On the Boundaries of Randomization for Throughput-Optimal Scheduling in Switches

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***Abstract*—It is well-known that numerous Queue-Length-Based (QLB) schedulers, both deterministic and randomized, can achieve the maximum possible throughput region of wireless networks. While randomization is useful in allowing flexibilities in the design and implementation of the schedulers, it may lead to throughput loss if it is not within limits. In this work, we focus on the $N \times N$ input-queued switch topology to identify the boundaries of randomization in QLB scheduling for achieving throughput-optimality. To that end, we introduce a class of randomized QLB schedulers that are characterized by a wide range of functions. Then, we identify necessary and sufficient conditions on the number of switch ports N and the class of functions that can guarantee throughput-optimality of our class of randomized schedulers. Our results show that while our randomized QLB schedulers are throughput-optimal when $N = 2$, they cannot be throughput-optimal when $N \geq 3$ for a large set of functional forms. For $N \geq 3$, we further characterize an achievable rate region described via l_2 and l_∞ norms in an N^2 dimensional space that extends the existing achievable rate region descriptions. For $N = 2$, we also study the delay performance of various randomized QLB schedulers through simulations. This preliminary work reveals the sensitivity of throughput-optimal scheduling to the topological characteristics of the network and the functional characteristics of the randomization.**

I. Introduction

Efficient utilization of the network resources calls for careful scheduling of transmissions over time, subject to interference constraints. A first-order measure of efficiency of a scheduler is the achievable throughput it can provide. Those schedulers that can provide the largest set of possible throughput levels are commonly called *throughput-optimal*, and are of particular interest. The seminal works of Tassiulas and Ephremides [19], [20] and related works (e.g. [5], [12], [17]; see [16] for an overview) have established the throughput-optimality of a variety of *Queue-Length-Based (QLB) Scheduling* strategies, which prioritize activation of links with the greatest backlog awaiting service, also called *Maximum Weight Scheduling (MWS)*.

Subsequently, numerous QLB schedulers have been proposed with a range of complexity, distributiveness, and throughput characteristics (e.g. [18], [4], [15], [10], [8], [2], [11], [7], [14], [13]). In particular, *randomization* has been widely utilized to create flexibilities in the operation of many of these schedulers. Yet, to the best of our knowledge, there is no framework in which a variety of QLB randomized schedulers can be studied in terms of their throughput-optimality characteristics in interference-limited networks.

In this work, we aim to fill this gap by developing a common framework for the modeling and analysis of randomized schedulers, and then by establishing necessary and sufficient conditions on the throughput-optimality of a large functional class of QLB schedulers for switch topologies. Our framework is based on the modeling of randomized schedulers as a probabilistic mapping of queue-length vector space to the set of feasible schedules. Specifically, given the existing queue-lengths of the links, each scheduling strategy can be viewed as a particular probability distribution over the set of feasible schedules. While the means with which this random assignment may vary in its distributiveness or complexity, this perspective allows us to model a large set of existing and an even wider set of potential randomized schedulers within a common framework. Hence, these results are expected to assist the development of new randomized schedulers with favorable implementability and/or higher-order performance gains for switch networks.

The following list highlights our contributions and provides an outline of the remainder of the paper:

- In Section II, we introduce a functional class of randomized queue-length-based scheduling strategies (see Definitions 1), where the steepness of the function f determines the weight given to the heavily loaded *link*.
- We find sufficient (in Section III) and necessary (in Section IV) conditions on the number of ports in a switch for the throughput-optimality of these randomized schedulers as a function of the class of functions used in their operation.
- Then, for an $N \times N$ ($N \geq 3$) switch where the randomized scheduler cannot be throughput-optimal, we characterize an achievable rate region (in Section V) described by l_2 and l_∞ norms in an N^2 dimensional space. This result extends the results in [3] both in terms of the achievable rate region and the considered functional forms.
- For 2×2 switches where the randomized scheduler is throughput-optimal, we also study the delay performance of the proposed scheduler through simulations (in Section VI). These results highlight the sensitivity of different functional forms to traffic load asymmetries.

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II. System Model

Consider an $N \times N$ input-queued switch. Each input has infinite buffer for holding packets prior to switching them to their corresponding outputs. We assume a time-slotted system, where all packets are transmitted at the beginning of each time slot. During each time slot, at most one packet can be transferred from each input and at most one packet can be transferred to each output. For ease of exposition, we assume that a successful transmission over any link achieves a unit rate measured in packets per slot.

We use *complete bipartite graphs* to capture the switch constraint in an $N \times N$ switch. In a *complete bipartite graph* $\mathcal{BG} = (\mathcal{N}, \mathcal{E})$, the nodes can be divided into two sets: one set including all input ports and the other containing all output ports, where there is no edge between nodes in the same set and each node in one set connects with any node in another set. For convenience, we use (i,j) $(i,j = 1, ..., N)$ to denote the link in $\mathcal{E}$. In each time slot, we can successfully transmit over links in a subset of $\mathcal{E}$ that form a *matching* (i.e., no two edges share the same input or output port. We call each such matching as a *feasible schedule*, and denote it as $\mathbf{S} = (S_{ij})_{(i,j)\in\mathcal{E}} \in \{0,1\}^{|\mathcal{E}|}$, where $S_{ij} = 1$ if link (i,j) is active and $S_{ij} = 0$ if link (i,j) is inactive in the schedule. We further call a feasible schedule as *maximal* if no more links in $\mathcal{BG}$ can be added without violating the interference constraint. As maximal schedules represent extreme points in the space of feasible schedules, we collect them in the set $\mathcal{S}$. Then, we can define the *capacity region* Λ as the convex hull of $\mathcal{S}$, which will give the upper bound on the achievable link rates in packets per slot that can be supported by the network under stability for the given interference model.

In its simplest form, a *scheduler* determines a maximal feasible schedule $\mathbf{S}[t] \in \mathcal{S}$ at each time slot t. This selection may be influenced by the earlier experiences of each transmitter, and may be performed through a variety of strategies. Here, we are not interested in the means of selecting schedules, but in the eventual selection modeled as a probabilistic function of the state of the network. Before we define the randomized scheduler we consider more explicitly, we need to establish the traffic model.

