Basic Equations

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1 Basic Equations

Denote the spatial coordinates by x_i; the velocity components by u_i; the density by ρ and the pressure by p. The continuity equation is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0 \tag{1.1a}$$

where the Einstein summation convention is used for repeated indices. The conservation of momentum gives

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(p\delta_{ij} + \rho u_i u_j - \sigma_{ij}) = 0 \tag{1.1b}$$

where the viscosity stress tensor is given by

$$\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\,\delta_{ij}\,\frac{\partial u_k}{\partial x_k}\right) + \eta\,\delta_{ij}\,\frac{\partial u_k}{\partial x_k} \tag{1.1c}$$

The viscosity parameters, μ and η are material properties of the fluid.

For the flow of air over water, the densities of the air and water may be assumed constants. Then, (1.1a) implies the condition of incompressibility,

$$\frac{\partial u_j}{\partial x_j} = 0 \tag{1.2a}$$

and the viscous stress tensor (1.1c) takes a simpler form,

$$\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.2b}$$

Consequently, (1.1b) may be written as

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{1.2c}$$

2 Interfacial conditions

Suppose there is a sharp interface between two fluids given by $X_j(\xi, \eta, t)$. Appendix A contains a derivation of the conditions that must hold across the interface for a general conservation law. This condition (A.7) must now be applied to the two conservation laws (1.1).

For (1.1a),

$$n_j \left(\rho_1 u_j^{(1)} - \rho_2 u_j^{(2)} \right) = n_j \frac{\partial X_j}{\partial t} (\rho_1 - \rho_2) \tag{2.1}$$

Here the normal n_j points from region 2 to region 1. The superscripts indicate the limiting values of quantities at the interface. From kinematics considerations, we require the normal component of the velcoity to be continuous across the interface (otherwise the interface would lose its definition of sharpness).

$$n_j \, u_j^{(1)} = n_j \, u_j^{(2)} = n_j \frac{\partial X_j}{\partial t} \tag{2.2}$$

where the last equality follows by substituting the first equality into (2.1).

For (1.1b), we first add a surface force, the surface tension T.

$$\frac{\partial}{\partial t} (\rho u_i) + \frac{\partial}{\partial x_j} (p\delta_{ij} + \rho u_i u_j - \sigma_{ij}) = n_i 2T\kappa\delta(n) \tag{2.3}$$

where κ is the mean curvature, T the surface tension coefficient, and n is the normal distance from the interface. Application of (A.7) gives

$$n_j \left(p^{(1)}\delta_{ij} - p^{(2)}\delta_{ij} - \sigma_{ij}^{(1)} + \sigma_{ij}^{(2)} \right) = n_i 2T\kappa \tag{2.4}$$

where we used the fact that

$$n_j \left(\rho_1 u_i^{(1)} u_j^{(1)} - \rho_2 u_i^{(2)} u_j^{(2)} \right) = n_j \frac{\partial X_j}{\partial t} \left(\rho_1 u_i^{(1)} - \rho_2 u_i^{(2)} \right) \tag{2.5}$$

(see (2.2)).

To complete the specification of conditions at the interface, we add the continuity of the tangential speeds on physical grounds. Thus,

$$u_j^{(1)} = u_j^{(2)} \equiv u_j^{(I)} \tag{2.6}$$

Notice that the inclusion of gravity effects, a body force, does not change the jump conditions at the interface (see Appendix A).

3 Basic formulation

Here we express the equations of motion and the interfacial conditions in a specific form by choosing $\mathbf{x} = (x, y, z)$ and $\mathbf{u} = (u, v, w)$. The equations of motion are:

$$\rho u_t + \rho u u_x + \rho v u_y + \rho w u_z = -p_x + \mu \left(u_{xx} + u_{yy} + u_{zz} \right) \tag{3.1a}$$

$$\rho v_t + \rho u v_x + \rho v v_y + \rho w v_z = -p_y + \mu \left(v_{xx} + v_{yy} + v_{zz} \right) \tag{3.1b}$$

$$\rho w_t + \rho u w_x + \rho v w_y + \rho w w_z = -p_z + \mu \left(w_{xx} + w_{yy} + w_{zz} \right) - \rho g \tag{3.1c}$$

with the incompressibility condition,

$$u_x + v_y + w_z = 0 \tag{3.1d}$$

Here we use subscripts to refer to partial differentiation with respect to the subcript variable.

