Distributed Detection of Binary Decisions with Collisions in a Large, Random Network

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ABSTRACT

We consider the problem of distributed detection in a large network of sensors. A random number of sensor nodes are randomly deployed. Sensor nodes perform local detection tests and communicate detections over a multiple-access channel to a fusion center. The fusion center can recognize both successful communications and communication collisions in the channel. We derive decision rules for both perfect communications and a delay-constrained communications protocol, and show that each are functions of count statistics only. We derive analytical expressions that characterize the performance of the system in terms of detection performance. We analyze performance with respect to sensor density and with respect to communications delay. Simulation examples validate theoretical predictions with numerical results. We show that the detection performance improves with network density despite increasing communication collisions. In addition, we show that detection performance using the protocol model, with imperfect communications, rapidly converges to the perfect communications case as the number of communication slots increase.

Index Terms—Distributed detection, random access, censoring, random sensor network

I. INTRODUCTION

Wireless sensor networks consist of many spatially-distributed sensor nodes that observe their local environment and communicate among one another or to a fusion center in order to aggregate sensed information. A fusion center infers the state of the environment, hopefully with better accuracy than that of a single node. The sensor nodes are typically energy-constrained and therefore must operate accordingly.

In this paper we consider the problem of distributed detection in a large, random network of sensor nodes. We consider the problem of detecting a localized event, and focus on the case where sensor nodes communicate local binary decisions over a single-hop wireless network to a common fusion node. The fusion node combines the received information in order to make a global decision on the presence or absence of the event.

Our approach is related to and draws from several recent works in distributed detection. Similar to [1], [2], [3], [4], and [5], we adopt a spatial Poisson point process model of the sensor network. When the number of nodes in the network is known, Niu et al. [2] show that, if local decisions are communicated perfectly, the global decision rule for independent and identically distributed (i.i.d.) binary observations, conditioned on the true hypothesis, simplifies to counting the number of detections. Niu et al. provide both analytic and approximate expressions for the counting rule in a large, random sensor network and a random target location [3]. Our work extends this analysis to include a communications protocol. Chang et al. [6] considers detection with imperfect communications by incorporating a random access protocol for a fixed number of sensor nodes; they derive a decision rule that is a weighted sum of the number of 1’s and 0’s successfully received at the fusion node. Similarly, [1], [4], and [5] incorporate a random access protocol, but for a random sensor network. Aldalahmeh et al. develop a real-time counting rule and provide an approximation to the system performance assuming that collisions are negligible [5].

In this paper, we build on the ideas above to develop distributed detection in an energy-efficient way. Similar to the approaches in [3], [4], and [5], we consider a large, random sensor network and incorporate a random access protocol. In contrast, however, we additionally incorporate censoring of nodes, so that a node transmits only when it declares a detection (i.e., when a node decides a target signal is present). Such a protocol reduces network traffic and therefore reduces the overall number of collisions from simultaneous communications. Censoring sensors in distributed detection was proposed in [7] for a fixed number of sensor nodes. In [7], globally optimal decision rules and local censoring regions were derived when local data-likelihoods are communicated to the fusion center under a resource constraint. In contrast, in this work the number of nodes is random and sensor nodes make binary decisions locally before transmission. This form of censoring was also considered in [1]. However, unlike in [1], we make use of the detection information inferred by communication collisions to improve performance and to permit better performance as the network scales in size. We extend the work in [1] by modeling communication collisions and incorporating a random target model. Furthermore, the model in [1] assumes the fusion node knows precisely the location of all the sensor nodes; we assume this information is unknown to the fusion node, requiring less information be known in order to make a detection decision.

Censoring of local node transmissions is motivated by two observations: (1) under perfect communications, the local non-detect declarations are not informative given the count of
detections (hence the counting rule of [3]), so there is no value in transmitting them to a fusion node, and (2) collisions due to simultaneous communications are mostly uninformative when both local detects and local non-detects are transmitted, but become informative when the only transmissions are local detections. Collisions imply that at least two sensor nodes are attempting to transmit messages. If both detect/no-detect messages are transmitted, then collisions are ambiguous observations relative to the underlying hypothesis. Under censoring, collisions now contain valuable information that is exploited in the global decision rule in this work.

The contributions of this paper are summarized as follows. First, we consider a variation on local binary transmissions in which only local detections are transmitted via a random access protocol and incorporate subsequent communication collisions into the decision process. We derive Neyman-Pearson optimal decision rules and provide analytical expressions for the global detection and false alarm receiver operating characteristic (ROC) curve. We do so for two communications protocols - an idealized communications model in which perfect communications with no collisions is assumed, and a more practical media access control (MAC) protocol in which local nodes communicate to a fusion center in one of a fixed set of slots. In the MAC protocol, message collisions will occur, and these collisions are included in the global decision rule and shown to contribute positively to performance in the ensuing performance analysis. We show that under specific conditions on local detection rules, these optimal tests are counting rules that count the number of received local detections, or in the case of the MAC protocol, a weighted sum of the number of received detections and the number of slots that have collisions.

Next, using the global performance expressions, we analyze performance trends as a function of key system parameters, namely, the sensor density of the network and, in the case of nonideal communications, the number of communication slots. In the case of sensor density, we require that local sensors transmit fewer detections as the density increases, in order to keep constant the communication traffic under $H_0$, i.e., when no target event is present. It can be shown that global performance improves with increasing sensor density if local detection probabilities remain constant; however, in this case the required communication bandwidth grows without bound, which would be an unfair resource advantage. The proof for the communication-resource constrained case is more subtle. Finally, we analyze the performance of the MAC protocol network as the number of communication slots increases, and show that the ideal communications performance is the same as the limiting MAC performance, as the number of communication slots increases to infinity.

The remainder of this paper is outlined as follows. In Section II we outline the sensor network model and the network communications model. In Section III we derive the global decision rules for the case of perfect communications and a given delay-constrained communications protocol. In Section IV we derive analytic expressions of global detection performance and investigate the performance as a function of the density of the sensor network and the number of communication slots. In Section V we provide a confidence interval on receiver operating characteristics. We present numerical examples in section VI that validate the analytic results and compare performance to previous results under perfect communications. Finally, conclusions are given in Section VII.

II. NETWORK SYSTEM MODEL

We first describe the local sensor operation model and the network communications model. We assume a large number of sensor nodes are randomly deployed independently and uniformly over a bounded, circular region. The sensor network is composed of a random number of nodes. Each node in the network senses its environment and computes a local decision. In addition, if the decision is a detection, the node communicates the detection to the fusion node using a delay-constrained MAC protocol. The model and processes are detailed below.

A. Sensor Network Model

Sensor nodes are deployed randomly, independently and uniformly over circular region $\mathcal{R} \subset \mathbb{R}^2$. Without loss of generality, we assume $\mathcal{R}$ is centered at $(0, 0)$ with radius $r_0 < \infty$. The locations of the target and sensor $i$ are denoted by $L_s$ and $L_i$, respectively, and denote the distance between them by $D_i = \|L_i - L_s\|$, where the norm is Euclidean\(^1\). Conditioned on $L_s$, the distances $\{D_i\}$ between the target and the sensors are conditionally i.i.d.

The sensor network is modeled as a homogeneous Poisson point process (PPP). Let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ be the set of non-negative integers, and $N \in \mathbb{N}_0$ be the number of sensor nodes during any given period. The number of sensor nodes is modeled according to

$$\Pr(N = n) = e^{-\lambda} \frac{\lambda^n}{n!}, \quad n \in \mathbb{N}_0,$$

\(^1\)Random variables are denoted with uppercase letters, specific outcomes are represented by lowercase letters, and underlined letters represent vectors. For example, $X = [X_1, \ldots, X_n]^T$ is a random vector and $\underline{x} = [x_1, \ldots, x_n]^T$ is a vector that represents a potential realization of $X$. \n
Fig. 1. One realization of a random sensor deployment. The image depicts the sensor network (circles), sensor nodes with detections (blue circles) with location $L_i$, and a localized target signal (red ‘x’) with location $L_s$. The distances between the sensors and target are denoted by $d_i$. Sensor nodes within circular region $\mathcal{R}$ with radius $r_0$ can communicate with the fusion node (black square) at the center of $\mathcal{R}$.
where $\lambda \in (0, \infty)$ is the average number of sensor nodes in region $\mathcal{R}$. Figure 1 depicts a realization of a sensor network deployment with a fusion node centered in a circular region.

