Multi-Rate Multi-Casting with Intra-Layer Network Coding

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***Abstract*— Multi-rate multi-casting is a generalization of single-rate multi-casting to prevent destinations with good connections from being limited by the capacity of bottleneck connections. While multi-rate multi-casting has been traditionally performed over fixed trees, advances in network coding theory have enabled higher throughput and have helped us move beyond the restriction of tree structures for routing the multi-cast data. In this work, we address the questions of optimal rate allocation and low-complexity network coding solutions to the problem of multi-rate multi-casting in general multi-hop networks. Our work considers intra-layer network coding capabilities, where the session is conceptually divided into layers optimally and coding is performed across packets belonging to the same layer. Our approach differs from earlier works in this domain in its separation of the problem into rate allocation and content distribution items, which allows a number of optimization and graphical techniques in their solution. Noting the complexities involved in the optimal rate allocation and content distribution solutions, we then propose and investigate two novel approaches for reducing the complexity of the original scheme for more practical implementation based on a *layered multi-casting mechanism* and *nested optimization approach*. We demonstrate the implementation advantages of these low-complexity schemes via extensive numerical studies.**

I. Introduction

Multi-casting is the transfer of information of common interest from a source to a set of receivers spread in the network. It finds applications in multi-media broadcasts, group communication in social networks, etc. The multi-cast traffic in networks can constitute a significant portion of the total traffic (e.g. as high as 80% in military communications), and hence it is imperative that they are served efficiently.

Traditionally, multi-casting refers to *Single-Rate Multi-Casting (SRMC)* where all receivers receive a common information stream at the same rate, hence the name. The celebrated max-flow min-cut theorem [1] immediately implies that the maximum possible rate of a SRMC session is upper-bounded by the maximum flow rate supportable between the source and the receiver with the *worst* connection. The seminal work of Ahslwede et al. [2] and subsequent findings [3], [4], [5] have established that this upper bound is achievable via a quite practical strategy of *random linear network coding*, whereby relay nodes within the network forward random linear combinations of the SRMC session packets that traverse them. This is called *intra-session* network coding since the mixing is restricted to packets of the same SRMC session (see Section II for more detail).

Yet, SRMC may yield low utilization of the network resources and low user satisfaction when a subset of the receivers creates a bottleneck for the whole multi-cast group. Such a deficiency is likely to happen under heterogeneous network conditions and for sessions with large receiver sets. This motivates us in this work to consider *Multi-Rate Multi-Casting* (MRMC) that allows different transmission rates to different receivers of the same session, leading to better network utilization and increased end-user satisfaction.

Contrary to SRMC, the full characterization of maximum achievable *multi-rate vectors* for MRMC and practical schemes that achieve these rate vectors proved to be extremely difficult as one needs to ensure the decodability of all the packets at the destination nodes. Algebraic conditions for the existence of a multi-rate multi-cast network code are derived in [4], and finding the optimal network code for a given network that maximizes the total amount of information flow is proved to be NP-hard in [6].

Previous works on multi-rate multi-casting lie between two extremes of multi-rate multi-casting over predefined trees without network coding capabilities, and multi-rate multi-casting over the full network graph with inter-session coding capabilities. While the achievable region expands from the former to the latter extreme, the resulting solutions also become more difficult to implement. Next, we briefly summarize these results to position our work within this context.

Works that focus on multi-rate multi-casting without network coding capabilities are restricted to fixed tree structures: [7] has proposed an optimization-based approach for a given fixed routing tree associated with each multi-cast session; [8] has developed multi-rate multi-casting algorithm based on scheduling virtual (shadow) "traffic" that "moves" in the opposite direction from destinations to sources; [9] has considered multi-casting over predefined M-Best trees to develop a strategy for dynamic switching between the trees depending on the observed congestion level. These works share a common attractive characteristic of dynamism in their solutions, while suffering from the limitations of routing over network coding.

In contrast, a different thread of works exist that assume a range of network coding capabilities. The main focus of these works has been under the banner of *Layered Multi-Cast* or *Multi-Resolution Coding Condition*, which requires separation of the information into successive layers with each layer requiring the decoding of lower layers (see Section II for more detail). Under such a condition, the design objective is to maximize the network throughput while also ensuring that each receiver receives only consecutive layers of data (including the base layer). Contributions of some of the previous works in this thread are: [6] has formulated the problem of *Rainbow Network Coding* (RNC), and showed that performance of linear network coding at intermediate nodes of a flow path can achieve higher network throughput than the original rainbow network flow solution; [10] has proposed a polynomial time algorithm for multi-casting to heteroge-

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neous receivers using network coding; [11] has considered the problem of multi-rate multi-cast in overlay networks; and [12] has provided network coding based multi-rate multimedia streaming in directed networks, aimed at maximizing the total layers received by all receivers. All these works operate with intra-layer network coding, i.e. they mix the packets belonging to the same layer. More recently, there have been works related to *inter-layer* network coding between packets belonging to different layers: [13] has proposed a layered multi-cast scheme with the capability of inter-session network coding across the different layers; [14] has proposed an algorithm which makes use of both intra-layer and inter-layer mixing between the packets guaranteeing the decodability of the base layer at all the receivers.

Noting the fairly complex nature of operation under inter-layer coding capabilities, in this work, we consider multi-rate multi-casting under easily implementable intra-layer coding capabilities, which nevertheless yield significant improvement over routing solutions. This scenario is further interesting as its achievable rate region lends itself to dynamic algorithm design. Our approach differs from earlier works in this domain that strictly enforce the multi-resolution coding condition. The following example illustrates that this requirement can lead to sub-optimal rate allocations if strictly enforced.

Example 1: For the butterfly network with antenna depicted in Figure 10, we can see that the max-flow min-cut specifies rates $(2, 2, 1)$ units as the maximum achievable to the destinations (d_1, d_2, d_3), respectively. In this example, we can see

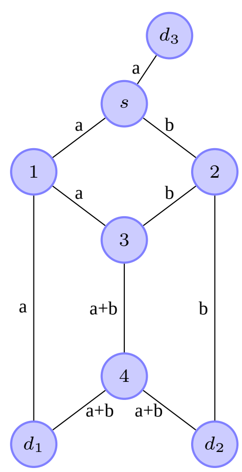

Fig. 1. Butterfly network with Antenna

that rates of $(2, 2, 1)$ is also achievable with intra-layer coding under our relaxation: send at rate 2 to destinations d_1 and d_2 over one layer, and send at rate 1 to destination d_3 over the other layer. The contents of these layers can be distributed as: packets (a, b) for subsession 1 and packet a for the subsession 2. The content can be distributed across the butterfly network as shown in the Figure 10. Note that to be able to decode packets at the destinations, we need to make the assumption of intra-layer coding and also relax the multi-resolution coding condition for rate allocation.

However, this multi-rate vector is not achievable with intra-layer coding if the layers are constructed iteratively with strict enforcement of multi-resolution coding condition in each iteration. Instead, rate vectors $(2, 2, 0)$ or $(1, 1, 1)$ can be achieved. Notice that in this case, destination d_3 cannot decode the packet of subsession 2 (packet a) without actually decoding the packets from subsession 1 (packet a, b) and hence achieves a rate of 0. An alternate scheme would be to send packet a to all the destinations so that each one achieves a rate of 1. In either of the two schemes, it can be seen that the network does not achieve its maximum utility.

This example demonstrates that in order to achieve the maximum possible rate, it is not practical to impose restrictions on the decodability between the layers from start. Hence, it is important that we formulate algorithms which do not strictly impose such restrictions. It also suggests that the problem of multi-rate multi-casting can be separated into steps of *rate allocation* and *information distribution* to obtain larger rates, both with or without multi-resolution coding condition.

With this motivation, in this work, we pursue an alternative direction towards the design of low-complexity rate control and coding algorithms that utilize simple random linear coding capabilities in the context of multi-rate multi-casting. Our contributions can be outlined as follows:

• In Section II, we propose a tractable framework based on the relaxation of the multi-resolution coding condition for the moment to separate the rate allocation and information content distribution problems. This separation is important as it enables the use of different approaches and tools in the resolution of these problems, summarized next.

• In Section III-A, we formulate the rate allocation problem under intra-layer coding capabilities to maximize user satisfaction, measured via appropriate utility functions. Then, we utilize primal-dual subgradient methods to develop an optimal rate allocation and intra-layer coding algorithm for MRMCing.

• In Section III-B, we consider the complementary problem of optimal information content distribution over the optimal rate allocation obtained in the previous step. The aforementioned separation allows the use of graphical models and dynamic-programming-based solutions to this problem.

• Using the above optimal solutions as a foundation, we then turn to more practical considerations in their implementation: (i) in Section IV-A, we re-impose the multi-resolution condition to develop a novel layered rate allocation policy that build on the pricing mechanism of the optimal policy; (ii) in Section IV-B, we develop a two-level optimization framework for simultaneous utility maximization and cost minimization to steer the rate allocation towards sparse solutions within the set of optimal allocations for easy implementation; (iii) in Section IV-C, we propose a greedy information distribution scheme that builds on the graphical model of Section III-B, and works together with the sparse solution of the previous item to significantly reduce the computational complexity of the optimal solution and render its operation more practical.

II. System Model and Objectives

Consider a fixed multihop network that is described by a graph $\mathcal{G} = (\mathcal{N}, \mathcal{L})$, where $\mathcal{N}$ is the set of nodes with cardinality N and $\mathcal{L}$ is the set of directed links. We assume that the network operates in a time slotted fashion, where each link $(i, j) \in \mathcal{L}$ has a fixed capacity of $c_{ij} \in \{1, 2, \cdots\}$ packets per slot. Each packet, in turn, is represented by a vector over a finite field $\mathbf{F}_z$, where z is assumed to be a large positive integer. The network is required to serve many applications that generate a flow of information from a source to a set of destinations, leading to the notion of a multi-cast session.

Definition 1 (Multi-Cast (MC) Session): A *multi-cast session* $(s, \mathcal{D})$ is described by its source node s and a set of destination nodes $\mathcal{D} \in \mathcal{N}\backslash\{s\}$ that are interested in the information generated at s. We denote the cardinality of $\mathcal{D}$ as D in the sequel. A session is called *unicast* if $D = 1$. $\diamond$

For some multi-cast sessions, it is required that all destinations in the session receive the exact same information. This occurs in scenarios where critical information must be transmitted to all intended parties without exception, and leads to the notion of single-rate multi-cast session we define next.

