On the Theory of Space–Time Codes for PSK Modulation

A. Roger Hammons, Jr., *Member, IEEE,* and Hesham El Gamal, *Member, IEEE*

***Abstract*—The design of space–time codes to achieve full spatial diversity over fading channels has largely been addressed by handcrafting example codes using computer search methods and only for small numbers of antennas. The lack of more general designs is in part due to the fact that the diversity advantage of a code is the minimum rank among the complex baseband differences between modulated codewords, which is difficult to relate to traditional code designs over finite fields and rings. In this paper, we present general binary design criteria for PSK-modulated space–time codes. For linear BPSK/QPSK codes, the rank of (binary projections of) the unmodulated codewords, as binary matrices over the binary field, is a sufficient design criterion: full binary rank guarantees full spatial diversity. This criterion accounts for much of what is currently known about PSK-modulated space-time codes. We develop new fundamental code constructions for both quasi-static and time-varying channels. These are perhaps the first general constructions—other than delay diversity schemes—that guarantee full spatial diversity for an arbitrary number of transmit antennas.**

***Index Terms*—Algebraic code design, fading channels, multiple transmit antennas, PSK modulation, space–time codes, transmit diversity, wireless communication.**

I. Introduction

FOR wireless communication systems, the principal radio design challenges arise from the harsh radio frequency (RF) propagation environment characterized by channel fading, due to diffuse and specular multipath, and cochannel interference (CCI), due to the aggressive reuse of radio resources. Interleaved coded modulation on transmit and multiple antennas on receive are standard methods used by wireless communication system designers to combat time-varying fading and to mitigate interference. Both are examples of *diversity techniques*. Simple transmit diversity schemes—in which, for example, a delayed replica of the transmitted signal is retransmitted through a second, spatially independent antenna and the two signals are coherently combined at the receiver by a channel equalizer—have also been considered within the wireless communications industry as a method to combat multipath fading. From a coding perspective, such transmit diversity schemes amount to repetition codes and lead to the natural question as to whether more sophisticated codes might be more effective. Information-theoretic studies by Foschini and Gans [7] and Teletar [16] have in fact demonstrated that the capacity of multiantenna systems significantly exceeds that of conventional single-antenna systems for fading channels. The challenge of designing channel codes for high-capacity multiantenna systems has led to the development of so-called "space–time codes," in which coding is performed across the spatial dimension (antenna channels) as well as time [9], [13].

Work by Guey *et al.* [9] and then independently by Tarokh *et al.* [13] on the problem of signal design for transmit diversity systems derived the fundamental performance parameters for space–time codes over quasi-static fading channels: 1) diversity advantage, which describes the exponential decrease of decoded error rate versus signal-to-noise ratio (asymptotic slope of the performance curve on a log–log scale) and 2) coding advantage, which does not affect the asymptotic slope but results in a shift of the performance curve. These parameters turn out to be, respectively, the minimum rank and minimum geometric mean of the nonzero eigenvalues among a set of complex-valued matrices associated with the differences between baseband modulated codewords. In [13], Tarokh *et al.* further present a number of interesting, handcrafted trellis codes for two antenna systems that provide maximum two-level diversity advantage and good coding advantage. Subsequent computer searches by Baro *et al.* [2] and Grimm *et al.* [8] have yielded new space–time trellis codes with improved coding advantage. Follow-on work by Tarokh *et al.* [14], [15] has looked at coding based on a novel space–time modulation format proposed by Alamouti [1] and its generalizations.

One of the fundamental difficulties of space–time codes, which has so far hindered the development of more general results, is the fact that the diversity and coding advantage design criteria apply to the complex domain of baseband-modulated signals rather than the binary or discrete domain in which the underlying codes are traditionally designed. In this paper, we develop general binary rank criteria, pertaining to the unmodulated codewords (or their projections) as binary matrices over the binary field, that guarantee that the space–time code achieves full spatial diversity. These results unify much of what is currently known about space–time code design for full spatial diversity and lead to new, more sophisticated construction methods. We show that many of the conventional convolutional codes of rate $1/L$ with optimal d_{free} can be formatted to provide space–time codes achieving full L-level spatial diversity. We exhibit an expurgated, punctured version of the Golay code

Manuscript received November 3, 1998; revised July 28, 1999. The material in this paper was presented in part at the 1999 IEEE International Workshop on Information Theory, Metsov, Greece, June 27–July 2, 1999, and at the Thirty-Seventh Annual Allerton Conference on Communication, Control, and Computing, University of Illinois at Urbana-Champaign, September 22–24, 1999.

A. R. Hammons, Jr. is with Hughes Network Systems, Germantown, MD 20876 USA (e-mail: rhammons@hns.com).

H. El Gamal was with the Electrical Engineering Department, University of Maryland, College Park, MD 20742 USA. He is now with Hughes Network Systems, Germantown, MD 20876 USA (e-mail: helgamal@hns.com).

Communicated by E. Soljanin, Associate Editor for Coding Techniques.

Publisher Item Identifier S 0018-9448(00)01666-7.

$\mathcal{G}_{23}$ that can be formatted as a BPSK-modulated space–time block code achieving full $L = 2$ spatial diversity and maximum bandwidth efficiency (rate 1 transmission). For $L = 3$ diversity, we derive by hand an explicit rate 1 space–time block code that achieves full spatial diversity for BPSK and QPSK modulation. By contrast, the space–time block codes derived from the complex generalized orthogonal designs in [14] for $L = 3$ and 4 antennas provide no better bandwidth efficiency than rate $3/4$. Since the space–time trellis codes of [13] fall within the scope of our general theory and have published simulation performance data, the current paper is devoted exclusively to the development of the general theory. Additional performance data comparing the new code constructions with previously known space–time codes will be presented separately.

In the body of this paper, Section II summarizes the basics of space–time coding as previously developed in the literature and establishes our perspective and notation. Section III is a fundamental one that provides a detailed development and discussion of the new binary rank criteria for BPSK- and QPSK-modulated space–time codes. Sections IV and V expand on these results to develop a comprehensive theory. In Section IV, the BPSK section, we present new fundamental constructions that encompass as special cases transmit delay diversity schemes, rate $1/L$ convolutional codes, and certain concatenated coding schemes. We also discuss the general problem of formatting existing binary codes into full-diversity space–time codes. Specific space–time block codes of rate 1 for $L = 2$ and $L = 3$ antennas are given that provide coding gain as well as achieve full spatial diversity. In Section V, the QPSK section, we present the $\mathbb{Z}_4$ analogs of the binary theory. We show that full diversity BPSK designs lift to full diversity QPSK designs. In Section VI, we show how the existing body of space–time trellis codes fits within the new theory. Extension to time-varying channels is considered in Section VII, which shows that the new multistacking constructions provide a general class of so-called "smart–greedy" space–time codes for such channels. Section VIII addresses the applicability of the binary rank criteria to multilevel constructions for higher order constellations. Section IX presents our conclusions.

II. Space–Time Coding

In this section, we lay out the main concepts of space–time coding for quasi-static, flat Rayleigh fading channels and the prior knowledge as to how to design them.

A general block diagram for the systems of interest is shown in Fig. 1. In this system, the source generates k information symbols from the discrete alphabet $\mathcal{X}$, which are encoded by the error control code C to produce codewords of length $N = nL_t$ over the symbol alphabet $\mathcal{Y}$. The encoded symbols are parsed among L_t transmit antennas and then mapped by the modulator into constellation points from the discrete complex-valued signaling constellation Ω for transmission across the channel. The modulated streams for all antennas are transmitted simultaneously. At the receiver, there are L_r receive antennas to collect the incoming transmissions. The received baseband signals are subsequently decoded by the space–time decoder. Each spatial channel (the link between one transmit antenna and one receive antenna) is assumed to experience statistically independent flat Rayleigh fading. Receiver noise is assumed to be additive white Gaussian noise (AWGN).

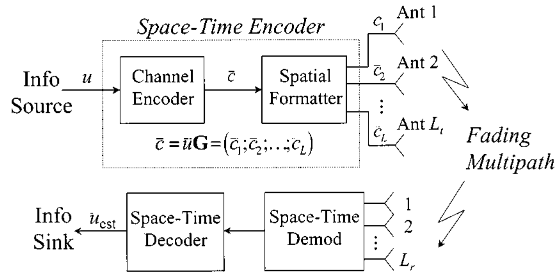

Fig. 1. Reference block diagram for space–time encoding and decoding.

We formally define a space–time code to consist of an underlying error control code together with the spatial parsing format.

Definition 1: An $L \times n$ space–time code $\mathcal{C}$ of size M consists of an (Ln, M) error control code C and a spatial parser σ that maps each codeword vector $\bar{c} \in C$ to an $L \times n$ matrix $\boldsymbol{c}$ whose entries are a rearrangement of those of $\bar{c}$. The space–time code $\mathcal{C}$ is said to be linear if both C and σ are linear.

Except as noted to the contrary, we will assume the standard parser that maps

$$\bar{c} = \Big(c_1^1, c_1^2, \cdots, c_1^{L_t}, c_2^1, c_2^2, \cdots, c_2^{L_t}, \cdots, c_n^1, c_n^2, \cdots, c_n^{L_t}\Big) \in C$$

to the matrix

$$\boldsymbol{c} = \begin{bmatrix} c_1^1 & c_2^1 & \cdots & c_n^1 \\ c_1^2 & c_2^2 & \cdots & c_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ c_1^{L_t} & c_2^{L_t} & \cdots & c_n^{L_t} \end{bmatrix}.$$

In this notation, it is understood that c_t^i is the code symbol assigned to transmit antenna i at time t.