It is well known that the Poisson distribution is an accurate approximation of a binomial distribution, with parameters $n$ and probability $q$ both given, with $\lambda = nq$ and $n \approx 20$ or more. The probability $q$ in this case could, for example, represent the duty cycling of $n$ sensor nodes to extend runtime or model the reliability of communication links to the fusion node. In general, $q$ could model the scaling of average per-node quality of a fixed-cost sensor network. Regardless of the mechanism, for the remainder of this work we assume the sensor network is accurately modeled as a homogeneous PPP with $\lambda$ average nodes in circular region $\mathcal{R}$.

Each sensor node senses its surrounding environment and makes a local detection decision of whether or not a target signal is present in the scene. When present, the target signal is embedded in noisy sensor observations. We do not explicitly model the embedding of the signal. Instead, we discuss an abstracted stochastic model that, in the presence of a signal, the sensor observations are statistically dependent, while i.i.d. in absence of the target signal for a given number of sensor nodes. Furthermore, a parametric model of the target signal is reduced to just two random parameters.

We denote the vector of sensor observations by $X = [X_1, \ldots, X_n]^T$ for $N = n$ sensor nodes. Under the null hypothesis $H_0$, no signal is present and sensors measure only noise. We assume the observations are i.i.d. given $H = H_0$. The conditional joint distribution of the sensor observations under $H_0$ is given by

$$f_0(x) = \prod_{i=1}^{n} f_0(x_i),$$

where $\mathbf{x} = [x_1, \ldots, x_n]^T$ with $x_i \in \mathbb{R}$ for $i = 1, 2, \ldots, n$, and $\mathbf{x}$ is a realization of $\mathbf{X}$. In this paper we consider a general sensor signal processing model as described by the probability distribution $f_0$ under hypothesis $H_0$ and $f_1$ under $H_1$. Here, $x_i$ could, for example, represent scalar or vector measurements, or a general function of these measurements. Note that $H$ is not random in this work; the index of the hypothesis is capitalized as is done traditionally. Since the sensor network is modeled as a homogeneous PPP, indexing of the sensor nodes, e.g. $x_i$ for node $i$, is arbitrary and done only to establish statistical properties of the observations.

Under the alternative hypothesis $H_1$, in which a target is present, each sensor measures a target signal embedded in noise. We model the observations under $H_1$ to be conditionally i.i.d. according to

$$f_1(x|\omega, n) = \prod_{i=1}^{n} f_1(x_i|\omega_i),$$

conditioned on $N = n$ sensor nodes and the signal vector $\omega = [\omega_1, \ldots, \omega_n]^T$, where $\omega$ is a realization of the random signal vector $\mathbf{A} = [A_1, \ldots, A_n]^T$. Each $A_i$ is an attenuated version of the scalar signal level $A_s > 0$ originating from the source. We assume a homogeneous medium that attenuates the signal in an isotropic manner. We model the signal at sensor $i$ as $A_i := A_s g(D_i) > 0$, where $g: [0, \infty) \rightarrow (0, \infty)$ models the attenuation as the signal propagates from the source and is a non-increasing function of distance. The attenuation function $g$ can model, for example, spherical or cylindrical spreading of a signal. As in the $H_0$ case, we consider a general local sensor processing model as described by the distribution $f_1$, which under $H_1$ depends also on the (attenuated) signal amplitude.

We assume the target’s location $L_s$ and signal level $A_s$ are random with known distributions and are mutually independent and independent of $N$, $\{L_i\}$, and measurement noise. Since $\{L_i\}$ are i.i.d., it follows that the $\{A_i\}$ are identically distributed, although the $\{A_i\}$ are not necessarily mutually independent due to their common dependence on $A_s, L_s$.

Let $(R_s, \Phi_s)$ represent the polar coordinates of the target located at $L_s$. It was shown in [8] that $\{A_i\}$ are conditionally independent of the angle $\Phi_s$ and therefore i.i.d. given $R_s$, $A_s$, and $N$. The attenuated amplitudes are conditionally i.i.d. and angle-independent since the signal propagation is isotropic and the target and sensors are spread independently and uniformly in the region. We define a set $\mathcal{P} = (0, \infty) \times [0, r_0]$ and signal parameter vector $\Theta = [A_s, R_s]^T \in \mathcal{P}$ with probability distribution $f_2(\cdot)$. The conditional joint distribution of the (attenuated but noiseless) signal at the sensors is then given by

$$f_1(x|\Theta, n) = \prod_{i=1}^{n} f_1(x_i|\Theta),$$

The vector $\Theta$ captures the target signal components that are shared through observations at all sensor nodes. It follows that the sensor observations are conditionally i.i.d. according to

$$f_1(x|\Theta, n) = \prod_{i=1}^{n} f_1(x_i|\Theta),$$

where $f_1(x_i|\Theta) = E_{A|\Theta}(f_1(x_i|a, \Theta))$ and $E(\cdot)$ denotes expectation. This sensor-level probability distribution is the result of marginalizing the randomness in a sensor node’s location — the randomness that remains in $A_s$ after conditioning by $\Theta$. The probability distribution $f_1(x_i|\Theta)$ corresponds to the conditional distribution of an observation at sensor node $i$ under $H_1$ given $\Theta = \Theta_i$. Note that the $\{X_i\}$ are conditionally independent of $\Theta$ under $H_0$, so $f_0(x_i|\Theta) = f_0(x_i)$. It follows that under both hypotheses, the observations are conditionally i.i.d. across the sensor network for a given signal parameter $\Theta = \Theta_i$ and given $N = n$ sensor nodes. On the other hand, the observations are not statistically independent conditioned on the true hypothesis alone.

### B. Distributed Detection

We assume each sensor node first makes a local binary decision and, if the decision is a detection, then communicates that detection to a common fusion node. In what follows, we consider two different communication channel models: a perfect communication model and a delay-constrained random access communication protocol.

The local decision rules are not given explicitly, but instead are parameterized in terms of probabilities of false alarm and positive detection relative to the signal parameter $\theta$. Latter sections provide conditions on these probabilities and therefore conditions on the local tests.
The local decisions are determined by an \( \alpha_j \)-level decision rule, denoted by \( \delta_j \). The local decision rules \( \delta_j \) are not necessarily identical across the sensor network in optimal fusion of binary decisions, even when observations are conditionally i.i.d. [9]. However, we assume each sensor shares a common \( \alpha \)-level decision rule \( \delta : \mathbb{R} \to \{0, 1\} \). This choice dramatically simplifies the design and deployment of the sensor network, and, in some cases, proves to be asymptotically optimal for binary decisions [10].

We denote the local decisions by \( Y_i \) for \( i = 1, 2, \ldots, N \); where each \( Y_i \in \{ H_0, H_1 \} \). We refer to \( Y_i = H_1 \) as a local detection, whether it is a true \( (H = H_1) \) or a false \( (H = H_0) \) detection. Each \( Y_i \) conditioned on \( \Theta \), \( H \) can be seen as a Bernoulli random variable with conditional success probability \( p_j(\delta|\theta) \) defined by

\[
p_j(\delta|\theta) = E_{X|\Theta}(\delta(X); H_j),
\]

for \( j \in \{0, 1\} \). The success probabilities can be viewed as the conditional probabilities of false alarm \( (j = 0) \) and detection \( (j = 1) \) for decision rule \( \delta \) and for signal parameter \( \Theta = \theta \).

When convenient, we suppress the dependence on \( \delta \) by the local probabilities of detection and false alarm and simply write \( p_1(\theta) = p_1(\delta|\theta) \) and \( p_0(\theta) = p_0(\delta|\theta) \), respectively.

Note that \( p_0(\theta) = \alpha \) and does not depend on \( \theta \), while the decision region for \( \delta \) may depend on \( \theta \). In general, all the sensor nodes need to know the distribution \( f_\theta \) in order to make a local decision. On the other hand, if the decision region for \( \delta \) does not depend on \( \theta \), then sensor nodes can make their local decisions without knowledge of the distribution \( f_\theta \). For example, for the additive noise model considered in [3] the decision threshold \( \tau \) can be set without knowledge of the distribution of the target signal.