We assume each link $(i,j) \in \mathcal{E}$ maintains a queue for each input and output (i.e. the buffer at an input is partitioned into N Virtual Output Queues (VOQs), each of infinite capacity. The virtual output queue VOQ_{ij} $(i,j = 1, ..., N)$ holds packets arriving at input i destined for output j). $A_{ij}[t]$ arrivals occur to link (i,j) in slot t that are independently distributed over links and identically distributed over time with mean λ_l, and $A_l[t] \le K$ for some $K < \infty$ [1]. We let $Q_{ij}[t]$ denote the queue length of queue (i,j) at time t. Recall from above that $S_{ij}[t]$ denotes the number of potential departures at time t. Further, we let $U_{ij}[t]$ denote the unused service for Queue (i,j) in slot t. If the queue (i,j) is empty and is scheduled, then $U_{ij}[t]$ is equal to 1; otherwise, it is equal to 0. Then, the evolution of the Queue (i,j) is described as follows:

$$Q_{ij}[t+1] = Q_{ij}[t] + A_{ij}[t] - S_{ij}[t] + U_{ij}[t], \forall l \in \mathcal{E}. \quad (1)$$

We say that Queue (i,j) is *f-stable* if there exists a non-negative valued, non-decreasing and divergent function f satisfying $\limsup_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} \mathbb{E}[f(Q_{ij}[t])] < \infty$. We note that this is an extended form of the more traditional strong stability condition (see [6]) that coincide when $f(x) = x$. Moreover, it is easy to show that f-stability implies strong stability when f is also a convex function. We say that the *network is f-stable* if all its queues are f-stable. Accordingly, we say that a scheduler is *f-throughput-optimal* if it achieves f-stability of the network for any arrival rate vector $\lambda = (\lambda_{ij})_{(i,j)\in\mathcal{E}}$ that lies strictly inside the capacity region Λ. Again, in the special case of $f(x) = x$, the notion of f-throughput-optimality reduces to traditional throughput-optimality, and when f is also convex, f-throughput-optimality implies throughput-optimality. The capacity region for an $N \times N$ switch is

$$\Lambda = \left\{ \lambda \ge 0 : \sum_{i=1}^{N} \lambda_{ij} < 1 \text{ and } \sum_{j=1}^{N} \lambda_{ij} < 1 \right\} \quad (2)$$

Starting with the seminal work [19], there is a vast literature on the design of throughput-optimal schedulers that utilize queue-length information in the selection of the schedules (see e.g. [6], [16]). Of special interest in this class of throughput-optimal schedulers are those that employ probabilistic assignments (e.g. [18], [10], [11], [7], [14], [4]). This is not only because they model possible errors in the scheduling process, but also because they allow significant flexibilities in the development of low-complexity and distributed implementations. Yet, randomization causes inaccurate operation and may be hurtful if not performed within limitations.

The aim of this work is to identify the limitations of randomization for a wide class of randomized dynamic schedulers that utilize functions of queue-lengths to schedule transmissions. To that end, we study a functional class of randomized schedulers that tends to select schedules with higher buffer occupancy levels. Before we describe them, let us define a basic set of functions we consider:

$\mathcal{F}$:= the set of nondecreasing and differentiable functions $f(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{x\to\infty} f(x) = \infty$.

Definition 1 (RSOF Scheduler): : For a given $f \in \mathcal{F}$ and queue-length vector $\mathbf{Q}$, the Ratio-of-Sum-of-Functions (RSOF) Scheduler picks a schedule $\mathbf{S} \in \mathcal{S}$ in that slot such that

$$P_{\mathbf{S}}(\mathbf{Q}) := \frac{\sum_{i\in\mathbf{S}} f(Q_i)}{\sum_{\mathbf{S}':\mathbf{S}'\in\mathcal{S}} \sum_{j\in\mathbf{S}'} f(Q_j)} \quad (3)$$

Note that the RSOF Scheduler is more likely to pick a

[1] We note that the boundedness assumption on the arrival process simplifies the technical arguments, but can be relaxed (see [5]) to the more common assumption of $E[A_l^2(t)] < \infty$.

schedule with the larger queue length, but with different distributions based on the form of $f \in \mathcal{F}$. In particular, the steepness of the function f determines the weight given to the heavily loaded *link* in the RSOF Scheduler.

It is important to understand the variety of functional forms that may achieve throughput-optimality since they are likely to possess differences in their implementation complexity and distributiveness characteristics. In particular, we identify the following class of functions.

Definition 2: We consider a subset of the space of functions $\mathcal{F}$:

$$\mathcal{A} := \{f \in \mathcal{F}: \lim_{x\to\infty} \frac{f(x+a)}{f(x)} = 1, \text{ for any } a \in \mathbb{R}\}.$$

Example of functions $f \in \mathcal{A}$ is the functions $f(x) = (\log(x+1))^{\alpha}$, $f(x) = x^{\alpha}$ ($\alpha > 0$) and $f(x) = \frac{1}{x^{\beta}} e^{x^{\alpha}}$ ($0 < \alpha < 1$, $\beta \geq 0$).

Remarks 1: In $\mathcal{A}$, if $\lim_{x\to\infty} \frac{f(x+a)}{f(x)}$ exists for any $a \in \mathbb{R}$, then this limit should be equal to 1. Indeed, let $\lim_{x\to\infty} \frac{f(x+a)}{f(x)} = b$ for any $a \in \mathbb{R}$, where $b > 0$. Then $b = \lim_{x\to\infty} \frac{f(x+2)}{f(x)} = \lim_{x\to\infty} \frac{f(x+2)}{f(x+1)} \cdot \frac{f(x+1)}{f(x)} = b^2$. Thus, $b = 1$.

III. f-THROUGHPUT-OPTIMALITY IN A 2×2 SWITCH

In this section, we establish the f-throughput-optimality of RSOF Scheduler in a 2×2 switch for any $f \in \mathcal{A}$. Thus, this result yields a sufficient condition for the f-throughput-optimality of RSOF Scheduler in switches. We will complement this result by a necessary result that shows its tightness in Section IV.

Lemma 1: In an $N \times N$ switch, if for any $\lambda \in \Omega \subseteq \Lambda$, there exist a $\delta > 0$ and $a_{ij} > 0$ such that, for all $\mathbf{Q}$,

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} f(Q_{ij})(\lambda_{ij} - P_{ij}) \leq -\delta \sum_{i=1}^{N}\sum_{j=1}^{N} f(Q_{ij}), \quad (4)$$

where P_{ij} is the probability serving the link (i, j) under the RSOF Scheduler with $f \in \mathcal{A}$, then RSOF is f-stable in the region Ω.