Next, we represent the surface in the form,

$$\mathbf{X} = ((x, y, h(x, y, t)) \tag{3.2a}$$

which means we have made the choice $\xi = x$ and $\eta = y$. Then, (2.2) and (2.6) give us,

$$h_t + u^{(I)} h_x + v^{(I)} h_y = w^{(I)} \tag{3.2b}$$

where we have used the components of the normal are determined after (C.6).

Since the components of the stress tensor are

$$\sigma_{11} = 2\, \mu u_x \tag{3.3a}$$

$$\sigma_{22} = 2\, \mu v_y \tag{3.3b}$$

$$\sigma_{33} = 2\, \mu w_z \tag{3.3c}$$

$$\sigma_{12} = \sigma_{21} = \mu (u_y + v_x) \tag{3.3d}$$

$$\sigma_{13} = \sigma_{31} = \mu (u_z + w_x) \tag{3.3e}$$

$$\sigma_{23} = \sigma_{32} = \mu (v_z + w_y) \tag{3.3f}$$

we find

$$\sqrt{1 + h_x^2 + h_y^2}\, \frac{n_j \sigma_{1j}}{\mu} = -2\, h_x\, u_x - h_y \left(u_y + v_x \right) + \left(u_z + w_x \right) \tag{3.4a}$$

$$\sqrt{1 + h_x^2 + h_y^2}\, \frac{n_j \sigma_{2j}}{\mu} = -h_x \left(u_y + v_x \right) - 2 h_y\, v_y + \left(v_z + w_y \right) \tag{3.4b}$$

$$\sqrt{1 + h_x^2 + h_y^2}\, \frac{n_j \sigma_{3j}}{\mu} = -h_x \left(u_z + w_x \right) - h_y \left(v_z + w_y \right) + 2 w_z \tag{3.4c}$$

Using these results, (2.4) becomes

$$\begin{aligned}-h_x\left(p^{(1)}-p^{(2)}\right)+2h_x\left[\mu_1 u_x^{(1)}-\mu_2 u_x^{(2)}\right]+h_y\left[\mu_1(u_y^{(1)}+v_x^{(1)})-\mu_2(u_y^{(2)}+v_x^{(2)})\right]\\ -\left[\mu_1(u_z^{(1)}+w_x^{(1)})-\mu_2(u_z^{(2)}+w_x^{(2)})\right]=-2h_xT\kappa\end{aligned} \tag{3.5a}$$

$$\begin{aligned}-h_y\left(p^{(1)}-p^{(2)}\right)+h_x\left[\mu_1(u_y^{(1)}+v_x^{(1)})-\mu_2(u_y^{(2)}+v_x^{(2)})\right]+2h_y\left[\mu_1 v_y^{(1)}-\mu_2 v_y^{(2)}\right]\\ -\left[\mu_1(v_z^{(1)}+w_y^{(1)})-\mu_2(v_z^{(2)}+w_y^{(2)})\right]=-2h_yT\kappa\end{aligned} \tag{3.5b}$$

$$\begin{aligned}\left(p^{(1)}-p^{(2)}\right)+h_x\left[\mu_1(u_z^{(1)}+w_x^{(1)})-\mu_2(u_z^{(2)}+w_x^{(2)})\right]\\ +h_y\left[\mu_1(v_z^{(1)}+w_y^{(1)})-\mu_2(v_z^{(2)}+w_y^{(2)})\right]-2\left[\mu_1 w_z^{(1)}-\mu_2 w_z^{(2)}\right]=2T\kappa\end{aligned} \tag{3.5c}$$