Sensor nodes with detections represents a thinning of the PPP [11]. Let \( Z = \sum_{i=1}^{N} 1(Y_i = H_1) \) be the number of detections from sensors in region \( R \). Under \( H_1 \) and conditioned on \( \Theta = \theta \), it is straightforward to show that the conditional probability distribution of \( Z \) is given by

\[
Pr(Z = z; H_j|\theta) = e^{-\lambda_j(\theta)} \frac{(\lambda_j(\theta))^z}{z!},
\]

for \( z \in \mathbb{N}_0 \). Note that the dependence on \( \Theta \) in equations (1) and (2) drops out under hypothesis \( H_0 \).

C. Network Communication: Protocol Model

In this section, we describe a delay-constrained MAC protocol model. The local decisions \( Y_i \), for \( i = 1, 2, \ldots, N \), are communicated to a fusion node. The fusion node then combines received information to make a global decision as to the presence or absence of a target signal. This global decision rule must account for the effects of imperfect communications due to potential collisions.

Sensor nodes sense their environment and transmit detections in a given activity period, which is formed by a sensing period followed by a communicating period. The sensing and communicating periods are synchronized so that each sensor node observes the environment and makes a local decision during the sensing period. We assume the target event is localized in space and time and any given sensing period is aligned with the target signal. Sensor nodes attempt to transmit their decisions to a common fusion node during the communicating period. Figure 2 graphically depicts the sensing and communicating periods. Note that there could be overlap between a communicating period and the next sensing period. We model the system as being time-invariant. Thus, we need only model a single activity period.

As part of the communication protocol, sensor nodes transmit only detections (i.e., when \( Y_i = H_1 \)); this both saves transmission energy and reduces communication collisions. In addition, sensor nodes do not retransmit messages in the event of a collision. Since transmitted messages only represent detections, collisions provide valuable information at the fusion node. Similar distributed detection models were considered in [4], [5], but there local nodes transmit both decisions (i.e., \( Y_i = H_0 \) or \( Y_i = H_1 \)). Our protocol differs from [4], [5] in that transmitted messages represent only \( Y_i = H_1 \) decisions.

During the communicating period, each sensor with a detection attempts to transmit a single detection message via Slotted ALOHA. The communicating period is broken into \( M \) equal-duration slots, with \( 1 \leq M < \infty \). On the receiving side, the fusion node does not know which slots will be used, so it must monitor all \( M \) slots for messages during the communicating period before making a decision. Thus, the communications delay is directly proportional to \( M \). The delay, and therefore the number of slots \( M \), is specified according to the needs of the detection application. Note that \( M \), though capitalized, is not random.

We assume the fusion node can detect collisions in any of the \( M \) slots. A message is considered to be received successfully in a slot if exactly one sensor node transmits in that slot. Otherwise, a slot is either unused or contains collisions. A slot containing two or more transmissions is counted as a collision. We denote by \( T_m \) the number of sensor nodes that select slot \( m \) for transmitting a message, and let \( U_m \) represent the observation in slot \( m \) at the fusion node. We also refer to \( T_m \) as the slot occupancy (i.e., the number of nodes attempting a transmission in the slot). The observations at the fusion node are then related to the slot occupancies according to:

\[
U_m = \begin{cases} 
0, & \text{if } T_m = 0 \\
1, & \text{if } T_m = 1 \\
c, & \text{if } T_m > 1,
\end{cases}
\]

for \( m = 1, 2, \ldots, M \) and where \( c \) signifies a collision in the slot. The vector \( \bar{U} = [U_1, U_2, \ldots, U_M]^{T} \) represents the collection of observations at the fusion node at the end of the communicating period, and \( \bar{u} = [u_1, u_2, \ldots, u_M]^{T} \) is a particular realization of \( \bar{U} \in \{0, 1, c\}^{M} \).

Sensor nodes with detections randomly choose one out
of \( M \) slots, independently of other nodes, according to a uniform probability distribution. The number of sensor nodes is unknown to the fusion node and the number of transmitting nodes are uniformly spread across slots. Thus, sensor nodes with detections per communication slot represents another thinning of the PPP. It is straightforward to show that the slot occupancies \( T_m \), for \( m = 1, 2, \ldots, M \), are conditionally i.i.d., and each are distributed according to

\[
\Pr(T = k; H_j) = e^{-\lambda} \frac{\lambda^k p_j(\theta)}{k!},
\]

for \( k \in \mathbb{N}_0 \). The fusion node observes \( \mathcal{U} \in \{0, 1, c\}^M \) with slot occupancies mapped according to equation (3). The conditional joint probability distribution of the observations at the fusion node is then given by

\[
\Pr(\mathcal{U} = u; H_j) = (\pi_{0,j})^{M-n_u-n_c} (\pi_{1,j})^{n_1} (\pi_{c,j})^{n_c},
\]

where \( n_1 = \sum_{m=1}^{M} 1(u_m = 1) \) and \( n_c = \sum_{m=1}^{M} 1(u_m = c) \) are just the counts of 1’s and c’s across slots, respectively, and

\[
\begin{align*}
p_{0,0} &= \Pr(T = 0; H_0), & p_{0,1}(\theta) &= \Pr(T = 0; H_1|\theta) \\
p_{1,0} &= \Pr(T = 1; H_0), & p_{1,1}(\theta) &= \Pr(T = 1; H_1|\theta) \\
p_{c,0} &= \Pr(T > 1; H_0), & p_{c,1}(\theta) &= \Pr(T > 1; H_1|\theta).
\end{align*}
\]

The probabilities \( p_{u,0} \) and \( p_{u,1} \) depend on the local decision rule \( \delta \), and \( p_{u,1} \) additionally depends on the signal parameter \( \Theta \) under \( H_1 \).

We define the three random counts \( N_0, N_1, \) and \( N_c \), as

\[
N_u = \sum_{m=1}^{M} 1(U_m = u); \text{ these are the number of slots containing 0, 1, and } > 1 \text{ transmissions, respectively. From equation (5), the triplet } \{N_0, N_1, N_c\} \text{ is clearly a trinomial with respective probabilities defined in (6). Since } N_0 + N_1 + N_c = M, \text{ any two of these counts determines the third; we use } \{N_1, N_c\} \text{ in what follows and infer } N_0. \text{ It is clear from equation (5) that the set } \{N_1, N_c\} \text{ is a sufficient statistic for determining } H. \text{ The observations at the fusion node are then just the counts of the 1’s and c’s (i.e., } N_1 \text{ and } N_c). \text{ Formally, the joint conditional of the counts from the MAC protocol after observing all } M \text{ slots is given by}
\]

\[
\Pr(N_1 = n_1, N_c = n_c; H_j) = \binom{M}{n_c} \binom{M-n_u-n_c}{n_1} (\pi_{0,j})^{M-n_1-n_c} (\pi_{1,j})^{n_1} (\pi_{c,j})^{n_c},
\]

where \( \binom{n}{k} = n!/(n-k)!k! \).

### III. Global Decision Rules

In this section, we derive the global decision rules for the sensor network. First, we derive the global decision rule assuming the local decisions are perfectly communicated to the fusion node. Second, we derive the global decision rule for the delay-constrained MAC protocol outlined in section II-C. The global decision rules are determined by composite likelihood-ratio tests [12, Chapter II.E].

#### A. Perfect Channel

We first consider fusing the binary decisions assuming unconstrained and error-free communications. In other words, local detection decisions are received perfectly at the fusion node. We consider the Neyman-Pearson (NP) optimal global decision rule \( \delta^{\text{NP}} \) which satisfies the objective

\[
\min_{\delta^{\text{NP}}} \beta^{\text{NP}} \text{ s.t. } \alpha^{\text{NP}} = \bar{\alpha},
\]

where \( \bar{\alpha} \in (0, 1) \) is a desired global probability of false alarm and where

\[
\alpha^{\text{NP}} = E_Z (\delta^{\text{NP}}(z); H_0) \text{ and } \beta^{\text{NP}} = 1 - E_Z (\delta^{\text{NP}}(z); H_1)
\]

are the probabilities of false alarm and missed detection at the fusion center, respectively.