Proof: See Appendix A for the proof. ■

Theorem 1: The RSOF Scheduler with $f \in \mathcal{A}$ is f-throughput-optimal in a 2×2 switch.

Proof: By Lemma 1, if for any $\lambda \in \Lambda$, there exists a $\delta > 0$ such that

$$\sum_{i=1}^{2}\sum_{j=1}^{2} \frac{1}{\lambda_{ij}} f(Q_{ij})(\lambda_{ij} - P_{ij}) < -\delta \sum_{i=1}^{2}\sum_{j=1}^{2} f(Q_{ij}) \quad (5)$$

then, the RSOF Scheduler with $f \in \mathcal{A}$ is f-stable for any $\lambda \in \Lambda$ and thus is throughput-optimal. Indeed, in a 2×2 switch, $P_{11} = P_{22} = P_{S_1} = \frac{f(Q_{11})+f(Q_{22})}{\Delta}$ and $P_{12} = P_{21} = P_{S_2} = \frac{f(Q_{12})+f(Q_{21})}{\Delta}$, where $\Delta = \sum_{i=1}^{2}\sum_{j=1}^{2} f(Q_{ij})$. Since

$$\begin{aligned}
&\sum_{i=1}^{2}\sum_{j=1}^{2} \frac{1}{\lambda_{ij}} f(Q_{ij})(\lambda_{ij} - P_{ij}) \\
=&\frac{1}{\Delta}\Big[\Delta^2 - (\frac{f(Q_{11})}{\lambda_{11}} + \frac{f(Q_{22})}{\lambda_{22}})(f(Q_{11}) + f(Q_{22})) \\
&- (\frac{f(Q_{12})}{\lambda_{12}} + \frac{f(Q_{21})}{\lambda_{21}})(f(Q_{12}) + f(Q_{21}))\Big]
\end{aligned} \quad (6)$$

and

$$\begin{aligned}
&\left(\frac{f(Q_{11})}{\lambda_{11}} + \frac{f(Q_{22})}{\lambda_{22}}\right)(f(Q_{11}) + f(Q_{22})) \\
&+ \left(\frac{f(Q_{12})}{\lambda_{12}} + \frac{f(Q_{21})}{\lambda_{21}}\right)(f(Q_{12}) + f(Q_{21})) \\
\geq&\frac{1}{\lambda_1}(f(Q_{11}) + f(Q_{22}))^2 + \frac{1}{\lambda_2}(f(Q_{12}) + f(Q_{21}))^2 \\
=&\frac{1}{\lambda_1+\lambda_2}\Big[\frac{\lambda_1+\lambda_2}{\lambda_1}(f(Q_{11}) + f(Q_{22}))^2 \\
&+ \frac{\lambda_1+\lambda_2}{\lambda_2}(f(Q_{12}) + f(Q_{21}))^2\Big] \\
=&\frac{1}{\lambda_1+\lambda_2}\Big[(f(Q_{11}) + f(Q_{22}))^2 + (f(Q_{12}) + f(Q_{21}))^2 \\
&+ \frac{\lambda_2}{\lambda_1}(f(Q_{11}) + f(Q_{22}))^2 + \frac{\lambda_1}{\lambda_2}(f(Q_{12}) + f(Q_{21}))^2\Big] \\
\geq&\frac{1}{\lambda_1+\lambda_2}\Big[(f(Q_{11}) + f(Q_{22}))^2 + (f(Q_{12}) + f(Q_{21}))^2 \\
&+ 2\sqrt{\frac{\lambda_2}{\lambda_1}\frac{\lambda_1}{\lambda_2}(f(Q_{11}) + f(Q_{22}))^2(f(Q_{12}) + f(Q_{21}))^2}\Big] \\
=&\frac{1}{\lambda_1+\lambda_2}(f(Q_{11}) + f(Q_{22}) + f(Q_{12}) + f(Q_{21}))^2 \\
>&(f(Q_{11}) + f(Q_{22}) + f(Q_{12}) + f(Q_{21}))^2(1+\delta)
\end{aligned} \quad (7)$$

Where $\lambda_1 := \max\{\lambda_{11}, \lambda_{22}\}$, $\lambda_2 := \max\{\lambda_{12}, \lambda_{21}\}$ and there exists a $\delta > 0$ such that $\frac{1}{\lambda_1+\lambda_2} > 1 + \delta$. Hence, by combining the inequalities (6) and (7), we can see that inequality (5) is true for any $\lambda \in \Lambda$. ■

Note that RSOF Scheduler with $f \in \mathcal{F} - \mathcal{A}$ is not necessarily non-f-throughput-optimal in 2×2 switches. In fact, we conjecture that RSOF Scheduler with the function f steeper than any function in A is f-throughput-optimal. We validate this conjecture through simulations in section VI.

IV. A NECESSARY CONDITION FOR THROUGHPUT-OPTIMALITY IN AN $N \times N$ SWITCH

We have shown that RSOF Scheduler with the function $f \in \mathcal{A}$ is f-throughput-optimal in the 2×2 switch. However, the next result establishes that the RSOF Scheduler with any function $f \in \mathcal{F}$ cannot be throughput-optimal in an $N \times N$ switch when $N \geq 3$, which provides the necessary condition for throughput optimality of RSOF Scheduler in an $N \times N$ switch.

Theorem 2: In an $N \times N$ switch, where $N \geq 3$, the RSOF Scheduler is not throughput-optimal for any $f \in \mathcal{F}$.

Proof: We prove this claim by considering an arrival process that is inside the capacity region, but that is not supportable by the RSOF Scheduler. To that end, let's consider a maximal schedule $\mathbf{S}_1 = \{(1,1), (2,2), ..., (N,N)\}$.

We assume that arrivals only happen to those N links at rates $\lambda_{11}, ..., \lambda_{NN}$ with the constraint that $\lambda_{ii} \in [0, 1)$ for all $i = 1, ..., N$, which clearly can be supported by a simple policy that always serves the schedule $\mathbf{S}_1$. Thus, setting λ_{ii} arbitrarily close to one for each i, this simple policy can achieve a sum rate of $\sum_{i=1}^{N} \lambda_{ii} < N$.