Definition 2 (Single-Rate Multi-Cast (SRMC) Session): A *single rate multi-cast session* $(s, \mathcal{D}, r)$ with $r \in \mathbb{R}_+$ is a multi-cast session $(s, \mathcal{D})$ where all destinations must receive the common information generated by the source at a rate of r packets per slot. $\diamond$

Hence, maximum rate $r^\star$ achievable by a SRMC session $(s, \mathcal{D}, r)$ is restricted by the smallest rate achievable by all the destinations in $\mathcal{D}$. It is shown in the seminal work [2] and subsequent literature (e.g. [3], [4]) that this maximum common rate $r^\star$ is given by the minimum (over $d \in \mathcal{D}$) of the maximal cut value (over all cuts that separate s from d), and that it is achievable by linear network coding.

However, there are many applications for which the goal should, more appropriately, be to achieve, not the maximum common information flow rate to all destinations, but the maximum possible rate of information flow to each destination while trying to keep the common information to a maximum. A typical example of such applications would be multi-media streaming, where different destinations can receive information at different video or voice quality levels depending on the heterogeneity in the topology and the resource availability of the underlying network. Moreover, we would like to distinguish the preferences of different destinations since they may be willing to value the flow rate non-homogeneously. To address such considerations, we next define the notion of a multi-rate multi-cast session.

Definition 3 (Multi-Rate Multi-Cast (MRMC) Session): A *multi-rate multi-cast session* $(s, \mathcal{D}, \mathbf{y})$ with $\mathbf{y} = (y_d)_{d\in\mathcal{D}} \in \mathbb{R}_+^D$ is a multi-cast session $(s, \mathcal{D})$ where destination $d \in \mathcal{D}$ must receive the information generated by the source at a rate of y_d packets per slot. $\diamond$

Thus, an MRMC allows different rate of information flow from the source to different destinations of the session. Notice that the above definition of MRMC intentionally avoids imposing any restrictions on the common information structure amongst the data received by the destinations. Instead, we will next define a multi-resolution coding condition separately that imposes a rather strict condition on the multi-layer encoding/decoding structure of an MRMC. We shall see that this separation will allow us to distinguish the maximal rate allocation problem from minimal information distribution problem.

Definition 4 (Multi-Resolution Coding Condition): An MRMC session $(s, \mathcal{D}, \mathbf{y})$ is said to satisfy the *multi-resolution coding condition* if the information generated at the source node s can be encoded into distinct *layers* $L_1, L_2, \cdots$ of information flows with associated appropriate[1] rates $r_1, r_2, \cdots$, where for any $m > 1$, the information contained in Layers $L_1, \cdots, L_{m-1}$ is necessary to decode the information contained in Layer L_m. $\diamond$

It is straight-forward to see that, if an MRMC $(s, \mathcal{D}, \mathbf{y})$ satisfies the multi-resolution coding condition, then the rate of information flow generated by the source equals $\max_{d\in\mathcal{D}} y_d$. [2]

Multi-resolution coding condition is appealing due to the nested nature of encoding/decoding it enables: layers $L_1, L_2, \cdots$ are conceptually linked linearly as $L_1 \to L_2 \to \cdots$, where $L_i \to L_j$ indicates that the information of Layer L_i is necessary to decode Layer L_j. However, this condition can also be restrictive in reducing the achievable rates from what is achievable by more sophisticated encoding/decoding capabilities. This motivates us to relax the multi-resolution coding condition in the rate allocation, and impose it later on if it is required.

Once source rate vector is divided into layers with their corresponding rates, we allow nodes to perform linear *intra-layer* (or *intra-subsession*) network coding across packets that traverse them. More precisely, each node can transmit a linear combination over the finite field $\mathbf{F}_z$ of any packets it has previously received at a given layer ([2]). The achievable MRMC rate region with the above intra-layer coding capabilities vectors is denoted as $\mathcal{Y}$. We postpone the precise description of this region to after we introduce the notion of *subsessions* in this section.

Acknowledging the fact that destinations of the same MRMC session may differ in their preferences, we assume that the utilization achieved at destination $d \in \mathcal{D}$ is measured by a utility function $U_d(\cdot)$ of its mean rate y_d. We make the following standard assumptions on the utility function (e.g. [15], [16]).

Assumption 1: For each destination $d \in \mathcal{D}^s$, the utility function $U_d(\cdot)$ satisfies the following conditions:
- $U_d(.)$ is a strictly concave [3], twice differentiable, non-decreasing function of mean rate per session y_d.
- For every m and M satisfying $0 < m < M < \infty$, there exists constants $\tilde{c}$ and $\tilde{C}$ satisfying $0 < \tilde{c} < \tilde{C} < \infty$ such that

$$\tilde{c} \le -\frac{1}{U_d''(y_d)} \le \tilde{C}, \quad \forall y_d \in [Km, KM],$$

where $K = 2^D$.

We note that these conditions hold for a large class of utility functions, including $U_d(y_d) = (y_d)^{(1-\phi)}/(1-\phi)$ for $\phi > 0$ that is known to capture a large class of fairness criteria (see [17]). [4].

As the content of different layers do not get mixed with intra-layer coding, the source must decide how to distribute information across these layers in a favorable manner. This

[1] A flow rate vector $\mathbf{r} = (r_i)_i$ with r_i denoting the flow rate at Layer L_i is appropriate if, for each destination $d \in \mathcal{D}$, a consecutive sequence of layers $L_1, L_2, \cdots, L_{l(d)}$ are decodable with $\sum_{i=1}^{l(d)} r_i = y_d$.

[2] Yet, having the rate of flow generated by the source to equal $\max_{d\in\mathcal{D}} y_d$ does not imply the multi-resolution coding condition.

[3] Function f is strictly concave if for any two points x and y in its domain C and any $t \in (0, 1)$, we have $f(tx + (1-t)y) > tf(x) + (1-t)y$

[4] We remark that the strictness of the concavity of $U_d(\cdot)$ can be relaxed without too much complication

allows a conceptual separation between rates and content as: (1) layering/rate allocation for maximum utilization; and (2) information content distribution over layers for maximum information packing.

Thus, the goals of the rate allocation and information distribution can be separately stated as follows.
(i) The goal of rate allocation is to solve the optimization problem given by

$$\mathbf{y}^\star \triangleq \underset{\{\mathbf{y}\geq\mathbf{0}\}}{\arg\max} \sum_{d\in\mathcal{D}} U_d(y_d) \quad s.t. \quad \mathbf{y}\in\mathcal{Y}, \tag{1}$$

where we recall that $\mathcal{Y}$ denotes the convex[5] set of all end-to-end rates achievable via intra-layer network coding, and $\mathbf{y}^\star$ is the *unique* solution due to the strictly concave nature of $U_d(\cdot)$. In Section III-A, we provide an iterative and distributed solution to this problem by characterizing the problem in terms of subsessions and providing an allocation through primal-dual optimization methods.
(ii) The goal of information content distribution is to distribute the maximum possible amount of common information to all the destinations at the rates imposed by the rate allocation solution. In Section III-B, we attack this problem using a novel graphical model and a recursive scheme that can be operated at the source of each MRMC session independently.

We note several potential complications associated with the above solutions. First, the multi-resolution coding condition is not necessarily satisfied at its conclusion. While this may be acceptable for some applications, some applications may strictly require such layering. Second, the rate allocation may lead to the use of many layers with small rates, which causes implementational and architectural complexity. Instead, we prefer the allocated rates to be sparse, i.e., distributed over a small number of layers for guaranteeing implementation and architectural complexity. Third, the dynamic-programming (DP) based information distribution solution suffers from the well-known curse-of-dimensionality of DP. We address all of these issues by developing novel schemes for each, listed next.
• In Section IV-A, we utilize the pricing strategy of our earlier developed rate allocation policy to develop a strategy that operates in rounds to generate *nested layers* that guarantee the multi-resolution coding condition (see Definition 4).
• In Section IV-B, we introduce a two-level optimization framework and propose a dynamic scheme to minimize any given convex cost metric within the class of utility maximizing solutions. By using the proposed cost function within this framework, we steer the rate allocation towards a sparse solution with favorable implementation characteristics.
• In Section IV-C, we propose a recursive scheme for information distribution that significantly reduces the computational complexity of DP-based solution. The sparsity of allocation established in the previous items fits perfectly with this recursive solution to render it practical for many scenarios.

III. Optimal Rate Allocation and Information Distribution

In this section, we attack the rate allocation and information content distribution problems listed in the previous section. The conceptual separation of these two problems introduced in the previous section allows us to solve these through different methods. Thus, in Section III-A we utilize optimization tools to solve the rate allocation problem, and in Section III-B we utilize graphical models to solve the information content distribution problem.

A. Optimal Subsession Rate Allocation

In this section, we propose a decentralized, iterative solution to the MRMC rate allocation problem (1). We recall that $\mathcal{Y}$ denotes the set of end-to-end rates $\mathbf{y}$ achievable by an MRMC session $(s,\mathcal{D},\mathbf{y})$ with intra-layer coding capabilities. This region is difficult to characterize directly because its structure depends on how the layers are constructed. Next, we provide a decomposition that enables a simple rate region characterization and facilitates our solution.

We decompose a given MRMC session $(s,\mathcal{D},\mathbf{y})$ into many SRMC subsessions, where each *Subsession* k is described by the same source node s as the MRMC session, a subset of destinations $\mathcal{D}_k\subset\mathcal{D}$, and a single-rate $x_k\in\mathbb{R}_+$. Accordingly, we denote the SRMC Subsession k as $(s,\mathcal{D}_k,x_k)$. Noting the potential number of subsessions to be 2^D, the set of subsessions is denoted by $\mathcal{K}\triangleq\{1,2,...,2^D\}$ with cardinality $K\triangleq 2^D$. With this decomposition, the MRMC rate vector $\mathbf{y}$ and the subsession rates $\mathbf{x}:=(x_k)_{k=1}^K$ are related as

$$y_d(\mathbf{x})=\sum_{k\ni d} x_k, \qquad \text{for all } d\in\mathcal{D}, \tag{2}$$

where $k\ni d$ is henceforth used to denote $\{k\in\mathcal{K}: d\in\mathcal{D}_k\}$, i.e., all subsessions that have d as a destination. Also, we denote the cardinality of set $\mathcal{D}_k$ by the notation D_k.