**Theorem 1.** Under perfect communications, the fusion node observes the total number of local detections from the sensor network. The NP-optimal global decision rule for a perfect channel is given by

\[
\delta^{\text{NP}}(z) = \begin{cases} 
1, & \text{if } \ell_{\text{NP}}(z) > \eta_{\text{NP}} \\
\zeta_{\text{NP}}, & \text{if } \ell_{\text{NP}}(z) = \eta_{\text{NP}} \\
0, & \text{if } \ell_{\text{NP}}(z) < \eta_{\text{NP}}
\end{cases}
\]

where \( \ell_{\text{NP}} \) is the likelihood ratio given by

\[
\ell_{\text{NP}}(z) = \frac{Pr(Z = z; H_1)}{Pr(Z = z; H_0)} = E_\Theta \left( e^{-\lambda|p_1(\theta) - \alpha|} \left| p_1(\theta) \right| \right).
\]

**Proof.** See [12] on optimality of composite likelihood-ratio tests. The likelihood ratio follows from equation (2) and the fact that the likelihood under \( H_0 \) does not depend on \( \Theta \). \( \square \)

Because \( Z \) is discrete-valued, the global decision rule yields global performance defined as discrete points (vertices) on the receiver operating characteristic (ROC) curve, with linear interpolation between these vertices realized by a randomized test. Specifically, the threshold \( \eta_{\text{NP}} \) and the randomization parameter \( \zeta_{\text{NP}} \) satisfy the constraint \( \alpha_{\text{NP}} = \bar{\alpha} \). More precisely [12, Chapter II.D], \( \eta_{\text{NP}} \) is the smallest number such that \( Pr(\ell_{\text{NP}}(Z) > \eta_{\text{NP}}; H_0) \leq \bar{\alpha} \). For that \( \eta_{\text{NP}} \), if \( Pr(\ell_{\text{NP}}(Z) > \eta_{\text{NP}}; H_0) < \bar{\alpha} \), then the randomization parameter is set according to

\[
\zeta_{\text{NP}} = \bar{\alpha} - Pr(\ell_{\text{NP}}(Z) > \eta_{\text{NP}}; H_0)/Pr(\ell_{\text{NP}}(Z) = \eta_{\text{NP}}; H_0).
\]

Under mild conditions, the decision rule reduces to a simple thresholding of the count statistic, as we show next.

**Corollary 1.** If \( p_1(\theta) > \alpha \) for \( \alpha \in (0, 1) \) and for all \( \theta \) in the support of \( f_\Theta \), then an equivalent test to (8) is given by

\[
\delta^{\text{NP}}(z) = \begin{cases} 
1, & \text{if } z > \eta'_{\text{NP}} \\
\zeta'_{\text{NP}}, & \text{if } z = \eta'_{\text{NP}} \\
0, & \text{if } z < \eta'_{\text{NP}}
\end{cases}
\]

where \( \eta'_{\text{NP}} \) and \( \zeta'_{\text{NP}} \) satisfy the \( \bar{\alpha} \)-level test.

**Proof.** If \( p_1(\theta) > \alpha \), the likelihood ratio \( \ell_{\text{NP}}(z) \) is strictly increasing in \( z \). Thus, there is a one-to-one mapping between decision regions defined by (8) and (9). \( \square \)

Under the condition of Corollary 1, setting the threshold and randomization parameter simplifies to evaluating a few points of the cumulative distribution of a Poisson random variable with mean \( \lambda \alpha \). As a consequence, no knowledge of the signal or its probability distribution are needed in order to evaluate the global decision rule. Accordingly, \( \eta'_{\text{NP}} \) is the
under the MAC protocol described in section II-C, the fusion node observes counts of detection messages and message collisions after monitoring all M communication slots. The NP-optimal global decision rule for the MAC protocol is given by

\[
\delta_{\text{MAC}} (n_1, n_c) = \begin{cases} 
1, & \text{if } \ell_{\text{MAC}} (n_1, n_c) > \eta_{\text{MAC}} \\
\zeta_{\text{MAC}}, & \text{if } \ell_{\text{MAC}} (n_1, n_c) = \eta_{\text{MAC}} \\
0, & \text{if } \ell_{\text{MAC}} (n_1, n_c) < \eta_{\text{MAC}},
\end{cases}
\]  

(12)

where \( \ell_{\text{MAC}} \) is the likelihood ratio given by

\[
\ell_{\text{MAC}} (n_1, n_c) = E_\Theta \left( \frac{\pi_{0,1}(\theta)}{\pi_{0,0}} \right) M - n_1 - n_c \left[ \frac{\pi_{1,1}(\theta)}{\pi_{1,0}} \right]^{n_1} \left[ \frac{\pi_{c,1}(\theta)}{\pi_{c,0}} \right]^{n_c},
\]  

(13)

for \( \pi_{u,0} \neq 0 \). The threshold \( \eta_{\text{MAC}} \) and the randomization parameter \( \zeta_{\text{MAC}} \) satisfy the constraint \( \alpha_{\text{MAC}} = \bar{\alpha} \in (0, 1) \).

Proof. The likelihood ratio of the NP-optimal test follows from equation (7).

The threshold \( \eta_{\text{MAC}} \) is the smallest number such that

\[
\Pr \left( \ell_{\text{MAC}} (N_1, N_c) > \eta_{\text{MAC}} ; H_0 \right) \leq \bar{\alpha}.
\]

For that \( \eta_{\text{MAC}} \), if \( \Pr \left( \ell_{\text{MAC}} (N_1, N_c) > \eta_{\text{MAC}} ; H_0 \right) < \bar{\alpha} \), then the randomization parameter is set according to

\[
\zeta_{\text{MAC}} = \bar{\alpha} - \Pr \left( \ell_{\text{MAC}} (N_1, N_c) > \eta_{\text{MAC}} ; H_0 \right) \cdot \Pr \left( \ell_{\text{MAC}} (N_1, N_c) = \eta_{\text{MAC}} ; H_0 \right).
\]

Similar to Theorem 1, Theorem 2 shows that the global decision rule depends on counts; in the MAC protocol case, the rule is a function of the count pair \((n_1, n_c)\). The rule is in general a nonlinear function that requires multiple integrals to be calculated numerically. Let \( \mathcal{X} = \{ (i, j) \in \{0, 1, \ldots, M \}^2 ; i + j \leq M \} \) denote the feasible set of points. There are \((M+1)(M+2)/2\) discrete feasible points in the set \( \mathcal{X} \). The ROC of the NP-optimal decision rule (12) is continuous and piecewise linear, with at most \((M+1)(M+2)/2\) vertices (fewer if multiple points have the same likelihood ratio). Given \( \{ \lambda, \alpha, \bar{\alpha} \} \), the likelihood ratios could be precomputed offline for each feasible point and stored as a lookup table. Similarly, the threshold and randomization parameter for the \( \bar{\alpha} \)-level test can be precomputed.

The decision rule can be reduced to thresholding of a weighted sum of the count statistics, as we show below.

Corollary 2. If \( \Theta = \theta \) w.p. 1, then an equivalent test to (12) is given by [13]

\[
\delta_{\text{MAC}} (n_1, n_c) = \begin{cases} 
1, & \text{if } n_1 + w' n_c > \eta_{\text{MAC}} \\
\zeta_{\text{MAC}}, & \text{if } n_1 + w' n_c = \eta_{\text{MAC}} \\
0, & \text{if } n_1 + w' n_c < \eta_{\text{MAC}},
\end{cases}
\]  

(14)

where

\[
w' = \frac{\log \left( e^{\lambda n_1(n_0)} - 1 - \lambda p_1(n_0) M \right) - \log \left( e^{\lambda n_0} - 1 - \lambda p_1(n_0) M \right)}{\log p_1(n_0) - \log \alpha}.
\]

Proof. If \( \Theta = \theta \) w.p. 1, then we can write the likelihood ratio as

\[
\ell_{\text{MAC}} (n_1, n_c) = \left[ \frac{\pi_{0,1}(\theta)}{\pi_{0,0}} \right]^{M - n_1 - n_c} \left[ \frac{\pi_{1,1}(\theta)}{\pi_{1,0}} \right]^{n_1} \left[ \frac{\pi_{c,1}(\theta)}{\pi_{c,0}} \right]^{n_c},
\]

since \( \log \) is monotone increasing, we can take the logarithm of the order relations in (12) without affecting the decision rule. After substituting in expressions for \( \pi_{u,0} \) and \( \pi_{u,1}(\theta) \) from equations (4) and (6) and after some simplification, we arrive at global decision rule (14).

The procedure for finding the parameters \( \eta_{\text{MAC}} \) and \( \zeta_{\text{MAC}} \) is the same as that in Theorem 2 with the expectation with respect to \( \Theta \) greatly simplified.