Let $f: \mathcal{Y} \rightarrow \Omega$ be the modulator mapping function. Then $\boldsymbol{s} = f(\boldsymbol{c})$ is the baseband version of the codeword as transmitted across the channel. For this system, we have the following baseband model of the received signal:

$$y_t^j = \sum_{i=1}^{L_t} \alpha_{ij} s_t^i \sqrt{E_s} + n_t^j \tag{1}$$

where y_t^j is the signal received at antenna j at time t; α_{ij} is the complex path gain from transmit antenna i to receive antenna j; $s_t^i = f(c_t^i)$ is the transmitted constellation point corresponding to c_t^i; and n_t^j is the AWGN noise sample for receive antenna j at time t. The noise samples are independent samples of a zero-mean complex Gaussian random variable with variance $N_0/2$ per dimension. The fading channel is quasi-static in the sense that, during the transmission of n codeword symbols across any one of the links, the complex path gains do not change with time t, but are independent from one codeword transmission to the next. In matrix notation, we have

$$\overline{Y} = \sqrt{E_s}\,\overline{A}\boldsymbol{D_c} + \overline{N} \tag{2}$$

where (see the top of the following page).

$$\overline{Y} = [y_1^1 \quad y_2^1 \quad \cdots \quad y_n^1 \quad y_1^2 \quad y_2^2 \quad \cdots \quad y_n^2 \quad \cdots \quad y_1^{L_r} \quad y_2^{L_r} \quad \cdots \quad y_n^{L_r}]$$
$$\overline{N} = [n_1^1 \quad n_2^1 \quad \cdots \quad n_n^1 \quad n_1^2 \quad n_2^2 \quad \cdots \quad n_n^2 \quad \cdots \quad n_1^{L_r} \quad n_2^{L_r} \quad \cdots \quad n_n^{L_r}]$$
$$\overline{A} = [\alpha_{11} \quad \alpha_{21} \quad \cdots \quad \alpha_{L_t1} \quad \alpha_{12} \quad \alpha_{22} \quad \cdots \quad \alpha_{L_t2} \quad \cdots \quad \alpha_{1L_r} \quad \alpha_{2L_r} \quad \cdots \quad \alpha_{L_tL_r}]$$
$$\boldsymbol{D_c} = \begin{bmatrix} f(\boldsymbol{c}) & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & f(\boldsymbol{c}) & \cdots & \boldsymbol{0} \\ \vdots & \ddots & \ddots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & f(\boldsymbol{c}) \end{bmatrix}_{L_rL_t \times L_rn}.$$

Let codeword $\boldsymbol{c}$ be transmitted. Then the pairwise error probability that the decoder will prefer the alternate codeword $\boldsymbol{e}$ to $\boldsymbol{c}$ is given by

$$P(\boldsymbol{c} \to \boldsymbol{e} | \{\alpha_{ij}\}) = P(V < 0 | \{\alpha_{ij}\})$$

where

$$V = \|\overline{A}(\boldsymbol{D}_c - \boldsymbol{D}_e) + \overline{N}\|^2 - \|\overline{N}\|^2$$

is a Gaussian random variable with mean

$$\mathrm{E}\{V\} = \|\overline{A}(\boldsymbol{D}_c - \boldsymbol{D}_e)\|^2$$

and variance

$$\mathrm{Var}\{V\} = 2N_0 \mathrm{E}\{V\}.$$

Thus

$$P(V < 0 | \{\alpha_{ij}\}) = Q\left(\frac{\|\overline{A}(\boldsymbol{D}_c - \boldsymbol{D}_e)\|}{\sqrt{2N_0}}\right) \tag{3}$$

$$\leq \frac{1}{2} \exp\left\{-\frac{1}{4N_0} \left\|\overline{A}(\boldsymbol{D}_c - \boldsymbol{D}_e)\right\|^2\right\}. \tag{4}$$

For the quasi-static, flat Rayleigh fading channel, (4) can be manipulated [9], [13] to yield the fundamental bound

$$P(\boldsymbol{c} \to \boldsymbol{e}) \leq \left(\frac{1}{\prod_{i=1}^{r} (1 + \lambda_i E_s / 4N_0)}\right)^{L_r} \tag{5}$$

$$\leq \left(\frac{\eta E_s}{4N_0}\right)^{-rL_r} \tag{6}$$

where $r = \mathrm{rank}\,(f(\boldsymbol{c}) - f(\boldsymbol{e}))$ and $\eta = (\lambda_1 \lambda_2 \cdots \lambda_r)^{1/r}$ is the geometric mean of the nonzero eigenvalues of

$$\boldsymbol{A} = (f(\boldsymbol{c}) - f(\boldsymbol{e}))(f(\boldsymbol{c}) - f(\boldsymbol{e}))^{\mathrm{H}}.$$

This leads to the rank and equivalent product distance criteria for space–time codes, which were first presented in [9] and later more extensively investigated in [13].

- *Rank Criterion*: Maximize the diversity advantage

$$r = \mathrm{rank}\,(f(\boldsymbol{c}) - f(\boldsymbol{e}))$$

over all pairs of distinct codewords $\boldsymbol{c}, \boldsymbol{e} \in \mathcal{C}$.

- *Product Distance Criterion*: Maximize the coding advantage $\eta = (\lambda_1 \lambda_2 \cdots \lambda_r)^{1/r}$ over all pairs of distinct codewords $\boldsymbol{c}, \boldsymbol{e} \in \mathcal{C}$.

The rank criterion is the more important of the two as it determines the asymptotic slope of the performance curve as a function of E_s/N_0. The product distance criterion is of secondary importance and, ideally, should be optimized after the diversity advantage is maximized. For an $L \times n$ space–time code $\mathcal{C}$, the maximum possible rank is L. Consequently, full spatial diversity is achieved if all baseband difference matrices corresponding to distinct codewords in $\mathcal{C}$ have full rank L.

In [13], Tarokh *et al.* proposed simple design rules for space–time trellis codes for $L = 2$ spatial diversity (see also the discussion in [8]):

Rule 1) Transitions departing from the same state differ only in the second symbol.

Rule 2) Transitions merging at the same state differ only in the first symbol.

When these rules are followed, the codeword difference matrices are of the form

$$f(\boldsymbol{c}) - f(\boldsymbol{e}) = \begin{bmatrix} \cdots & x_1 & \cdots & 0 & \cdots \\ & 0 & & x_2 & \end{bmatrix}$$

with x_1, x_2 nonzero complex numbers. Thus every such difference matrix has full rank, and the space–time code achieves two-level spatial diversity. Two good codes that satisfy these design rules, and several others that do not, were handcrafted using computer search methods.

In [8], Grimm *et al.* introduced the notion of "zeros symmetry" as a generalization of the Tarokh design rules for higher levels of diversity $L \geq 2$. A space–time code has zeros symmetry if every baseband codeword difference $f(\boldsymbol{c}) - f(\boldsymbol{e})$ is upper and lower triangular (and has appropriate nonzero entries to ensure full rank). As noted in [8], the zeros symmetry property is sufficient for full rank but not necessary; nonetheless, it is useful in constraining computer searches for good space–time codes. Grimm *et al.* presented the results of a computer search undertaken to identify full diversity space–time codes with best possible coding advantage. They provide a small table of short constraint length space–time trellis codes that achieve full spatial diversity ($L = 2, 3$, and 5 for BPSK modulation; $L = 2$ for QPSK modulation) and have the best known value of coding advantage. They discuss the difficulties involved in evaluating coding advantage for general space–time trellis codes and consider the notion of geometric uniformity with respect to coding advantage (rather than Euclidean distance). As a general space–time code construction, they show that trellis-coded delay diversity schemes achieve full diversity for all $L \geq 2$ with the fewest possible number of states.

Recently, Baro *et al.* [2] have conducted a computer search similar to that of Grimm *et al.* for optimal $L = 2$ QPSK space–time trellis codes of short constraint length. Their results agree with Grimm *et al.* regarding the optimal product distances but the given codes have different generators, indicating that at least for $L = 2$ there is a multiplicity of optimal codes.

In [1], Alamouti introduced a simple transmitter diversity scheme for two antennas that provides two-level diversity gain with modest decoder complexity. In the Alamouti scheme, independent signaling constellation points x_1, x_2 are transmitted simultaneously by different transmit antennas during a given symbol interval. On the next symbol interval, the conjugated signals $-x_2^*$ and x_1^* are transmitted by the respective antennas. This elegant scheme has the interesting property that the two transmissions are orthogonal in both time and the spatial dimension. Tarokh *et al.* [14], [15] use the Hurwitz–Radon theory of real and complex orthogonal designs to generalize Alamouti's scheme to multiple transmit antennas. The orthogonal designs of Alamouti and Tarokh are not space–time codes as we have defined them since, depending on the constellation, the complex conjugate operation that is essential to these designs may not have a discrete algebraic interpretation. (The complex generalized designs for $L = 3$ and 4 antennas also involve division by $\sqrt{2}$.) Nonetheless, if the PSK modulation format is chosen so that the constellation is closed under complex conjugation, one may derive interesting nonlinear space–time codes of very short block length from certain of these schemes.

III. Binary Rank Criteria for Space–Time Codes

The design of space–time codes is hampered by the fact that the rank criterion applies to the complex-valued differences between the baseband versions of the codewords. It is not easy to transfer this design criterion into the binary domain where the problem of code design is relatively well understood. In this section, we give general binary design criteria that are sufficient to guarantee that a space–time code achieves full spatial diversity.

In the rank criterion for space–time codes, the sign of the differences between modulated codeword symbols is important. On the other hand, it is difficult to see how one would be able to preserve that information in the binary domain. Thus we are led to investigate what can be said in the absence of such specific structural knowledge. To this end, we introduce the following definition.