By substituting (3.5c) into (3.5a) and (3.5b), we may replace this set with

$$\begin{aligned}\left(h_x^2-1\right)\left[\mu_1\left(u_z^{(1)}+w_x^{(1)}\right)-\mu_2\left(u_z^{(2)}+w_x^{(2)}\right)\right]+h_y\left[\mu_1\left(u_y^{(1)}+v_x^{(1)}\right)-\mu_2\left(u_y^{(2)}+v_x^{(2)}\right)\right]\\ +h_xh_y\left[\mu_1\left(v_z^{(1)}+w_y^{(1)}\right)-\mu_2\left(v_z^{(2)}+w_y^{(2)}\right)\right]+2h_x\left[\mu_1\left(u_x^{(1)}-w_z^{(1)}\right)-\mu_2(u_x^{(2)}-w_z^{(2)})\right]=0\end{aligned} \tag{3.6a}$$

$$\begin{aligned}\left(h_y^2-1\right)\left[\mu_1(v_z^{(1)}+w_y^{(1)})-\mu_2(v_z^{(2)}+w_y^{(2)})\right]+h_xh_y\left[\mu_1(u_z^{(1)}+w_x^{(1)})-\mu_2(u_z^{(2)}+w_x^{(2)})\right]\\ 2h_y\left[\mu_1(v_y^{(1)}-w_z^{(1)})-\mu_2(v_y^{(2)}-w_z^{(2)})\right]+h_x\left[\mu_1(u_y^{(1)}+v_x^{(1)})-\mu_2(u_y^{(2)}+v_x^{(2)})\right]=0\end{aligned} \tag{3.6b}$$

$$\begin{aligned}\left(p^{(1)}-p^{(2)}\right)+h_x\left[\mu_1(u_z^{(1)}+w_x^{(1)})-\mu_2(u_z^{(2)}+w_x^{(2)})\right]\\ +h_y\left[\mu_1(v_z^{(1)}+w_y^{(1)})-\mu_2(v_z^{(2)}+w_y^{(2)})\right]-2\left[\mu_1 w_z^{(1)}-\mu_2 w_z^{(2)}\right]=2T\kappa\end{aligned} \tag{3.6c}$$

A Jump conditions

Consider the general conservation equation

$$\frac{\partial A}{\partial t}+\frac{\partial B_j}{\partial x_j}=C\delta(n)+F \tag{A.1}$$

valid in a region that contains an interface. The normal n is the distance along the normal from an interface given in parametric form as $X_i(\xi,\eta,t)$. The quantities C and F represent surface and body forces respectively.

Multiply by the smooth test function ϕ and integrate over a spatial region D that includes the interface ∂D and over a small interval of time.

$$\iint_D\left(\phi\,\frac{\partial A}{\partial t}+\phi\,\frac{\partial B_j}{\partial x_j}\right)\,d\mathbf{x}\,dt=\iint_{\partial D}\phi\,C\,dS\,dt+\iint_D\phi\,F\,d\mathbf{x}\,dt \tag{A.2}$$

After integrating by parts,

$$\iint_D\left(A\,\frac{\partial\phi}{\partial t}+B_j\,\frac{\partial\phi}{\partial x_j}\right)\,d\mathbf{x}\,dt=-\iint_{\partial D}\phi\,C\,dS\,dt-\iint_D\phi\,F\,d\mathbf{x}\,dt \tag{A.3}$$

