Pricing for the Optimal Coordination of Opportunistic Agents

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***Abstract*—We consider a system where a load aggregator (LA) serves a large number of small-sized, economically-driven consumers with deferrable demand, as envisioned in future smart electricity grid and data networks. In these systems, consumers can behave opportunistically by deferring their demand in response to the prices, in order to obtain economic gains and reduce their payments. However, if not controlled properly, such opportunistic behavior can be detrimental to the system by creating aggregate effects that lead to undesirable fluctuations in price and total load.**

To avoid the unwanted effects of demand-side flexibilities and to reap system-wide benefits from them, we propose two novel real-time dynamic pricing algorithms. The first algorithm communicates individual prices to consumers by adding small random perturbations to a common price. The second algorithm introduces a secondary price that penalizes the change in users' consumption in time. The common feature of both algorithms is creating differentiation among consumers and thus regulating the aggregate load. We conduct comprehensive numerical investigations and show that both the LA and the consumers economically benefit under the proposed pricing schemes.

I. Introduction

In this work, we aim to design real-time dynamic pricing strategies for large systems where the demands possess various types of flexibilities. Demand-side flexibilities arise in systems such as smart electricity grids and cloud computing services. In these systems, flexibilities can materialize in various forms including, but not limited to, shifting or deferring service in time, giving intermittent service, and controlling the service amount. Consumers, who are inherently self-interested and economically-driven, will naturally want to alter their consumption behavior to take advantage of these flexibilities. Such consumer behavior induced by demand-side flexibilities brings both opportunities and challenges, and necessitates the design of novel management techniques.

We consider a system where a large number of *small consumers with flexible demand* are served by a *load aggregator* (LA). Specifically, we use the smart electricity grid as an example. The consumers can be households with smart electrical devices, or small manufacturers, whereas the LA can be an electricity retailer. Moreover, the type of demand-side flexibility that is considered in this paper is the ability to defer demand in time. For instance, smart air conditioners and washing machines can operate in this fashion. We further note that the model in this paper can be applied to various scenarios such as a cloud computing center serving customers with computational tasks.

From the LA's perspective, demand-side flexibilities can be utilized to the advantage of the system operation. For example, in a smart electricity grid [1], or in a computer network [2], consumer demand can be deferred to a later time to cut peak load and to reduce service and maintenance costs. On the other hand, from consumer's perspective, flexibilities can be exploited to obtain economic benefits by reducing payments [3]. Towards this end, this work aims to design real-time pricing schemes to be implemented by an LA that *incentivize* economically-driven agents to *defer* their flexible demand so that system-wide benefits can be obtained.

However, pricing-based dynamic control of self-interested users also raises challenges. Under a time-dependent pricing scheme, consumers will likely defer their service to the periods of time with lower price in an opportunistic manner. Indeed, works such as [4], [5] establish optimality of threshold-based consumption policies, that have this opportunistic flavor, under different flexibility and cost structures. But, aggregate response of a large consumer base employing such threshold policies can potentially lead to highly- and abruptly-fluctuating total load and price (as will be demonstrated in Section III). In most systems, this volatile behavior is undesirable, because it increases service costs, puts stress on the network, and endangers the stability of the infrastructure [6]. Thus, in designing new pricing mechanisms, we aim to mitigate the mentioned effects of opportunistic behavior of flexible consumers.

There has been much research on demand-side management, especially in the context of electricity network and the related wholesale and retail markets. At the retail level, at which this paper is also positioned, the literature is abundant. One line of work studies the direct control of time-shiftable demand [7]–[9]. Another line of research, which this paper also pursues, focuses on designing day-ahead and real-time pricing mechanisms to coordinate flexible demand [10]. Under day-ahead pricing, users are informed of the hourly prices a day ahead. But, this removes much of the uncertainty in flexible user load, because delaying tasks can be done more dynamically on a finer time-scale [11]–[13]. On the other hand, real-time pricing is more dynamic and adaptive, hence we focus our investigation on real-time pricing schemes.

Previous related work and their fundamental differences from this work are listed below. In [14]–[16], the welfare maximization problem of an LA is studied under utility maximization framework at day-ahead and real-time scales. However, consumers are assumed to have concave utility functions, which result in smoother user behavior, and hence do not capture the opportunistic decision making expected from self-interested users. Game theoretic approaches are discussed in [17], [18] for incentivizing time-shifting of energy consumption, but the resulting mechanisms require the knowledge of consumer demand and utility. Furthermore, the authors in [19] consider an LA's problem of renewable supply integration via day-ahead and real-time load scheduling, and

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formulate a Markov Decision problem.

A key difference of this work from the aforementioned literature is that threshold policies are considered instead of modeling the consumer flexibility and utility with smooth functions. First, threshold policies are optimal in many scenarios as shown in [4], [5]. Second, they capture the opportunistic consumer behavior, and they easily raise the volatility and stability issues [6]. Furthermore, this paper proposes price differentiation via randomization, which has never been studied in the context of the smart grid to the best of our knowledge.

In this paper, we present two novel real-time dynamic pricing algorithms. Preliminary investigations (Section III) show that consumption decisions of flexible consumers get synchronized under a common price signal. Motivated by this observation, the key idea in designing our algorithms is to create differentiation either in price or in agents' internal states. Our first algorithm creates information asymmetry among users by sending *individual prices* to users obtained by creating small perturbations around a common price (Section IV). This scheme is appropriate when there are a large number of consumers, and it preserves fairness among users although each user sees different prices. On the other hand, the second algorithm introduces heterogeneity among users by communicating a secondary price for the *change in consumption* of each user in time (Section V). This scheme is appropriate for both small and large systems, and it does not differentiate users based on the price they see.

Technically, both of the proposed algorithms solve different versions of the same cost minimization problem, which are obtained by augmenting the objective of the original problem with convex terms. Furthermore, our results (Section VI) convey the prominent message that introducing differentiation among opportunistically behaving agents alleviates the detrimental effects of the feedback loop between aggregate load and price. In particular, under the proposed algorithms (i) high volatility and instability problems are alleviated; (ii) a flatter load pattern, which is less costly to supply, is achieved; (iii) flexible consumers obtain economic benefits.

II. System Model and Problem Formulation

We consider a system where a load aggregator (LA) serves a large number of small consumers. In the following, we present a generic real-time model and introduce the system participants. Then, we focus on the smart electricity grid and formulate the control and pricing problem.

A. System Model

The system is operated over discrete time periods, $t = 0, 1, \ldots$, and at each time period the participants make their control decisions. The system comprises an LA and a large number of consumers as depicted in Figure 1. The LA sets the real-time prices for its consumers, and ensures that consumers' loads are served upon their request. The goal of the LA is to maximize its profit. On the other hand, consumers seek to satisfy their demand with the aim of making the lowest payment for consumption. There are two types of consumers in the system. *Flexible* consumers have deferrable demand, i.e. they can delay their consumption. *Inflexible* consumers, however, cannot delay their consumption and must serve their demand at the time the demand is realized. The system model described is a closed-loop feedback stochastic dynamical system. Consumers react to the price generated by the LA in real-time, and the LA adjusts the price based on the total load. Next, we present the system participants and the overall operation of the system in detail, using electricity system as a specific example.