Definition 2: Two complex matrices $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ will be said to be ω*-equivalent* if $\boldsymbol{r}_1$ can be transformed into $\boldsymbol{r}_2$ by multiplying any number of entries of $\boldsymbol{r}_1$ by powers of the complex number ω.

We are primarily interested in the ω-equivalence of matrices when ω is a generator for the signaling constellation Ω. Since BPSK and QPSK are of particular interest, we introduce the following special notation:

BPSK ($\omega = -1$):

$\boldsymbol{r}_1 \doteq \boldsymbol{r}_2$ denotes that $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are (-1)-equivalent.

QPSK ($\omega = i = \sqrt{-1}$):

$\boldsymbol{r}_1 \ddot{=} \boldsymbol{r}_2$ denotes that $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are i-equivalent.

Using this notion, we derive binary rank criteria for space–time codes that depend only on the unmodulated codewords themselves. The binary rank criterion provides a complete characterization for BPSK-modulated codes (under the assumption of lack of knowledge regarding signs in the baseband differences). It provides a highly effective characterization for QPSK-modulated codes that, although not complete in the same way, provides a fertile new framework for space–time code design.

The BPSK and QPSK binary rank criteria simplify the problem of code design and the verification that full spatial diversity is achieved. They apply to both trellis and block codes and for arbitrary numbers of transmit antennas. In a sense, these results show that the problem of achieving full spatial diversity is relatively easy. Within the large class of space–time codes satisfying the binary rank criteria, code design is reduced to the problem of product distance or coding advantage optimization as in [2] and [8].

A. BPSK-Modulated Codes

For BPSK modulation, the natural discrete alphabet is the field $\mathbb{F} = \{0, 1\}$ of integers modulo 2. Modulation is performed by mapping the symbol $x \in \mathbb{F}$ to the constellation point $s = f(x) \in \{-1, 1\}$ according to the rule $s = (-1)^x$. Note that it is possible for the modulation format to include an arbitrary phase offset $e^{i\phi}$, since a uniform rotation of the constellation will not affect the rank of the matrices $f(\boldsymbol{c}) - f(\boldsymbol{e})$ nor the eigenvalues of the matrices $\boldsymbol{A} = (f(\boldsymbol{c}) - f(\boldsymbol{e}))(f(\boldsymbol{c}) - f(\boldsymbol{e}))^{\mathrm{H}}$. Notationally, we will use the circled operator $\oplus$ to distinguish modulo 2 addition from real- or complex-valued $(+, -)$ operations. It will often be convenient to identify the binary digits $0, 1 \in \mathbb{F}$ with the complex numbers $0, 1 \in \mathbb{C}$. We will do this without special comment or notation.

Theorem 3: Let $\mathcal{C}$ be a linear $L \times n$ space–time code with $n \geq L$. Suppose that every nonzero binary codeword $\boldsymbol{c} \in \mathcal{C}$ has the property that every real matrix (-1)-equivalent to $\boldsymbol{c}$ is of full rank L. Then, for BPSK transmission, $\mathcal{C}$ satisfies the space–time rank criterion and achieves full spatial diversity L.

Proof: It is enough to note that

$$[(-1)^{\boldsymbol{c}_1} - (-1)^{\boldsymbol{c}_2}]/2 \doteq \boldsymbol{c}_1 \oplus \boldsymbol{c}_2. \qquad \square$$

It turns out that (-1)-equivalence has a simple binary interpretation. We first need the following lemma.

Lemma 4: Let $\boldsymbol{M}$ be a matrix of integers. Then the matrix equation $\boldsymbol{M}\overline{x} = 0$ has nontrivial real solutions if and only if it has a nontrivial integral solution $\overline{x} = [d_1, d_2, \cdots, d_L]$ in which the integers $d_1, d_2, \cdots, d_L$ are jointly relatively prime—that is, $\gcd(d_1, d_2, \cdots, d_L) = 1$.

Proof: Applying Gaussian elimination to the matrix $\boldsymbol{M}$ yields a canonical form in which all entries are rational. Hence, the null space of $\boldsymbol{M}$ has a basis consisting of rational vectors. By multiplying and dividing by appropriate integer constants, any rational solution can be transformed into an integral solution of the desired form. $\square$

Theorem 5: The $L \times n$ $(n \geq L)$ binary matrix $\boldsymbol{c} = [\overline{c}_1 \ \overline{c}_2 \ \cdots \ \overline{c}_L]^T$ has full rank L over the binary field $\mathbb{F}$ if and only if every real matrix $\boldsymbol{r} = [\overline{r}_1 \ \overline{r}_2 \ \cdots \ \overline{r}_L]^T$ that is (-1)-equivalent to $\boldsymbol{c}$ has full rank L over the real field $\mathbb{R}$.

Proof:

$(\Rightarrow)$ Suppose that $\boldsymbol{r}$ is not of full rank over $\mathbb{R}$. Then there exist real $\alpha_1, \alpha_2, \cdots, \alpha_L$, not all zero, for which

$$\alpha_1\overline{r}_1 + \alpha_2\overline{r}_2 + \cdots + \alpha_L\overline{r}_L = 0.$$

By the lemma, one may assume that the α_i are integers and are jointly relatively prime. Given the assumption on $\boldsymbol{r}$ and $\boldsymbol{c}$, we have $\overline{r}_i \equiv \overline{c}_i \,(\mathrm{mod}\, 2)$. Therefore, reducing the integral equation modulo 2 produces a binary linear combination of the $\overline{c}_i$ that sums to zero. Since the α_i are not all divisible by 2, the binary linear combination is nontrivial. Hence, $\boldsymbol{c}$ is not of full rank over $\mathbb{F}$.

$(\Leftarrow)$ Suppose that $\boldsymbol{c}$ is not of full rank over $\mathbb{F}$. Then there are rows $\overline{c}_{i_1}, \overline{c}_{i_2}, \cdots, \overline{c}_{i_\nu}$ such that

$$\overline{c}_{i_1} \oplus \overline{c}_{i_2} \oplus \cdots \oplus \overline{c}_{i_\nu} = \overline{0}.$$

Each column of $\boldsymbol{c}$ therefore contains an even number of ones among these ν rows. Hence, the $+$ and $-$ signs in each column can be modified to produce a real-valued summation of these ν rows that is equal to zero. This modification produces a real-valued matrix that is (-1)-equivalent to $\boldsymbol{c}$ but is not of full rank. □

The binary design criterion for linear space–time codes now follows immediately.

Theorem 6—Binary Rank Criterion: Let $\mathcal{C}$ be a linear $L \times n$ space–time code with $n \geq L$. Suppose that every nonzero binary codeword $\boldsymbol{c} \in \mathcal{C}$ is a matrix of full rank over the binary field $\mathbb{F}$. Then, for BPSK transmission, the space–time code $\mathcal{C}$ achieves full spatial diversity L.

The binary rank criterion makes it possible to develop algebraic code designs for which one can prove that full spatial diversity is achieved without resorting to brute-force verification. Although the binary rank criterion and associated theorems are stated for linear codes, it is clear from the proofs that they work in general, even if the code is nonlinear, provided the results are applied to the modulo 2 differences between codewords instead of the codewords themselves.

B. QPSK-Modulated Codes

For QPSK modulation, the natural discrete alphabet is the ring $\mathbb{Z}_4 = \{0, \pm 1, 2\}$ of integers modulo 4. Modulation is performed by mapping the symbol $x \in \mathbb{Z}_4$ to the constellation point $s \in \{\pm 1, \pm i\}$ according to the rule $s = i^x$, where $i = \sqrt{-1}$. Again, the absolute phase reference of the QPSK constellation could have been chosen arbitrarily without affecting the diversity advantage or coding advantage of a $\mathbb{Z}_4$-valued space–time code. Notationally, we will often use subscripts to distinguish modulo 4 operations $(\oplus_4, \ominus_4)$ from binary $(\oplus)$ and real- or complex-valued $(+, -)$ operations.

For the $\mathbb{Z}_4$-valued matrix $\boldsymbol{c}$, we define the binary component matrices $\alpha(\boldsymbol{c})$ and $\beta(\boldsymbol{c})$ to satisfy the expansion

$$\boldsymbol{c} = \beta(\boldsymbol{c}) + 2\alpha(\boldsymbol{c}).$$

Thus $\beta(\boldsymbol{c})$ is the modulo 2 projection of $\boldsymbol{c}$ and

$$\alpha(\boldsymbol{c}) = [\boldsymbol{c} \ominus_4 \beta(\boldsymbol{c})]/2.$$

We also introduce the following special matrices that are useful in the analysis of QPSK-modulated space–time code:

- complex-valued $\zeta(\boldsymbol{c}) = \boldsymbol{c} + i\beta(\boldsymbol{c})$;
- binary-valued indicant projections: $\Xi(\boldsymbol{c})$ and $\Psi(\boldsymbol{c})$.

The indicant projections are defined based on a partitioning of $\boldsymbol{c}$ into two parts, according to whether the rows (or columns) are or are not multiples of two, and serve to indicate certain aspects of the binary structure of the $\mathbb{Z}_4$ matrix in which multiples of two are ignored.

A $\mathbb{Z}_4$-valued matrix $\boldsymbol{c}$ of dimension $L \times n$ will be said to be of *type* $1^\ell 2^{L-\ell} \times 1^m 2^{n-m}$ if it consists of exactly ℓ rows and m columns that are not multiples of two. It will be said to be of *standard type* $1^\ell 2^{L-\ell} \times 1^m 2^{n-m}$ if it is of type $1^\ell 2^{L-\ell} \times 1^m 2^{n-m}$ and the first ℓ rows and first m columns in particular are not multiples of two. When the column (row) structure of a matrix is not of particular interest, the matrix will be said to be of *row type* $1^\ell 2^{L-\ell}$ (*column type* $1^m 2^{n-m}$) or, more specifically, *standard row* (*column*) *type*.