Though the condition of Corollary 2 can be viewed as restrictive, the form of the rule remains unchanged, for example, as a generalized likelihood ratio test (GLRT), with \( \theta \) replaced by a maximum likelihood estimate \( \hat{\theta} \).

The likelihood ratios for both perfect communications and the MAC protocol are summarized in Table I. Under perfect...
communications, a “counting rule” was derived in [2] for a known number of sensor nodes and analyzed in [3] for a large, random sensor network. Similarly, the global decision rule (14) for the MAC protocol is simply a weighted sum of the counts of successful detection messages and collisions, which was first reported in [13]. In Theorems 1 and 2, we generalize these results for both communications models to a random target model.

IV. SYSTEM PERFORMANCE

In this section, we investigate the system performance as a function of the density of the sensor network and the number of communication slots. First, we analyze the detection performance as we increase the sensor density in the deployed region. As we increase density, we correspondingly raise the local detection thresholds in such a way as to maintain the same average number of false alarms being transmitted to the fusion node. The underlying assumption is the target signal is mostly absent, thus communications are dominated by local false positives. Under this constraint, we show that the probability of missed detection, for a given global probability of false alarm at the fusion center, decreases monotonically with increased node density. Second, we analyze performance under the MAC protocol as the number of transmission slots, \( M \), increases. We show that the detection performance under the MAC protocol asymptotically, as \( M \rightarrow \infty \), achieves the same performance as assuming perfect communications.

The following results are nontrivial since sensor observations are correlated through a shared random target model. To make the analysis more manageable, the following condition is assumed:

**Condition 1.** The local detection tests are such that the local conditional probability of detection \( p_{1}(\delta|\theta) \) is strictly concave and increasing in \( \alpha \in (0, 1) \), for all signal parameters \( \theta \) in the support of \( f_{\Theta} \).

Roughly speaking, Condition 1 means the local sensor node detection tests perform better than chance. The following results can be extended to the case when concavity is not necessarily strict, although the proofs are more cumbersome.

A. Error Probability Versus Sensor Network Density

We first analyze the global probability of missed detection while increasing sensor network density. Since the deployed region \( \mathcal{R} \) remains fixed, the sensor density is directly proportional to the average number of sensor nodes, \( \lambda \), in the network. As we increase density, we hold the average number of false alarms, \( \lambda_{\alpha} \), transmitted in the network fixed; doing so results in a fair comparison, as otherwise (if the local sensor false alarm probabilities \( \alpha \) were kept constant) the number of total sensor transmissions in the network would increase - requiring more communication bandwidth - and also the average energy allocation to the network, as measured by number of transmissions per measurement slot, would also increase. By keeping \( \lambda_{\alpha} \) constant, sensor nodes are desensitized as the density of nodes increases. In other words, the probability of detecting a target at a sensor node, whether true or false, is degraded.

We first consider the missed-detection probability for perfect communications as a function of the sensor network density.

**Theorem 3.** Under perfect communications, Condition 1, and keeping \( \lambda_{\alpha} \) constant, the global probability of missed detection \( \beta_{\lambda} \) is decreasing in the average number of nodes \( \lambda \).

**Proof.** Under Condition 1, \( p_{1}(\theta) > \alpha \). Thus, the global decision rule is given by (9). Meaning, the test simplifies to thresholding the count of local detections. Let \( \tilde{\alpha} \) denote the desired global probability of false alarm at the fusion center. The decision region for this \( \tilde{\alpha} \)-level test does not depend on \( \theta \).

Therefore, the decision region for rule \( \delta_{\lambda}(z|\theta) \) (i.e., the global rule when conditioned on a specific signal parameter) that achieves \( \tilde{\alpha} \) under \( H_{0} \) does so also for composite rule \( \delta_{\lambda}(z) \).

Since the decision region for \( \delta_{\lambda}(\theta) \) does not depend on \( \theta \), we have \( \beta_{\lambda} = E_{\Theta}(\beta_{\lambda}(\theta)) \), where \( \beta_{\lambda}(\theta) \) is the conditional probability of missed detection of rule \( \delta_{\lambda}(z|\theta) \) given \( \Theta = \theta \).

Let \( \Theta = \theta \) be given from the support of \( f_{\Theta} \). The conditional probability of missed detection is given by

\[
\beta_{\lambda}(\theta) = \sum_{\delta \in \Theta} \eta_{\lambda}(\delta) e^{-\lambda p_{1}(\delta|\theta)} \zeta_{\lambda}(\delta) e^{-\lambda p_{1}(\delta|\theta)} \gamma_{\lambda}(\delta|\theta). \tag{15}
\]

If \( \tilde{\alpha} \) is fixed and \( \lambda_{\alpha} \) is held constant as \( \lambda \) increases, then it is clear from equations (10) and (11) both \( \eta_{\lambda}(z) \) and \( \zeta_{\lambda}(z) \) are fixed. Then any change in \( \beta_{\lambda}(\theta) \) is due to a change in \( \lambda p_{1}(\delta|\theta) \).

It is straightforward to show that equation (15) is decreasing in \( \mu = \lambda p_{1}(\delta|\theta) \). Thus, if \( \mu \) is increasing in \( \lambda \) for every \( \theta \) in the support of \( f_{\Theta} \), then \( \beta_{\lambda}(\theta) \) and therefore \( \beta_{\lambda} \) is decreasing in \( \lambda \). It remains to show that \( \lambda p_{1}(\delta|\theta) \), the average number of positive local detections, is increasing in \( \lambda \) with the constraint that \( \lambda_{\alpha} \) remains fixed.

Let \( 0 < \lambda_{0} < \lambda_{1} < \lambda_{2} < \infty \) and \( 0 < \alpha_{2} < \alpha_{1} < \alpha_{0} < 1 \) be such that \( \lambda_{0} \alpha_{0} = \lambda_{1} \alpha_{1} = \lambda_{2} \alpha_{2} \). Additionally, let \( \delta_{0}, \delta_{1}, \) and \( \delta_{2} \) be the local detection rules associated with \( \alpha_{0}, \alpha_{1}, \) and \( \alpha_{2}, \) respectively.

The change in the average number of detections from sensor nodes is then

\[
\lambda_{1} p_{1}(\delta_{1}|\theta) - \lambda_{0} p_{1}(\delta_{0}|\theta) = \lambda_{0} \left( \frac{\alpha_{0}}{\alpha_{1}} p_{1}(\delta_{1}|\theta) - p_{1}(\delta_{0}|\theta) \right) = \lambda_{0} \lambda_{0} \left( \frac{p_{1}(\delta_{1}|\theta)}{\alpha_{1}} - \frac{p_{1}(\delta_{0}|\theta)}{\alpha_{0}} \right). \tag{16}
\]

where the first equality follows since \( \lambda_{1} \alpha_{1} = \lambda_{0} \alpha_{0} \). Since \( \lambda_{0} \alpha_{0} > 0 \), it remains to show that the term in parenthesis in equation (16) is strictly positive.

Since the local tests are strictly concave for any given \( \theta \) in the support of \( f_{\Theta} \) and non-decreasing in \( \alpha \), we have

\[
\frac{p_{1}(\delta_{1}|\theta) - p_{1}(\delta_{2}|\theta)}{\alpha_{1} - \alpha_{2}} \geq \frac{p_{1}(\delta_{1}|\theta) - p_{1}(\delta_{0}|\theta)}{\alpha_{0} - \alpha_{2}}.
\]

Let \( \alpha_{2} \to 0 \). The inequality becomes

\[
\frac{p_{1}(\delta_{1}|\theta) - \epsilon}{\alpha_{1}} \geq \frac{p_{1}(\delta_{0}|\theta) - \epsilon}{\alpha_{0}}
\]

where \( \epsilon = p_{1}(\delta_{2}|\theta) \geq 0 \). After rearranging terms,

\[
\frac{p_{1}(\delta_{1}|\theta)}{\alpha_{1}} - \frac{p_{1}(\delta_{0}|\theta)}{\alpha_{0}} \geq \epsilon \left( \frac{1}{\alpha_{1}} - \frac{1}{\alpha_{0}} \right) \geq 0,
\]

where the last inequality follows since \( \epsilon \geq 0 \) and \( 0 < \alpha_{1} < \alpha_{0} \).