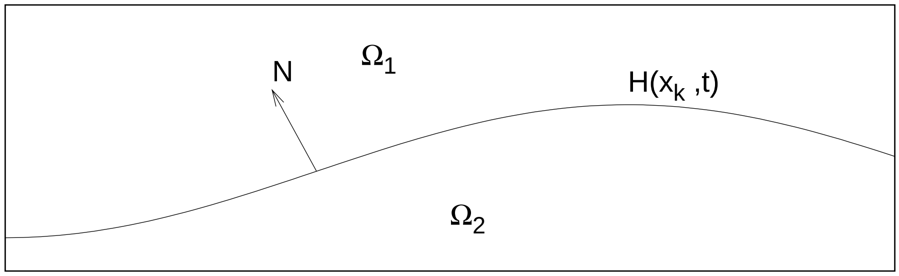

Figure 1: An illustration of the surface

The test function ϕ is chosen to vanish on the boundary of D. In the sense of distributions, (A.3) is what we mean by (A.1). Note that (A.3) allows jumps in value for A and B_j. We seek such weak solutions by using (A.3) and breaking up the region of integration.

The form of (A.1) suggests the divergence of the vector (B_1, B_2, B_3, A) in the space (x_1, x_2, x_3, t). In this extended space, the moving interface is a stationary surface Γ, separating two regions, Ω_1 and Ω_2, as shown in figure 1. Away from the interface, we assume the solutions are smooth, so we may rewrite the integrals as divergences.

$$\int_{\Omega_1} \left(A \frac{\partial \phi}{\partial t} + B_j \frac{\partial \phi}{\partial x_j} \right) d\mathbf{x}\, dt = \int_{\Omega_1} \left(\frac{\partial}{\partial t}(\phi A) + \frac{\partial}{\partial x_j}(\phi B_j) \right) d\mathbf{x}\, dt \tag{A.4a}$$

$$\int_{\Omega_2} \left(A \frac{\partial \phi}{\partial t} + B_j \frac{\partial \phi}{\partial x_j} \right) d\mathbf{x}\, dt = \int_{\Omega_2} \left(\frac{\partial}{\partial t}(\phi A) + \frac{\partial}{\partial x_j}(\phi B_j) \right) d\mathbf{x}\, dt \tag{A.4b}$$

The divergence theorem as described in Appendix B allows these integrals to be written as

$$\int_{\Omega_1} \left(\frac{\partial}{\partial t}(\phi A) + \frac{\partial}{\partial x_j}(\phi B_j) \right) d\mathbf{x}\, dt = -\iint_{\partial D} \phi \left(B_j^{(1)} - A^{(1)} \frac{\partial X_j}{\partial t} \right) N_j \, d\xi\, d\eta\, dt \tag{A.5a}$$

$$\int_{\Omega_2} \left(\frac{\partial}{\partial t}(\phi A) + \frac{\partial}{\partial x_j}(\phi B_j) \right) d\mathbf{x}\, dt = \iint_{\partial D} \phi \left(B_j^{(2)} - A^{(2)} \frac{\partial X_j}{\partial t} \right) N_j \, d\xi\, d\eta\, dt \tag{A.5b}$$

where the integration over the boundaries other than Γ are assumed to vanish, if necessary, through an appropriate choice of ϕ. The superscripts $1, 2$ refer to the limiting values along ∂D and $\mathbf{N}$ is the vector normal as defined in (B.3c),(B.8d). The result has been written for the case where the normal points from Ω_2 to Ω_1.

Since the integral over the combined region $\Omega_1 + \Omega_2$ must vanish, we find

$$\iint_{\partial D} \phi \left[\left(B_j^{(2)} - B_j^{(1)} \right) - \frac{\partial X_j}{\partial t} \left(A^{(2)} - A^{(1)} \right) \right] N_j \, d\xi\, d\eta\, dt = \\ - \iint_{\partial D} \phi\, C\, \Lambda\, d\xi\, d\eta\, dt - \iint_D \phi\, F\, d\mathbf{x}\, dt \tag{A.6}$$

where we have used the result $dS = \Lambda\, d\xi\, d\eta$ as derived in (B.4). This result (A.6) must be true for any choice of ϕ. In particular, choose a family of test functions of the form

$$\phi \sim \exp\left(-\frac{n^2}{\varepsilon}\right)$$

where ε can be made arbitrarily small. This means the volume integral of the body force F will vanish, while the surface integrals still provide a non-trivial result. Thus,

$$n_j \left[\left(B_j^{(1)} - B_j^{(2)}\right) - \frac{dX_j}{dt}\left(A^{(1)} - A^{(2)}\right)\right] = C \tag{A.7}$$