Given this construction, we next prove the following claim:

Claim 1: If $\sum_{i=1}^{N} \lambda_{ii} \geq 2$, the RSOF Scheduler with any function $f \in \mathcal{F}$ is unstable for an $N \times N$ switch, where $N \geq 3$.

Proof of Claim: Based on the above model, the RSOF Scheduler becomes

$$P_{\mathbf{S}} = \frac{\sum_{i=1}^{N} \sum_{(i,i) \in \mathbf{S}} f(Q_{ii}) + (N - \sum_{i=1}^{N} \sum_{(i,i) \in \mathbf{S}} 1) f(0)}{\sum_{\mathbf{S}'} \left(\sum_{j=1}^{N} \sum_{(j,j) \in \mathbf{S}'} f(Q_{jj}) + (N - \sum_{j=1}^{N} \sum_{(j,j) \in \mathbf{S}'} 1) f(0) \right)}$$

Then,

$$\begin{aligned} \sum_{i=1}^{N} P_{ii} &= \sum_{i=1}^{N} \sum_{\{\mathbf{S}:(i,i) \in \mathbf{S}\}} P_{\mathbf{S}} \\ &= \frac{\overbrace{\sum_{i=1}^{N} \sum_{\{\mathbf{S}:(i,i) \in \mathbf{S}\}} \left(\sum_{j=1}^{N} \sum_{(j,j) \in \mathbf{S}} f(Q_{jj}) + (N - \sum_{j=1}^{N} \sum_{(j,j) \in \mathbf{S}} 1) f(0) \right)}^{=:Y}}{\underbrace{\sum_{\mathbf{S}'} \left(\sum_{j=1}^{N} \sum_{(j,j) \in \mathbf{S}'} f(Q_{jj}) + (N - \sum_{j=1}^{N} \sum_{(j,j) \in \mathbf{S}'} 1) f(0) \right)}_{=:X}} \end{aligned}$$

We can expand X and Y as follows

$$\begin{aligned} X &= \sum_{\mathbf{S}} \sum_{j=1}^{N} \sum_{(j,j) \in \mathbf{S}} (f(Q_{jj}) - f(0)) + N f(0) \sum_{\mathbf{S}} 1 \\ &= \sum_{j=1}^{N} (f(Q_{jj}) - f(0)) \sum_{\{\mathbf{S}:(j,j) \in \mathbf{S}\}} 1 + N \cdot N! f(0) \\ &= (N-1)! \sum_{j=1}^{N} f(Q_{jj}) + (N-1) N! f(0) \end{aligned} \tag{8}$$

$$\begin{aligned} Y &= \sum_{i=1}^{N} \sum_{\{\mathbf{S}:(i,i) \in \mathbf{S}\}} \sum_{j=1}^{N} \sum_{(j,j) \in \mathbf{S}} (f(Q_{jj}) - f(0)) \\ &\quad + N f(0) \sum_{i=1}^{N} \sum_{\{\mathbf{S}:(i,i) \in \mathbf{S}\}} 1 \\ &= \sum_{\mathbf{S}} (\sum_{i=1}^{N} \sum_{(i,i) \in \mathbf{S}} 1)(\sum_{j=1}^{N} \sum_{(j,j) \in \mathbf{S}} (f(Q_{jj}) - f(0))) \\ &\quad + N \cdot N! f(0) \\ &= \sum_{j=1}^{N} (f(Q_{jj}) - f(0)) \sum_{\mathbf{S}:(j,j) \in \mathbf{S}} (\sum_{i=1}^{N} \sum_{(i,i) \in \mathbf{S}} 1) + N \cdot N! f(0) \end{aligned}$$

Since fixing j, $\sum_{\mathbf{S}:(j,j) \in \mathbf{S}} (\sum_{i=1}^{N} \sum_{(i,i) \in \mathbf{S}} 1)$ means the number of links (i, i) $(i = 1, ..., N)$ appearing in the schedulers having the link (j, j). We know that link (j, j) appears in the $(N-1)!$ schedules and all other links (k, k) $(k \neq j)$ appears in the $(N-2)!$ schedules. Thus, $\sum_{\mathbf{S}:(j,j) \in \mathbf{S}} (\sum_{i=1}^{N} \sum_{(i,i) \in \mathbf{S}} 1) = (N-1)! + (N-1) \cdot (N-2)! = 2(N-1)!$. Hence, $Y = 2(N-1)! \sum_{j=1}^{N} f(Q_{jj}) + (N-2) N! f(0)$ and thus

$$\begin{aligned} \sum_{i=1}^{N} P_{ii} &= \frac{2(N-1)! \sum_{j=1}^{N} f(Q_{jj}) + (N-2) N! f(0)}{(N-1)! \sum_{j=1}^{N} f(Q_{jj}) + (N-1) N! f(0)} \\ &\leq \frac{2(N-1)! \sum_{j=1}^{N} f(Q_{jj}) + 2(N-1) N! f(0)}{(N-1)! \sum_{j=1}^{N} f(Q_{jj}) + (N-1) N! f(0)} \\ &= 2 \end{aligned} \tag{9}$$

Consider the Lyapunov function $L(\mathbf{Q}) := \sum_{i=1}^{N} Q_{ii}$, then

$$\begin{aligned} \Delta L :=& \mathbb{E} \left[L(\mathbf{Q}[t+1]) - L(\mathbf{Q}[t]) | \mathbf{Q}[t] = \mathbf{Q} \right] \\ \geq& \sum_{i=1}^{N} E \left[A_{ii}[t] - S_{ii}[t] | \mathbf{Q}[t] = \mathbf{Q} \right] \\ =& \sum_{i=1}^{N} \lambda_{ii} - \sum_{i=1}^{N} P_{ii} \geq \sum_{i=1}^{N} \lambda_{ii} - 2 \end{aligned} \tag{10}$$

If $\sum_{i=1}^{N} \lambda_{ii} \geq 2$, then $\Delta L \geq 0$. Hence, by the Theorem 20 of [9], the RSOF Scheduler is unstable if $\sum_{i=1}^{N} \lambda_{ii} \geq 2$. ■