Note that, in general, the decision regions for \( \delta_{\lambda}(z|\theta) \)
and composite rule $\delta_{\alpha}(z)$ are not necessarily the same. The decision region for $\delta_{\alpha}(z)\theta)$ may depend on $\theta$ without the condition of Corollary 1 (i.e., $p_1(\theta) > \alpha$).

Next we consider the probability of missed detection for the MAC protocol as a function of the sensor network density.

**Theorem 4.** Under the MAC protocol, Condition 1, and keeping $\lambda \alpha$ constant, the global probability of missed detection $\beta_{\text{MAC}}$ is decreasing in the average number of nodes $\lambda$ for sufficiently large $\lambda$.

**Proof.** Let the number of slots $M$ be fixed and finite. Under Condition 1, $p_1(\theta) > \alpha$. Then, it is straightforward to show that the conditional likelihood ratio $\ell(\omega) i,j(\theta)$ is strictly increasing in each coordinate, and thus $\ell(\omega) i,j(\theta)$ is strictly increasing in each coordinate. Also, the joint probability distribution of $(N_1, N_c)$, under $H_0$, remains fixed with respect to $\lambda$ and does not depend on $\theta$. As before, let $\alpha$ denote the global false alarm probability at the fusion center. The $\alpha$-level decision region therefore does not depend on $\alpha$ or $\theta$.

Recalling that $\chi$ is the feasible set of points, let $A_{\alpha} = \{(i,j) \in \chi; \ell(i,j) \leq \eta_{\text{MAC}}(\alpha)\}$ denote the $\alpha$-level acceptance region for $H_0$. Also recall that the threshold $\eta_{\text{MAC}}$ and randomization $\zeta_{\text{MAC}}$ satisfy the constraint $\alpha_{\text{MAC}} = \bar{\alpha}$. Under the MAC protocol, the global probability of false alarm is given by

$$\alpha_{\text{MAC}} = \sum_{(i,j) \in \chi} \delta_{\text{MAC}}(i,j) \Pr(N_1 = i, N_c = j; H_0).$$

The global probability of missed detection, conditioned on $\theta = \bar{\theta}$, is given by

$$\beta_{\text{MAC}}(\bar{\theta}) = \sum_{(i,j) \in \chi} (1 - \delta_{\text{MAC}}(i,j)) \Pr(N_1 = i, N_c = j; H_1|\theta).$$

It is straightforward to show that $\beta_{\text{MAC}}(\bar{\theta})$ is continuous and piecewise linear with respect to $\bar{\alpha} \in (0,1)$, with a finite number of vertices that satisfy $\Pr(\ell_{\text{MAC}}(N_1, N_c) > \eta_{\text{MAC}}; H_0) = \bar{\alpha}$. The linear segments are just convex combinations of the bordering vertices. Thus, we need only prove the theorem for global probabilities of false alarm that correspond vertices of the probability of missed detection. The theorem then follows for the remaining false alarm probabilities.

Let $\alpha$ be the smallest positive false alarm probability such that $\Pr(\ell_{\text{MAC}}(N_1, N_c) > \eta_{\text{MAC}}; H_0) = \bar{\alpha}$. This $\alpha$ corresponds to the largest vertex of $\beta_{\text{MAC}}(\bar{\theta}) < 1$. We claim the point $(0, M)$ maximizes the likelihood ratio $\ell_{\text{MAC}}$. Assuming the claim to be true, we have $A_{\alpha} = \chi \setminus (0, M)$ for this $\alpha$. Considering $\alpha$ large enough ensures the point $(0, M)$, the maximizer of the likelihood ratio $\ell_{\text{MAC}}$, is not in the acceptance region $A_{\alpha}$. The conditional probability of missed detection for this $\alpha$ is then

$$\beta_{\text{MAC}}(\bar{\theta}) = \sum_{(i,j) \in A_{\alpha}} \Pr(N_1 = i, N_c = j; H_1|\theta).$$

Differentiating $\beta_{\text{MAC}}(\bar{\theta})$ with respect to $\lambda$, we have

$$\frac{d\beta_{\text{MAC}}(\bar{\theta})}{d\lambda} = \sum_{(i,j) \in A_{\alpha}} \Pr(N_1 = i, N_c = j; H_1|\theta) \times \frac{1}{\lambda} \left( -\lambda p_1(\theta) + i + j \frac{\lambda p_1(\theta)}{M} e^{\lambda p_1(\theta)/M} - 1 - \lambda p_1(\theta)/M \right).$$

(18)

The probability masses in (18) are strictly positive since $p_{u,1} > 0$, for $u \in \{0, 1, c\}$. For the derivative in (18) to be negative, it suffices that all of the probability weights in (18) are negative. Let $x = \lambda p_1(\theta)/M$. Thus, we wish to show that

$$0 > -x M + i + j x e^{x - 1} + e^x e^{-1}. \quad (19)$$

It can be shown that the right-hand side of (19) is maximized by $(i,j) = (1, M - 1)$ for points in $A_\delta$ when $x > 1$ (i.e., $\lambda p_1(\theta) > M$). Thus, any $x > 1$ that satisfies the inequality in (19) with $(i,j) = (1, M - 1)$ also satisfies the inequality for any $(i,j) \in A_\delta$. For $x > 1$, we have the inequality

$$-x M + i + j x e^{x - 1} + e^x e^{-1} < -x M + 1 + (M - 1) \left( x + \frac{6}{x^3} \right). \quad (20)$$

The inequality (20) follows since

$$x e^{x - 1} - x e^{-1} + e^x e^{-1} = x + \frac{6}{x^3} + 3 \frac{1}{x^3}.$$

From equation (20) and since $x + 3 > 0$, we need to find the roots of the polynomial that satisfies $x^2 + 2x - 6M + 3 = 0$. The roots of this polynomial are $-1 \pm \sqrt{6M - 2}$. Therefore, we require $x > \sqrt{6M - 2} - 1$ for the right-hand-side of equation (20) to be negative. This implies that the average number of local detections be $\lambda p_1(\theta) > M(\sqrt{6M - 2} - 1)$ for $\beta_{\text{MAC}}(\bar{\theta})$ to be strictly decreasing.

The acceptance region $A_{\delta}$ contains points for all other vertices of $\beta_{\text{MAC}}(\bar{\theta})$. Thus, all other vertices of $\beta_{\text{MAC}}(\bar{\theta})$ are also decreasing for sufficiently large $\lambda$. Hence, everywhere in $\alpha \in (0, 1)$.

It remains to show that the point $(0, M)$ maximizes the likelihood ratio $\ell_{\text{MAC}}$. Since the likelihood ratio is strictly increasing in its coordinates, we need only consider the $M + 1$ points that satisfy $n_1 + n_c = M$. From equation (13), the likelihood ratio for the points that satisfy $n_1 + n_c = M$ simplifies to

$$\ell_{\text{MAC}}(n_1, n_c) = E_{\overline{\theta}} \left[ \left( \frac{\pi_{1,1}(\overline{\theta})}{\pi_{1,0}} \right)^{n_1} \left( \frac{\pi_{c,1}(\overline{\theta})}{\pi_{c,0}} \right)^{n_c} \right].$$

Now, we only need to show that the ratio $[\pi_{c,1}(\overline{\theta})/\pi_{c,0}] / [\pi_{1,1}(\overline{\theta})/\pi_{1,0}]$ is greater than unity. This ratio can be written as

$$\frac{\pi_{1,1}(\overline{\theta})}{\pi_{1,0}} \frac{\pi_{c,1}(\overline{\theta})}{\pi_{c,0}} = \frac{(\lambda M)^{\sum_{k=2}^{\infty} (\lambda p_1(\theta)/M)^{k-1}}}{\sum_{k=2}^{\infty} (\lambda M)^{k-1}} > 1. \quad (21)$$

The inequality in (21) follows since $f(x) = \sum_{k=2}^{\infty} x^{k-1}$ is increasing in $x \in \mathbb{R}^+$ and $p_1(\theta) > \alpha$ from Condition 1. Thus, the point $(0, M)$ is the unique feasible point that maximizes $\ell_{\text{MAC}}$. □

**B. Detection Probability Versus Communication Slots**

In this section, we analyze the performance of the global decision rule under the MAC protocol while increasing the
number of communication slots \( M \), for a fixed sensor density \( \lambda \) and a fixed local false alarm probability \( \alpha \). Intuitively, as the number of slots increases, the probability of a collision per slot vanishes, resulting in a binomial distribution of only 0’s and 1’s in each slot. In the following we show that this binomial distribution converges to the Poisson distribution of detections for perfect communication case as the number of communication slots increase. As a result a network with the MAC protocol approaches the performance of a network with perfect communication.