Subsession k injects packets with an average rate of x_k to be multi-cast to its destination group $\mathcal{D}_k$, where each packet is represented by a vector over a finite field $\mathbf{F}_z$. Following the intra-layer coding capabilities, we allow each subsession to perform *intra-subsession network coding*, where the incoming packets of the same subsession can be linearly combined for transmission at each node. In particular, we assume a random linear combination of the packets $\{\mathbf{P}_1,\cdots,\mathbf{P}_J\}$ given by $\sum_{j=1}^J \theta_j\mathbf{P}_j$, where $\{\theta_j\}$ are randomly selected coefficients from the finite field. Such random intra-subsession coding operations are known to maximize the achievable throughput region under intra-coding capabilities (see e.g. [2], [5]). We aim to exploit such network coding capabilities for subsessions while optimizing the subsession rates to yield the best multi-rate session performance.

The advantage of our decomposition is that it enables a tractable achievable rate region characterization of $\mathcal{Y}$ in terms of subsession rates, denoted $\mathcal{X}$. The characterization of set $\mathcal{X}$, provided next, follows directly from previous works (e.g. [18] and [19]) that discuss the flow level characterization of achievable rate region of intra-session network coding.

Definition 5 (Achievable Subsession Rate Region, $\mathcal{X}$): Under intra-subsession network coding capabilities, the achievable subsession rate region $\mathcal{X}$ for an MC session $(s,\mathcal{D})$ contains all subsession rate vectors $\mathbf{x}=(x_k)_{k\in\mathcal{K}}\in\mathbb{R}_+^K$ for which there exist a non-negative-valued *information flow rate* vector $\mathbf{f}\triangleq\left(f_{ij}^{(d,\mathcal{D}_k)}\right)_{\{(i,j)\in\mathcal{L},d\in\mathcal{D}_k,k\in\mathcal{K}\}}$, and a non-negative-valued *physical flow rate* vector $\mathbf{r}\triangleq\left(r_{ij}^{\mathcal{D}_k}\right)_{\{(i,j)\in\mathcal{L},k\in\mathcal{K}\}}$

[5]That $\mathcal{Y}$ is convex follows trivially from the ability to achieve any convex combination of two rates in it via proper time sharing.

satisfying

$$f_{out(i)}^{(d,\mathcal{D}_k)} - f_{in(i)}^{(d,\mathcal{D}_k)} = x_i^k, \qquad \forall i \neq d, \forall k, \forall d \in \mathcal{D}_k \tag{3}$$

$$f_{ij}^{(d,\mathcal{D}_k)} \leq r_{ij}^{\mathcal{D}_k}, \quad \forall (i,j) \in \mathcal{L}, \forall k, \forall d \in \mathcal{D}_k, \tag{4}$$

$$\sum_{k=1}^{K} r_{ij}^{\mathcal{D}_k} \leq c_{ij}, \qquad \forall (i,j) \in \mathcal{L}, \tag{5}$$

where, $x_i^k \triangleq \begin{cases} x_k, & \text{if } i = s, \\ 0, & \text{otherwise.} \end{cases}$, and $f_{out(i)}^{(d,\mathcal{D}_k)}$ and $f_{in(i)}^{(d,\mathcal{D}_k)}$ respectively denote the information flow rate out of and into node i for Subsession k and Destination $d \in \mathcal{D}_k$, expressed explicitly in terms of $\mathbf{f}$ as,

$$f_{out(i)}^{(d,\mathcal{D}_k)} \triangleq \sum_{j:(i,j)\in\mathcal{L}} f_{ij}^{(d,\mathcal{D}_k)}, \qquad f_{in(i)}^{(d,\mathcal{D}_k)} \triangleq \sum_{j:(j,i)\in\mathcal{L}} f_{ij}^{(d,\mathcal{D}_k)}.$$

c_{ij} denotes the maximum capacity for the link (i,j). Here, (3) is the information flow balance condition, assuring that the information influx to a node be no more than the information outflux, unless the node is a destination node. Similarly, (4) assures that the physical flow rate is large enough to accommodate the information flow rate on each link through network coding, and (5) assures that the total promised physical flow rate does not exceed the available capacity for each link. $\diamond$

This equivalent characterization of the achievable rates allows us to recast (1) over the space of subsession rates as

$$\hat{\mathcal{X}} \triangleq \underset{\{\mathbf{x}\geq\mathbf{0}\}}{\arg\max} \sum_{d\in\mathcal{D}} U_d(y_d(\mathbf{x})) \quad s.t. \quad \mathbf{x} \in \mathcal{X}. \tag{6}$$

The polyhedral nature of the region $\mathcal{X}$ (cf. (3)-(5)) and the assumed utility function structure (cf. Assumption 1) render this a convex optimization problem without a duality gap ([20]), and motivate the use of dual methods for its distributed and iterative solution. Before we pose this solution, we emphasize that although $U_d(\cdot)$ is a strictly concave function of y_d, it is only concave in $\mathbf{x}$ since $\mathbf{y}$ and $\mathbf{x}$ are related as in (2). Thus, we denote the solution to the optimization problem by the set of subsession rates $\hat{\mathcal{X}}$ that can be alternatively expressed as a function of $\mathbf{y}^\star$ defined in (1) as $\hat{\mathcal{X}} = \{\hat{\mathbf{x}} \in \mathcal{X} : y_d^\star = y_d(\hat{\mathbf{x}}),\ \forall d \in \mathcal{D}\}$.

It is important to note that some optimal subsession rate vectors are more desirable than others in terms of satisfying the multi-resolution coding condition (see Definition 4) and in terms of low-complexity implementation. In this section, we ignore such considerations to propose a basic flow control and resource allocation mechanism that solves (6) through primal-dual methods. Then, in Sections IV-A and IV-B, we will build on this basic solution to incorporate the multi-resolution coding and complexity considerations.

Definition 6 (Basic (Subsession) Rate Allocation Algorithm): The algorithm operates iteratively with index t by updating the subsession rate vector $\mathbf{x}[t] = (x_k[t])_k$ and a price vector $\mu[t] := \left(\mu_i^{(d,\mathcal{D}_k)}[t]\right)_{i,d,k}$ as follows:

Rate Control: The flow rate of each subsession $k \in \mathcal{K}$ is updated according to the congestion level it observes through the price level at its source as

$$x_k[t+1] = \left(x_k[t] + \alpha \sum_{d\in\mathcal{D}_k} \left(U_d'(y_d[t]) - \mu_s^{(d,\mathcal{D}_k)}[t]\right)\right)_m^M, \tag{7}$$

where $U_d'(y) = \frac{dU_d(y)}{dy}$; $y_d[t] \triangleq \sum_{k\ni d} x_k[t]$; and $(z)_a^b$ denotes the projection of z onto the interval $[a,b]$. Further, $\alpha > 0$ is a small step-size parameter, $0 < m < \min_k \hat{x}_k$ where $\hat{\mathbf{x}}$ is the optimal solution, and M is a finite constant that is greater than $\sum_{(i,j)\in\mathcal{L}} c_{ij}$.

Resource Allocation and Intra-Subsession Coding: For each link (i,j), two parameters $w_{i,j}^*[t]$ and $k_{i,j}^*[t]$ are computed as follows.

$$w_{i,j}^*[t] = \max_{k\in K_s} \sum_{d\in\mathcal{D}_k} (\mu_i^{(d,\mathcal{D}_k)}[t] - \mu_j^{(d,\mathcal{D}_k)}[t])^+$$

$$k_{i,j}^*[t] = \underset{k\in K_s}{\arg\max} \sum_{d\in\mathcal{D}_k} (\mu_i^{(d,\mathcal{D}_k)}[t] - \mu_j^{(d,\mathcal{D}_k)}[t])^+$$

where $(z)^+ := \max(0,z)$. Here, $w_{i,j}^*[t]$ can be interpreted as the weight of the link (i,j), and $k_{i,j}^*[t]$ denotes the index of the session and subsession data to be transmitted over link (i,j). This means that the physical rate $r_{ij}^{\mathcal{D}_{k^*}}[t]$ of subsession $k_{i,j}^*[t]$ of session $s_{ij}^*[t]$ is equal to c_{ij}, and all the other subsessions are unscheduled.

The subsession $k_{i,j}^*[t]$ is served as follows:

1) Let

$$\mathcal{D}^+_{\{k_{i,j}^*[t]\}} = \{d \in \mathcal{D}_{\{k_{i,j}^*[t]\}} : \mu_i^{(d,\mathcal{D}_k)^*}[t] - \mu_j^{(d,\mathcal{D}_k)^*}[t] \geq 0\}.$$

Then, for each $d \in \mathcal{D}^+_{\{k_{i,j}^*[t]\}}$ take $c_{i,j}$ packets from its associated queue. If the queue empties, continue to the next step.

2) Form c_{ij} independent random linear combinations of these selected packets and transmit them over link (i,j).

3) At the receiver end, enqueue a copy of the incoming packets at each of the queues $\mu_j^{(d,\mathcal{D}_{k^*})}$ with $d \in \mathcal{D}^+_{\{k_{i,j}^*[t]\}}[t]$.

Therefore effectively, $f_{ij}^{(d,\mathcal{D}_{k^*})}[t] = c_{i,j}$ for $d \in \mathcal{D}^+_{k_{i,j}^*[t]}$. Finally, update the $\mu[t]$ vector as follows: for all i,k,

$$\mu_i^{(d,\mathcal{D}_k)}[t+1] = \left(\mu_i^{(d,\mathcal{D}_k)}[t] + \beta(x_i^k[t] + f_{in(i)}^{(d,\mathcal{D}_k)}[t] - f_{out(i)}^{(d,\mathcal{D}_k)}[t])\right)^+, \tag{8}$$

where $\beta > 0$ is a small step size parameter. $\diamond$

Before we proceed further, we note that the system model and the rate allocation algorithm explained in this section considers a single MRMC session across the network. However, the system model can be trivially extended to include multiple MRMC sessions and dividing each MRMC session into multiple SRMC subsessions. The rate control and resource allocation algorithm will result in appropriately dividing the link bandwidths between the MRMCs according to the utility function chosen. Throughout this work, in order to keep the notations simple, we consider the presence of just a single MRMC session in the network.