V. The Throughput Performance of RSOF Scheduler for an $N \times N$ switch

Even though the RSOF Scheduler with any function $f \in \mathcal{F}$ cannot be throughput-optimal in an $N \times N$ switch ($N \geq 3$), it can still achieve stability within some region. In this section, we specify the region of supportable arrival rates under the RSOF Scheduler with $f \in \mathcal{A}$ for an $N \times N$ switch and prove its stabilizing properties. Theorem 3 implies that the stable region that RSOF Scheduler with $f \in \mathcal{A}$ can achieve is:

$$\Gamma = \Gamma_1 \bigcup \Gamma_2 \tag{11}$$

Where $\Gamma_1 := \left\{ \lambda \geq 0 : \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_{ij}^2 < \frac{1}{N} \right\}$ and $\Gamma_2 := \left\{ \lambda \geq 0 : \lambda_{ij} < \frac{1}{N} \ \forall i, j = 1, ..., N \right\}$. Note that Γ_1 doesn't coincide with Γ_2. Consider a 2×2 switch. Assuming $\lambda_{22} = 0$, Figure 1 illustrates the relationship among Γ_1, Γ_2 and Λ in a three dimensional space. We observe that Γ_1 and Γ_2 are captured by ball and cube forms, respectively. Also, we can observe that Γ_1 and Γ_2 partially overlap.

Lemma 2: In an $N \times N$ switch, for any non-negative valued vector $\mathbf{Q}$, we have

$$(1) \ \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} f^2(Q_{ij}) \leq \left(\sum_{i=1}^{N} \sum_{j=1}^{N} f(Q_{ij}) P_{ij} \right)^2$$

$$(2) \ \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} f(Q_{ij}) \leq \sum_{i=1}^{N} \sum_{j=1}^{N} f(Q_{ij}) P_{ij}$$

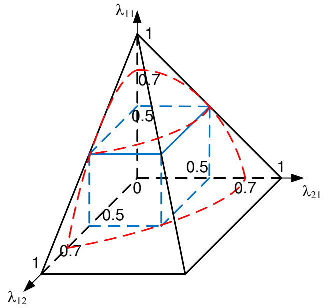

Fig. 1. The relationship among Γ_1,Γ_2 and Λ

where $P_{ij} = \sum_{\{\mathbf{S}:(i,j)\in\mathbf{S}\}} P_{\mathbf{S}}(\mathbf{Q})$, where $P_{\mathbf{S}}(\mathbf{Q})$ is given by (3).

Proof: See Appendix B for the proof. ■

Theorem 3: The RSOF Scheduler with $f \in \mathcal{A}$ stabilizes the $N \times N$ switch for any arrival rate $\lambda \in \Gamma$.

Proof: By Lemma 1, if inequality (4) holds for any $\lambda \in \Gamma$, then the RSOF Scheduler with $f \in \mathcal{A}$ is f-stable for any $\lambda \in \Gamma$.

(1) If $\lambda \in \Gamma_1$, that is, $\sum_{i=1}^N \sum_{j=1}^N \lambda_{ij}^2 < \frac{1}{N}$, then there exists a $\delta > 0$, likely a function of N, such that

$$\sum_{i=1}^N \sum_{j=1}^N \lambda_{ij}^2 \le \frac{1}{N} - 2N^4\delta \tag{12}$$

Thus, by Cauchy-Schwarz inequality, we have

$$\begin{aligned}(\sum_{i=1}^N \sum_{j=1}^N \lambda_{ij} f(Q_{ij}))^2 &\le (\sum_{i=1}^N \sum_{j=1}^N \lambda_{ij}^2) \sum_{i=1}^N \sum_{j=1}^N f^2(Q_{ij}) \\ &\le (\frac{1}{N} - 2N^4\delta) \sum_{i=1}^N \sum_{j=1}^N f^2(Q_{ij})\end{aligned}$$

By Lemma 2, we have

$$\begin{aligned}&(\sum_{i=1}^N \sum_{j=1}^N \lambda_{ij} f(Q_{ij}))^2 \\ \le& -2N^4\delta \sum_{i=1}^N \sum_{j=1}^N f^2(Q_{ij}) + (\sum_{i=1}^N \sum_{j=1}^N P_{ij} f(Q_{ij}))^2\end{aligned}$$

Thus,

$$\begin{aligned}&(\sum_{i=1}^N \sum_{j=1}^N \lambda_{ij} f(Q_{ij}) + \sum_{i=1}^N \sum_{j=1}^N P_{ij} f(Q_{ij})) \\ &\cdot (\sum_{i=1}^N \sum_{j=1}^N \lambda_{ij} f(Q_{ij}) - \sum_{i=1}^N \sum_{j=1}^N P_{ij} f(Q_{ij})) \\ =&(\sum_{i=1}^N \sum_{j=1}^N \lambda_{ij} f(Q_{ij}))^2 - (\sum_{i=1}^N \sum_{j=1}^N P_{ij} f(Q_{ij}))^2 \\ \le& -2N^4\delta \sum_{i=1}^N \sum_{j=1}^N f^2(Q_{ij})\end{aligned} \tag{13}$$

Hence, we have

$$\begin{aligned}&\sum_{i=1}^N \sum_{j=1}^N \lambda_{ij} f(Q_{ij}) - \sum_{i=1}^N \sum_{j=1}^N P_{ij} f(Q_{ij}) \\ \le& \frac{-2N^4\delta \sum_{i=1}^N \sum_{j=1}^N f^2(Q_{ij})}{\sum_{i=1}^N \sum_{j=1}^N \lambda_{ij} f(Q_{ij}) + \sum_{i=1}^N \sum_{j=1}^N P_{ij} f(Q_{ij})} \\ \le& -N^4\delta \frac{\sum_{i=1}^N \sum_{j=1}^N f^2(Q_{ij})}{\sum_{i=1}^N \sum_{j=1}^N f(Q_{ij})}\end{aligned} \tag{14}$$

In addition,

$$\begin{aligned}&(\sum_{i=1}^N \sum_{j=1}^N f(Q_{ij}))^2 \le (N^2 \max_{ij}\{f(Q_{ij})\})^2 \\ &= N^4 \max_{ij}\{f^2(Q_{ij})\} \le N^4 \sum_{i=1}^N \sum_{j=1}^N f^2(Q_{ij})\end{aligned} \tag{15}$$

Hence, by combining inequalities (14) and (15), we can see that inequality (4) is true for any $\lambda \in \Gamma_1$.

