Figure 11: Profiles of the modified pressure and viscous contribution to the normal stresses in water: $\varepsilon_1 = 0.1$.

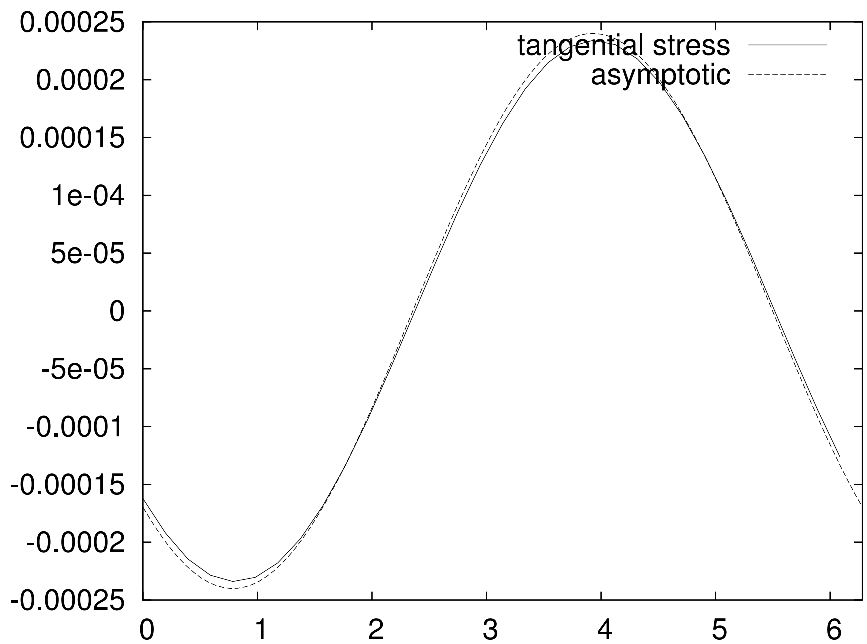

Figure 12: Profiles of the tangential stress in water: $\varepsilon_1 = 0.1$.

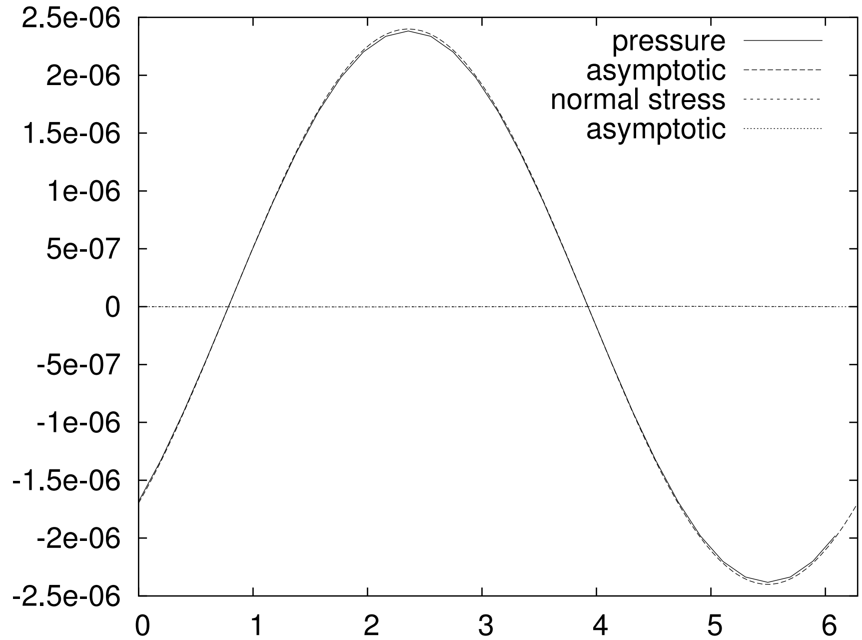

Figure 13: Profiles of the modified pressure and viscous contribution to the normal stresses in air: $\varepsilon_1 = 0.001$.