Next, we remark on several aspects of this algorithm before we study its optimality properties. First, we note that the structure of the resource allocation and coding algorithm is similar to the backpressure schemes that are extensively studied in the literature (e.g. [21], [22], [23], [24], [25]). Yet, the price parameter $\mu[t]$ is used as a measure of achievability of the existing rate allocation through intra-subsession coding as opposed to a measure of congestion levels. Also, in our formulation the objective function is defined in terms of "information flow rates" rather than the traditional physical link rates due to the network coding capability within subsessions. Thus, the implementation of the algorithm requires the description of network coding operations to be performed at the nodes (see [5] and references therein).

Moreover, contrary to earlier works in the literature (e.g. [26], [15], [27], [28], [29], [30], [31]), the rate allocation problem does not readily lend itself to complete decomposition across subsession rates. This is because, due to the network coding capabilities and the destination-based utility functions, the optimal subsession rates are interdependent. This motivates us to use first-order optimization methods to solve for the rate allocation (primal) problem, and iteratively update the rates in the subgradient direction as in (7).

We further remark that the dual update equation (8) in the cross-layer mechanism is tightly related to the actual queue-length evolution. In particular, if we let $q_i^{(d,\mathcal{D}_k)}[t]$ denote the length of the queue at node i that contains packets destined to node $d \in \mathcal{D}_k$ as part of the subsession k, then, we approximately have $q[t] \approx \beta\mu[t]$. This means that a scaled version of the actual queue-lengths can also be used in the scheduling algorithm, and no extra price maintenance is necessary.

We analyze the primal dual based algorithm under two different aspects. First, we focus on the optimality of the solution obtained by the iterative updates. We prove the global optimality of our algorithm for a heuristic fluid model approximation, whereby time evolves in continuous-time, and the difference equations (7), (8) describing the rate allocation and price updates are replaced by their respective differential equation counterparts:

$$\dot{x}(t) = \alpha\big(U_d'(y_d(t)) - \mu_s^{(d,\mathcal{D}_k)}(t))\big)\Big)^+_{x(t)}, \quad (9)$$

$$\dot{\mu}(t) = \beta(x_i^k(t) + f_{in(i)}^{(d,\mathcal{D}_k)}(t) - f_{out(i)}^{(d,\mathcal{D}_k)}(t))^+_{\mu(t)}, \quad (10)$$

where we used lower-case letters and (t) to highlight the continuous-time fluid nature of the system evolution. Also, we used the notation $(z)_y^+$ equals 0 if $y = 0$ and $z > 0$ and equals z, otherwise. Such fluid approximations have been used to establish the optimality of the actual stochastic operations in many earlier works (e.g. [32], [33], [24], [25]). Next, we provide this result in our context, which is proven in the Appendix.

Proposition 1: Starting from any initial state $\mathbf{x}(0)$, the rate allocation $\mathbf{x}(t)$ under the fluid evolution (9)-(10) of the basic rate allocation algorithm converges to an element $\hat{\mathbf{x}}$ in the optimal set of rate $\hat{\mathcal{X}}$ defined in (6), i.e., $\mathbf{x}(t) \to \hat{\mathbf{x}} \in \hat{\mathcal{X}}$ as $t \to \infty$.

Secondly, we discuss the architectural and computational complexity of the algorithm. For the primal dual based algorithm, one needs to maintain $\sum_{i=1}^{2^D} ND_i$ virtual queue information (pricing information, recall that the there are 2^D number of subsessions in the network). Note that the architectural complexity in terms of the pricing information is exponential in the number of destination nodes. In terms of computational complexity, note that the rate update and subsession resource allocation are based on simple calculations such as addition of the pricing information and comparison operation which do not incur significant overhead.

B. Optimal Information Distribution over the Subsessions

In Section III-A, we have provided a solution to the MRMC rate allocation problem (1) through subsession rate decomposition followed by a primal-dual type iterative algorithm (Definition 6). This algorithm is shown to converge to an optimal subsession rate vector $\hat{\mathbf{x}}$ from the set $\hat{\mathcal{X}}$ of all optimal subsession rates. Recalling that each Subsession k is associated with a subset $\mathcal{D}_k$ of the destination set $\mathcal{D}$ of the MRMC session, different subsessions can contain the same destination in their destination sets. Thus, information received by a given destination from different subsessions should have different content to add up. Yet, we also want to send maximum possible amount of common information to all destinations, as expected from a multi-cast session. Next, we pose and solve this problem of optimal distribution of information content over any given vector of optimal rates $\hat{\mathbf{x}}$.

Problem 1 (Optimal Information Distribution Problem): For a given subsession rate vector $\hat{\mathbf{x}}$, find the rates and subsession assignments of independent information streams with rates $b_1(\hat{\mathbf{x}}), b_2(\hat{\mathbf{x}}), ...$ that minimize $B(\hat{\mathbf{x}}) \triangleq \sum_i b_i(\hat{\mathbf{x}})$ such that:

(i) the sum of rates assigned to Subsession k adds up to $\hat{x}_k$, and
(ii) for each destination d, a *different* stream is assigned to each positive-rate subsession involving that destination.

Thus, the problem aims to maximize the amount of common information sent to the destinations, while assuring that the information content to any destination is not repetitive. Next, we provide a solution to this problem through dynamic programming after reformulating it in graphical context. Then, noting the computational complexity of the solution, we point to lower complexity iterative schemes. For clarity of exposition, we first consider binary-valued rate vectors and then relax this assumption to rational-valued rate vectors.

Information Distribution for Binary-Valued Vectors: We first solve Problem 1 with an assumption that limits the optimal rate vector $\hat{\mathbf{x}}$ to binary values. Later on, we will relax this assumption to all rational values, which is enough to approach any real value arbitrarily closely.

Assumption 2: Assume $\hat{\mathbf{x}}$ to be a binary $\{0, 1\}$-valued vector of rates. Then, we call Subsession k to be *active* if $\hat{x}_k = 1$ and *inactive* if $\hat{x}_k = 0$.

This assumption allows us to assume the rate of each stream to be 1, hence allowing us to refer to each stream as a *bit* in this context. With this terminology, we can rephrase Problem 1 as the assignment of the smallest total number of bits $b_1, b_2, \cdots$ to active subsessions in $\hat{\mathbf{x}}$ such that: (i) a single bit is assigned to each active subsession; and (ii) for each destination, a different bit is assigned to each active subsession involving that destination. Next, we construct a graphical representation of this problem.

We start by noting that each active Subsession k can be represented by a $D-$dimensional binary $\{0, 1\}$-valued vector with its d^{th} entry, $d \in \{1, 2, \cdots, D\}$ is one if Destination $d \in \mathcal{D}_k$, and zero otherwise. Thus, the representation captures the set of destinations the subsession serves. Then, we construct a graph $G(\hat{\mathbf{x}})$ where each active subsession of $\hat{\mathbf{x}}$ is a node, and an undirected link exists between two active subsessions that *do not* share a destination (see Figure 2 for an example).

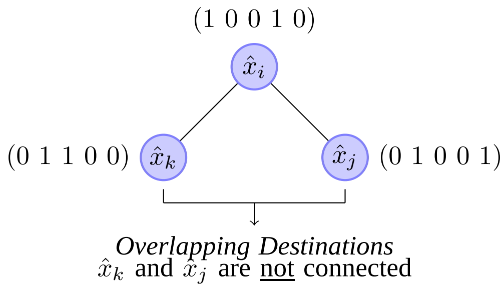

Fig. 2. Construction of Max Clique

The rationale behind this construction is that the same bit can be assigned to subsessions that are connected in this graph. Yet, the order in which such assignment amongst connected nodes is yet to be decided. Next, we propose such a scheme that solves Problem 1 under Assumption 2.

Definition 7: (Optimal Content Distribution (Binary Case)) Let $V(G)$ be an optimal solution to Problem 1 for the graph G, i.e., it is the smallest number of bits required to solve Problem 1. Also, let $\mathcal{CQ}(G)$ denote the set of cliques[6] in the graph G, i.e., each element of $\mathcal{CQ}$ is a collection of vertices of G that are fully connected. Thus, for the graph $G(\hat{\mathbf{x}})$, each clique can be represented by a binary vector $\mathbf{z}$ with 1's indicating the nodes (i.e. subsessions) in the clique. With this notation, $V(G(\hat{\mathbf{x}}))$ must satisfy:

$$\begin{aligned} V(G(\hat{\mathbf{x}})) &= \min_{\mathbf{z}\in\mathcal{CQ}(G(\hat{\mathbf{x}}))} \left(1 + V(G(\hat{\mathbf{x}} - \mathbf{z}))\right), \qquad (11) \\ V(G(\mathbf{0})) &= 0. \end{aligned}$$

Also, the bit assignment of the subsessions is provided by the optimizing $z^*(G)$ of the above minimization. $\diamond$

The optimality of this procedure follows from dynamic programming, which is stated next for completeness.

Proposition 2: The solution of (11) yields the optimum bit distribution over the active subsessions, and therefore solves Problem 1 under Assumption 2.

While this dynamic programming based scheme optimally solves Problem 1, it suffers from high computational complexity, even for a centralized decision center. In particular, the complexity of the DP based solution is exponential in nature. This motivates us to propose a simpler but satisfactory solution in Section IV-C. However, note that the complexity of the DP based solution is lesser than that of complete enumeration due to the structure involved in the choice of the set of optimal cliques.

Information Distribution for Rational-Valued Vectors: Having established an optimal procedure to distribute content for binary-valued rate vectors, we can extend the solutions to rational-valued rate vectors that can be used to approximate any real-valued rate level.

Assumption 3: Assume $\hat{\mathbf{x}}$ to be a rational-valued vector of subsession rates. Then, the values of each subsession rate $\hat{x}_k$ can be written as integer multiples of a constant rational value, which we will refer to as *an information bit* in the following discussion.

Under this assumption, we can assign information rates by integer multiples of the associated information bit. Then, our goal is to assign integer number of these bits to positive-valued subsessions to solve Problem 1. Once this connection is made, the optimal solution can again be characterized by dynamic programming as in Definition 7. Detailed description of this characterization is omitted avoid repetition and unnecessary new notation.

Note that the proposed method of information distribution for rational-valued subsession rate might suffer from an additional complexity namely the resulting integer value (which multiplies the constant rational value) might be large, resulting in higher number of nodes in the equivalent graph. Hence, it is important to emphasize that the implementation of this extended scenario will be even more impractical due to curse of dimensionality of dynamic programming. In Section IV-C, we will propose a remedy to this deficiency via a lower complexity alternative scheme.

IV. Improvements for Low-Complexity Implementation

So far, our focus has been on the development of optimal rate allocation and information distribution strategies to serve MRMC sessions in Sections III-A and III-B.