(2) If $\lambda \in \Gamma_2$, that is, $\lambda_{ij} < \frac{1}{N}$ for $\forall i,j = 1,...,N$, then there exist a $\delta > 0$ such that $\lambda_{i,j} \le \frac{1}{N} - \delta$. Thus,

$$\begin{aligned}&\sum_{i=1}^N \sum_{j=1}^N f(Q_{ij})(\lambda_{ij} - P_{ij}) \\ \le& \sum_{i=1}^N \sum_{j=1}^N f(Q_{ij})(\frac{1}{N} - \delta - P_{ij}) \\ =& -\delta \sum_{i=1}^N \sum_{j=1}^N f(Q_{ij}) + \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^N f(Q_{ij}) \\ &- \sum_{i=1}^N \sum_{j=1}^N f(Q_{ij}) P_{ij}\end{aligned}$$

By Lemma 2, we can see that inequality (4) holds for any $\lambda \in \Gamma_2$. ■

VI. Simulation Results

In this section, we perform numerical studies to evaluate the delay performance of RSOF Schedulers with different functions in a 2×2 switch and throughput performance of RSOF Scheduler in a 3×3 switch.

A. The impact of steepness of function on delay performance

In a 2×2 switch, there are 2 maximal schedules, each containing 2 links. We consider arrival rate $\lambda = \rho\mathbf{H}$, where $\mathbf{H} = [H_{ij}]$ is a doubly-stochastic matrix with H_{ij} denoting the fraction of the total rate from input port i that is destined to output port j. Here, $\rho \in (0,1)$ represents the average arrival intensity, where the larger the ρ, the more heavily loaded the switch is. Due to limited space, we present two cases: symmetric arrival process ($\mathbf{H}_1 = [0.5\ 0.5; 0.5\ 0.5]$) and asymmetric arrival process ($\mathbf{H}_2 = [0.1\ 0.9; 0.9\ 0.1]$) under the high arrival intensity $\rho = 0.99$. We run the simulation 100 times and get the average queue length per link for RSOF

Schedulers with different functions, which are shown in Table I.

TABLE I
AVERAGE QUEUE LENGTH PER LINK

Functions	RSOF: $\mathbf{H}_1 = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$	RSOF: $\mathbf{H}_2 = \begin{bmatrix} 0.1 & 0.9 \\ 0.9 & 0.1 \end{bmatrix}$
e^{x^2}	21.36	7.85
e^x	21.40	8.07
$e^{\sqrt{x}}$	20.75	11.06
$(x+1)^2$	20.02	10.06
$x+1$	20.71	17.62
$\log(x+e)$	19.48	1126

From Table I, we can observe that under symmetric arrival traffic, the delay performance is highly insensitive to the choice of the functional form being used in the RSOF Scheduler. So, there is a wide class of choices under which the RSOF Scheduler can yield good performance. On the other hand, under asymmetric arrival traffic, it appears that the steepness of f needs to be high enough for RSOF Scheduler to yield good delay performance. In addition, we can see that RSOF Schedulers with either e^x or e^{x^2} are also f-stable in symmetric and asymmetric arrival processes. Thus, we conjecture that RSOF Scheduler with the function steeper than the function in Class $\mathcal{A}$ is f-throughput-optimal.

B. *The throughput performance in an $N \times N$ switch*

In an $N \times N$ switch ($N \geq 3$), the RSOF Scheduler with any function $f \in \mathcal{F}$ is non-throughput-optimal. In section V, we show that the RSOF Scheduler with the function $f \in \mathcal{A}$ is f-stable in the region Γ. However, the RSOF Scheduler can achieve a larger rate region than Γ. To see this, for example, consider a RSOF Scheduler with the linear function under the arrival rate $\lambda = [0.0368\ 0.2201\ 0.0686; 0.4308\ 0.2656\ 0.0498; 0.1266\ 0.0464\ 0.6309]$ in a 3×3 switch. While $\lambda \in \Lambda - \Gamma$, the simulation result shown in Figure 2 indicates that the given rate is actually supportable. Thus, the largest possible achievable rate region of RSOF schedulers for general $N \times N$ switches is an open problem. From Figure 2, we can observe that RSOF Scheduler is still f-stable outside the region Γ.

VII. Conclusions

We proposed and explored the limitations of a functional class of queue-length-based randomized schedulers in $N \times N$ switch networks. Our study revealed the sensitivity of the randomized schedulers to the number of ports of the switch by establishing that $N = 2$ is necessary and sufficient for the throughput-optimality a large class of functional forms considered. We also characterized an achievable rate region for $N \geq 3$ and studied the delay performance under the throughput-optimal scenario. These results not only revealed the strengths and weaknesses of randomization in dynamic scheduling but also helped us establish important insights for the study of a wider class of schedulers in more general network topologies, which has formed the basis of our ongoing work.

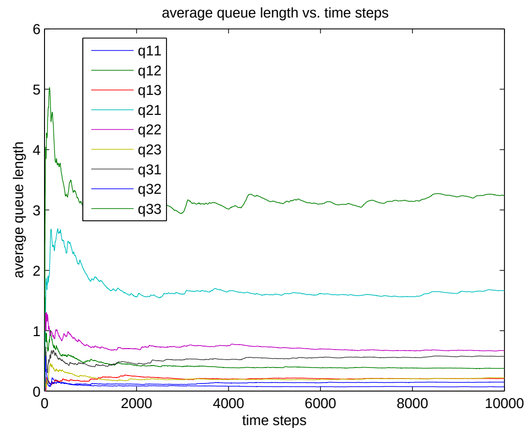

Fig. 2. The average queue length vs. time steps

VIII. Appendix A
Proof of Lemma 1

Consider the Lyapunov function

$$V(\mathbf{Q}) := \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} h(Q_{ij}) \tag{16}$$

where $h'(x) = f(x)$. Then

$$\Delta V := \mathbb{E}\left[V(\mathbf{Q}[\mathbf{t}+\mathbf{1}]) - V(\mathbf{Q}[\mathbf{t}]) | \mathbf{Q}[t] = \mathbf{Q}\right]$$
$$= \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\mathbb{E}\left[h(Q_{ij}[t+1]) - h(Q_{ij}[t]) | \mathbf{Q}[t] = \mathbf{Q}\right]$$