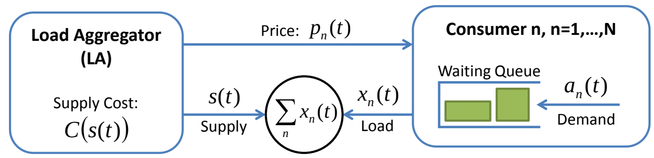

Fig. 1. The system model depicting the participants and their interactions.

1) *Load Aggregator (LA):* The LA serves its customers by procuring electricity via purchasing from a wholesale market or a distributor. The procurement of s watts of power incurs cost $C(s)$ to the LA. Note that, the cost function C encapsulates the payments for purchasing electricity as well as maintenance and capital costs [12]–[15], [17], [18]. We assume that $C : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuously differentiable and increasing function of s. We also assume that $\ddot{C}(s) > 0$, and hence C is strictly convex and $\dot{C}$ is invertible.

The LA intends to coordinate its customers by setting the real-time prices at each time period, i.e. $p(t)$ for $t = 0, 1, \ldots$. The price is generated *ex-ante*, meaning that the amount of consumption is unknown at the time the price is set. Based on the price it sets, the LA receives the payment $\omega(t) \triangleq p(t)s(t)$. Hence, the LA's goal is to maximize its profit $\omega(t) - C(s(t))$. Furthermore, the LA does not have the knowledge of consumer valuations and their control strategies.

2) *Consumers:* There are N flexible and N_i inflexible consumers. At period t, consumer n generates demand $a_n(t)$. $a_n(t)$ is a random variable that is assumed to be independent among consumers and i.i.d. over time. The average demand generation rate is λ_n, and $\mathbb{E}\left[A_n(t)\right] = \lambda_n$ for all t. We assume that demand is bounded such that $A_n(t) \in [0, a_{n,max}]$. The energy consumption, namely *load* [1], by user n at period t is denoted by $x_n(t) \in [0, x_n^m]$. For inflexible consumers, $x_n(t) = a_n(t)$ because the realized demand must be served immediately. We define $S_i(t) \triangleq \sum_{n=1}^{N_i} x_n(t)$, with mean $\lambda_S = \sum_{n=1}^{N_i} \lambda_n$, to be the total load of inflexible users. For flexible consumers, the amount of electric energy consumption is not necessarily equal to the amount of realized demand; Realized demand can be deferred and served later as load.

The waiting queue for flexible consumer n's deferrable demand at time t is $q_n(t)$ and its evolution is given by

$$q_n(t+1) = \left[q_n(t) + a_n(t) - x_n(t)\right]^+ \tag{1}$$

These queues are required to be stable, otherwise the delay experienced by the demand will approach infinity. The goal of flexible consumer n is to minimize its payment, $r_n(t) \triangleq$

[1] To be precise: "Demand" is externally generated according to a, but can be delayed. "Load" is the actual consumption at each time period.

$p(t)x_n(t)$, under the queue stability constraint. Inflexible consumers do not have such objective since they do not have control on their load.

B. Problem Formulation

In the paper, we use boldface letters to denote vectors, e.g. $\mathbf{x} = (x_1, \ldots, x_N)$ is the N dimensional vector of the scalar quantities x_n for $n = 1, \ldots, N$. We use $\{.\}$ to denote a set of quantities whose size should be understood from context.

The optimization problem we consider is the LA's cost minimization problem:

$$\min_{\{\mathbf{x}(t)\}\{s(t)\}} \quad \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\left[C\left(s(t)\right)\right] \tag{2}$$

$$\text{s.t.} \quad \sum_{n=1}^{N} x_n(t) + S_i(t) = s(t), \ \forall t = 0, 1, \ldots \tag{3}$$

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\left[x_n(t)\right] \geq \lambda_n, \ \forall n. \tag{4}$$

In problem (2), objective is the time averaged expected cost of electricity procurement. Constraint (3) ensures that consumer load is served completely, constraint (4) ensures that the flexible consumers experience finite delay.

Instead of Problem (2), we will consider the following static (one time-period) problem:

$$\min_{\mathbf{x},s} \quad C\left(s\right) \quad \text{s.t.} \quad \sum_{n=1}^{N} x_n + \lambda_S \leq s \tag{5}$$
$$\lambda_n \leq x_n, \ \forall n.$$

It can be shown that the optimum objective value of (5) is a lower bound for that of (2) due to the convexity of C. Hence, by solving problem (5), whether exactly or approximately, we can obtain a solution whose objective value is close to the optimum value of problem (2). Furthermore, problem (5) has a simpler structure, thus we can apply well-known techniques such as duality to derive iterative algorithms and modify them to satisfy certain performance criteria while still being able to provide analytic results on the performance of the system.

Problem (5) is easy to solve and various iterative algorithms can be developed to achieve the optimum solution. However, such algorithms may dictate undesirable control-rules on the consumer side that do not align with flexible consumers' objective of minimizing their payments. On the other hand, as we will show in Section III-B, allowing flexible consumers to fully exhibit their opportunistic behavior may cause instability and inefficiency by generating abrupt changes and fluctuations in power generation . Therefore, our goal is to design control and real-time pricing schemes that will give flexible consumers the freedom to opportunistically consume electricity for their own interest, and that will also achieve the minimum or close-to-minimum electricity procurement cost.

III. Flexible Consumer Behavior and Benchmark Real-time Pricing Schemes

In the following, we will first characterize the flexible consumer behavior, and then discuss its impact on the system performance. To demonstrate the detrimental effects of consumer-side flexibility, we present two simple and intuitive real-time pricing schemes that will also serve as benchmarks when assessing our own pricing schemes' performance.

A. Flexible Consumer Behavior

In our model, consumers are price-taking; At period t, each consumer receives a price $p(t)$ for consuming unit amount of power, and then decides on his load. Thus, the optimization problem faced by a flexible consumer can be formulated as

$$\min_{\{x_n(t)\}} \quad \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\left[p(t)x_n(t)\right] \tag{6}$$
$$\text{s.t.} \quad \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\left[x_n(t)\right] \geq \lambda_n, \ \forall n.$$

From a single consumer's perspective, his individual load decisions have negligible effect on the future prices when the number of users is large. Hence, we assume that $p(t)$ is exogenous; It is independent of $x_n(t)$ in problem (6). Under this assumption, the following policy asymptotically achieves the optimal value of (6) as the design parameter κ_n gets small:

$$x_n(t) = x_n^m \mathbb{1}\{p(t) \leq \kappa_n q_n(t)\} \tag{7}$$

We note that this threshold policy and similar threshold-based policies have been shown to be asymptotically optimal when the prices are exogenous [4], [7], [20].

The policy in (7) results in an opportunistic behavior. Users consume electricity only when price is below a certain threshold, and when they consume they put their maximum load x_n^m to take full advantage of the low price. However, this behavior, when aggregated over a large consumer base, will cause very high (low) load when price is low (high). Thus, as we will see subsequently, the resulting load pattern will not be flat and will be costly to supply. Furthermore, supply and price will be highly fluctuating since the price is adjusted in real-time by the LA as a response to the changes in load.