However, each of these optimal solutions possess deficiencies that render them unattractive for immediate implementation. Specifically: (i) the optimal rate allocator does not provide any guarantees for the satisfaction of multi-resolution coding (cf. Definition 4) that may be crucial to some applications, such as multi-resolution multimedia traffic; (ii) the optimal rate allocator requires the maintenance and and update of dual variables that grows exponentially with the number of destinations of each session, which may be impractical under computational and storage limitations; (iii) the optimal information distribution requires the solution of a high-dimensional dynamic program (cf. (11)) and the identification of cliques of a graph.

In this section, we highlight and provide low-complexity solutions for these deficiencies. These improvements utilize the flexibility of our setup, and extend it to incorporate practical considerations via appropriate cost functions. In Section IV-A, we build on the pricing mechanism to propose a layered strategy for rate allocation that not only guarantees MRC but also restricts the number of active subsessions to the number of destinations, thus reducing the exponential scaling to linear. In Section IV-B, we propose an analytical framework and a novel iterative algorithm for simultaneously optimizing rate and minimizing a convex cost function over the subsession rates. This allows the rate allocation to evolve towards implementationally more favorably solutions by selecting proper cost functions that capture implementational considerations, for example sparsity of the allocation. Finally, in Section IV-C, we propose a greedy information distribution scheme that significantly reduces the complexity of the dynamic-programming-based optimal solution.

A. Layered Rate Allocation with Multi-Resolution Coding

The Cross-Layer Algorithm obtains optimality characteristics which is important and certainly attractive, it must be noted that our cross-layer algorithm is based on decomposing a single multi-rate session with D number of destinations into 2^D single-rate multi-cast subsessions, one for each subset of

[6]A clique is a fully connected subgraph of the given graph G.

destinations. Thus, the coding and rate allocation complexity of the algorithm grows exponentially with the size of the multi-cast session. Next, we will propose a low-complexity implementation by utilizing the special structure of the problem.

1) Intuition and Algorithm Description: In this section, we propose a novel strategy to reduce the exponential nature of the architectural complexity of the cross-layer mechanism with respect to the number of destination nodes. The main idea behind our approach can be described by a horizontal water-filling analogy as follows. Consider the maximum achievable rates in the multi-rate multi-cast session as a function of the destinations, such illustrated in Figure 3, where we can see that Destination 4 has the best channel, while Destination 5 has the worst channel.

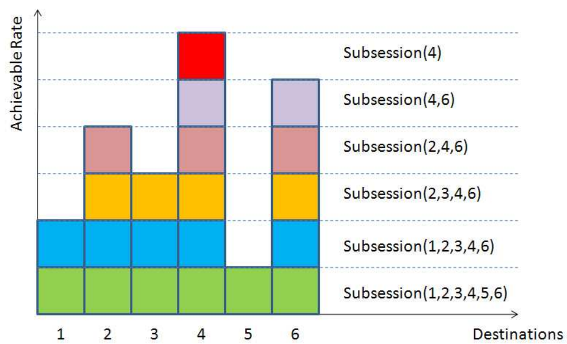

Fig. 3. Determining the optimal subsessions with a embedded structure.

The idea is to start by serving a single subsession containing all the destinations and then adding more subsessions by gradually eliminating the bottleneck destinations by utilizing the pricing information at the sources. In the example of Figure 3, we see that Destination 5 must be eliminated in the first round as it is the bottleneck link of the first (i.e. bottom) subsession. Then, in the second round the first two subsessions are served, leading to the identification of Destination 1 as the bottleneck destination in the second subsession. Thus, Destination 1 is eliminated in the third round to construct the third subsession, and so on. In order to identify the bottleneck destination at the end of each iteration, we make use of level of prices at the end of each round. In particular, in each round, the receiver whose price converges to the largest value in its subsession is marked as the bottleneck destination. To the best of our knowledge, such use of prices is novel and has not been exploited in earlier works. The details of our proposed procedure is described next.

Definition 8 (Low-Complexity Mechanism): Perform the following steps.

- Initialize the number of subsessions R to 1, with $\mathcal{D}_1 = \mathcal{D}$.
- In the R^{th} round, $R = 1, \cdots, K-1$, with $\{\mathcal{D}_1, \cdots, \mathcal{D}_R\}$ denoting the existing multi-cast subsessions, do:

(i) Run the cross-layer mechanism of Definition 6 only for the subsessions with destinations $\{\mathcal{D}_1, \cdots, \mathcal{D}_R\}$, such that $x_r(t) \rightarrow \tilde{x}_r$, for each $r = 1, \cdots, R$; and $\mu_s^{(d,\mathcal{D}_r)}(t) \rightarrow \tilde{\mu}_s^{(d,\mathcal{D}_r)}$, for each $r = 1, \cdots, R$, and $d \in \mathcal{D}_r$.
(ii) Let $d_R^\star = \arg\max_{d \in \mathcal{D}_R} \tilde{\mu}_s^{(d,\mathcal{D}_R)}$, which implies that $d_R^\star$ is the bottleneck destination in the multi-cast subsession $\mathcal{D}_R$ since its price is the maximum.
(iii) Construct the $(R+1)^{st}$ subsession as $\mathcal{D}_{R+1} = \mathcal{D}_R \backslash d_R^\star$. If $(R+1) = K$, exit. Otherwise start the $(R+1)^{st}$ round.

$\diamond$

It can be seen in Step (ii) of the iterative procedure that the limit price levels at the source are used to identify the bottleneck link in the most recent, i.e. R^{th}, multi-cast subsession, and in Step (iii), the next subsession is created by eliminating the identified bottleneck, $d_R^\star$.

In terms of complexity, for the layered algorithm, one needs to maintain $\sum_{i=1}^{D} N(D-i)$ virtual queue information in the network. Hence, the architectural complexity is significantly reduced as compared to the original primal dual based algorithm. In the following subsection, we provide a few simulation results to demonstrate that the low complexity scheme continues to achieve rates very close to optimal in addition to the complexity gains it provides.

B. Nested Optimization

The layered algorithm developed in the previous subsection reduces the number of subsessions from 2^D to D and hence drastically decreases the architectural complexity. . However it operates in rounds and requires ”approaching the limit” to start a new round. In large network scenarios where such convergence requirements may be impractical, we next propose an alternative called the nested optimization framework.

1) Nested Optimization Framework: Suppose we are primarily interested in finding an optimal solution to the problem:

$$P_{\mathbf{x}} : \max_{\mathbf{x} \in \mathcal{X}} U(h(\mathbf{x}))$$

where h(.) is a many-to-one mapping and U(.) is the utilization factor. Since h(.) is a many to one mapping function, there might be more than one points in $\mathcal{X}$ which maximizes the utility function. Let $\hat{\mathcal{X}}$ denote the set of vectors in $\mathcal{X}$ which maximizes the utilities. Now we are interested in finding a point $\hat{\mathbf{x}} \in \hat{\mathcal{X}}$ that minimizes the cost function while simultaneously maximizing the utility. Therefore we define the secondary problem given by

$$S_{\mathbf{x}} : \min_{\hat{\mathbf{x}} \in \hat{\mathcal{X}}} C(\hat{\mathbf{x}})$$

Such a scenario is shown in Figure 4.

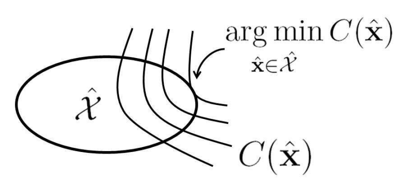

Fig. 4. Nested Optimization Framework

We are interested in approximately solving the two problems in parallel using a duality framework. In particular, suppose the constraint set $\mathcal{X}$ is described as

$$\mathcal{X} = \{\mathbf{x} \in M : g(\mathbf{x}) \leq 0\}$$

for some closed convex set M and inequality constraints. Let $\mathbf{q}$ be the Lagrange multiplier vector associated with the inequality constraints. Then, the Lagrangian function associated with $P_{\mathbf{x}}$ is given by

$$L(\mathbf{x}, \mathbf{q}) = U(h(\mathbf{x})) - \mathbf{q}^T g(\mathbf{x})$$

In the absence of a secondary problem, we could have solved the optimization problem by using the dual technique of updating the rate and Lagrange multiplier. However, in this case we have to include the cost minimization into this framework as well. We now present the nested optimization algorithm:

Definition 9 (Nested Optimization): In each step, perform **Primary problem Update:** Maximize the utilization achieved as a function of the rate $\mathbf{x}(t)$ and sets a value for $h(\mathbf{x}(t))$. in every time slot $[t]$ as follows:

$$\mathbf{x}(t) = \arg\max_{\mathbf{x}\in M}(U(h(\mathbf{x})) - \mathbf{q}^T g(\mathbf{x}))$$

Secondary Problem Update: Find a point $\bar{\mathbf{x}}(t) \in \mathcal{X}$ as follows while preserving the $h(\mathbf{x}(t))$ value set by the primary problem update:

$$\bar{\mathbf{x}}(t) = \arg\min_{\{\bar{\mathbf{x}}\in M: h((\bar{\mathbf{x}}))(t)=h(\mathbf{x}(t))\}} (C(\bar{\mathbf{x}}) - \mathbf{q}^T g(\bar{\mathbf{x}}))$$

Lagrange Multiplier Update: The Lagrange multipliers $\mathbf{q}(t)$ are updated based on the $\bar{\mathbf{x}}(t)$ according to:

$$\mathbf{q}(t+1) = \mathbf{q}(t) + \alpha(t) g(\bar{\mathbf{x}}(t))$$

◇

The working of the nested optimization structure is pictorially depicted in Figure 5, where we see that the primary and secondary problems are intricately related via the variables $\{\mathbf{x}, \bar{\mathbf{x}}, \mathbf{q}\}$.