Let $\boldsymbol{c}$ be a $\mathbb{Z}_4$-valued matrix of type $1^\ell 2^{L-\ell} \times 1^m 2^{n-m}$. Then, after suitable row and column permutations if necessary, it has the following row and column structure:

$$\boldsymbol{c} = \begin{bmatrix} \overline{c}_1 \\ \overline{c}_2 \\ \vdots \\ \overline{c}_\ell \\ 2\overline{c}'_{\ell+1} \\ 2\overline{c}'_{\ell+2} \\ \vdots \\ 2\overline{c}'_L \end{bmatrix} = [\overline{h}_1^T \quad \overline{h}_2^T \quad \cdots \quad \overline{h}_m^T \quad 2\overline{h}'^T_{m+1} \quad 2\overline{h}'^T_{m+2} \quad \cdots \quad 2\overline{h}'^T_n].$$

Then the row-based indicant projection (Ξ-projection) is defined as

$$\Xi(\boldsymbol{c}) = \begin{bmatrix} \beta(\overline{c}_1) \\ \beta(\overline{c}_2) \\ \vdots \\ \beta(\overline{c}_\ell) \\ \beta(\overline{c}'_{\ell+1}) \\ \beta(\overline{c}'_{\ell+2}) \\ \vdots \\ \beta(\overline{c}'_L) \end{bmatrix}$$

and the column-based indicant projection (Ψ-projection) is defined as shown at the top of the following page. Note that

$$[\Psi(\boldsymbol{c})]^T = \Xi(\boldsymbol{c}^T). \tag{7}$$

$$\Psi(\boldsymbol{c}) = [\,\beta(\overline{h}_1^T) \quad \beta(\overline{h}_2^T) \quad \cdots \quad \beta(\overline{h}_m^T) \quad \beta(\overline{h}'^T_{m+1}) \quad \beta(\overline{h}'^T_{m+2}) \quad \cdots \quad \beta(\overline{h}'^T_n)\,]$$

The first result shows that the baseband difference of two QPSK-modulated codewords is directly related to the $\mathbb{Z}_4$-difference of the unmodulated codewords.

Proposition 7: Let $\mathcal{C}$ be a $\mathbb{Z}_4$ space–time code. For $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{C}$, let $i^{\boldsymbol{x}} - i^{\boldsymbol{y}}$ denote the baseband difference of the corresponding QPSK-modulated signals. Then

$$i^{\boldsymbol{x}} - i^{\boldsymbol{y}} \doteq \zeta(\boldsymbol{x} \ominus_4 \boldsymbol{y}).$$

Furthermore, any complex matrix $\boldsymbol{z} = \boldsymbol{r} + i\boldsymbol{s}$ that is (-1)-equivalent to $i^{\boldsymbol{x}} - i^{\boldsymbol{y}}$ has the property that

$$\boldsymbol{r} \equiv \boldsymbol{s} \equiv \beta(\boldsymbol{x} \ominus_4 \boldsymbol{y}) \equiv \boldsymbol{x} \oplus \boldsymbol{y} \pmod 2.$$

Proof: We note that any component of $i^{\boldsymbol{x}} - i^{\boldsymbol{y}}$ can be written as

$$i^x - i^y = -i^y \cdot (1 - i^\delta)$$

where $\delta = x \ominus_4 y$. Since

$$1 - i^\delta = \begin{cases} 0, & \delta = 0 \\ 1 - i = -i \cdot (1+i), & \delta = 1 \\ 2, & \delta = 2 \\ 1 + i = -i \cdot (-1+i), & \delta = -1 \end{cases}$$

the entry $i^x - i^y$ can be turned into the complex number $(x \ominus_4 y) + i(x \oplus y)$ by multiplying by ± 1 or $\pm i$ as necessary. Thus

$$i^{\boldsymbol{x}} - i^{\boldsymbol{y}} \doteq (\boldsymbol{x} \ominus_4 \boldsymbol{y}) + i(\boldsymbol{x} \oplus \boldsymbol{y}) = \zeta(\boldsymbol{x} \ominus_4 \boldsymbol{y})$$

as claimed.

For (-1)-equivalence, multiplication by $\pm i$ is not allowed. Under this restriction, it is no longer possible to separate $\boldsymbol{z}$ into the terms $\boldsymbol{x} \ominus_4 \boldsymbol{y}$ and $\boldsymbol{x} \oplus \boldsymbol{y}$ so cleanly; the discrepancies, however, amount to additions of multiples of 2. Hence, if

$$\boldsymbol{z} = \boldsymbol{r} + i\boldsymbol{s} \doteq i^{\boldsymbol{x}} - i^{\boldsymbol{y}}$$

we have

$$\boldsymbol{r} \equiv \boldsymbol{x} \ominus_4 \boldsymbol{y} \pmod 2$$

and

$$\boldsymbol{s} \equiv \boldsymbol{x} \oplus \boldsymbol{y} \pmod 2. \qquad \square$$

Theorem 8: Let $\mathcal{C}$ be a linear, $L \times n$ $(n \geq L)$ space–time code over $\mathbb{Z}_4$. Suppose that every nonzero codeword $\boldsymbol{c} \in \mathcal{C}$ has the property that every complex matrix i-equivalent to $\zeta(\boldsymbol{c})$ is of full rank L. Then, for QPSK transmission, $\mathcal{C}$ satisfies the space–time rank criterion and achieves full spatial diversity L.

Proof: Since $\mathcal{C}$ is linear, the $\mathbb{Z}_4$-difference between any two codewords is also a codeword. The result then follows immediately from the previous proposition. $\square$

It turns out that the indicant projections of the $\mathbb{Z}_4$-valued matrix $\boldsymbol{c}$ provide a great deal of information regarding the singularity of $\zeta(\boldsymbol{c})$ and any of its i-equivalents. Thus the indicants provide the basis for our binary rank criterion for QPSK-modulated space–time codes.

Theorem 9: Let $\boldsymbol{c} = [\overline{c}_1\ \overline{c}_2\ \cdots\ \overline{c}_L]^T$ be a $\mathbb{Z}_4$-valued matrix of dimension $L \times n$ $(n \geq L)$. If the row-based indicant $\Xi(\boldsymbol{c})$ or the column-based indicant $\Psi(\boldsymbol{c})$ has full rank L over $\mathbb{F}$, then every complex matrix $\boldsymbol{z}$ that is i-equivalent to $\zeta(\boldsymbol{c})$ has full rank L over the complex field $\mathbb{C}$.

Proof: We give the proof for the row-based indicant. The proof for the column-based indicant is similar.

By rearranging the rows of $\boldsymbol{c}$ if necessary, we can assume any row that is a multiple of 2 appears as one of the last rows of the matrix. Thus there is an ℓ for which $\beta(\overline{c}_i) \neq 0$ whenever $1 \leq i \leq \ell$ and $\beta(\overline{c}_i) = 0$ for $\ell < i \leq L$. The first ℓ rows will be called the 1-*part* of $\boldsymbol{c}$; the last $L - \ell$ rows will be called the 2-*part*.

Suppose that

$$\boldsymbol{z} = \begin{bmatrix} \overline{z}_1 \\ \overline{z}_2 \\ \vdots \\ \overline{z}_L \end{bmatrix} = \begin{bmatrix} \overline{r}_1 + i\overline{s}_1 \\ \overline{r}_2 + i\overline{s}_2 \\ \vdots \\ \overline{r}_L + i\overline{s}_L \end{bmatrix}$$

is singular and is i-equivalent to $\zeta(\boldsymbol{c})$. Then there exist complex numbers $\alpha_1 = a_1 + ib_1, \alpha_2 = a_2 + ib_2, \cdots, \alpha_L = a_L + ib_L$, not all zero, for which

$$\begin{aligned} \alpha_1\overline{z}_1 + \alpha_2\overline{z}_2 + \cdots + \alpha_L\overline{z}_L &= \sum_{i=1}^{L} (a_i\overline{r}_i - b_i\overline{s}_i) \\ &\quad + i\sum_{i=1}^{L} (b_i\overline{r}_i + a_i\overline{s}_i) \\ &= 0. \end{aligned} \tag{8}$$

Without loss of generality, we may assume that the a_i, b_i are integers having greatest common divisor equal to 1. Hence, there is a nonempty set of coefficients having real or imaginary part an odd integer. The coefficient α_i will be said to be *even* or *odd* depending on whether two is or is not a common factor of a_i and b_i. It will be said to be of *homogeneous parity* if a_i and b_i are of the same parity; otherwise, it will be said to be of *heterogeneous parity*.

There are now several cases to consider based on the nature of the coefficients applied to the 1-part and 2-part of $\boldsymbol{z}$.

Case i): There is an odd coefficient of heterogeneous parity applied to the 1-part of $\boldsymbol{z}$.

In this case, taking the projection of (8) modulo 2, we find that

$$\sum_{i=1}^{\ell} (\beta(a_i) \oplus \beta(b_i))\,\beta(\overline{c}_i) = 0$$

since $\beta(\overline{r}_i) = \beta(\overline{s}_i) = \beta(\overline{c}_i)$ by the proposition. By assumption, at least one of the binary coefficients $\beta(a_i) \oplus \beta(b_i)$ is nonzero. Hence, this is a nontrivial linear combination of the first ℓ rows of $\Xi(\boldsymbol{c})$, and so $\Xi(\boldsymbol{c})$ is not of full rank over $\mathbb{F}$.