B. Benchmark Real-time Pricing Schemes

i) *Scheme I (Real-time Pricing With Zero Flexible Consumer Penetration):* In this scheme, *all consumers are inflexible*, so they do not have the ability to defer their loads; Arriving demand is served immediately, i.e. $x_n(t) = a_n(t)$ for all n. On the other hand, the LA uses $\sum_n x_n(t)$ as the prediction of the load on the next time period, and sets the price to the total marginal procurement cost, i.e. $p(t+1) = \dot{C}\left(s(t)\right)$ subject to $s(t) = \sum_n x_n(t)$. This choice of price maximizes the LA's profit assuming that the load prediction is accurate. To summarize, Scheme 1 is given as follows:

Scheme 1. *At time t:*
- *Consumer n sets $x_n(t) = a_n(t)$.*
- *The LA computes:*

$$p(t+1) = \dot{C}\left(s(t)\right), \ \text{s.t.} \ s(t) = \sum_n x_n(t) + S_i(t)$$

We note that Scheme 1 serves as a base setup, which will be useful in assessing both the advantages and disadvantages of consumer-side flexibility.

ii) Scheme II (Gradual Real-time Price Update under Flexible Consumer Presence): Under his scheme, a percentage of users have flexible demand. We assume that these users implement the threshold policy (7). Due to (7), we expect the aggregate load to become either very large or too small, since the consumers use the maximum amount x_n^m or nothing based on the common price. Thus, in order to prevent fluctuations in price in response to the total load, this scheme iteratively updates the price instead of setting it to the marginal cost of the total load. Scheme 2, which updates the price based on the dual of problem (5), is presented below.

Scheme 2. *At time t:*

- *Consumer n computes (1) and*

$$x_n(t) = x_n^m \mathbb{1}\left\{p(t) \le \kappa_n q_n(t)\right\} \tag{8}$$

- *The LA must meet the real load $\sum_n x_n(t)+S_i(t)$. Further, it computes $s(t) = \dot{C}^{-1}\left(p(t)\right)$, and updates the price:*

$$p(t+1) = \left[p(t) + \kappa^s \left(\sum_n x_n(t) + S_i(t) - s(t)\right)\right]^+ \tag{9}$$

Under Scheme 2, although the price exhibits relatively small oscillations due to the dampening effect of κ^s, the total load abruptly fluctuates as seen in Figure 2. Figure 2 also depicts the same amount of total load under Scheme 1. Note that although Scheme 1 does not have the fluctuation problem as Scheme 2, it does not take advantage of the demand flexibilities either. Under the presence of demand flexibilities, the key problem appears to be that the customers, who implement the threshold policy (7), respond to a common price in a synchronous manner. In the following sections, we will propose pricing schemes that will resolve this synchronization problem by introducing differentiation among consumers.

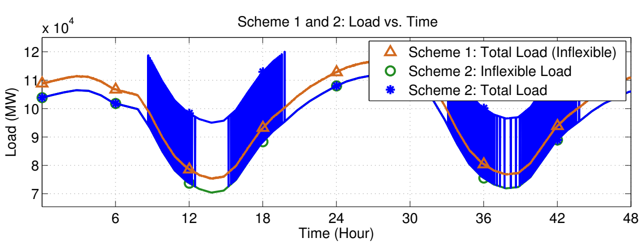

Fig. 2. Load under Scheme 1 and 2: Flexible consumers receive Poisson distributed demand arrivals, and their load constitute 5% of the total load.

IV. Randomized Pricing (RP) Algorithm

RP algorithm aims to mitigate the volatility and instability problems that can arise due to the opportunistic behavior of flexible consumers. Towards this goal, the underlying motivation in the design of RP is twofold. First, we consider updating the common price incrementally so that sudden changes in load do not directly translate as large fluctuations in price. Second, in order to prevent flexible consumers' load decisions from aligning together (which creates peaks and valleys in the aggregate load), we differentiate the price over the consumer base. In particular, each consumer receives an individual price that is randomly differentiated from the common price.

Algorithm RP Randomized Pricing Algorithm

At iteration t:

- Consumer n receives an individual price $p_n(t)$. Then, it computes its queue as in (1) and load as

$$x_n(t) = x_n^m \mathbb{1}\{p_n(t) \le \kappa_n q_n(t)\} \tag{10}$$

- The LA must meet the real load $\sum_n x_n(t) + S_i(t)$. Further, it computes $s(t) = \dot{C}^{-1}(p(t))$, and updates the common price:

$$p(t+1) = \left[p(t) + \alpha\left(\sum_{n=1}^{N} x_n(t) + S_i(t) - s(t)\right)\right]^+$$

Then, the LA generates individual prices that are communicated to each consumer separately:

$$p_n(t+1) = p(t+1) + \epsilon_n(t+1)$$

where $\epsilon_n(t)$ are i.i.d. random variables over time and consumers with the CDF (Cumulative Distribution Function) F_ϵ.

The real-time randomized pricing algorithm is given in Algorithm RP. In Algorithm RP, individual prices are generated by adding i.i.d. random noise $\epsilon_n(t)$ to the common price. The noise can have an arbitrary distribution as long as it satisfies Assumption 1 which is not restrictive.

Assumption 1. *F_ϵ is continuous on its domain and strictly increasing from 0 to 1 on an interval $[\epsilon_{min}, \epsilon_{max}]$.*

Under Algorithm RP, users pay for their consumption at the individual price that is privately communicated to them by the LA. Since this price is generated by adding a random disturbance to the common price, the revenue obtained at each time period will be different from the revenue anticipated by the LA. Hence, it is not surprising that RP does not achieve the optimal solution to problem (2). Instead, we will show that RP achieves the optimal solution to a welfare maximization problem that is closely related to the original problem. The basic idea is that communicating randomized prices to consumers induces a utility-function based decision at the consumer side. To demonstrate this, we present a continuous-time fluid approximation of RP which will also be instrumental in analyzing its optimality and convergence.

A. Continuous-time Fluid Approximation Model and Utility-Maximization-Based Formulation

In this section we derive a continuous-time fluid approximation for algorithm RP [21]. Then, we relate the model to a utility maximization problem with modified consumer utility functions induced by price randomization.

The aggregate flexible consumer load is the sum of N binary variables, i.e. $X(t) \triangleq \sum_n x_n^m \mathbb{1}\{\epsilon_n(t) \le \kappa_n q_n(t) - p(t)\}$. Moreover, conditioned on $p(t)$ and $q_n(t)$, each $x_n(t)$ is independent since ϵ_n are independent. Applying the Law of Large Numbers based on this assumption, we obtain the following expression for the aggregate load

$$X(t) \approx \sum_n x_n^m F_\epsilon\left(\kappa_n q_n(t) - p(t)\right). \tag{11}$$

The above expression is the mean behavior for the aggregate load, and when the number of users is large it will well approximate the dynamics of the load.

We define $u_n(x) \triangleq x_n^m F_\epsilon(-x)$, and write

$$x_n(t) \approx u_n\left(p(t) - \kappa_n q_n(t)\right),$$

which approximates the mean behavior of individual users. Next, we present below a continuous-time approximation to RP.