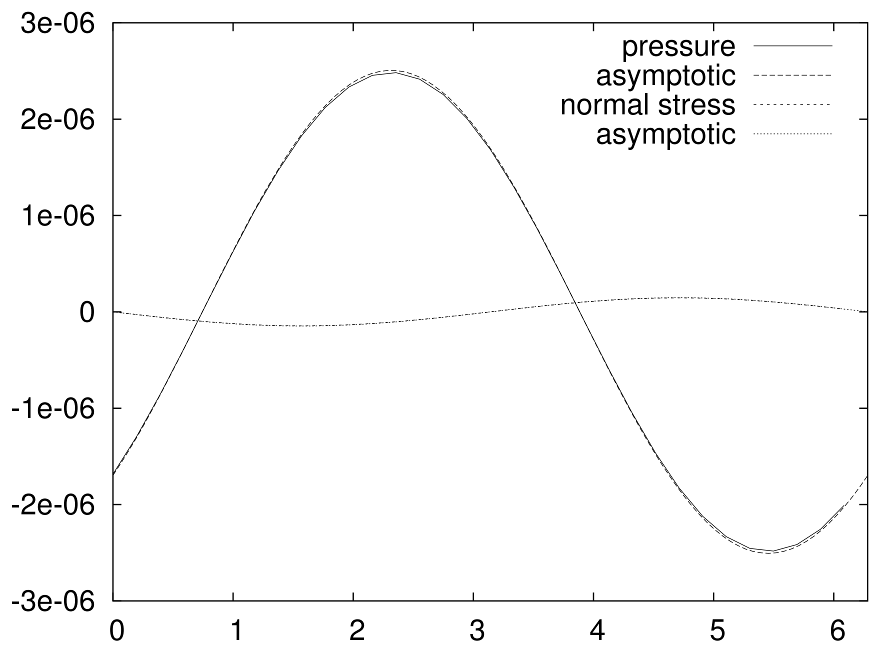

Figure 14: Profiles of the modified pressure and viscous contribution to the normal stresses in water: $\varepsilon_1 = 0.001$.

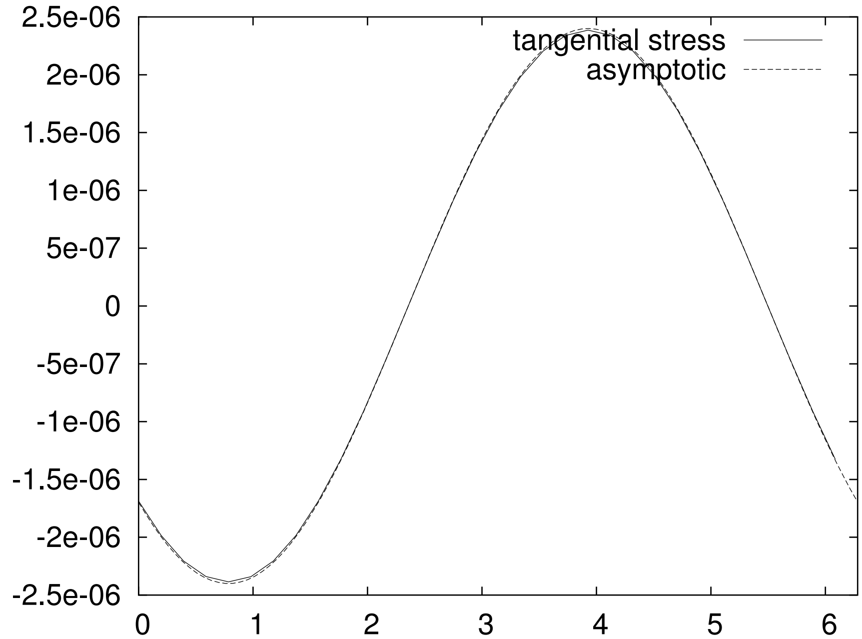

Figure 15: Profiles of the tangential stress in water: $\varepsilon_1 = 0.001$.

A Harrison's Form

Our result, (4.14), is equivalent to Harrison's result, given in Chandrashka's book on p.443 as

$$-\left\{\frac{gk}{\sigma^2}\left[(\alpha_2-\alpha_1)+\frac{k^2T}{g(\rho_1+\rho_2)}\right]+1\right\}(\alpha_1 q_2+\alpha_2 q_1-k)-4k\alpha_1\alpha_2 + \frac{4k^2}{\sigma}(\alpha_2\nu_2-\alpha_1\nu_1)\{(\alpha_1 q_2-\alpha_2 q_1)+k(\alpha_2-\alpha_1)\} + \frac{4k^2}{\sigma^2}(\alpha_2\nu_2-\alpha_1\nu_1)^2(q_2-k)(q_1-k)=0 \quad \text{(A.1)}$$

where

$$\alpha_1 = \frac{\rho_1}{\rho_1+\rho_2} \quad \text{(A.2a)}$$

$$\alpha_2 = \frac{\rho_2}{\rho_1+\rho_2} \quad \text{(A.2b)}$$

$$q_1 = \frac{\Omega_1}{\sqrt{\nu_1}} \quad \text{(A.2c)}$$

$$q_2 = \frac{\Omega_2}{\sqrt{\nu_2}} \quad \text{(A.2d)}$$

First note that

$$\left\{\frac{gk}{\sigma^2}\left[(\alpha_2-\alpha_1)+\frac{k^2T}{g\,(\rho_1+\rho_2)}\right]+1\right\}=\frac{1}{\sigma^2(\rho_1+\rho_2)}\left\{\sigma^2(\rho_1+\rho_2)+(\rho_2-\rho_1)gk+Tk^3\right\} \tag{A.3a}$$

and

$$\alpha_1 q_2+\alpha_2 q_1-k=\frac{1}{\sqrt{\nu_1}\,\sqrt{\nu_2}\,(\rho_1+\rho_2)}\left\{\rho_1\sqrt{\nu_1}\,(\Omega_2-\sqrt{\nu_2}\,k)+\rho_2\sqrt{\nu_2}\,(\Omega_1-\sqrt{\nu_1}\,k)\right\} \tag{A.3b}$$