**Theorem 5.** For a given sensor density \( \lambda \), local false alarm probability \( \alpha \), and signal parameter \( \theta \), the fusion center message distribution under the MAC protocol is asymptotically given by:

\[
\lim_{M \to \infty} \Pr(N_1 = i, N_c = j; H_\theta) = \delta_j e^{-\lambda p_{\theta}(\theta)} \left[ \frac{\lambda p_{\theta}(\theta)}{i!} \right]^i,
\]

where \( \delta_j \) is the Kronecker delta with \( \delta_j = 1 \) if \( j = 0 \) and \( \delta_j = 0 \) if \( j \neq 0 \).

**Proof.** We show that under \( H_1 \) and a signal parameter \( \theta \), the probability distribution of \( \{N_1, N_c\} \) converges to a Poisson distribution with the same conditional mean as the perfect channel case.

The conditional probability for all feasible points for \( N_c > 0 \) and for any finite \( M \geq 1 \) is given by

\[
\Pr(N_1 = i, N_c = j; H_\theta) = \sum_{j=1}^{M} \left( \frac{\lambda p_{\theta}(\theta)}{i!} \right)^j \binom{M}{i} (\pi_{c,1}(\theta))^j (\pi_{0,1}(\theta))^{M-j} (\pi_{0,1}(\theta))^j = \sum_{j=1}^{M} \left( \frac{\lambda p_{\theta}(\theta)}{i!} \right)^j (1 - \pi_{c,1}(\theta))^{M-j}.
\]

This implies that all the probability masses of \( \{N_1, N_c\} \) under \( H_1 \) for all \( N_c \neq 0 \) converge to 0 as \( M \to \infty \). On the other hand, for \( N_c = 0 \), any \( 0 \leq i < M \), and any finite \( M \geq 1 \), the probability mass simplifies to

\[
\Pr(N_1 = i, N_c = 0; H_\theta) = \frac{M!}{(M-i)!} (1 - \pi_{c,1}(\theta))^{M-j} \leq e^{-\mu M} \frac{\mu^j}{j!},
\]

where \( \mu = \lambda p_{\theta}(\theta) \). For any fixed and finite \( j \), we clearly have

\[
\Pr(N_1 = i, N_c = 0; H_\theta) \to e^{-\mu} \frac{\mu^j}{j!} \text{ as } M \to \infty.
\]

The result is the same under \( H_0 \) with \( \mu = \lambda \alpha \). Thus, the conditional distributions of the observations at the fusion node under the MAC protocol, and under both hypothesis, converge to Poisson distributions with the same conditional means as the case of perfect communications.

**Corollary 3.** For a given sensor density \( \lambda \), local false alarm probability \( \alpha \), and global false alarm probability \( \tilde{\alpha} \), the global confidence interval of the ROC is asymptotically given by:

\[
\lim_{M \to \infty} \beta_{asc} = \beta_{sc}
\]

**Proof.** This result follows directly from Theorem 5. Theorem 5 holds true for any realization of the signal parameter \( \theta \in \Theta \), and the prior \( f_\theta \) does not depend on \( M \). As a result, for large \( M \), composite likelihoods at the fusion center under the MAC protocol converges to the composite likelihoods for the perfect communication case.

**V. CONFIDENCE INTERVAL OF ROC**

In this section, we derive a simple bound on the confidence interval of the receiver operating characteristic. The global decision rules (8) and (12) are designed for the average case. The performance for a particular realization, given an average-case decision rule, will vary from the expected value. The confidence interval captures the variability of the ROC across realizations of the sensor network and of the target signal.

The confidence interval of the ROC is derived using Hölder’s integral inequality. It is a bound and not necessarily tight, but applies in very general cases where the uncertainty in the false alarm probability (i.e., horizontal variation in the ROC) is insignificant compared that of the detection probability.

Since the sensor observations are independent of the target under the null hypothesis, there is no variation in the global false alarm probability due to a random target. Similarly, there is no variation on the false alarm probability due to the randomness of sensor node locations. So the only variability in the global false alarm probability arises from the randomness in the number of sensor nodes. For a given number of sensor nodes, the count statistics at the fusion node are binomial (under perfect communications). Binomial distributions are well approximated by Poisson distributions as the density of the network increases.

For random variable \( W \in \mathcal{W} \) and mapping \( g : \mathcal{W} \to [a, b] \), with \( -\infty < a < b < \infty \), the variance of \( g(W) \) can be upper bounded by

\[
\text{var}(g(W)) = E_W(g(W)^2) - E_W(g(W))^2 \leq E_W(g(W)g(W))^{1/p} E_W(g(W)g(W))^{1/q} - E_W(g(W)^2).
\]

where the inequality follows from Hölder’s integral inequality with \( p, q \in (1, \infty) \), and \( 1/p + 1/q = 1 \).

Let \( P_{\alpha,w} \) represent the probability of detection for a specific realization, indexed by \( w \in \mathcal{W} \), of the sensor network and target for an \( \alpha \)-level decision rule \( \delta \), which was designed for the average case. The probability of detection varies across realizations due to the randomness in the number of sensor nodes, the locations of the sensor nodes and the target, and the signal intensity. Thus, \( P_{\alpha,w} \) could be viewed as a random variable constrained to the interval \( [0, 1] \). The variance of the probability of detection is then bounded above by

\[
\text{var}(P_{\alpha,w}) \leq P_{\alpha} (1 - P_{\alpha}),
\]

where \( P_{\alpha} = E_W(P_{\alpha,w}) \). Equation (26) follows from (25) by noting that \( P_{\alpha,w} \leq 1 \) and letting \( p \to 1 \). We use (26) to upper and lower bound the 1-\( \sigma \) confidence interval of the ROC about its average.
Notice that as the average probability of detection tends toward the extremes, the confidence interval on the detection probability shrinks. Additionally, it can be shown that the lower bound on the 1-σ confidence interval is negative when the average probability of detection is less than 0.5, so a tighter lower bound is the trivial bound of zero. Similarly, the upper bound on the 1-σ confidence interval is $P_\alpha > 0.5$, so a tighter upper bound here is the trivial bound of unity.

VI. NUMERICAL RESULTS

We illustrate system performance of this sensor network model through several numerical examples. The examples show performance in terms of global ROC curves and global probability of detection versus sensor network density and number of communication slots.

In these simulations, the sensor locations are uniformly distributed on a disk of radius $r_0 = 20$ units. The range and amplitude of the target signal are modeled as discrete random variables. The distributions of the target’s range $R_\alpha$ from the center of the disk and amplitude $A_\alpha$ are discrete uniform on the intervals $[0, 20]$ and $[2, 200]$, respectively. The average number of false alarms from the network per activity period is $\lambda \alpha = 2$. The signal attenuation in the propagation medium is given by

$$g(d) = \frac{1}{1 + d^2}$$

where $d$ is the distance between the sensor and target. This signal propagation loss model is very similar to those used in [2], [3], [4], [5]. The sensor observations under $H_0$ are central $\chi^2$ distributed with 10 degrees of freedom. Under $H_1$, the observations are (conditionally) noncentral $\chi^2$ distributed with 10 degrees of freedom and noncentrality parameter $10(a_\alpha g(d))^2$, where the signal-to-sensor distance is $d = \sqrt{r_x^2 + r^2 - 2r_x r \cos(\phi)}$, the target signal amplitude is $a_\alpha$, with distance $r_x$ from the center of the disk, and for a sensor node located at $(r, \phi)$. It can be shown, at the sensor level, that the NP-optimal test does not depend on the signal parameter vector $\Theta$, and the local test is simply to compare the sensor observation to a threshold, which satisfies

$$\int_\tau^\infty \frac{1}{2^{5}4!} x^4 e^{-x^2/2} \, dx = \alpha.$$ 

The conditional probability $p_1(\tilde{\theta})$ is then given by

$$p_1(\tilde{\theta}) = \int_0^\infty \int_0^{2\pi} \frac{e^{x}}{\pi r^2} f(x, 10, \frac{x}{r_0}) \, dx \, d\phi \, dr,$$

where $f(x, k, h)$ is a noncentral chi-squared distribution with $k$ degrees of freedom and noncentrality parameter $h$. For this sensor-level observation model, the sensor node does not need to know the target distribution $f_{\tilde{\theta}}$ in order to make a local decision. The global decision rules were evaluated using rules (9) and (12) for perfect communication and the MAC protocol, respectively.