Algorithm RP-C Continuous-time Approximation to RP

$$x_n(t) = x_n^m F_\epsilon\left(\kappa_n q_n(t) - p(t)\right) \tag{12}$$
$$s(t) = \dot{C}^{-1}(p(t)) \tag{13}$$
$$\dot{q}_n(t) = \begin{cases} \lambda_n - x_n(t) & \text{if } {}^{q_n(t)>0, \text{ or}}_{\lambda_n - x_n(t) \geq 0} \\ 0 & \text{otherwise} \end{cases}$$
$$\dot{p}(t) = \begin{cases} \alpha\left(\sum_{n=1}^N x_n(t) + \lambda_S - s(t)\right) \\ \quad \text{if } {}^{p(t)>0, \text{ or}}_{\sum_{n=1}^N x_n(t) + \lambda_S - s(t) \geq 0} \\ 0 \quad \text{otherwise} \end{cases}$$

In (RP-C), consumer loads are computed via the smooth functions u_n. Since F_ϵ is continuous and strictly increasing on $[\epsilon_{min}, \epsilon_{max}]$, u_n is continuous and strictly decreasing, and it has an inverse u_n^{-1}. The domain of u_n^{-1} is the interval $[0, x_n^m]$, possibly with $u_n^{-1}(0) = \infty$ and $u_n^{-1}(x_n^m) = -\infty$, depending on the values of ϵ_{min} and ϵ_{max}. We define function U_n such that $\dot{U}_n(x) \triangleq u_n^{-1}(x)$ on $(0, x_n^m)$, which exists since $u_n^{-1}(x)$ is continuous, and hence integrable. If u_n^{-1} is bounded at the end points of its domain, i.e. 0 and x_n^m, we set $\dot{U}_n(x) = u_n^{-1}(x)$ at these points. Otherwise, we set $U_n(0) = -\infty$ if $u_n^{-1}(0) = \infty$, and $U_n(x_n^m) = -\infty$ if $u_n^{-1}(x_n^m) = -\infty$. Note that by this definition, U_n is strictly concave on $[0, x_n^m]$.

Having defined the functions U_n, we consider the following social welfare maximization problem

$$\min_{\mathbf{x}, s} \quad C(s) - \sum_{n=1}^N U_n(x_n) \tag{14}$$
$$\text{s.t.} \quad \sum_{n=1}^N x_n + \lambda_S \leq s \tag{15}$$
$$\lambda_n \leq x_n, \ \forall n \tag{16}$$

In (14), U_n can be interpreted as a consumer utility function. U_n is strictly concave since u_n^{-1} is strictly decreasing and $\dot{U}_n(x) = u_n^{-1}(x)$. Note that we do not restrict U_n to be a monotone function. Problem (14) is quite similar to problem (5) only with a change in the objective function, where the utility of consumption is amended.

Define p to be the dual variable corresponding to (15), and q_n to be the dual variables corresponding to (16). Let $(\hat{\mathbf{x}}, \hat{s}, \hat{p}, \hat{\mathbf{q}})$ be the optimal primal-dual solution to problem (14). The next theorem shows that RP-C converges to the optimal solution of (14)-(16).

Theorem 1. *The continuous-time approximation algorithm RP-C converges to the optimal solution $(\hat{\mathbf{x}}, \hat{s}, \hat{p}, \hat{\mathbf{q}})$ of Problem (14).*

Proof. See Appendix A □

Note that Theorem 1 does not prove the convergence of the discrete-time algorithm RP. However, it gives insights on the *average* behavior of RP as we will see in Section VI.

V. Change-of-Use Pricing (COUP) Algorithm

In this section, we take a different approach and propose a new pricing scheme. The key idea is to penalize large variations in each consumer's load by introducing a secondary price. In particular, consumers are charged for an extra penalty based on the amount of change in their loads between consecutive time periods, while they still pay for their consumption at each time period at the primary price.

Pricing the change in load can be interpreted as another sort of differentiation among users. In this case, the secondary price introduces heterogeneity among consumption decisions of users. Intuitively, users will prefer changing their consumption more gradually depending on their internal states instead of consuming either the maximum x_n^m or 0. We will show that our pricing algorithm coordinates users' consumption decisions in an asynchronous manner such that changes in users' loads cancel out to create a total load that is flat.

Under the new algorithm COUP, at each time t the common price $p(t)$ and the secondary price γ are announced. The payment at time t for a consumer with load $x_n(t)$ is

$$p(t)x_n(t) + \gamma(x_n(t) - x_n(t-1))^2.$$

Here, the second term is the new component that incurs a penalty (uniform across users and constant over time) on the change of load. Intuitively, this penalty encourages the users to smooth out their loads, and reduces the potential volatility. Having discussed the new pricing scheme, we present the new pricing mechanism in Algorithm COUP below.

Algorithm COUP Change-of-Use Pricing Algorithm

At iteration t:

• Consumer n receives the common price $p(t)$ and the penalty price γ. Then, it computes its queue as in (1) and load as

$$x_n(t) = \left[x_n(t-1) + \frac{1}{2\gamma}\left(\kappa_n q_n(t) - p(t)\right)\right]_0^{x_n^m} \tag{17}$$

• The LA must meet the real load $\sum_n x_n(t) + S_i(t)$. Further, it computes $s(t) = \dot{C}^{-1}(p(t))$, and updates the price:

$$p(t+1) = \left[p(t) + \alpha\left(\sum_{n=1}^N x_n(t) + S_i(t) - s(t)\right)\right]^+$$

In COUP, (17) corresponds to the solution of the following optimization problem given the user's consumption in the previous period $t-1$:

$$\min_{x_n} \left\{(p - \kappa_n q_n)x_n + \gamma(x_n - x_n(t-1))^2\right\}. \tag{18}$$

Drawing direct comparison to RP and the threshold rule (10), we observe that without the second term, (18) is the same problem that a consumer solves under RP. Hence, the second term can be seen as the addition due to the penalty on the change in consumption. Furthermore, the price update rule in COUP is still the same as that in RP. Following a similar method as in the analysis of RP, we will show next that the continuous-time approximation of COUP achieves the optimal objective value of the original problem (5) by solving a closely related welfare maximization problem. In particular, the new welfare maximization problem differs from the original one in its objective, which involves the augmentation of a *proximal term* to the original objective due to the penalty term we introduced in the pricing mechanism.

A. The Continuous-Time Fluid Approximation Model and Welfare-Maximization-Based Formulation

The continuous-time fluid approximation model for COUP is straightforward to obtain and it is presented in Algorithm COUP-C. Similar to what we noted before for the discrete-time algorithms, COUP-C differs from RP-C only in the description of user consumption $x_n(t)$.