B The divergence theorem

Suppose there is a closed region of space Ω in n dimensions. Let the surface $\partial\Omega$ be given in parametric form $X_i(\xi_1, \ldots, \xi_{n-1})$. The divergence theorem stated in this space is

$$\int \cdots \int_D \frac{\partial F_i}{\partial x_i}\, dx_1, \cdots, dx_n = \int \cdots \int_{\partial D} F_i\, dS_i \tag{B.1a}$$

where

$$dS_i = \varepsilon_{i,j_1,\ldots j_{n-1}} \frac{\partial X_{j_1}}{\partial \xi_1} \cdots \frac{\partial X_{j_{n-1}}}{\partial \xi_{n-1}}\, d\xi_1 \cdots d\xi_{n-1} \tag{B.1b}$$

The permutation symbol $\varepsilon_{i,j,\ldots}$ is the one associated with the vector product.

As an illustration, consider $n = 3$, and let $X_1 = X$, $X_2 = Y$ and $X_3 = Z$. Also, $\xi_1 = \xi$ and $\xi_2 = \eta$. Then,

$$dS_1 = \left(\frac{\partial Y}{\partial \xi}\frac{\partial Z}{\partial \eta} - \frac{\partial Z}{\partial \xi}\frac{\partial Y}{\partial \eta}\right) d\xi\, d\eta \tag{B.2a}$$

$$dS_2 = \left(\frac{\partial Z}{\partial \xi}\frac{\partial X}{\partial \eta} - \frac{\partial X}{\partial \xi}\frac{\partial Z}{\partial \eta}\right) d\xi\, d\eta \tag{B.2b}$$

$$dS_3 = \left(\frac{\partial X}{\partial \xi}\frac{\partial Y}{\partial \eta} - \frac{\partial Y}{\partial \xi}\frac{\partial X}{\partial \eta}\right) d\xi\, d\eta \tag{B.2c}$$

To compare with a direct geometrical approach, we first determine the tangent vectors,

$$\mathbf{t}_1 = \left(\frac{\partial X}{\partial \xi}, \frac{\partial Y}{\partial \xi}, \frac{\partial Z}{\partial \xi}\right) \tag{B.3a}$$

$$\mathbf{t}_2 = \left(\frac{\partial X}{\partial \eta}, \frac{\partial Y}{\partial \eta}, \frac{\partial Z}{\partial \eta}\right) \tag{B.3b}$$

then

$$\mathbf{t}_1 \times \mathbf{t}_2 = \mathbf{i}\left(\frac{\partial Y}{\partial \xi}\frac{\partial Z}{\partial \eta} - \frac{\partial Z}{\partial \xi}\frac{\partial Y}{\partial \eta}\right) + \mathbf{j}\left(\frac{\partial Z}{\partial \xi}\frac{\partial X}{\partial \eta} - \frac{\partial X}{\partial \xi}\frac{\partial Z}{\partial \eta}\right) + \mathbf{k}\left(\frac{\partial X}{\partial \xi}\frac{\partial Y}{\partial \eta} - \frac{\partial Y}{\partial \xi}\frac{\partial X}{\partial \eta}\right) \tag{B.3c}$$

Let

$$\Lambda = \|\mathbf{t}_1 \times \mathbf{t}_2\| \tag{B.4}$$

then the elemental surface area is $dS = \Lambda \, d\xi \, d\eta$ and the outward normal to the surface is $\mathbf{n} = (\mathbf{t}_1 \times \mathbf{t}_2)/\Lambda$. Thus,

$$\int_{\partial D} F_i \, \varepsilon_{i,j,k} \, \frac{\partial X_j}{\partial \xi} \, \frac{\partial X_k}{\partial \eta} \, d\xi \, d\eta = \int_{\partial D} F_i \, n_i \, dS \tag{B.5}$$