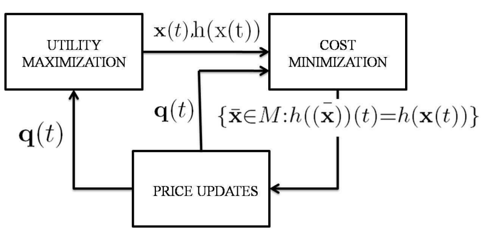

Fig. 5. Nested Optimization Loop

2) Application of Nested Optimization to the MRMC Problem: We now define the primary and secondary optimization problems associated with our multi-rate multi-cast problem. The primary problem is maximizing the destination utilities subject to the network constraints, P_{NEST} is given by

$$\max_{\{\mathbf{x},\mathbf{f},\mathbf{r}\geq\mathbf{0}\}} \sum_{d\in\mathcal{D}} U_d(y_d(\mathbf{x})) \qquad (12)$$
$$s.t. \quad (3) - (5)$$

In our problem $h(\mathbf{x})$ is of the form $\sum_{k\ni d} x_k$. The quantity specifies the net rate received by each destination from all the subsessions. The secondary optimization problem S_{NEST} is given by

$$\min_{\{\bar{\mathbf{x}},\mathbf{f},\mathbf{r}\geq\mathbf{0}\}} C(\bar{\mathbf{x}}) \qquad (13)$$
$$s.t. \quad (3) - (5)$$
$$\sum_{k\ni d} \bar{x}_k = y_d$$

The secondary problem tries to minimize the cost subject to the network constraint and also preserving the destination rates specified by the the utility maximization problem.

We adopt the primal dual algorithm to solve both the optimization problems. The dynamics of the nested optimization structure are as follows:

$$x_k(t+1) = \Big(x_k(t) + \alpha \sum_{d\in\mathcal{D}_k} \big(U_d'(y_d[t]) - \mu_s^{(d,\mathcal{D}_k)}[t]\big)\Big)_m^M, \qquad (14)$$

$$\bar{x}_k(t+1) = \bar{x}_k(t) - \gamma\big(C^{'}(\bar{\mathbf{x}}) + \sum_{d\in\mathcal{D}_k} \mu_s^{(d,\mathcal{D}_k)} + \lambda_d\big) \qquad (15)$$

$$\mu_i^{(d,\mathcal{D}_k)}[t+1] = \Big(\mu_i^{(d,\mathcal{D}_k)}[t] + \beta(\bar{x}_i^k[t] + f_{in(i)}^{(d,\mathcal{D}_k)}[t] - f_{out(i)}^{(d,\mathcal{D}_k)}[t])\Big)^+, \qquad (16)$$

$$\lambda_d(t+1) = \lambda_d(t) + \delta(\sum_{k\ni d} \bar{x}_k - y_d(t)) \qquad (17)$$

Note that $\alpha, \gamma, \beta, \lambda$ are step sizes for the updates.

The first equation updates x in accordance with the utility maximization problem and sets up the destination rate $(Y_d^s(t))$ levels in each iteration. The second equation updates $\bar{x}$ according to the cost minimization problem while trying to preserve the destination rate levels set by the primary problem update. It must be noted that the two problems are related by the common prices that are the Lagrange multipliers associated with the network constraint equations of both the problems. The third equation gives the queue-length updates. The last equation gives λ updates. The Lagrange multiplier λ evolves in such a way that the destination rates set by the variables x and $\bar{x}$ becomes equal at convergence.

While these observations indicate the intuition behind the desirable characteristics of the algorithm, the intricate and coupled dynamics of it render its performance analysis intractable. Instead, we use extensive numerical investigations to understand and demonstrate the performance of this original design.

In terms of complexity, the worst case architectural complexity for the nested optimization problem is given by $\sum_{i=1}^{2^D} ND_i$ (the number of subsessions having non zero rates is upper bounded by 2^D, however there might be subsessions which can have zero rates).

C. Low-Complexity Information Distribution

In this section, we propose a low-complexity alternative to the optimal information distribution solution of Section III-B by utilizing the graphical model introduced in that section. In the same spirit as before, we will first propose the solution for binary-valued subsession rate allocations, and then extend to rational-valued allocations.

Information Distribution for Binary-Valued Vectors: We consider a *binary-valued* subsession rate allocation $\hat{\mathbf{x}}$, and construct its associated graph $G(\hat{\mathbf{x}})$ as described in Section III-B to capture the active subsessions and their inter-dependencies. We next propose a low-complexity alternative to Definition 7 to efficiently distribute content across the sessions.

Definition 10: (Low-Complexity Content Distribution (Binary Case)) Construct the graph $G(\hat{\mathbf{x}})$ as described before. Then, repeat the following steps to iteratively reduce the graph to a set of isolated nodes:

(1) Select a *maximum size clique*[7] of the current graph, and

[7]A maximum size clique is a clique with the maximum number of nodes.

assign the a new bit to all the subsessions of the maximum size clique.

(2) Merge all the nodes in the selected maximum size clique into a new super-node that is disconnected from all the other remaining nodes.

(3) Repeat steps (1) and (2) until no links are left.

Thus, the above procedure assigns a single bit to each active subsession, and yields a graph of isolated super-nodes. ◇

Note that the steps consist of finding a maximum size clique of a graph, which is a canonical NP-hard problem. Yet, there exist many heuristic algorithms for solving the maximum clique problems based on techniques such as branch and bound, sequential greedy search, local search algorithms and this stream of works have been well researched (see [34] and references therein). This scheme is hence significantly more practical than the dynamic programming based solution of Definition 7. Moreover, since the size of the graph will sharply decrease with each reduction of the above procedure, the computational strain of a maximum clique search will also reduce.

Information Distribution for Rational-Valued Vectors: Next, we extend the scheme for binary-valued allocations to rational-valued allocations by first extending the graphical model, and then utilizing the proposed solution for the binary case. We refer the reader to Section III-B for the intuition behind this extension.

Definition 11: (Low-Complexity Content Distribution (Rational Case)) First express each $\hat{x}_k$ as an integer-valued vector written in terms of the rational-valued common bit amount. Then, construct a new graph for $\hat{\mathbf{x}}$ as follows: *split* each subsession with integer-valued rate $r \geq 2$ into r unirate subsessions that are all disconnected from each other. This yields a graph exactly as in the binary scenario. Then, follow the same procedure as in Definition 10. When this terminates in a set of isolated *super-nodes*, assign an independent information stream of the above constant rate to all subsessions in each super-node (see Figure 6 for an illustration). ◇

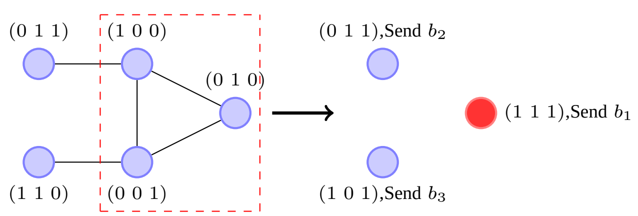

Fig. 6. Construction of Max-Clique for the Generalized Case

As a note, we would like to mention that due to the structure of the rate vector obtained from the low complexity rate allocation scheme of Section IV-A, the equivalent graph consists of a set of isolated nodes (since every node shares a common destination). Hence the formation of super-nodes is not necessary in this scenario.

V. Numerical Results

A. Numerical Results for Cross Layer Algorithm

In this section we provide some simulation results to study the performance of the original cross-layer algorithm of Definition 6. We start with the canonical butterfly network.

1) Butterfly Network: We start with the canonical butterfly network that excludes the d_3 node of the network in Figure 10, and assume that there is a single multi-rate multi-cast session having node s as the source node and nodes d_1 and d_2 as the destination nodes. We decompose this multi-rate multi-cast session into three single-rate multi-cast subsessions each having source node s. The subsessions have the following destination set, $\mathcal{D}_1 = \{d_1\}$, $\mathcal{D}_2 = \{d_2\}$ and $\mathcal{D}_3 = \{d_1, d_2\}$. We interpret subsessions 1 and 2 to be unicast subsessions and 3 to be the multi-cast subsession. We assume that the utilization achieved at each destination $d \in \mathcal{D}$, is measured by $U_d(y_d) = \log(y_d)$, which leads to the proportionally fair allocation. We choose the α and β parameters to be 0.01, and examine the network under two different scenarios to show the effectiveness of our cross-layer algorithm.

Case(a): Initially we analyze the network with link capacity of all links to be 1 unit. The subsession rates at convergence point are tabulated in Table IV and the time evolution of the rates is plotted in Figure 7. The rows of the table indicate different subsessions and their corresponding destination set. “ON” indicates the node is in the destination set and “OFF” indicates it is not. We see that the rate values at convergence point are $0, 0$ and 2 respectively. Each destination receives a data rate of 2 which equals the theoretically maximum achievable rate specified by the max-flow-min-cut theorem. Also, it can be observed that rates for the unicast subsessions are 0. Hence when the link capacities are uniform, all the data can be transmitted through multi-cast taking full advantage of network coding.

Case(b): In the second simulation scenario, we introduce a bottleneck link in the paths of one of the destinations, i.e. we reduce the average capacity over link $(1, d_1)$ to 0.1 units. We achieve this average capacity through a Bernoulli random process with parameter 0.1. The subsession rates are again tabulated in Table V and evolution of rates in Figure 8. The net achieved rate for destination d_1 is 1.1 units where as destination d_2 receives a rate of 2 units. Case (b) clearly shows the effectiveness of our algorithm: while single-rate multi-cast with fixed coding subgraphs would have resulted in a lower data rate to both the destinations due to receiver d_1 creating the bottleneck for the whole multi-cast group, our cross layer mechanism still achieves rates very close to theoretical value for either destinations.

TABLE I
Case (a)

	d_1	d_2	x_k
S {10}	OFF	ON	0
S {01}	ON	OFF	0
S {11}	ON	ON	2

TABLE II
Case (b)

	d_1	d_2	x_k
S {10}	OFF	ON	0
S {01}	ON	OFF	0.9
S {11}	ON	ON	1.1

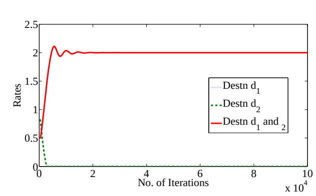

Fig. 7. Case (a)

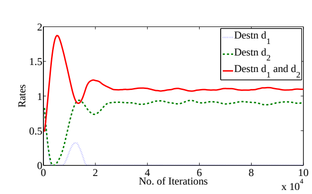

Fig. 8. Case (b)

2) Butterfly Network with Antenna: For the butterfly network with antenna (cf. Figure 10), we consider the case of unit capacity links, yielding 7 non-empty subsessions. We assume the utilization function to be $U_d(y_d) = \log(y_d)$, and choose the α and β parameters to be 0.01, sufficiently small for convergence. The session rate evolutions of the optimal rate allocation algorithm are depicted in Figure 9.