Algorithm COUP-C Continuous-time Approximation to COUP

$$\dot{x}_n(t) = \begin{cases} \frac{1}{2\gamma}(\kappa_n q_n(t) - p(t)) & \text{if } {\scriptstyle x_n(t)>0, \text{ or } \atop \scriptstyle \kappa_n q_n(t)-p(t)\geq 0} \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

$$\dot{q}_n(t) = \begin{cases} \lambda_n - x_n(t) & \text{if } {\scriptstyle q_n(t)>0, \text{ or } \atop \scriptstyle \lambda_n - x_n(t)\geq 0} \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

$$s(t) = \dot{C}^{-1}(p(t)) \tag{21}$$

$$\dot{p}(t) = \begin{cases} \alpha\left(\sum_{n=1}^{N} x_n(t) + \lambda_S - s(t)\right) \\ \qquad \text{if } {\scriptstyle p(t)>0, \text{ or } \atop \scriptstyle \left(\sum_{n=1}^{N} x_n(t)+\lambda_S - s(t)\right)>0} \\ 0 \qquad \text{otherwise} \end{cases} \tag{22}$$

Next, we will show that COUP-C converges to a stationary regime where it achieves the optimal objective of the original problem (5). To this end, we augment the objective of problem (5) with an additional cost term motivated by the proximal optimization algorithm [22]. The resulting welfare-maximization problem is

$$\min_{\mathbf{x},\mathbf{y},s} \quad C(s) + \gamma \sum_{n=1}^{N} (x_n - y_n)^2 \tag{23}$$

$$\text{s.t.} \quad \sum_{n=1}^{N} x_n + \lambda_S \leq s \tag{24}$$

$$\lambda_n \leq x_n, \ \forall n, \tag{25}$$

where γ is a positive constant and $y_n \in \mathbb{R}$ are auxiliary variables. It is easy to see that if x_n^* and s^* are the optimal solution to problem (5), then $x_n = x_n^*$, $y_n = x_n^*$, and $s = s^*$ are trivially the optimal solution to problem (23). However, the quadratic term in (23) makes the problem strictly convex in x_n, which helps to alleviate volatility as we will see shortly. As in other proximal optimization algorithms [22], at each iteration we first fix $y_n(t)$ and optimize the objective of (23) over x_n. Let the corresponding optimal solution be $x_n(t)$. We then set $y_n(t+1) = x_n(t)$ and continue with the next iteration. By setting $y_n(t+1) = x_n(t)$, the quadratic term in (23) becomes $\gamma \sum_{n=1}^{N} (x_n(t) - x_n(t-1))^2$, which penalizes the difference in load between periods t and $t-1$.

We consider the Lagrangian function for problem (23) for fixed $y_n = x_n(t-1)$, and obtain the dual function as

$$D(p,\mathbf{q}) = \sum_{n=1}^{N} \min_{x_n} \left\{(p - \kappa_n q_n)x_n + \gamma(x_n - y_n)^2\right\} + \min_{s\geq 0}\{C(s) - ps\} + p\lambda_S + \sum_{n=1}^{N} \lambda_n q_n, \tag{26}$$

where $p \geq 0$ and $\mathbf{q} \triangleq [q_n \geq 0, n = 1, 2, \ldots, N]$ are the dual variables corresponding to the constraints (24) and (25), respectively. The first optimization in (26) is the users' optimization problem (18) whose solution gives the consumption update rule (17) of COUP. The second optimization in (26) is the profit maximization for the LA. Furthermore, inspecting the dual problem reveals that COUP-C corresponds to the dual algorithm for problem (23). Specifically, the primal variables $\mathbf{x}$, s and the dual variables p, $\mathbf{q}$ are updated at each iteration first with $\mathbf{y}$ kept fixed, and then $\mathbf{y}$ is updated at the end of each iteration by setting $\mathbf{y}(t+1) = \mathbf{x}(t)$. Thus, $y(t)$ is dropped from the algorithm description and is replaced with $x(t-1)$.

Having established the relation between COUP-C and problems (5) and (23), we can study the convergence and optimality of COUP-C. Before doing so, we give the definition of the stationary point for the sum of variables.

Definition 1. *Define* $\Phi(t) \triangleq (\sum_n x_n(t), \sum_n q_n(t), p(t), s(t))$. $\Phi^* \triangleq (X^*, Q^*, p^*, s^*)$ *is a stationary point of COUP and COUP-C in the sum sense, if* $\Phi(t_0) = \Phi^*$ *for some* $t_0 < \infty$ *and* $\Phi(t) = \Phi(t_0)$ *for all* $t > t_0$.

Observe that if Φ^* satisfies $p^* = \dot{C}(s^*)$, $s^* = X^* + \lambda_S$, $X^* = \sum_n \lambda_n$, then Φ^* achieves the optimal objective of problem (5). Note that, Φ^* may not achieve the optimal objective of problem (23) since we use $x_n(t-1)$ in place of $y_n(t)$. The next theorem shows that the system of equations given in COUP-C converges to the stationary state as described in Definition 1, where Φ^* achieves the optimal objective of (5).

Theorem 2. *In the system characterized by Algorithm COUP-C,* $\Phi(t)$ *converges to a stationary point* Φ^**, which achieves the optimal objective value of problem (5).*

Proof. See Appendix B □

Theorem 2 shows that the oscillations in price and total load converge to zero under COUP-C. However, COUP-C does not converge for each consumer's load x_n. The following corollary states that each $x_n(t)$ exhibits a sinusoid-like behavior at its stationary operating regime Φ^*.

Corollary 1. *If the initial state of Algorithm COUP-C is* $(\vec{x}(0), \vec{q}(0), s(0), p(0)) = (\vec{x}, \vec{q}, s^*, p^*)$ *such that* $\sum_n x_n =$

X^* and $\sum_n q_n = Q^*$, then the trajectories of $q_n(t)$ and $x_n(t)$ are characterized by

$$q_n(t) = p^* + (q_n(0) - p^*)\cos(\sqrt{\kappa_n/\gamma}t) - \sqrt{\kappa_n\gamma}(x_n(0) - \lambda_n)\sin(\sqrt{\kappa_n/\gamma}t)$$
$$x_n(t) = \lambda_n + (x_n(0) - \lambda_n)\cos(\sqrt{\kappa_n/\gamma}t) + \frac{1}{\sqrt{\kappa_n\gamma}}(q_n(0) - p^*)\sin(\sqrt{\kappa_n/\gamma}t).$$

From Corollary 1, we observe that once the system reaches its stationary operating regime, users' consumption and backlogs follow sinusoidal trajectories. Furthermore, summing $q_n(t)$ and $x_n(t)$ over users reveals that individual trajectories' phases add up to 0, and thus the aggregate consumption and backlog remain constant. Note that Corollary 1 becomes more accurate for COUP as the step sizes κ_n get smaller. But, numerical investigations in Section VI demonstrate that for $\kappa_n = 1$, the system state still shows a similar behavior; Aggregate load and total backlog change smoothly and remain close to the stationary state (X^*, Q^*, p^*, s^*). In Figure 3, this behavior is depicted in discrete-time for Algorithm COUP. Although the load amount $x_n(t)$ of each consumer depends on his own queue length $q_n(t)$, they evolve asynchronously in time as seen in Figure 3 such that the total load is constant.

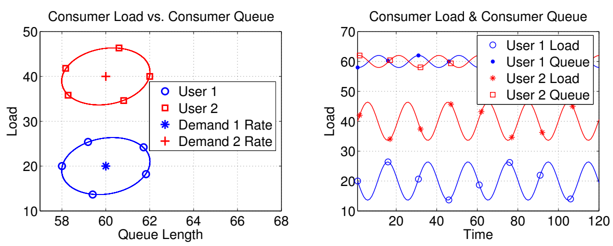

Fig. 3. Equilibrium behavior of Algorithm COUP with 2 consumers having constant demand arrivals, $\gamma = 1$, $\kappa_n = 0.1$, $\alpha = 0.01$, $C(s) = \frac{s^2}{2}$.