Now let us consider a surface integral in space-time with coordinates (x, y, z, t) . Let the surface by given as $X(\xi, \eta, t)$, $Y(\xi, \eta, t)$, $Z(\xi, \eta, t)$ and $T(t)$. In other words, the surface has a parametric form in terms of the variables ξ, η and t. The divergence theorem takes the form

$$\int_{\Omega} \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} + \frac{\partial G}{\partial t} \right) dx \, dy \, dz \, dt = \int_{\partial \Omega} F_i \, dS_i + \int_{\partial \Omega} G \, dS_4 \tag{B.6}$$

where the index i runs over 1,2,3, and

$$dS_1 = \left(\frac{\partial Y}{\partial \xi} \frac{\partial Z}{\partial \eta} - \frac{\partial Z}{\partial \xi} \frac{\partial Y}{\partial \eta} \right) d\xi \, d\eta \, dt \tag{B.7a}$$

$$dS_2 = \left(\frac{\partial Z}{\partial \xi} \frac{\partial X}{\partial \eta} - \frac{\partial X}{\partial \xi} \frac{\partial Z}{\partial \eta} \right) d\xi \, d\eta \, dt \tag{B.7b}$$

$$dS_3 = \left(\frac{\partial X}{\partial \xi} \frac{\partial Y}{\partial \eta} - \frac{\partial Y}{\partial \xi} \frac{\partial X}{\partial \eta} \right) d\xi \, d\eta \, dt \tag{B.7c}$$

$$\begin{aligned} dS_4 = \Bigg[& -\frac{\partial X}{\partial t} \left(\frac{\partial Y}{\partial \xi} \frac{\partial Z}{\partial \eta} - \frac{\partial Z}{\partial \xi} \frac{\partial Y}{\partial \eta} \right) + \frac{\partial Y}{\partial t} \left(\frac{\partial X}{\partial \xi} \frac{\partial Z}{\partial \eta} - \frac{\partial Z}{\partial \xi} \frac{\partial X}{\partial \eta} \right) \\ & + \frac{\partial Z}{\partial t} \left(\frac{\partial Y}{\partial \xi} \frac{\partial X}{\partial \eta} - \frac{\partial X}{\partial \xi} \frac{\partial Y}{\partial \eta} \right) \Bigg] d\xi \, d\eta \, dt \end{aligned} \tag{B.7d}$$

If we define $\mathbf{N} = \mathbf{t}_1 \times \mathbf{t}_2$ as given in (B.3c), we may write (B.7) as

$$dS_1 = N_1 \, d\xi \, d\eta \, dt \tag{B.8a}$$

$$dS_2 = N_2 \, d\xi \, d\eta \, dt \tag{B.8b}$$

$$dS_3 = N_3 \, d\xi \, d\eta \, dt \tag{B.8c}$$

$$\begin{aligned} dS_4 &= -\left(\frac{\partial X}{\partial t} N_1 + \frac{\partial Y}{\partial t} N_2 + \frac{\partial Z}{\partial t} N_3 \right) d\xi \, d\eta \, dt \\ &= -\frac{\partial X_i}{\partial t} N_i \, d\xi \, d\eta \, dt \end{aligned} \tag{B.8d}$$

and (B.6) can be written as

$$\iint_D \left(\frac{\partial F_i}{\partial x_i} + \frac{\partial G}{\partial t} \right) d\mathbf{x} \, dt = \iint_{\partial D} \left(F_i - G \frac{\partial X_i}{\partial t} \right) N_i \, d\xi \, d\eta \, dt \tag{B.9}$$