By the mean-value theorem, we have $h(Q_{ij}[t+1]) - h(Q_{ij}[t]) = f(R_{ij}[t])(Q_{ij}[t+1] - Q_{ij}[t]) = f(R_{ij}[t])(A_{ij}[t] - S_{ij}[t] + U_{ij}[t])$, where $R_{ij}[t]$ lies between $Q_{ij}[t]$ and $Q_{ij}[t+1]$. Due to the space limitation, we will drop the time index $[t]$ from the quantities $A_{ij}[t]$,$S_{ij}[t]$,$U_{ij}[t]$ and $R_{ij}[t]$ in the following proof. Hence, we get

$$\Delta V = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\mathbb{E}\left[f(R_{ij})(A_{ij} - S_{ij} + U_{ij}) | \mathbf{Q}[t] = \mathbf{Q}\right]$$
$$= \underbrace{\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\mathbb{E}\left[f(R_{ij})U_{ij} | \mathbf{Q}[t] = \mathbf{Q}\right]}_{=:\Delta V_1}$$
$$+ \underbrace{\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\mathbb{E}\left[f(R_{ij})(A_{ij} - S_{ij}) | \mathbf{Q}[t] = \mathbf{Q}\right]}_{=:\Delta V_2}$$

For ΔV_1, if $Q_{ij}[t] = Q_{ij} > 0$, then $U_{ij}[t] = 0$. If $Q_{ij}[t] = Q_{ij} = 0$, then $U_{ij}[t]$ may be equal to 1. But in this case,

$Q_{ij}[t+1] \leq K$ (since $A_{ij}[t] \leq K$). Hence, $f(R_{ij}[t]) \leq f(K) < \infty$. Thus,

$$\begin{aligned}
\Delta V_1 =& \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\mathbb{E}\left[f(R_{ij})U_{ij}|\mathbf{Q}[t]=\mathbf{Q}\right]\mathbf{1}_{\{Q_{ij}>0\}}+ \\
& \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\mathbb{E}\left[f(R_{ij})U_{ij}|\mathbf{Q}[t]=\mathbf{Q}\right]\mathbf{1}_{\{Q_{ij}=0\}} \\
=& \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\mathbb{E}\left[f(R_{ij})U_{ij}|\mathbf{Q}[t]=\mathbf{Q}\right]\mathbf{1}_{\{Q_{ij}=0\}} \\
\leq& D\sum_{i=1}^{N}\sum_{j=1}^{N} f(K)
\end{aligned} \tag{17}$$

Where $D := \max\{a_{ij}\}$ and $\mathbf{1}_{\{\cdot\}}$ denotes the indicator function.

Next, let's focus on ΔV_2. We know that $f(R_{ij}[t]) = f(Q_{ij}[t]+b_{ij})$ ($|b_{ij}| \leq K$). According to the definition of function $f \in \mathcal{A}$, given $\epsilon > 0$, there exists $M > 0$, such that for any $Q_{ij}[t] = Q_{ij} > M$, we have $\left|\frac{f(R_{ij})}{f(Q_{ij})} - 1\right| < \epsilon$, that is, $(1-\epsilon)f(Q_{ij}) < f(R_{ij}) < (1+\epsilon)f(Q_{ij})$. Thus, we have

$$\begin{aligned}
& f(R_{ij})(A_{ij}-S_{ij}) \\
=& f(R_{ij})\left[(A_{ij}-S_{ij})_+ - (A_{ij}-S_{ij})_-\right] \\
<& (1+\epsilon)f(Q_{ij})(A_{ij}-S_{ij})_+ - (1-\epsilon)f(Q_{ij})(A_{ij}-S_{ij})_- \\
=& f(Q_{ij})\left[(A_{ij}-S_{ij})_+ - (A_{ij}-S_{ij})_-\right] \\
& + \epsilon f(Q_{ij})\left[(A_{ij}-S_{ij})_+ + (A_{ij}-S_{ij})_-\right] \\
=& f(Q_{ij})(A_{ij}-S_{ij}) + \epsilon f(Q_{ij})\left|A_{ij}-S_{ij}\right| \\
\leq& f(Q_{ij})(A_{ij}-S_{ij}) + K\epsilon f(Q_{ij})
\end{aligned} \tag{18}$$

Where $(x)_+ = \max\{x,0\}$, $(x)_- = -\min\{x,0\}$ and $|A_{ij} - S_{ij}| \leq |A_{ij}| \leq K$. Thus, we divide ΔV_2 into two parts:

$$\begin{aligned}
\Delta V_2 =& \underbrace{\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\mathbb{E}\left[f(R_{ij})(A_{ij}-S_{ij})|\mathbf{Q}[t]=\mathbf{Q}\right]\mathbf{1}_{\{Q_{ij}>M\}}}_{=:\Delta V_3} \\
& + \underbrace{\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\mathbb{E}\left[f(R_{ij})(A_{ij}-S_{ij})|\mathbf{Q}[t]=\mathbf{Q}\right]\mathbf{1}_{\{Q_{ij}\leq M\}}}_{=:\Delta V_4}
\end{aligned}$$

For ΔV_3, by using (18), we have

$$\begin{aligned}
\Delta V_3 \leq& \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\mathbb{E}\left[f(Q_{ij})(A_{ij}-S_{ij})|\mathbf{Q}[t]=\mathbf{Q}\right]\mathbf{1}_{\{Q_{ij}>M\}} \\
& + \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}K\epsilon f(Q_{ij})\mathbf{1}_{\{Q_{ij}>M\}} \\
\leq& \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}f(Q_{ij})(\lambda_{ij}-P_{ij})\mathbf{1}_{\{Q_{ij}>M\}} \\
& + DK\epsilon\sum_{i=1}^{N}\sum_{j=1}^{N} f(Q_{ij})\mathbf{1}_{\{Q_{ij}>M\}}
\end{aligned} \tag{19}$$

Since inequality (4) holds, we have

$$\begin{aligned}
& \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}f(Q_{ij})(\lambda_{ij}-P_{ij})\mathbf{1}_{\{Q_{ij}>M\}} \\
\leq& -\delta\sum_{i=1}^{N}\sum_{j=1}^{N} f(Q_{ij})\mathbf{1}_{\{Q_{ij}>M\}} - \delta\sum_{i=1}^{N}\sum_{j=1}^{N} f(Q_{ij})\mathbf{1}_{\{Q_{ij}\leq M\}} \\
& - \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}f(Q_{ij})(\lambda_{ij}-P_{ij})\mathbf{1}_{\{Q_{ij}\leq M\}} \\
<& -\delta\sum_{i=1}^{N}\sum_{j=1}^{N} f(Q_{ij})\mathbf{1}_{\{Q_{ij}>M\}} + \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}f(Q_{ij})P_{ij}\mathbf{1}_{\{Q_{ij}\leq M\}} \\
<& -\delta\sum_{i=1}^{N}\sum_{j=1}^{N} f(Q_{ij})\mathbf{1}_{\{Q_{ij}>M\}} + D\sum_{i=1}^{N}\sum_{j=1}^{N} f(M)
\end{aligned} \tag{20}$$