VI. Performance and Numerical Results

We now provide numerical results that demonstrate the desirable features of the proposed algorithms. The performance metrics that we consider are the payments made by the consumers and the cost of generation. In terms of these metrics, we compare the performance of RP and COUP to the benchmark schemes. In the rest of this section, the following simulation setup is used unless otherwise is stated: LA cost is set to be $C(s) = \frac{s^2}{2}$. Flexible consumers receive Poisson distributed random arrivals. In RP, $\epsilon_n(t) \sim \mathcal{U}(-\epsilon^m, \epsilon^m)$ for all t. In COUP, $\kappa_n = 1$ and $\alpha = 0.01$.

First, we present the algorithms' behavior over time. In Figure 4, load evolutions obtained by running RP and COUP are plotted for two days. There are 1000 flexible consumers and their total average load is set to be 5% of the total load. Historical metered load data from PJM is used as inflexible load [23]. In Figure 4, we observe a *waterfilling* behavior that results in a smoother load pattern (c.f. Figure 2); Flexible users consume electricity when the inflexible demand is low (i.e. when the average price is low) and fill the valleys in the daily pattern. The effect of price differentiation by randomization among flexible users shows up as the random zigzag pattern for RP. On the other hand, total load is smoother under COUP because of the asynchronous consumption pattern which is discussed in Corollary 1 and plotted in Figure 3.

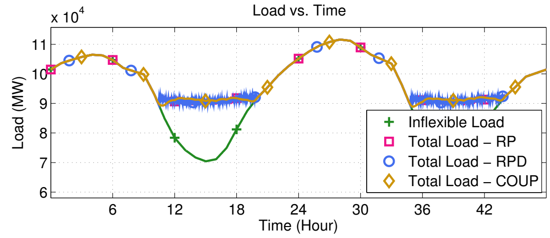

Fig. 4. Waterfilling behavior of the aggregate grid load. In RP, ϵ_m is set to be 1% of the average price. In COUP, γ is set to be 1% of the average price.

A. Consumer and LA Payments

The payment made by consumer n in RP-C is given by

$$r_n(t) = p(t)x_n(t) + x_n^m\mathbb{E}\left[\epsilon_n\mathbb{1}\left\{\epsilon_n \le -\dot{U}(x_n(t))\right\}\right]$$

which follows from (12). On the other hand, the LA anticipates the payment $w(t) = p(t)s(t)$, since it sets $p(t)$ based on the computed value of $s(t)$. At the equilibrium of Algorithm RP-C, the supplier and total consumer payments are given by

$$\hat{r} = \hat{p}\sum_n \hat{x}_n + \sum_n x_n^m\mathbb{E}\left[\epsilon_n\mathbb{1}\left\{\epsilon_n \le -\dot{U}(\hat{x}_n)\right\}\right] \tag{27}$$
$$\hat{\omega} = \hat{p}\hat{s} = \hat{p}\sum_n \hat{x}_n \tag{28}$$

where (28) follows from the KKT condition in (32).

On the other hand, under COUP-C, an individual consumer's payment does not achieve a constant equilibrium value due to the sum convergence result in Theorem 2. Instead, consumer payment has the following time-varying limit when the system is in the stationary regime given in Definition 1:

$$r_n(t) = \hat{p}x_n(t) + \gamma\left(x_n(t) - x_n(t-1)\right)^2 \tag{29}$$

Therefore, total consumer payment is also time-varying, and is given by

$$r(t) = \hat{p}X^* + \gamma\sum_n\left(x_n(t) - x_n(t-1)\right)^2 \tag{30}$$

Besides, supplier payment under COUP-C achieves a constant value in the stationary state as it does under RP-C

$$\hat{\omega} = \hat{p}\hat{s} = \hat{p}X^* \tag{31}$$

Comparing (27) to (28), and (30) to (31), we observe that the amount of payment received from flexible consumers and the amount of anticipated payment computed by the supplier do not necessarily match. The differences between consumer and supplier payments are given by the second terms in (27) and (30). We call this difference the *LA deficit*. Note that the LA deficit is always positive for COUP-C because of the secondary price γ, whereas under RP-C, the deficit can be either negative or positive depending on the distribution of ϵ_n.

Naturally, one wants to make the LA deficit as close as possible to 0 so that the system actually clears in terms of payments. As an example, for algorithm RP-C, consider the case where $\lambda_n = \lambda$, $x_n^m = x_n^m$ for all n, and ϵ_n's have the identical uniform distribution over the interval $[\epsilon, \epsilon + a]$, i.e. $F_\epsilon(x) = \frac{x-\epsilon}{a}$. Then, setting $\epsilon = -\frac{a\lambda}{x^m}$ ensures that the deficit is 0. On the other hand, for COUP-C, decreasing κ_n decreases the frequency of oscillations in individual consumer loads due to the trajectories defined in Corollary 1. Hence, the difference between consecutive time periods decreases, and consequently the secondary term in (30) decreases. In fact, our simulations show that the LA deficit is fairly small for both RP and COUP. For RP, naively setting $\epsilon_n \sim \mathcal{U}(-\epsilon^m, \epsilon^m)$, where ϵ^m is approximately 1% of the average price, ensures that the deficit is no larger than 0.5% of the payment anticipated by the LA. On the other hand for COUP, setting κ_n to 1 and γ to 1% of the average price increases the consumers' payments by only 0.01% while achieving the desired flat load.

B. Impact of Flexible Consumer Penetration on Supply Cost and Payments

We demonstrate the impact of flexible consumer penetration on supply cost and flexible consumer payments in Figure 5. Towards this goal, we vary the number of flexible consumers in the system while keeping the total load constant. In Figure 5, we observe that a flexible consumer's payment is greatly reduced compared to the case where it has to serve its demand immediately (i.e. Scheme 1). Furthermore, compared to the amount they pay under Scheme 2, flexible consumers pay less under RP, and they pay similar or slightly higher under COUP. Thus, we can conclude that consumers significantly benefit from having flexible demand and they will be willing to participate in the new pricing mechanisms to further reduce their payments. Another observation is that payments increase as the number of flexible consumers increases. This is because lower price periods are filled with flexible load, and consequently prices in these periods are not as low as before for the consumers to take advantage of.

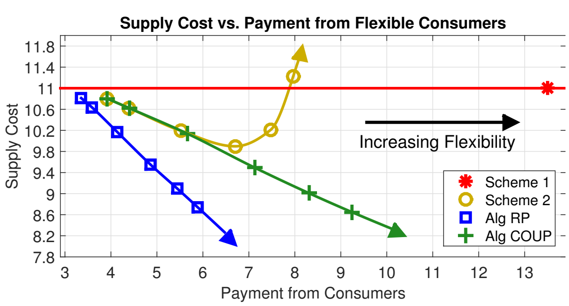

Fig. 5. Algorithms RP and COUP perform better with increasing flexible consumer penetration in the system. In RP, $\epsilon_n(t) \sim \mathcal{U}(-\epsilon_m, \epsilon_m)$, where ϵ_m is set to be 1% of the average of the common price. In COUP, γ is set to be 10% of the average of the common price.