TABLE III

SUBSESSION RATES

$S\{d_1 d_2 d_3\}$	x_k
S $\{100\}$	0
S $\{010\}$	0
S $\{001\}$	0
S $\{110\}$	1
S $\{011\}$	0
S $\{101\}$	0
S $\{111\}$	1

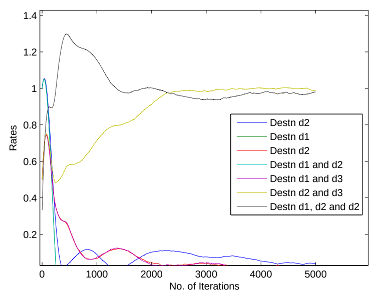

Fig. 9. Subsession Rates of Butterfly Network with Antenna

The result of Table III shows that the resulting subsession rates, confirming that our policy achieves destination rates of $(2, 2, 1)$, as expected from the discussion of Example 1.

Finally, we illustrate the applicability of our algorithm to include multiple MRMC through the simulation result below. Consider the two butterfly networks in Figure 10 which share a common link Node-2 to Node-d_2. We consider 2 MRMCs with source nodes s_1 and s_2 respectively. MRMC 1 targets destinations d_1 and d_2 where as the MRMC 2 targets destinations d_2 and d_3. The two MRMCs share a common destination d_2. The subsession rates are provided in the table below. It can be seen that the bandwidth of the shared link Node-2 to Node-d_2 is appropriately shared by the MRMCs according to the rate allocation algorithm.

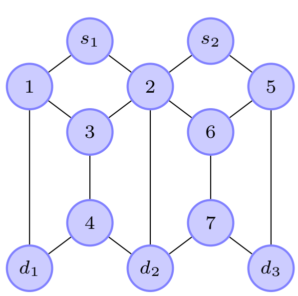

Fig. 10. Two butterfly networks with a common link

The convergence speed of the primal dual algorithm is identical to other primal dual based rate allocation algorithms

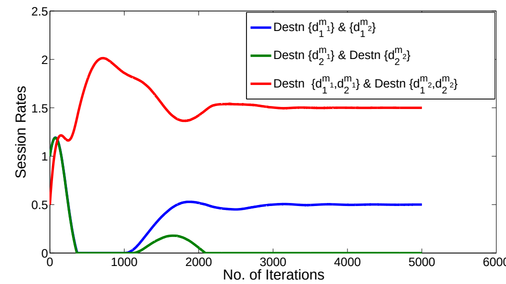

Fig. 11. Multiple Multirate Multicast Sessions - Rate Evolution

TABLE IV

MULTIRATE MULTICAST SESSON 1

	$\{d_1\}$	$\{d_2\}$	x_k
S $\{10\}$	ON	OFF	0.5
S $\{01\}$	OFF	ON	0
S $\{11\}$	ON	ON	1.5

TABLE V

MULTIRATE MULTICAST SESSON 2

	$\{d_3\}$	$\{d_2\}$	x_k
S $\{10\}$	ON	OFF	0.5
S $\{01\}$	OFF	ON	0
S $\{11\}$	ON	ON	1.5

existing in literature [35]. From an implementation point of view, the time scale of operation of the iterative algorithm depends in turn on the rate at which information can be exchanged between the nodes of the network. Recall that the only information exchange needed to implement the primal dual algorithm is the exchange of pricing information between the neighboring nodes. Hence, the convergence time required for the primal dual algorithm is linked to this rate of information exchange. Also, note that the system converges to within a small neighborhood of the optimal, since we have chosen constant step sizes. The choice of the step-size parameter determines the trade off between optimality and convergence rate. In particular, the smaller the step size, the slower the convergence and the closer to the optimal, which is a general characteristic of any gradient based method. Recently, there have been promising advances in the design of distributed second-order methods for utility maximization problems (e.g. [36], [37]). These indicate the promise of developing faster strategies by adapting them to the MRMC problem.

B. Numerical Results for Layered Algorithm

In this section, we illustrate the effectiveness of the low complexity multi-casting strategy of Definition 8 in reducing the algorithm complexity and achieve fast convergence.

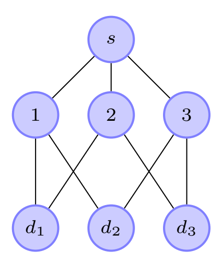

Fig. 12. Network serving a session with 3 destinations.

The network considered for illustrating complexity reduction is shown in Figure 12. The multi-rate session has three destinations, and all the links are assumed to have unit capacity, except for link $(1, d_1)$, which has an average capacity of 0.1 units. The rate values at convergence for different subsessions at different rounds are tabulated in the rightmost column of Table VI The numbers below the destinations in Table VI indicate the convergent price values at the source node for the corresponding destination. The boxed numbers in each round indicate that the corresponding destination has the highest price and is the bottleneck receiver in the subsession. Hence it is eliminated in the subsequent rounds.

TABLE VI
SUBSESSION RATES AT DIFFERENT ROUNDS FOR THE LAYERED MECHANISM

Round #	d_1	d_2	d_3	x_k
Round 1	$\boxed{2.7}$	0.0	0.0	1.1
Round 2	0.9	0.6	0.4	1.1
	-	0.5	$\boxed{0.5}$	0.9
Round 3	1.0	0.7	0.3	1.1
	-	0.5	0.5	0.9
	-	0.5	-	0

C. Numerical Results for Nested Optimization Algorithm

We now provide the simulation results for the Nested Algorithm of Section IV-B. The simulation results are implemented for the network shown in Figure 12. In these simulation results, the primary problem is the utility maximization problem using the log utilization function. In the secondary problem, which is the cost minimization problem, we consider linear cost function. Hence the linear cost function in terms of subsession rates can be given by $\sum_{s\in K_s} x_k^s$. We consider two cases. In case(a), the link capacities of all the links of the network is 1 unit. In the second case, case(b), we introduce one bottleneck link of capacity 0.1 units for the link joining node 1 and destination d_1. The subsession rates at convergence are provided in Table VII and Table VIII for case(a) and case(b) respectively.

TABLE VII
CASE (A)

S $\{d_1 d_2 d_3\}$	x_k
S {100}	0
S {010}	0
S {001}	0
S {110}	0
S {011}	0
S {101}	0
S {111}	2

TABLE VIII
CASE (B)

S $\{d_1 d_2 d_3\}$	x_k
S {100}	0
S {010}	0
S {001}	0
S {110}	0
S {011}	0.9
S {101}	0
S {111}	1.1

It can be seen in case(a) that the subsession rates converges to exactly the same point as the layered structure. Also, it can be seen that this is the solution which incurs the least cost on the network since there is just one subsession which has a non-zero subsession rate. Similarly, the results also holds for case(b).

The Figures 13 and 14 make a comparative study between the distribution of subsession rates for the original and nested algorithm case and it can be seen that the two solutions converge to different rate vectors inside the set of optimal rate vectors. This can be explained as follows. While solving the original optimization problem converges to an arbitrary rate

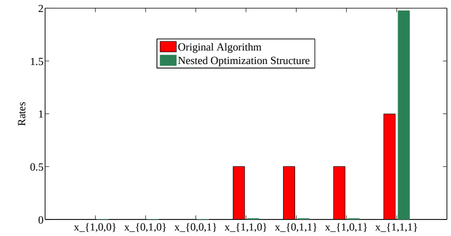

Fig. 13. Comparative study of distribution of data rates among subsessions for the original and nested optimization algorithm for case(a).

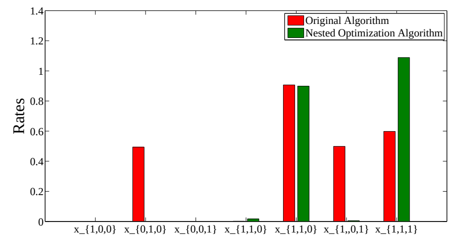

Fig. 14. Comparative study of distribution of data rates among subsessions for the original and nested optimization algorithm for case(b).

allocation vector inside the set $\hat{\mathcal{X}}$, the solution provided by the nested optimization algorithm tries to find the rate allocation vector which also minimizes the sum of rate of all the sessions. This particular solution has the advantage that it reduces the total number of packets injected into the network and also the implementation complexity.

It must be noted that linear cost function is just one of the possible function for evaluating the cost. The nested optimization algorithm gives us the flexibility of choosing a different cost function (as secondary objective functions) that can minimize other network parameters like aggregate queue-length and delay.

VI. CONCLUSIONS

In this work, we proposed a utility maximization formulation for the multi-rate multi-casting problem with in the presence of network coding capabilities. After suggesting a particular subsession decomposition, we developed the optimal rate allocation and information distribution strategies for its solution. Noting the computational complexities involved in obtaining the optimal solution, we further proposed two novel approaches with attractive implementation advantages and demonstrated their effectiveness via extensive numerical studies.

VII. ACKNOWLEDGMENT

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APPENDIX

Proof: [Proposition 1] Let $\mu := \left(\mu_i^{(d,\mathcal{D}_k)}\right)_{i,d,k}$ be the nonnegative Lagrange multiplier vector associated with the first constraint of (3). The KKT conditions (see [20]) for any optimal solution $(\hat{\mathbf{x}}, \hat{\mu}, \hat{\mathbf{f}})$ of our problem (6) are:

$$(\hat{x}_i^k + \hat{f}_{in(i)}^{(d,\mathcal{D}_k)} - \hat{f}_{out(i)}^{(d,\mathcal{D}_k)}) \le 0, \quad \forall i \in N, k \in K \quad (18)$$

$$\hat{\mu}_i^{(d,\mathcal{D}_k)} \ge 0, \quad \forall i \in N, k \in K \quad (19)$$

$$\hat{\mu}_i^{(d,\mathcal{D}_k)}(\hat{x}_i^k + \hat{f}_{in(i)}^{(d,\mathcal{D}_k)} - \hat{f}_{out(i)}^{(d,\mathcal{D}_k)}) = 0, \quad \forall i \in N, k \in K \quad (20)$$

$$\sum_{d \in \mathcal{D}_k} (U_d^{'}(\hat{y}_d) - \hat{\mu}_s^{(d,\mathcal{D}_k)}) = 0, \quad \forall k \in K \quad (21)$$

Then, consider the Lyapunov function $W_t(\mathbf{x}, \mu)$ given by

$$\sum_{k \in K} \frac{(x_k(t) - \hat{x}_k)^2}{2\alpha} + \sum_{i,k,d \in \mathcal{D}_k} \frac{\left(\mu_i^{(d,\mathcal{D}_k)}(t) - \hat{\mu}_i^{(d,\mathcal{D}_k)}\right)^2}{2\beta}$$