We first compare the analytic ROC curves against sample estimates from $10^5$ Monte Carlo trials for cases of perfect communications and of the MAC protocol. The ROC curves represent the global probability of detection versus global probability of false alarm for the sensor network. The purpose is to validate the analytic expressions since their evaluation involves numerical integration (due to the expectations with respect to random sensor locations and a random target). Figures 3-5 are plots of ROC curves, each with four different average number of sensor nodes in the disk $\mathbb{R}$: $\lambda = 10$ (blue), 20 (green), 40 (red), 80 (cyan) nodes. Figure 3 contains ROC curves for the perfect communications case. The solid lines represent the analytic performance curves and the circles are sample estimates of the ROC curves. Figure 4 contains ROC curves for the MAC protocol with $M = 4$ slots. The dashed lines in Figure 4 represent the analytic curves and the circles are sample estimates of the ROC curves from $10^5$ Monte Carlo trials. As seen in Figures 3 and 4, the sample estimates align well with the analytic results. Each also demonstrate how the ROC curves improve with average number of nodes.

Figure 5 overlays the analytic ROC curves of perfect communications (solid lines) and the MAC protocol (dashed lines) with $M = 4$ slots. Figure 5 contains the same information as Figures 3 and 4 but without the Monte Carlo estimates. The ROC curves in Figure 5 are plotted on a logarithmic false alarm scale to better show the performance gap at lower global false alarm probabilities resulting from a communications protocol with collisions. We see very good performance using the imperfect channel for only $M = 4$ slots; one reason such good performance is achieved with few slots is that the global decision rule for the proposed MAC protocol in this paper attempts to retain information from packet collisions. Global decision rules in the literature typically disregard slots with collisions since both positive and negative detections are transmitted, so collisions provide no definitive local detection information to the fusion center.

Figure 6 demonstrates the improvement in the probability of detection as the average number of sensor nodes $\lambda$ increases. Plotted are the probability of detection for perfect communications (solid line) and for the MAC protocol (dashed lines) for $M = 3$ (red), 4 (green), and 5 (blue) slots, and with a global probability of false alarm $\bar{\alpha} = 10^{-3}$. Interestingly, the probability of detection curve appears to flatten between 1000 and 3000 nodes. This is due to an approximate bifurcation of detection performance. This is in part due to the choice of a discrete distribution on the target signal and the PPP model of the network. Roughly, about 90% of the target signal and sensor network realizations result in a probability of detection that increases from nearly 0 to 1 for networks having between 10 and 1000 nodes, while the remaining 10% have low probability of detection until more than about 3000 nodes are present. The realizations with target signals that are mostly interior to the sensor network perform better; the sensor network provides better coverage in those cases. On the other hand, the PPP model permits sensor network realizations with N=0 sensor nodes, but with decreasing probability as the density increases. Nonetheless, the detection probability still exhibits improvement as the average number of sensor nodes increases, despite an increase in the probability of a collision in each slot of the MAC protocol.

Figure 7 shows the probability of detection as a function of the number of communication slots $M$ for the MAC protocol (squares) for $\lambda = 10$ (blue), 20 (green), 40 (red), 80 (cyan) nodes. The global probability of false alarm is $\bar{\alpha} = 10^{-3}$. The solid lines represent the probability of detection assuming
perfect communications, which does not depend on the number of slots. Figure 7 demonstrates that performance approaches that of the ideal communications case for increasing number of slots. Increasing the number of slots proportionally increases the delay at the fusion node to make a global decision, so a designer would want to select a minimum number of transmission slots that provides the desired detection probability.

Finally, we demonstrate the confidence interval bound of the global ROC. We compare the bound to the sample estimate of the 1-σ confidence interval for perfect communications; results for the MAC protocol are similar. The simulation parameters remain the same as above but with the number of Monte Carlo trials increased to $10^6$ to better estimate the variance across realizations at low false alarm probabilities. Figure 8 shows the analytic ROC for perfect communications (red solid) for $\lambda = 40$, the analytical bound on the 1-σ confidence interval (red dash-dot), and sample estimates of the 1-σ confidence interval (blue dash-dash). Note the upper limit of the confidence interval and the analytical bound are $\geq 1$, and therefore not shown. Figure 9 displays the same information as Figure 8, but on a logarithmic scale to better show the bounds in the low false alarm regions. The simulations demonstrate that the bound on the confidence interval reasonably captures the uncertainty in the ROC for a wide range of false alarm probabilities.

VII. CONCLUSION

We derived the global decision rules for the case of sensor nodes communicating local binary decisions to a common fusion node for (1) perfect communications and (2) communications through a delay-constrained random-access MAC on a single-hop wireless network. The fusion node combines received information to make a global decision on the presence or absence of a target signal. Under the MAC protocol, some detection messages collide with other messages, since the communications medium is shared. Sensor nodes transmit only local detections (instead of all decisions); this reduces both communications resource needs and lowers energy use of local sensors, and in addition means that collisions contain useful information about event occurrences. It was shown that the decision rule at the fusion center is a function of one or two simple count statistics, namely: (1) the number of local detections in the perfect communications case, or (2) the number of successful transmissions coupled with the number of collisions in the MAC protocol case. Additionally, conditions for reducing the computation complexity of the decision rules were provided.

We showed that the performance of the MAC protocol case converges to that of the perfect communications case as number of communication slots increases (i.e., as $M \rightarrow \infty$). We also showed that as the sensor network density increases, and keeping the average number of network transmissions constant under $H_0$, that detection performance improves. Under the MAC protocol, performance was shown to improve for sufficiently large network densities despite increases in communication collisions.

It was shown that the performance under the MAC protocol is somewhat degraded relative to the case of “ideal” communications. Simulations show that this gap in performance can be negligible, despite collisions (without retransmissions) in communicating detection messages. Additionally, the performance gap decreases as the number of MAC slots $M$ increases. Increasing $M$ results in a proportional delay in fused detection declarations, so one wishes to keep this number as small as possible.

Finally, we considered statistical variation of the global performance due to random sensor and target deployments. The ROC curve for a given sensor network and target signal will vary from the average case. We derived a bound on variance of the ROC curves. The bound is simple to compute and depends on few system parameters. Initial results showed good agreement between Monte Carlo results and the derived bound.

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Fig. 3. Global ROC curves for $\lambda = 10$ (blue), 20 (green), 40 (red), 80 (cyan) for perfect communications. Solid lines represent the analytic ROC and circles represent sample estimates.

Fig. 4. Global ROC curves for $\lambda = 10$ (blue), 20 (green), 40 (red), 80 (cyan) for the MAC protocol with $M = 4$ slots. Dashed lines represent the analytic ROC and circles represent sample estimates.

Fig. 5. Global ROC curves for $\lambda = 10$ (blue), 20 (green), 40 (red), 80 (cyan). Solid and dashed lines represent the analytic ROC for perfect communications and the MAC protocol with $M = 4$ slots, respectively.

Fig. 6. Global probability of detection versus number of sensor nodes $\lambda$ for $P_{fa} = 10^{-3}$. The curves represent $P_d$ for perfect communications (blue) and the MAC protocol with $M = 3$ (green), $M = 4$ (red), and $M = 5$ (cyan) slots.

Fig. 7. Global probability of detection versus number of communication slots $M$ for $P_{fa} = 10^{-3}$ and $\lambda = 10$ (blue), 20 (green), 40 (red), 80 (cyan). Solid lines represent perfect communication and squares represent the MAC protocol.

Fig. 8. Global ROC curve (red solid) for $\lambda = 40$, analytical confidence interval bound (red dash-dot), and sample 1-$\sigma$ confidence interval (blue dash-dash). Note the upper limit of both confidence intervals are $\geq 1$.

Fig. 9. Global ROC curve (red solid) for $\lambda = 40$, analytical confidence interval bound (red dash-dot), and sample 1-$\sigma$ confidence interval (blue dash-dash). Note the upper limit of both confidence intervals are $\geq 1$. 