Figure 5 demonstrates that for Scheme 2 there are two regimes in terms of supply cost. In the first regime, where flexible load is less than 20% of the total load, increasing flexibility decreases the cost although Scheme 2 exhibits abrupt fluctuations. A reason for the decrease in cost is the reduced peak load with the increased number of flexible consumers. We also note that our formulation is based on convex cost structure, hence it may not capture efficiently the effect of abrupt fluctuations such as the stress on the network and maintenance costs. However, in the second regime where flexible load is higher than 20%, fluctuations in load become too large and they are reflected directly in cost under Scheme 2. As far as RP and COUP are concerned, they clearly perform better than Scheme 2 in terms of supply cost. The real advantage of the proposed algorithms become more apparent when the flexible consumer penetration increases beyond 20% as seen in Figure 5.

VII. Conclusion

We proposed two novel real-time dynamic pricing schemes that attempt to solve the volatility problem in a system where economically-driven consumers have the flexibility to defer their demand. We demonstrated the destabilizing effect of opportunistic consumer behavior on the load and the price, when conventional real-time pricing methods are employed.

We propose two new pricing schemes to address this problem. In our first pricing scheme, individual consumers receive different prices that are created by adding small random perturbations to a common price. On the other hand, the second algorithm sets a secondary price for all consumers, along with the common price for consumption. The secondary price penalizes abrupt changes in individual users' consumption. The underlying idea in both proposed algorithms is to create differentiation among consumers so that their aggregate behavior is averaged out over the consumer base. The proposed pricing schemes are simple to implement since they do not require any knowledge on consumer strategies, and they can be employed in various systems other than the smart grid where demand has time flexibilities. Furthermore, in the paper, we numerically demonstrated that self-interested consumers economically benefit from deferring their demand while supply cost for the LA is kept low.

Appendix A
Proof of Theorem 1

First, consider the KKT conditions for problem (14):

$$\hat{s} = \sum_n \hat{x}_n + \lambda_S, \; \hat{p} = \dot{C}(\hat{s}), \tag{32}$$

$$\hat{x}_n = \dot{U}_n^{-1}\left(\hat{p} - \hat{q}_n/\kappa\right), \; \hat{x}_n \geq \lambda_n, \; \forall n \tag{33}$$

$$\hat{q}_n(\lambda_n - \hat{x}_n) = 0, \; \forall n \tag{34}$$

To establish the convergence of RP-C, we show that the following Lyapunov function is strictly decreasing:

$$V(t) = \frac{1}{2\alpha}(p(t) - \hat{p})^2 + \frac{1}{2}\sum_n \kappa_n (q_n(t) - \hat{q}_n)^2. \tag{35}$$

The drift is given by

$$\begin{aligned} \dot{V}(t) &= \frac{1}{\alpha}(p(t) - \hat{p})\dot{p}(t) + \sum_n \kappa_n (q_n(t) - \hat{q}_n)\dot{q}_n(t) \\ &\leq (p(t) - \hat{p})\left(\sum_n (x_n(t) - \hat{x}_n) + (\hat{s} - s(t))\right) \end{aligned}$$

$$+\sum_n \kappa_n (q_n(t)-\hat{q}_n)((\lambda_n-\hat{x})+(\hat{x}-x_n(t))) \quad (36)$$

where (36) is obtained by adding and subtracting $\hat{x}_n$ and $\hat{s}$, and noting from (32) that $\hat{s}=\sum_n \hat{x}_n+\lambda_S$. From complementary slackness condition given in (34), $\hat{q}_n(\lambda_n-\hat{x}_n)=0$ for all n. Also, $q_n(t)(\lambda_n-\hat{x}_n)\le 0$ due to dual and primal feasibility. Hence, $(q_n(t)-\hat{q}_n)(\lambda_n-\hat{x}_n)\le 0$. Using this in (36), we get

$$\begin{aligned}\dot{V}(t) \le &\sum_n \left(p(t)-\kappa_n q_n(t)-(\hat{p}-\kappa_n\hat{q}_n)\right)(x_n(t)-\hat{x}_n)\\ &-(p(t)-\hat{p})(s(t)-\hat{s})\end{aligned}$$

From (12), (33), and using the strict concavity of U_n we get

$$\begin{aligned}&\left(p(t)-\kappa_n q_n(t)-(\hat{p}-\kappa_n\hat{q}_n)\right)(x_n(t)-\hat{x}_n)\\ &=\left(\dot{U}_n(x_n(t))-\dot{U}_n(\hat{x}_n)\right)(x_n(t)-\hat{x}_n)\le 0,\end{aligned}$$

and the equality holds if and only if $x_n(t)=\hat{x}_n$. Similarly, due to strict convexity of C and from (13), $(p(t)-\hat{p})(s(t)-\hat{s})\ge 0$ where the equality holds if and only if $p(t)=\hat{p}$. Hence, we obtain $\dot{V}(t)<0$, and the equality holds if and only if $p(t)=\hat{p}$ and $x_n(t)=\hat{x}_n$ for all n. Thus, $\dot{V}(t)<0$ unless $(\mathbf{x}(t),s(t),p(t),\mathbf{q})=(\hat{\mathbf{x}},\hat{s},\hat{p},\hat{\mathbf{q}})$. As a result, the algorithm converges to the optimal point $(\hat{\mathbf{x}},\hat{s},\hat{p},\hat{\mathbf{q}})$.

APPENDIX B
PROOF OF THEOREM 2

Define $\Theta(t)\triangleq(\mathbf{x}(t),\mathbf{q}(t),p(t),s(t))$ to be the system state at t, and consider the following Lyapunov function:

$$\begin{aligned}V(\Theta(t))=&\frac{N}{2\alpha}\left(p(t)-p^*\right)^2\\ &+\gamma\left(\sum_n x_n(t)-X^*\right)^2+\frac{1}{2}\left(\sum_n \kappa_n q_n(t)-Q^*\right)^2 \quad (37)\end{aligned}$$

where Φ^* is given in Definition 1 and achieves the optimal objective of (5). The derivative of $V(\Theta(t))$ w.r.t. t is

$$\begin{aligned}\dot{V}(\Theta(t))\le&\left(\sum_n x_n(t)-X^*\right)\sum_n(\kappa_n q_n(t)-p(t))\\ &+\left(\sum_n \kappa_n q_n(t)-Q^*\right)\sum_n \kappa_n(\lambda_n-x_n(t))\\ &+N\left(p(t)-p^*\right)\left(\sum_n x_n(t)+\lambda_S-s(t)\right) \quad (38)\end{aligned}$$

Adding and subtracting the stationary values of Φ^*, we obtain

$$\begin{aligned}&\dot{V}(\Theta(t))\\ &\le\left(\sum_n x_n(t)-X^*\right)\left(\sum_n \kappa_n q_n(t)-\kappa_n Q^*+Np^*-Np(t)\right)\\ &+\left(\sum_n \kappa_n q_n(t)-Q^*\right)\left(\sum_n \lambda_n-X^*+X^*-\sum_n x_n(t)\right)\\ &+(p(t)-p^*)\left(\sum_n x_n(t)-X^*+s^*-s(t)\right) \quad (39)\\ &=\left(\sum_n x_n(t)-X^*\right)\kappa_n\left(\sum_n q_n(t)-Q^*\right)\\ &+\left(\sum_n x_n(t)-X^*\right)N\left(p^*-p(t)\right)\\ &+\kappa_n\left(\sum_n q_n(t)-Q^*\right)\left(X^*-\sum_n x_n(t)\right)\\ &+N(p(t)-p^*)\left(\left(\sum_n x_n(t)-X^*\right)+(s^*-s(t))\right) \quad (40)\\ &=N(p(t)-p^*)(s^*-s(t)) \quad (41)\end{aligned}$$