Case ii): All of the nonzero coefficients applied to the 1-part of $\boldsymbol{z}$ are homogeneous and at least one is odd; all of the coefficients applied to the 2-part of $\boldsymbol{z}$ are homogeneous (odd or even).

In this case, we first multiply (8) by $\alpha^*/2 = (a - ib)/2$, where $\alpha = a + ib$ is one of the coefficients applied to the 1-part of $\boldsymbol{z}$ having a and b both odd. Note that $\alpha^* \alpha_i$ is even if α_i is homogeneous (odd or even) and is odd homogeneous if α_i is heterogeneous. Hence, this produces a new linear combination, all coefficients of which still have integral real and imaginary parts. In this linear combination, one of the new coefficients is $|\alpha|^2/2 = (a^2 + b^2)/2$, which is an odd integer. The argument of Case i) now applies.

Case iii): All of the nonzero coefficients applied to the 1-part of $\boldsymbol{z}$ are homogeneous and at least one is odd; there is a heterogeneous coefficient applied to the 2-part of $\boldsymbol{z}$.

In this case, we normalize as in Case ii), using one of the odd homogeneous coefficients from the 1-part of $\boldsymbol{z}$, say $\alpha = a + ib$. Thus normalization produces the equation

$$\tilde{\alpha}_1 \overline{z}_1 + \cdots + \tilde{\alpha}_\ell \overline{z}_\ell + \tilde{\alpha}_{\ell+1} \overline{z}'_{\ell+1} + \cdots + \tilde{\alpha}_L \overline{z}'_L = 0 \qquad (9)$$

where $\tilde{\alpha}_i = \alpha^* \alpha_i/2$ for $i \leq \ell$ and $\tilde{\alpha}_i = \alpha^* \alpha_i$ for $i > \ell$.

Taking the projection modulo 2 of the real (or imaginary) part of (9) yields

$$\begin{aligned} 0 &= \sum_{i=1}^{\ell} \left[\beta(\tilde{a}_i) \oplus \beta(\tilde{b}_i)\right] \beta(\overline{c}_i) \\ &\quad \oplus \sum_{i=\ell+1}^{L} \beta(\tilde{a}_i)\beta(\overline{r}'_i) \oplus \sum_{i=\ell+1}^{L} \beta(\tilde{b}_i)\beta(\overline{s}'_i) \\ &= \sum_{i=1}^{\ell} \left[\beta(\tilde{a}_i) \oplus \beta(\tilde{b}_i)\right] \beta(\overline{c}_i) \\ &\quad \oplus \sum_{i=\ell+1}^{L} \beta(\tilde{a}_i) \left(\beta(\overline{r}'_i) \oplus \beta(\overline{s}'_i)\right). \end{aligned}$$

For $i \geq \ell + 1$, we note that $\beta(\overline{r}_i{}') \oplus \beta(\overline{s}'_i) = \beta(\overline{c}'_i)$, where $\overline{c}_i = 2\overline{c}'_i$ is the ith row of $\boldsymbol{c}$. By assumption, there is a nonzero coefficient in each of the three component sums. Hence, (9) establishes a nontrivial linear combination of the rows of $\Xi(\boldsymbol{c})$.

Case iv): All of the coefficients applied to the 1-part of $\boldsymbol{z}$ are even, and at least one of the coefficients applied to the 2-part of $\boldsymbol{z}$ is heterogeneous.

In this case, we divide (8) by two to get the modified dependence relation

$$\alpha'_1 \overline{z}_1 + \cdots + \alpha'_\ell \overline{z}_\ell + \alpha_{\ell+1} \overline{z}'_{\ell+1} + \cdots + \alpha_L \overline{z}'_L = 0 \qquad (10)$$

where $\alpha'_i = \alpha_i/2$ and $\overline{z}'_i = \overline{z}_i/2$. Projecting modulo 2, we get two independent binary equations corresponding to the real and imaginary parts of (10).

$$\sum_{i=1}^{\ell} [\beta(a'_i) \oplus \beta(b'_i)] \beta(\overline{c}_i) \oplus \sum_{i=\ell+1}^{L} \beta(a_i)\beta(\overline{r}'_i) \oplus \sum_{i=\ell+1}^{L} \beta(b_i)\beta(\overline{s}'_i) = 0$$

$$\sum_{i=1}^{\ell} [\beta(a'_i) \oplus \beta(b'_i)] \beta(\overline{c}_i) \oplus \sum_{i=\ell+1}^{L} \beta(b_i)\beta(\overline{r}'_i) \oplus \sum_{i=\ell+1}^{L} \beta(a_i)\beta(\overline{s}'_i) = 0.$$

Setting these two equal gives

$$\sum_{i=\ell+1}^{L} [\beta(a_i) \oplus \beta(b_i)] [\beta(\overline{r}'_i) \oplus \beta(\overline{s}'_i)] = 0$$

which is a nontrivial linear combination of the rows $\beta(\overline{c}'_i) = \beta(\overline{r}'_i) \oplus \beta(\overline{s}'_i)$, for $i \geq \ell + 1$, of $\Xi(\boldsymbol{c})$.

Case v): All of the coefficients applied to the 1-part of $\boldsymbol{z}$ are even, and all of the coefficients applied to the two part of $\boldsymbol{z}$ are homogeneous.

In this case, we start with (10) after dividing by two. Recalling that at least one of the coefficients $\alpha_{\ell+1}, \cdots, \alpha_L$ is odd, we take the modulo 2 projection of (10) to get (from either the real or imaginary parts) the equation

$$\sum_{i=1}^{\ell} [\beta(a'_i) \oplus \beta(b'_i)] \beta(\overline{c}_i) \oplus \sum_{i=\ell+1}^{L} \beta(a_i) [\beta(\overline{r}'_i) \oplus \beta(\overline{s}'_i)] = 0.$$

This is once again a nontrivial linear combination of the rows of $\Xi(\boldsymbol{c})$. □

The binary rank criterion for QPSK space–time codes now follows as an immediate consequence of the previous two theorems.

Theorem 10—QPSK Binary Rank Criterion I: Let $\mathcal{C}$ be a linear $L \times n$ space–time code over $\mathbb{Z}_4$, with $n \geq L$. Suppose that, for every nonzero $\boldsymbol{c} \in \mathcal{C}$, the row-based indicant $\Xi(\boldsymbol{c})$ or the column-based indicant $\Psi(\boldsymbol{c})$ has full rank L over $\mathbb{F}$. Then, for QPSK transmission, the space–time code $\mathcal{C}$ achieves full spatial diversity L.

In certain $\mathbb{Z}_4$ space–time code constructions, there may be no codeword matrices having isolated rows or columns that are multiples of two. (It is always possible of course for the entire codeword to be a multiple of two.) In this case, the following binary rank criterion is simpler yet sufficient.

Theorem 11—QPSK Binary Rank Criterion II: Let $\mathcal{C}$ be a linear $L \times n$ space–time code over $\mathbb{Z}_4$, with $n \geq L$. Suppose that, for every nonzero $\boldsymbol{c} \in \mathcal{C}$, the binary matrix $\beta(\boldsymbol{c})$ is of full rank over $\mathbb{F}$ whenever $\beta(\boldsymbol{c}) \neq 0$, and $\beta(\boldsymbol{c}/2)$ is of full rank over $\mathbb{F}$ otherwise. Then, for QPSK transmission, the space–time code $\mathcal{C}$ achieves full spatial diversity L.

Proof: Under the specified assumptions, we have either $\Xi(\boldsymbol{c}) = \beta(\boldsymbol{c})$ or $\Xi(\boldsymbol{c}) = \beta(\boldsymbol{c}/2)$, depending on whether $\beta(\boldsymbol{c}) = 0$ or not. □

There is a difference between the BPSK and QPSK binary rank criteria that is worth noting. Whereas in the BPSK case we were able to provide a complete binary characterization of the notion of (-1)-equivalence of real-valued matrices, we were not able to do the same for the i-equivalence of complex-valued matrices. Instead of being both necessary and sufficient, the QPSK binary rank criterion is only sufficient in ensuring the full rank of the complex-valued matrices i-equivalent to $\zeta(\boldsymbol{c})$.

Nonetheless, the QPSK binary rank criterion is a powerful tool in the design and analysis of QPSK-modulated space–time codes. As will be shown in Section VI, all of the QPSK-modulated space–time codes of [13] fall within the scope of our theory. The QPSK binary rank criterion makes the analysis of those handcrafted codes straightforward and points out the simple structure that enable them to achieve full spatial diversity. The binary rank criterion opens the door to more sophisticated code designs.

IV. Theory of BPSK Space–Time Codes

A. Stacking Construction

A general construction for $L \times n$ space–time codes that achieve full spatial diversity is given by the following theorem.