and

$$\begin{aligned}
&-4k\alpha_1\alpha_2+\frac{4k^2}{\sigma}\left(\alpha_2\nu_2-\alpha_1\nu_1\right)\left\{(\alpha_1 q_2-\alpha_2 q_1)+k\,(\alpha_2-\alpha_1)\right\}\\
&\qquad\qquad+\frac{4k^2}{\sigma}\left(\alpha_2\nu_2-\alpha_1\nu_1\right)^2(q_2-k)\,(q_1-k)=\\
&-\frac{4\rho_1\rho_2 k}{(\rho_1+\rho_2)^2}+\frac{4k^2}{\sigma}\,\frac{\rho_2\nu_2-\rho_1\nu_1}{\sqrt{\nu_1}\,\sqrt{\nu_2}\,(\rho_1+\rho_2)^2}\left\{\rho_1\sqrt{\nu_1}\,(\Omega_2-\sqrt{\nu_2}\,k)-\rho_2\sqrt{\nu_2}\,(\Omega_1-\sqrt{\nu_1}\,k)\right\}\\
&\qquad\qquad+\frac{4k^3}{\sigma^2}\frac{(\rho_2\nu_2-\rho_1\nu_2)^2}{\sqrt{\nu_1}\,\sqrt{\nu_2}\,(\rho_1+\rho_2)^2}\,(\Omega_1-\sqrt{\nu_1}\,k)\,(\Omega_2-\sqrt{\nu_2}\,k)
\end{aligned} \tag{A.3c}$$

By multiplying (A.1) with $-\sqrt{\nu_1}\,\sqrt{\nu_2}\,(\rho_1+\rho_2)^2$, we may write Harrison's result in the form $AF+B=0$ where

$$F=\sigma^2(\rho_1+\rho_2)+(\rho_2-\rho_1)gk+Tk^3 \tag{A.4a}$$

$$A=\frac{1}{\sigma^2}\left\{\rho_1\sqrt{\nu_1}\,(\Omega_2-\sqrt{\nu_2}\,k)+\rho_2\sqrt{\nu_2}\,(\Omega_1-\sqrt{\nu_1}\,k)\right\} \tag{A.4b}$$

$$\begin{aligned}
B&=4\sqrt{\nu_1}\,\sqrt{\nu_2}\,\rho_1\rho_2 k-\frac{4k^2}{\sigma}(\rho_2\nu_2-\rho_1\nu_1)\left\{\rho_1\sqrt{\nu_1}\,(\Omega_2-\sqrt{\nu_2}\,k)-\rho_2\sqrt{\nu_2}\,(\Omega_1-\sqrt{\nu_1}\,k)\right\}\\
&\quad-\frac{4k^3}{\sigma^2}(\rho_2\nu_2-\rho_1\nu_2)^2\,(\Omega_1-\sqrt{\nu_1}\,k)\,(\Omega_2-\sqrt{\nu_2}\,k)\\
&=-4k\left[\frac{k}{\sigma}(\rho_2\nu_2-\rho_1\nu_1)(\Omega_1-\sqrt{\nu_1}\,k)+\rho_1\sqrt{\nu_1}\right]\left[\frac{k}{\sigma}(\rho_2\nu_2-\rho_1\nu_1)(\Omega_2-\sqrt{\nu_2}\,k)-\rho_2\sqrt{\nu_2}\right]
\end{aligned} \tag{A.4c}$$

We now use the result,

$$(\Omega_1-\sqrt{\nu_1}\,k)\,(\Omega_1+\sqrt{\nu_1}\,k)=\sigma \tag{A.5}$$

by multiplying A and B by $\sigma(\Omega_1+\sqrt{\nu_1}\,k)\,(\Omega_2+\sqrt{\nu_2}\,k)$ to obtain

$$\begin{aligned}
C&=\sigma(\Omega_1+\sqrt{\nu_1}\,k)\,(\Omega_2+\sqrt{\nu_2}\,k)\,A\\
&=\rho_1\sqrt{\nu_1}\,(\Omega_1+\sqrt{\nu_1}\,k)+\rho_2\sqrt{\nu_2}\,(\Omega_2+\sqrt{\nu_2}\,k)
\end{aligned} \tag{A.6a}$$

$$\begin{aligned}
D&=\sigma(\Omega_1+\sqrt{\nu_1}\,k)\,(\Omega_2+\sqrt{\nu_2}\,k)\,B\\
&=4k\sigma(\rho_1\sqrt{\nu_1}\,\Omega_1+\rho_2\nu_2 k)\,(\rho_2\sqrt{\nu_2}\,\Omega_2+\rho_1\nu_1 k)
\end{aligned} \tag{A.6b}$$

The final result, $CF+D=0$, agrees with (4.14).