In (39), we used $\kappa_n Q^*=Np^*$, $s^*=X^*=\sum_n\lambda_n$. Noting that $p(t)=\dot{C}(s(t))$ and C is convex, we conclude

$$\dot{V}(\Theta(t))\le N(p(t)-p^*)(s^*-s(t))\le 0. \quad (42)$$

Now, we will show that $\lim_{t\to\infty}V(\Theta(t))=0$ and that the algorithm converges to Φ^*. First, we define an *invariant* set w.r.t. COUP-C to be the set of states such that any trajectory with an initial point in this set indefinitely remains in it. Define $\mathcal{S}\triangleq\left\{\Theta(t):\dot{V}(\Theta(t))=0\right\}$ and set $\mathcal{I}$ to be the largest invariant set in $\mathcal{S}$. Since $\dot{V}(\Theta(t))\le 0$ whenever Θ is outside of $\mathcal{S}$, by LaSalle's principle [24] any trajectory $\{\Theta(t),t\ge 0\}$ of the algorithm asymptotically approaches to the set $\mathcal{I}$.

It remains to show that the invariant set $\mathcal{I}$ consists of only the points which are stationary points of COUP, i.e.

$$\mathcal{I}=\left\{\Theta\in\mathcal{S}:\left(\sum_n x_n(t),\sum_n q_n(t),p(t),s(t)\right)=\Phi^*\right\}$$

Let $\Theta(0)\in\mathcal{I}$. Since $\dot{V}(\Theta(t))=0$ in $\mathcal{I}$, $p(t)=p^*$ and $s(t)=s^*$ for $t\ge 0$ due to (42). Hence, $\dot{p}(t)=0$ and $\sum_n x_n(t)=s^*=X^*$ for $t\ge 0$ from (22). Observing (37), and noting $\sum_n\dot{x}_n(t)=0$ and $\dot{p}(t)=0$, we have $\sum_n\dot{q}_n(t)=0$. Thus, it remains to show that $\sum_n q_n(t)=Q^*$.

Let $\mathcal{N}_1(t)\triangleq\{n:\dot{x}_n(t)\neq 0\}$ and $\mathcal{N}_2(t)\triangleq\{n:\dot{x}_n(t)=0\}$. We assume that $|\mathcal{N}_2(t)|>0$, and we will show a contradiction in the following. First, we note that $x_n(t)=0$ for $n\in\mathcal{N}_2(t)$, and then plug it in (20) to get $\sum_{\mathcal{N}_2(t)}\dot{q}_n(t)=\kappa_n\sum_{\mathcal{N}_2(t)}\lambda_n>0$. For sufficiently small κ_n we can assure, for $n\in\mathcal{N}_2(t)$, that $q_n(t+\delta t)<p^*$ and $x_n(t+\delta t)=0$, and consequently that $n\in\mathcal{N}_2(t+\delta t)$.

For $n\in\mathcal{N}_1(t)$ consider two cases: (i) $\mathcal{N}_1(t)=\mathcal{N}_1(t+\delta t)$, and (ii) $\mathcal{N}_1(t+\delta t)\subset\mathcal{N}_1(t)$, i.e. some $n\in\mathcal{N}_1(t)$ moves to $\mathcal{N}_2(t+\delta t)$. For case (i), summing (19) over all n we obtain $\sum_{\mathcal{N}_1(t)}q_n(t)=\sum_{\mathcal{N}_1(t+\delta t)}q_n(t+\delta t)=|\mathcal{N}_1(t)|p^*$, and hence $\sum_{\mathcal{N}_1(t)}\dot{q}_n(t)=0$, which is contradictory because $\sum_n\dot{q}_n(t)=\sum_{\mathcal{N}_1(t)}\dot{q}_n(t)+\sum_{\mathcal{N}_2(t)}\dot{q}_n(t)=-\kappa_n\sum_{\mathcal{N}_2(t)}\lambda_n\neq 0$. For case (ii), let $\mathcal{M}\triangleq\mathcal{N}_1(t)\setminus\mathcal{N}_1(t+\delta t)$. Then we have

$$\begin{aligned}&\sum_n q_n(t+\delta t)-\sum_n q_n(t)\\ &=\sum_{\mathcal{N}_1(t+\delta t)}q_n(t+\delta t)-\sum_{\mathcal{N}_1(t)}q_n(t)\\ &\quad+\sum_{\mathcal{N}_2(t+\delta t)}q_n(t+\delta t)-\sum_{\mathcal{N}_2(t)}q_n(t)\\ &=|\mathcal{N}_1(t+\delta t)|p^*-|\mathcal{N}_1(t)|p^*\\ &\quad+\sum_{\mathcal{N}_2(t)}(q_n(t+\delta t)-q_n(t))+\sum_{\mathcal{M}}q_n(t+\delta t) \quad (43)\end{aligned}$$

$$= -|\mathcal{M}|p^* + \sum_{\mathcal{N}_2(t)} (q_n(t+\delta t) - q_n(t))$$
$$+ \sum_{\mathcal{M}} (q_n(t+\delta t) - q_n(t)) + \sum_{\mathcal{M}} q_n(t)$$

where we used $\sum_{\mathcal{N}_1(t+\delta t)} q_n(t+\delta t) = |\mathcal{N}_1(t+\delta t)|p^* > 0$ to obtain (43). Dividing by δt, and taking the limit as $\delta t \to 0$:

$$\sum_n \dot{q}_n(t) = \sum_{\mathcal{N}_2(t)} \dot{q}_n(t) + \sum_{\mathcal{M}} \dot{q}_n(t) \tag{44}$$

Subtracting $\sum_n \dot{q}_n(t)$ from both sides of (44), we obtain $\sum_{\mathcal{N}_1(t+\delta t)} \dot{q}_n(t) = 0$.

From (19), $q_n(t) \geq p^*$ implies $\dot{x}_n(t) \geq 0$. However, if $x_n(t) > 0$ we have $n \in \mathcal{N}_1(t+\delta t)$; if $x_n(t) = 0$ we have $\dot{q}_n(t) > 0$ and thus $n \in \mathcal{N}_1(t+\delta t)$. Therefore $q_n(t) < p^*$ for $n \in \mathcal{M}$, and consequently $\sum_{\mathcal{N}_1(t)\setminus\mathcal{M}} q_n(t) = \sum_{\mathcal{N}_1(t+\delta t)} q_n(t) > |\mathcal{N}_1(t) \setminus \mathcal{M}|p^* = |\mathcal{N}_1(t+\delta t)|p^*$. However, $\sum_{\mathcal{N}_1(t+\delta t)} q_n(t+\delta t) = |\mathcal{N}_1(t+\delta t)|p^*$ implying $\sum_{\mathcal{N}_1(t+\delta t)} \dot{q}_n(t) < 0$, which is in contradiction with (44).

As a result $\mathcal{N}_2 = \emptyset$, and $|\mathcal{N}_1| = N$. Therefore, $\sum_n q_n(t) = \sum_{\mathcal{N}_1} q_n(t) = Np^* = Q^*$.

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