Differentiating the Lyapunov function with respect to time, we get the Lyapunov drift

$$\dot{W}_t(\mathbf{x}, \mu) = \frac{1}{\alpha} \sum_{k \in K} (x_k(t) - \hat{x}_k)\dot{x}(t) + \frac{1}{\beta} \sum_{i,k,d \in \mathcal{D}_k} (\mu_i^{(d,\mathcal{D}_k)}(t) - \hat{\mu}_i^{(d,\mathcal{D}_k)})\dot{\mu}_i^{(d,\mathcal{D}_k)}(t)$$

For notational simplicity, we drop the time index but all the rate and flow quantities must be interpreted as time dependent functions. Using equation (9) and (10) to determine the primal and dual gradients, we get

$$\begin{aligned}\dot{W}_t(\mathbf{x}, \mu) &= \sum_{k \in K} (x_k - \hat{x}_k)\Big(\sum_{d \in \mathcal{D}_k} (U_d^{'}(y_d) - \mu_s^{(d,\mathcal{D}_k)})\Big)^+ \\ &+ \sum_{i,k,d \in \mathcal{D}_k} (\mu_i^{(d,\mathcal{D}_k)} - \hat{\mu}_i^{(d,\mathcal{D}_k)})\Big(x_i^k + f_{in(i)}^{(d,\mathcal{D}_k)} - f_{out(i)}^{(d,\mathcal{D}_k)}\Big)^+ \\ &\le \sum_{k \in K} (x_k - \hat{x}_k)\Big(\sum_{d \in \mathcal{D}_k} (U_d^{'}(y_d) - \mu_s^{(d,\mathcal{D}_k)})\Big) \\ &+ \sum_{i,k,d \in \mathcal{D}_k} (\mu_i^{(d,\mathcal{D}_k)} - \hat{\mu}_i^{(d,\mathcal{D}_k)})\Big(x_i^k + f_{in(i)}^{(d,\mathcal{D}_k)} - f_{out(i)}^{(d,\mathcal{D}_k)}\Big),\end{aligned}$$

where the inequality follows from removing the projection operation from the rate and price gradients.

Optimality of $(\hat{\mathbf{x}}, \hat{\mu})$ implies that

$$\sum_{d\in\mathcal{D}_k} (U_d'(\hat{y}_d) - \hat{\mu}_s^{(d,\mathcal{D}_k)}) = 0 \qquad \forall k \in K$$

We add this quantity in the first summation and optimal rate in the second summation of the Lyapunov drift. Hence,

$$\begin{aligned} \dot{W}_t(\mathbf{x}, \mu) = &\sum_{k\in K} (x_k - \hat{x}_k) \\ &\quad (\sum_{d\in\mathcal{D}_k} [U_d'(y_d) - U_d'(\hat{y}_d) + \hat{\mu}_s^{(d,\mathcal{D}_k)} - \mu_s^{(d,\mathcal{D}_k)}) \\ &+ \sum_{i,k,d\in\mathcal{D}_k} (\mu_i^{(d,\mathcal{D}_k)} - \hat{\mu}_i^{(d,\mathcal{D}_k)}) \\ &\quad (x_i^k + f_{in(i)}^{(d,\mathcal{D}_k)} - f_{out(i)}^{(d,\mathcal{D}_k)} + \hat{x}_i^k - \hat{x}_i^k) \end{aligned} \tag{22}$$

Consider the quantity,

$$\begin{aligned} &\sum_{i,k,d\in\mathcal{D}_k} (\mu_i^{(d,\mathcal{D}_k)} - \hat{\mu}_i^{(d,\mathcal{D}_k)})(x_i^k - \hat{x}_i^k) \\ &= \sum_{k,d\in\mathcal{D}_k} (\mu_s^{(d,\mathcal{D}_k)} - \hat{\mu}_s^{(d,\mathcal{D}_k)})(x_k - \hat{x}_k), \end{aligned}$$

where the equality follows from the definition of x_i^k. This quantity cancels with the second part of (22) and hence the Lyapunov drift expression reduces to

$$\dot{W}(\mathbf{x}, \mu) = \sum_{k\in K} (x_k - \hat{x}_k)\big(\sum_{d\in\mathcal{D}_k} (U_d'(y_d) - U_d'(\hat{y}_d))\big) \tag{23}$$

$$+ \sum_{i,k,d\in\mathcal{D}_k} \hat{\mu}_i^{(d,\mathcal{D}_k)}(f_{out(i)}^{(d,\mathcal{D}_k)} - f_{in(i)}^{(d,\mathcal{D}_k)} - \hat{x}_i^k) \tag{24}$$

$$+ \sum_{i,k,d\in\mathcal{D}_k} \mu_i^{(d,\mathcal{D}_k)}(\hat{x}_i^k + f_{in(i)}^{(d,\mathcal{D}_k)} - f_{out(i)}^{(d,\mathcal{D}_k)}), \tag{25}$$

where (23) can be rearranged as

$$\sum_{d\in D} \Big((\sum_{k\ni d} x_k - \sum_{k\ni d} \hat{x}_k)(U_d'(\sum_{k\ni d} x_k) - U_d'(\sum_{k\ni d} \hat{x}_k)) \Big), \tag{26}$$

which, in turn, must be non-positive since the utility function is concave by assumption, and so (26) ≤ 0. Next, we will show that (24) and (25) in the Lyapunov drift are also non-positive. We start with (24):

$$\begin{aligned} &\sum_{i,k,d\in\mathcal{D}_k} \hat{\mu}_i^{(d,\mathcal{D}_k)}(f_{out(i)}^{(d,\mathcal{D}_k)} - f_{in(i)}^{(d,\mathcal{D}_k)}) \\ &\quad = \sum_{(i,j),k,d\in\mathcal{D}_k} f_{ij}^{(d,\mathcal{D}_k)}(\hat{\mu}_i^{(d,\mathcal{D}_k)} - \hat{\mu}_j^{(d,\mathcal{D}_k)}) \\ &\quad \overset{(a)}{\leq} \sum_{k\in K,d\in\mathcal{D}_k} \hat{x}_k \hat{\mu}_s^{(d,\mathcal{D}_k)}, \end{aligned}$$

where inequality (a) follows from summing the KKT condition (20) over all nodes and subsessions to get

$$\begin{aligned} \sum_{k\in K,d\in\mathcal{D}_k} \hat{\mu}_s^{(d,\mathcal{D}_k)} \hat{x}_k &= \sum_{i,k,d\in\mathcal{D}_k} \hat{\mu}_i^{(d,\mathcal{D}_k)}(\hat{f}_{out(i)}^{(d,\mathcal{D}_k)} - \hat{f}_{in(i)}^{(d,\mathcal{D}_k)}) \\ &= \sum_{(i,j),k,d\in\mathcal{D}_k} \hat{f}_{ij}^{(d,\mathcal{D}_k)}(\hat{\mu}_i^{(d,\mathcal{D}_k)} - \hat{\mu}_j^{(d,\mathcal{D}_k)}) \\ &\geq \sum_{(i,j),k,d\in\mathcal{D}_k} f_{ij}^{(d,\mathcal{D}_k)}(\hat{\mu}_i^{(d,\mathcal{D}_k)} - \hat{\mu}_j^{(d,\mathcal{D}_k)}) \end{aligned}$$

for any feasible f satisfying $0 \leq f_{ij}^{(d,\mathcal{D}_k)} \leq r_{ij}^{\mathcal{D}_k}$ and $\sum_{k\in K} r_{ij}^{\mathcal{D}_k} \leq c_{ij} \quad \forall (i,j)$. This establishes that (24) ≤ 0. Finally, to prove (25) ≤ 0, we multiply the flow balance equations (3) with corresponding subsession prices and sum them over all nodes and subsessions:

$$\begin{aligned} \sum_{k\in K,d\in\mathcal{D}_k} \hat{x}_k \mu_s^{(d,\mathcal{D}_k)} &\leq \sum_{i,k,d\in\mathcal{D}_k} (\hat{f}_{out(i)}^{(d,\mathcal{D}_k)} - \hat{f}_{in(i)}^{(d,\mathcal{D}_k)})\mu_i^{(d,\mathcal{D}_k)} \\ &= \sum_{(i,j),k,d\in\mathcal{D}_k} \hat{f}_{ij}^{(d,\mathcal{D}_k)}(\mu_i^{(d,\mathcal{D}_k)} - \mu_j^{(d,\mathcal{D}_k)}) \\ &\leq \sum_{(i,j),k,d\in\mathcal{D}_k} f_{ij}^{(d,\mathcal{D}_k)}(\mu_i^{(d,\mathcal{D}_k)} - \mu_j^{(d,\mathcal{D}_k)}), \end{aligned} \tag{27}$$

where the last inequality is true since the backpressure scheduler sets the information flow rates to maximize the expression in (27). There, from (27) it follows that (25) ≤ 0.

Combining the results, (23) + (24) + (25) ≤ 0, and hence the Lyapunov drift, $\dot{W}_t(\mathbf{x}, \mu) \leq 0$ and further,

$$\xi := \{(\mathbf{x}, \mu) : \dot{W}_t(\mathbf{x}, \mu) = 0\}$$

is contained in the set

$$S := \{(\mathbf{x}, \mu) : (23) = (24) = (25) = 0\}$$

Let M be the largest invariant set of the primal-dual algorithm contained in ξ. By LaSalle's invariance principle $(\mathbf{x}(t), \mu(t))$ converges to the set M as $t \to \infty$. Since $M \subset \xi \subset S$, as $t \to \infty$, the limit point of the pair $(\mathbf{x}(t), \mu(t))$ must also satisfy (23) $=$ (26) $= 0$. It must be noted that strict concavity of the utility functions implies (26) $= 0$ only when $U_d'(y_d) = U_d'(\hat{y}_d)$ and hence $\lim_{t\to\infty} \mathbf{x}(t) = \hat{\mathcal{X}}$.

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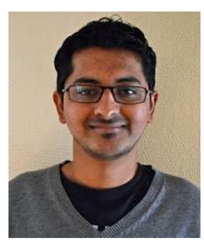

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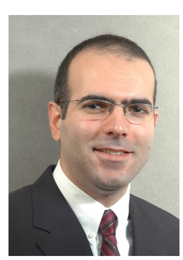

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