Theorem 12—Stacking Construction: Let $T_1, T_2, \cdots, T_L$ be linear vector-space transformations from $\mathbb{F}^k$ into $\mathbb{F}^n$, and let $\mathcal{C}$ be the $L \times n$ space–time code of dimension k consisting of the codeword matrices

$$\boldsymbol{c}(\overline{x}) = \begin{bmatrix} T_1(\overline{x}) \\ T_2(\overline{x}) \\ \vdots \\ T_L(\overline{x}) \end{bmatrix}$$

where $\overline{x}$ denotes an arbitrary k-tuple of information bits and $n \geq L$. Then $\mathcal{C}$ satisfies the binary rank criterion, and thus achieves full spatial diversity L, if and only if $T_1, T_2, \cdots, T_L$ have the property that

$$\begin{aligned} &\forall\, a_1, a_2, \cdots, a_L \in \mathbb{F}: \\ &\quad T = a_1T_1 \oplus a_2T_2 \oplus \cdots \oplus a_LT_L \text{ is nonsingular unless} \\ &\quad a_1 = a_2 = \cdots = a_L = 0. \end{aligned}$$

Proof:

($\Rightarrow$) Suppose $\mathcal{C}$ satisfies the binary rank criterion but that $T = a_1T_1 \oplus a_2T_2 \oplus \cdots \oplus a_LT_L$ is singular for some $a_1, a_2, \cdots, a_L \in \mathbb{F}$. Then there is a nonzero $\overline{x}_0 \in \mathbb{F}^k$ such that $T(\overline{x}_0) = 0$. In this case

$$T(\overline{x}_0) = a_1 \cdot T_1(\overline{x}_0) \oplus a_2 \cdot T_2(\overline{x}_0) \oplus \cdots \oplus a_L \cdot T_L(\overline{x}_0) = 0$$

is a dependent linear combination of the rows of $\boldsymbol{c}(\overline{x}_0) \in \mathcal{C}$. Since $\mathcal{C}$ satisfies the binary rank criterion, $a_1 = a_2 = \cdots = a_L = 0$.

($\Leftarrow$) Suppose $T_1, T_2, \cdots, T_L$ have the stated property but that $\boldsymbol{c}(\overline{x}_0) \in \mathcal{C}$ is not of full rank. Then there exist $a_1, a_2, \cdots, a_L \in \mathbb{F}$, not all zero, for which

$$T(\overline{x}_0) = a_1 \cdot T_1(\overline{x}_0) \oplus a_2 \cdot T_2(\overline{x}_0) \oplus \cdots \oplus a_L \cdot T_L(\overline{x}_0) = 0$$

where $T = a_1T_1 \oplus a_2T_2 \oplus \cdots \oplus a_LT_L$. By hypothesis, T is nonsingular; hence, $\overline{x}_0 = 0$ and $\boldsymbol{c} = \boldsymbol{0}$. □

Of course, we may implement the vector-space transformations T_i of the general stacking construction as binary $k \times n$ matrices $\boldsymbol{M}$. In this case, the spatial diversity achieved by the space–time code does not depend on the choice of basis used to derive the matrices.

There is a satisfying heuristic explanation of the constraints imposed on the stacking construction. One would expect that, in order to achieve spatial diversity L on a flat Rayleigh fading channel, the receiver would have to be able to recover from the simultaneous fading of any $L-1$ spatial channels and thus be able to extract the information vector $\overline{x}$ from any single, unfaded spatial channel (at least at high enough signal-to-noise ratio). This requires that each matrix $\boldsymbol{M}_i$ be invertible. That each linear combination of the $\boldsymbol{M}_i$ must also be invertible follows from similar reasoning and the fact that the transmitted symbols are effectively summed by the channel.

The use of *transmit delay diversity* provides an example of the stacking construction. In this scheme, the transmission from antenna i is a one-symbol-delayed replica of the transmission from antenna $i-1$. Let C be a linear $[n, k]$ binary code with (nonsingular) generator matrix $\boldsymbol{G}$, and consider the delay diversity scheme in which codeword $\overline{c} = \overline{x}\boldsymbol{G}$ is repeated on each transmit antenna with the prescribed delay. The result is a space–time code achieving full spatial diversity.

Theorem 13: Let $\mathcal{C}$ be the $L \times (n+L-1)$ space–time code produced by applying the stacking construction to the matrices

$$\begin{aligned} \boldsymbol{M}_1 &= [\boldsymbol{G} \quad \boldsymbol{0}_{k\times(L-1)}] \\ \boldsymbol{M}_2 &= [\boldsymbol{0}_{k\times 1} \quad \boldsymbol{G} \quad \boldsymbol{0}_{k\times(L-2)}], \cdots, \\ \boldsymbol{M}_L &= [\boldsymbol{0}_{k\times(L-1)} \quad \boldsymbol{G}] \end{aligned}$$

where $\boldsymbol{0}_{i\times j}$ denotes the all-zero matrix consisting of i rows and j columns and $\boldsymbol{G}$ is the generator matrix of a linear $[n, k]$ binary code. Then $\mathcal{C}$ achieves full spatial diversity L.

Proof: In this construction, any linear combination of the $\boldsymbol{M}_i$ has the same column space as that of $\boldsymbol{G}$ and thus is of full rank k. Hence, the stacking construction constraints are satisfied, and the space–time code $\mathcal{C}$ achieves full spatial diversity L. □

A more sophisticated example of the stacking construction is given by the class of binary convolutional codes. Let C be the binary, rate $1/L$, convolutional code having transfer function matrix [11]

$$\boldsymbol{G}(D) = [\, g_1(D) \quad g_2(D) \quad \cdots \quad g_L(D)\,].$$

The *natural space–time code* $\mathcal{C}$ *associated with* C is defined to consist of the codeword matrices $\boldsymbol{c}(D) = \boldsymbol{G}^T(D)x(D)$, where the polynomial $x(D)$ represents the input information bit stream. In other words, for the natural space–time code, we adopt the natural transmission format in which the output coded bits corresponding to $g_i(D)$ are transmitted via antenna i. As in [13], we assume the trellis codes are terminated by tail bits. Thus if $x(D)$ is restricted to a block of N information bits, then $\mathcal{C}$ is an $L \times (N+\nu)$ space–time code, where $\nu = \max_{1\leq i\leq L}\{\deg g_i(D)\}$ is the maximal memory order of the convolutional code C.

Theorem 14: The natural space–time code $\mathcal{C}$ associated with the rate $1/L$ convolutional code C satisfies the binary rank criterion, and thus achieves full spatial diversity L for BPSK transmission, if and only if the transfer function matrix $\boldsymbol{G}(D)$ of C has full rank L as a matrix of coefficients over $\mathbb{F}$.

Proof: Let

$$g_i(D) = g_{i0} + g_{i1}D + g_{i2}D^2 + \cdots + g_{i\nu}D^\nu$$

where $i = 1, 2, \cdots, L$. Then, the result follows from the stacking construction applied to the generator matrices

$$\boldsymbol{M}_i = \begin{bmatrix} g_{i0} & g_{i1} & \cdots & g_{i\nu} & 0 & \cdots & 0 \\ 0 & g_{i0} & g_{i1} & \cdots & g_{i\nu} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & g_{i0} & g_{i1} & \cdots & g_{i\nu} \end{bmatrix}$$

each of which is of dimension $N \times (N + \nu)$. □

Alternately, one can prove Theorem 14 by observing that

$$\sum_{1 \le i \le L} a_i g_i(D) x(D) = 0$$

for some $x(D) \neq 0$ iff

$$\sum_{1 \le i \le L} a_i g_i(D) = 0.$$

This proof readily generalizes to recursive convolutional codes.

Since the coefficients of $\boldsymbol{G}(D)$ form a binary matrix of dimension $L \times (\nu + 1)$ and the column rank must be equal to the row rank, the theorem provides a simple bound as to how complex the convolutional code must be in order to satisfy the binary rank criterion. Tarokh *et al.* [13] showed that the bound is, in fact, necessary for the trellis code to achieve full spatial diversity.

Corollary 15: In order for the corresponding natural space–time code to satisfy the binary rank criterion for spatial diversity L, a rate $1/L$ convolutional code C must have maximal memory order $\nu \geq L - 1$.

Standard coding theory texts [11], [12], [17] provide extensive tables of binary convolutional codes that achieve optimal values of free distance d_{free}. In Table I, we list the optimal rate $1/L$ convolutional codes whose natural space–time formatting achieves full spatial diversity L. The table covers the range of constraint lengths $\nu = 2$through 10 for $L = 2, 3, 4$ and constraint lengths $\nu = 2$ through 8 for $L = 5, 6, 7, 8$. It thus provides a substantial set of space–time codes of practical complexity and performance, well-suited for wireless communication applications.

From the table, one observes that it is relatively easy for low-complexity convolutional codes to satisfy the stacking construction constraints at low values of L. It becomes progressively harder as L increases since the available number of distinct connection polynomials is small. Thus there are no examples for $L = 6, 7$, or 8 within the complexity range covered by the table, whereas every optimal code of rate $1/2$ achieves full $L = 2$ spatial diversity. (In fact, any convolutional code of rate $1/2$ will achieve full spatial diversity as long as the two connection polynomials are distinct!) The table also shows gaps at $L = 4$ and 5, where the optimal convolutional codes of constraint length $\nu = 6$ fail to satisfy the binary rank criterion.

TABLE I
BINARY RATE $1/L$ CONVOLUTIONAL CODES WITH OPTIMAL d_{free} WHOSE NATURAL SPACE–TIME CODES ACHIEVE FULL SPATIAL DIVERSITY

L	ν	Connection Polynomials	d_{free}
2	2	5, 7	5
	3	64, 74	6
	4	46, 72	7
	5	65, 57	8
	6	554, 744	10
	7	712, 476	10
	8	561, 753	12
	9	4734, 6624	12
	10	4672, 7542	14
3	3	54, 64, 74	10
	4	52, 66, 76	12
	5	47, 53, 75	13
	6	554, 624, 764	15
	7	452, 662, 756	16
	8	557, 663, 711	18
	9	4474, 5724, 7154	20
	10	4726, 5562, 6372	22
4	4	52, 56, 66, 76	16
	5	53, 67, 71, 75	18
	7	472, 572, 626, 736	22
	8	463, 535, 733, 745	24
	9	4474, 5724, 7154, 7254	27
	10	4656, 4726, 5562, 6372	29
5	5	75, 71, 73, 65, 57	22
	7	536, 466, 646, 562, 736	28

Convolutional codes with near-optimal d_{free} could presumably be found to fill these gaps.