C Curvature

Assume a surface is known in parametric form $X(\xi,\eta)$, $Y(\xi,\eta)$ and $Z(\xi,\eta)$. Two tangent vectors to the surface are

$$\mathbf{t}_1 = (X_\xi, Y_\xi, Z_\xi) \tag{C.1a}$$
$$\mathbf{t}_2 = (X_\eta, Y_\eta, Z_\eta) \tag{C.1b}$$

where we have used subscripts to indicate partial deribatives. The normal to the surface is given by

$$\mathbf{n} = \frac{\mathbf{t}_1 \times \mathbf{t}_2}{\|\mathbf{t}_1 \times \mathbf{t}_2\|} \tag{C.1c}$$

Define

$$E = X_\xi^2 + Y_\xi^2 + Z_\xi^2 \tag{C.2a}$$
$$F = X_\xi\, X_\eta + Y_\xi\, Y_\eta + Z_\xi\, Z_\eta \tag{C.2b}$$
$$G = X_\eta^2 + Y_\eta^2 + Z_\eta^2 \tag{C.2c}$$

Then, $D = \|\mathbf{t}_1 \times \mathbf{t}_2\|$ may be determined from

$$\begin{aligned}\|\mathbf{t}_1 \times \mathbf{t}_2\|^2 &= EG - F^2 \\ &= Y_\xi^2\, Z_\eta^2 + Z_\xi^2\, Y_\eta^2 + Z_\xi^2\, X_\eta^2 + X_\xi^2\, Z_\eta^2 + X_\xi^2\, Y_\eta^2 + Y_\xi^2\, X_\eta^2 \\ &\quad - 2Y_\xi Y_\eta Z_\xi Z_\eta - 2X_\xi X_\eta Z_\xi Z_\eta - 2X_\xi X_\eta Y_\xi Y_\eta\end{aligned} \tag{C.3}$$

To obtain an expression for the mean curvature, we also need the following quantities:

$$DL = X_{\xi\xi}\,(Y_\xi\, Z_\eta - Z_\xi\, Y_\eta) + Y_{\xi\xi}\,(Z_\xi\, X_\eta - X_\xi\, Z_\eta) + Z_{\xi\xi}\,(X_\xi\, Y_\eta - Y_\xi\, X_\eta) \tag{C.4a}$$
$$DM = X_{\xi\eta}\,(Y_\xi\, Z_\eta - Z_\xi\, Y_\eta) + Y_{\xi\eta}\,(Z_\xi\, X_\eta - X_\xi\, Z_\eta) + Z_{\xi\eta}\,(X_\xi\, Y_\eta - Y_\xi\, X_\eta) \tag{C.4b}$$
$$DN = X_{\eta\eta}\,(Y_\xi\, Z_\eta - Z_\xi\, Y_\eta) + Y_{\eta\eta}\,(Z_\xi\, X_\eta - X_\xi\, Z_\eta) + Z_{\eta\eta}\,(X_\xi\, Y_\eta - Y_\xi\, X_\eta) \tag{C.4c}$$

From Stoker's book on differential geometry, the mean curvature κ is given by

$$2\kappa = \frac{EN - 2FM + GL}{D^2} \tag{C.5}$$

For the specific choice of surface representation,

$$\mathbf{X} = (\xi, \eta, h(\xi,\eta)) \tag{C.6}$$

we have

$$
\begin{gathered}
\mathbf{t}_1 = (1, 0, h_\xi) \\
\mathbf{t}_2 = (0, 1, h_\eta) \\
E = 1 + h_\xi^2 \\
F = h_\xi h_\eta \\
G = 1 + h_\eta^2 \\
EG - F^2 \equiv D^2 = 1 + h_\xi^2 + h_\eta^2 \\
D\mathbf{n} = (-h_\xi, -h_\eta, 1) \\
DL = h_{\xi\xi} \\
DM = h_{\xi\eta} \\
DN = h_{\eta\eta}
\end{gathered}
$$

which leads to

$$
2\kappa = \frac{(1 + h_\xi^2)\, h_{\eta\eta} - 2 h_\xi h_\eta h_{\xi\eta} + (1 + h_\eta^2)\, h_{\xi\xi}}{(1 + h_\xi^2 + h_\eta^2)^{3/2}} \tag{C.7}
$$