Thus, we have

$$\begin{aligned}
\Delta V_3 \leq& -\delta\sum_{i=1}^{N}\sum_{j=1}^{N} f(Q_{ij})\mathbf{1}_{\{Q_{ij}>M\}} \\
& + DK\epsilon\sum_{i=1}^{N}\sum_{j=1}^{N} f(Q_{ij})\mathbf{1}_{\{Q_{ij}>M\}} + D\sum_{i=1}^{N}\sum_{j=1}^{N} f(M)
\end{aligned}$$

We can choose ϵ small enough such that $\gamma = \delta - DK\epsilon > 0$ and thus we have

$$\Delta V_3 \leq -\gamma\sum_{i=1}^{N}\sum_{j=1}^{N} f(Q_{ij})\mathbf{1}_{\{Q_{ij}>M\}} + D\sum_{i=1}^{N}\sum_{j=1}^{N} f(M)$$

For ΔV_4, we have

$$\begin{aligned}
\Delta V_4 \leq& \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\mathbb{E}\left[f(R_{ij})\left|A_{ij}-S_{ij}\right||\mathbf{Q}[t]=\mathbf{Q}\right]\mathbf{1}_{\{Q_{ij}\leq M\}} \\
\leq& DK\sum_{i=1}^{N}\sum_{j=1}^{N} f(M+K)
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\Delta V \leq& -\gamma\sum_{i=1}^{N}\sum_{j=1}^{N} f(Q_{ij})\mathbf{1}_{\{Q_{ij}>M\}} + D\sum_{i=1}^{N}\sum_{j=1}^{N} f(K) \\
& + DK\sum_{i=1}^{N}\sum_{j=1}^{N} f(M+K) + D\sum_{i=1}^{N}\sum_{j=1}^{N} f(M)
\end{aligned} \tag{21}$$

By the Foster-Lyapunov theorem [1], we know that RSOF scheduler with $f \in \mathcal{A}$ is f-stable under the condition (4).

IX. Appendix B
Proof of Lemma 2

$$\begin{aligned}
& \sum_{i=1}^{N}\sum_{j=1}^{N} f(Q_{ij})P_{ij} \\
=& \frac{\sum_{i=1}^{N}\sum_{j=1}^{N} f(Q_{ij})\sum_{\{\mathbf{S}:(i,j)\in\mathbf{S}\}}\sum_{(k,l)\in\mathbf{S}} f(Q_{kl})}{\sum_{\mathbf{S}'}\sum_{(k,l)\in\mathbf{S}'} f(Q_{kl})}
\end{aligned}$$

$$=\frac{\sum_{\mathbf{S}}\left(\sum_{(i,j)\in\mathbf{S}} f(Q_{ij})\right)^2}{\sum_{\mathbf{S}}\sum_{(i,j)\in\mathbf{S}} f(Q_{ij})} \tag{22}$$

(1) We only need to show $\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} f^2(Q_{ij}) \leq \left(\frac{\sum_{\mathbf{S}}\left(\sum_{(i,j)\in\mathbf{S}} f(Q_{ij})\right)^2}{\sum_{\mathbf{S}}\sum_{(i,j)\in\mathbf{S}} f(Q_{ij})}\right)^2$. By Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left(\sum_{\mathbf{S}}\sum_{(i,j)\in\mathbf{S}} f(Q_{ij})\right)^2 &\leq \sum_{\mathbf{S}} 1 \sum_{\mathbf{S}}\left(\sum_{(i,j)\in\mathbf{S}} f(Q_{ij})\right)^2 \\ &= N!\sum_{\mathbf{S}}\left(\sum_{(i,j)\in\mathbf{S}} f(Q_{ij})\right)^2\end{aligned} \tag{23}$$

In addition, we have

$$\begin{aligned}\sum_{\mathbf{S}}\left(\sum_{(i,j)\in\mathbf{S}} f(Q_{ij})\right)^2 &\geq \sum_{\mathbf{S}}\sum_{(i,j)\in\mathbf{S}} f^2(Q_{ij}) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} f^2(Q_{ij}) \sum_{\{\mathbf{S}:(i,j)\in\mathbf{S}\}} 1 \\ &= (N-1)!\sum_{i=1}^{N}\sum_{j=1}^{N} f^2(Q_{ij})\end{aligned}$$

Then we have

$$\begin{aligned}&\left(\sum_{i=1}^{N}\sum_{j=1}^{N} f^2(Q_{ij})\right)\left(\sum_{\mathbf{S}}\sum_{(i,j)\in\mathbf{S}} f(Q_{ij})\right)^2 \\ \leq& N\left(\sum_{\mathbf{S}}\left(\sum_{(i,j)\in\mathbf{S}} f(Q_{ij})\right)^2\right)^2\end{aligned} \tag{24}$$

Thus, we have

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} f^2(Q_{ij}) \leq \left(\frac{\sum_{\mathbf{S}}\left(\sum_{(i,j)\in\mathbf{S}} f(Q_{ij})\right)^2}{\sum_{\mathbf{S}}\sum_{(i,j)\in\mathbf{S}} f(Q_{ij})}\right)^2$$

(2) Since

$$\begin{aligned}\sum_{\mathbf{S}}\sum_{(i,j)\in\mathbf{S}} f(Q_{ij}) &= \sum_{i=1}^{N}\sum_{j=1}^{N} f(Q_{ij}) \sum_{\mathbf{S}:(i,j)\in\mathbf{S}} 1 \\ &= (N-1)!\sum_{i=1}^{N}\sum_{j=1}^{N} f(Q_{ij})\end{aligned} \tag{25}$$

thus we only need to show

$$\frac{1}{N!}\left(\sum_{\mathbf{S}}\sum_{(i,j)\in\mathbf{S}} f(Q_{ij})\right)^2 \leq \sum_{\mathbf{S}}\left(\sum_{(i,j)\in\mathbf{S}} f(Q_{ij})\right)^2$$

This is true due to inequality (23).

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