One also notes in the table that the smallest code achieving full spatial diversity L has $\nu = L$ rather than $\nu = L - 1$. This is due to the fact the every optimal convolutional code under consideration for the table has all of its connection polynomials of the form $g_i(D) = 1 + \cdots + D^\nu$; hence, the first and last columns of $\boldsymbol{G}(D)$ are identical (all ones), so an additional column is needed to achieve rank L.

In the stacking construction, the information vector $\overline{x}$ is the same for all transmit antennas. This is necessary to ensure full rank in general. For example, if

$$T_1(\mathbb{F}^k) \cap T_2(\mathbb{F}^k) \neq \{\overline{0}\}$$

then the space–time code consisting of the matrices

$$\boldsymbol{c}(\overline{x}, \overline{y}) = \begin{bmatrix} T_1(\overline{x}) \\ T_2(\overline{y}) \end{bmatrix}$$

cannot achieve full spatial diversity even if T_1 and T_2 satisfy the stacking construction constraints. In this case, choosing $\overline{x}, \overline{y}$ so that $T_1(\overline{x}) = T_2(\overline{y}) \neq \overline{0}$ produces a codeword matrix having two identical rows. One consequence of this fact is that the natural space–time codes associated with noncatastrophic convolutional codes of rate k/L with $k > 1$ do not achieve full spatial diversity.

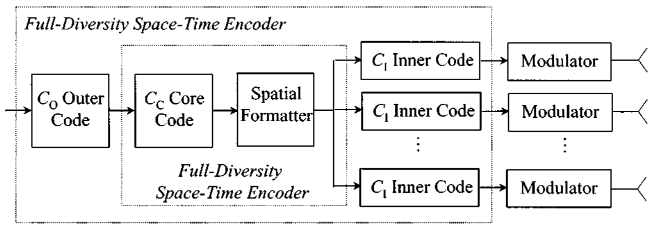

Fig. 2. General full-diversity space–time concatenated coding scheme.

The natural space–time codes associated with certain turbo codes illustrate a similar failure mechanism. In the case of a systematic, rate $1/3$ turbo code with two identical constituent encoders, the all-one input produces an output space–time codeword having two identical rows.

B. New Space–Time Codes from Old

We now look at transformations of space–time codes.

Theorem 16: Let $\mathcal{C}$ be an $L \times m$ space–time code satisfying the binary rank criterion. Given the linear vector-space transformation $T: \mathbb{F}^m \rightarrow \mathbb{F}^n$, we construct a new $L \times n$ space–time code $T(\mathcal{C})$ consisting of all codeword matrices

$$T(\boldsymbol{c}) = \begin{bmatrix} T(\overline{c}_1) \\ T(\overline{c}_2) \\ \vdots \\ T(\overline{c}_L) \end{bmatrix}$$

where

$$\boldsymbol{c} = [\overline{c}_1 \;\; \overline{c}_2 \;\; \cdots \;\; \overline{c}_L]^T \in \mathcal{C}.$$

Then, if T is nonsingular, $T(\mathcal{C})$ satisfies the binary rank criterion and, for BPSK transmission, achieves full spatial diversity L.

Proof: Let $\boldsymbol{c}, \boldsymbol{c}' \in \mathcal{C}$, and consider the difference $T(\boldsymbol{c}) \oplus T(\boldsymbol{c}') = T(\Delta \boldsymbol{c})$, where

$$\Delta \boldsymbol{c} = \boldsymbol{c} \oplus \boldsymbol{c}' = [\Delta\overline{c}_1 \;\; \Delta\overline{c}_2 \;\; \cdots \;\; \Delta\overline{c}_L]^T \neq \overline{0}.$$

Suppose

$$a_1 T(\Delta\overline{c}_1) \oplus a_2 T(\Delta\overline{c}_2) \oplus \cdots \oplus a_L T(\Delta\overline{c}_L) = 0.$$

Then $T(\Delta\overline{c}) = 0$ where

$$\Delta\overline{c} = a_1 \Delta\overline{c}_1 \oplus a_2 \Delta\overline{c}_2 \oplus \cdots \oplus a_L \Delta\overline{c}_L.$$

Since T is nonsingular, $\Delta\overline{c} = 0$. But since $\mathcal{C}$ satisfies the binary rank criterion, $a_1 = a_2 = \cdots = a_L = 0$. □

Certainly, column transpositions applied uniformly to all codewords in $\mathcal{C}$, for example, do not affect the spatial diversity of the code. A more interesting interpretation of the theorem is provided by the concatenated coding scheme of Fig. 2 in which T is a simple differential encoder or a traditional $[n, m]$ error control code that serves as a common inner code for each spatial transmission.

Given two full-diversity space–time codes that satisfy the binary rank criterion, it is easy to combine them into larger space–time codes that also achieves full spatial diversity. Let $\mathcal{A}$ be a linear $L \times n_A$ space–time code, and let $\mathcal{B}$ be a linear $L \times n_B$ space–time code, where $L \leq \min\{n_A, n_B\}$. Their concatenation is the $L \times (n_A + n_B)$ space–time code $\mathcal{C}_1 = |\mathcal{A}|\mathcal{B}|$ consisting of all codeword matrices of the form $\boldsymbol{c} = |\boldsymbol{a}|\boldsymbol{b}|$, where $\boldsymbol{a} \in \mathcal{A}$, $\boldsymbol{b} \in \mathcal{B}$.

A better construction is the space–time code $\mathcal{C}_2 = |\mathcal{A}|\mathcal{A} \oplus \mathcal{B}|$ consisting of the codeword matrices $\boldsymbol{c} = |\boldsymbol{a}|\boldsymbol{a} \oplus \boldsymbol{b}|$, where $\boldsymbol{a} \in \mathcal{A}$, $\boldsymbol{b} \in \mathcal{B}$. (Zero padding is used to perform the addition if $n_A \neq n_B$.) Thus $\mathcal{C}_2$ is an $L \times (n_A + \max\{n_A, n_B\})$ space–time code.

The following proposition regarding the full spatial diversity of these codes is easy to see.

Theorem 17: The space–time codes $\mathcal{C}_1 = |\mathcal{A}|\mathcal{B}|$ and $\mathcal{C}_2 = |\mathcal{A}|\mathcal{A} \oplus \mathcal{B}|$ satisfy the binary rank criterion if and only if the space–time codes $\mathcal{A}$ and $\mathcal{B}$ do.

As an application of the theorem, we note that codes built according to the stacking construction can also be "destacked."

Theorem 18—Destacking Construction: Let $\mathcal{C}$ be the $L \times n$ space–time code of dimension k consisting of the codeword matrices

$$\boldsymbol{c} = \begin{bmatrix} \overline{x}\boldsymbol{M}_1 \\ \overline{x}\boldsymbol{M}_2 \\ \vdots \\ \overline{x}\boldsymbol{M}_L \end{bmatrix}$$

where $\boldsymbol{M}_1, \boldsymbol{M}_2, \cdots, \boldsymbol{M}_L$ satisfy the stacking construction. Let $\ell = L/p$ be an integer divisor of L. Then the code $\mathcal{C}_\ell$ consisting of codeword matrices

$$\boldsymbol{c} = \begin{bmatrix} \overline{x}_1\boldsymbol{M}_1 & \overline{x}_2\boldsymbol{M}_{\ell+1} & \cdots & \overline{x}_p\boldsymbol{M}_{(p-1)\ell+1} \\ \overline{x}_1\boldsymbol{M}_2 & \overline{x}_2\boldsymbol{M}_{\ell+2} & \cdots & \overline{x}_p\boldsymbol{M}_{(p-1)\ell+2} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{x}_1\boldsymbol{M}_\ell & \overline{x}_2\boldsymbol{M}_{2\ell} & \cdots & \overline{x}_p\boldsymbol{M}_L \end{bmatrix}$$

is an $L/p \times pn$ space–time code of dimension pk that achieves full diversity L/p. Setting $\overline{x}_1 = \overline{x}_2 = \cdots = \overline{x}_p = \overline{x}$ produces an $L/p \times pn$ space–time code of dimension k that achieves full diversity.

More generally, we have the following construction.

Theorem 19—Multistacking Construction: Let $\mathcal{M} = \{\boldsymbol{M}_1, \boldsymbol{M}_2, \cdots, \boldsymbol{M}_L\}$ be a set of binary matrices of dimension $k \times n$, $n \geq k$, that satisfy the stacking construction constraints. For $i = 1, 2, \cdots, m$, let $(\boldsymbol{M}_{1i}, \boldsymbol{M}_{2i}, \cdots, \boldsymbol{M}_{\ell i})$ be an ℓ-tuple of distinct matrices from the set $\mathcal{M}$. Then, the space–time code $\mathcal{C}$ consisting of the codewords

$$\boldsymbol{c} = \begin{bmatrix} \overline{x}_1\boldsymbol{M}_{11} & \overline{x}_2\boldsymbol{M}_{12} & \cdots & \overline{x}_m\boldsymbol{M}_{1m} \\ \overline{x}_1\boldsymbol{M}_{21} & \overline{x}_2\boldsymbol{M}_{22} & \cdots & \overline{x}_m\boldsymbol{M}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{x}_1\boldsymbol{M}_{\ell 1} & \overline{x}_2\boldsymbol{M}_{\ell 2} & \cdots & \overline{x}_m\boldsymbol{M}_{\ell m} \end{bmatrix}$$

is an $\ell \times mn$ space–time code of dimension mk that achieves full spatial diversity ℓ. Setting $\overline{x}_1 = \overline{x}_2 = \cdots = \overline{x}_m = \overline{x}$ produces an $\ell \times mn$ space–time code of dimension k that achieves full spatial diversity.

These modifications of an existing space–time code implicitly assume that the channel remains quasi-static over the potentially longer duration of the new, modified codewords. Even when this implicit assumption is not true and the channel becomes more rapidly time-varying, however, these constructions are still of interest. In this case, the additional coding structure is useful for exploiting the temporal as well as spatial diversity available in the channel. Section VII discusses this in detail.

C. Space–Time Formatting of Binary Codes

We next investigate whether existing "time-only" binary error-correcting codes C can be formatted in such a fashion as to produce a full-diversity space–time code $\mathcal{C}$. It turns out that the maximum achievable spatial diversity of a code is not only limited by the code's least weight codewords but also by its maximal weight codewords.

Theorem 20: Let C be a linear binary code of length n whose Hamming weight spectrum has minimum nonzero value $d_{\min}$ and maximum value $d_{\max}$. Then, there is no BPSK transmission format for which the corresponding space–time code $\mathcal{C}$ achieves spatial diversity $L > \min\{d_{\min},\, n - d_{\max} + 1\}$.

Proof: Let $\boldsymbol{c}$ be a codeword of Hamming weight $d = \mathrm{wt}\,\boldsymbol{c}$. Then, in the baseband difference matrix $(-1)^{\boldsymbol{c}} - (-1)^{\mathbf{0}}$, between $\boldsymbol{c}$ and the all-zero codeword $\mathbf{0}$, the value -2 appears d times and the value 0 appears $n - d$ times. Thus the rank can be no more than d, since each independent row must have a nonzero entry, and can be no more than $n - d + 1$ since there must not be two identical rows containing only -2 entries. Therefore, the space–time code achieves spatial diversity at most

$$L \le \min_{\substack{\boldsymbol{c}\in\mathcal{C}\\ \boldsymbol{c}\neq 0}} \{\mathrm{wt}\,\boldsymbol{c},\, n - \mathrm{wt}\,\boldsymbol{c} + 1\} = \min\{d_{\min},\, n - d_{\max} + 1\}. \qquad \square$$

This leads to a general negative result useful in ruling out many classes of binary codes from consideration as space–time codes.

Corollary 21: If C is a linear binary code containing the all-1 codeword, then there is no BPSK transmission format for which the corresponding space–time code $\mathcal{C}$ achieves spatial diversity $L > 1$. Hence, the following binary codes admit no BPSK transmission format in which the corresponding space–time code achieves spatial diversity $L > 1$:

- *repetition codes*
- *Reed–Muller codes*
- *cyclic codes.*

As noted in the discussion of the stacking construction, it is possible to achieve full spatial diversity using repetition codes in a delay diversity transmission scheme. This does not contradict the corollary, however, since the underlying binary code in such a scheme is not strictly speaking a repetition code but a repetition code extended with extra zeros.

D. Special Cases

In this section, we look at special cases of the general theory for two- and three-antenna systems, exploring alternative space–time transmission formats and their connections to different partitionings of the generator matrix of the underlying binary code.

$L = 2$ Diversity: Let $\boldsymbol{G} = [\boldsymbol{I}\ \boldsymbol{P}]$ be a left-systematic generator matrix for a $[2k,\, k]$ binary code C, where $\boldsymbol{I}$ is the $k \times k$ identity matrix. Each codeword row vector $\overline{c} = (\overline{a}_I\ \overline{a}_P)$ has first half $\overline{a}_I$ consisting of all the information bits and second half $\overline{a}_P$ consisting of all the parity bits, where

$$\overline{a}_P = \overline{a}_I \boldsymbol{P}.$$

Let $\mathcal{C}$ be the space–time code derived from C in which the information bits are transmitted on the first antenna and the parity bits are transmitted simultaneously on the second antenna. The space–time codeword matrix corresponding to $\overline{c} = (\overline{a}_I\ \overline{a}_P)$ is given by

$$\boldsymbol{c} = \begin{bmatrix} \overline{a}_I \\ \overline{a}_P \end{bmatrix}.$$

The following proposition follows immediately from the stacking construction theorem.

Proposition 22: If the binary matrices $\boldsymbol{P}$ and $\boldsymbol{I} \oplus \boldsymbol{P}$ are of full rank over $\mathbb{F}$, then the space–time code $\mathcal{C}$ achieves full $L = 2$ spatial diversity.

As a nontrivial example of a new space–time block code achieving $L = 2$ spatial diversity, we note that both

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

and $\boldsymbol{I} \oplus \boldsymbol{P}$ are nonsingular over $\mathbb{F}$. Hence, the stacking construction produces a space–time code $\mathcal{C}$ achieving full $L = 2$ spatial diversity. The underlying binary code C, with generator matrix $\boldsymbol{G} = [\boldsymbol{I}|\boldsymbol{P}]$, is an expurgated and punctured version of the Golay code $\mathcal{G}_{23}$ (see [17, Fig. 6-2]). This is the first example of a space–time block code that achieves the highest possible bandwidth efficiency and provides coding gain as well as full spatial diversity.

The following proposition shows how to derive other $L = 2$ space–time codes from a given one.

Proposition 23: If the binary matrix $\boldsymbol{P}$ satisfies the conditions of the above theorem, so do the binary matrices $\boldsymbol{P}^2$, $\boldsymbol{P}^T$, and $\boldsymbol{U}\boldsymbol{P}\boldsymbol{U}^{-1}$, where $\boldsymbol{U}$ is any change of basis matrix.

We now reconsider $(a|a + b)$ constructions for the special case $L = 2$. Let A and B be systematic binary $[2k,\, k]$ codes with minimum Hamming distances d_A and d_B and generator matrices $\boldsymbol{G}_A = [\boldsymbol{I}\ \boldsymbol{P}_A]$ and $\boldsymbol{G}_B = [\boldsymbol{I}\ \boldsymbol{P}_B]$, respectively. From the stacking construction, the corresponding space–time codes $\mathcal{A}$ and $\mathcal{B}$ have codeword matrices

$$\boldsymbol{c}_A = \begin{bmatrix} \overline{a}_I \\ \overline{a}_I \boldsymbol{P}_A \end{bmatrix} \qquad \boldsymbol{c}_B = \begin{bmatrix} \overline{b}_I \\ \overline{b}_I \boldsymbol{P}_B \end{bmatrix}.$$

The $|a|a \oplus b|$ construction produces a binary $[4k, 2k]$ code C with minimum Hamming distance $d_C = \min\{2d_A, d_B\}$. A nonsystematic generator matrix for C is given by

$$\boldsymbol{G}_C = \begin{bmatrix} \boldsymbol{G}_A & \boldsymbol{G}_A \\ 0 & \boldsymbol{G}_B \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{P}_A & \boldsymbol{I} & \boldsymbol{P}_A \\ 0 & 0 & \boldsymbol{I} & \boldsymbol{P}_B \end{bmatrix}.$$

Applying the stacking construction using the left and right halves of $\boldsymbol{G}_C$ gives the space–time code $\mathcal{C} = |\mathcal{A}|\mathcal{A} \oplus \mathcal{B}|$ of Theorem 22, in which the codeword matrices are of nonsystematic form

$$\boldsymbol{c}_C = \begin{bmatrix} \overline{a}_I & \overline{a}_I \oplus \overline{b}_I \\ \overline{a}_I \boldsymbol{P}_A & \overline{a}_I \boldsymbol{P}_A \oplus \overline{b}_I \boldsymbol{P}_B \end{bmatrix}.$$

One may also derive a systematic version.

Proposition 24: Let $\mathcal{A}$ and $\mathcal{B}$ be $2 \times k$ space–time codes satisfying the binary rank criterion. Let $\mathcal{C}_s$ be the $2 \times 2k$ space–time code consisting of the codeword matrices

$$\boldsymbol{c} = \begin{bmatrix} \overline{a}_I & \overline{b}_I \\ \overline{a}_I \boldsymbol{P}_A & \overline{a}_I \boldsymbol{P}_A \oplus (\overline{a}_I \oplus \overline{b}_I) \boldsymbol{P}_B \end{bmatrix}.$$

Then $\mathcal{C}_s$ also satisfies the binary rank criterion and achieves full $L = 2$ spatial diversity.

Proof: Applying Gaussian elimination to $\boldsymbol{G}_C$ and reordering columns produces the systematic generator matrix

$$\boldsymbol{G}_C = [\boldsymbol{I}_{2k \times 2k} \quad \boldsymbol{P}_C]$$

where

$$\boldsymbol{P}_C = \begin{bmatrix} \boldsymbol{P}_A & \boldsymbol{P}_A \oplus \boldsymbol{P}_B \\ \boldsymbol{0} & \boldsymbol{P}_B \end{bmatrix}.$$

Note that $\boldsymbol{P}_C$ is nonsingular since $\boldsymbol{P}_A$ and $\boldsymbol{P}_B$ are both nonsingular. Likewise

$$\boldsymbol{I}_{2k \times 2k} \oplus \boldsymbol{P}_C = \begin{bmatrix} \boldsymbol{I} \oplus \boldsymbol{P}_A & \boldsymbol{P}_A \oplus \boldsymbol{P}_B \\ \boldsymbol{0} & \boldsymbol{I} \oplus \boldsymbol{P}_B \end{bmatrix}$$

is nonsingular since $\boldsymbol{I} \oplus \boldsymbol{P}_A$ and $\boldsymbol{I} \oplus \boldsymbol{P}_B$ are. The rest follows from the stacking construction. □

We now consider an alternate transmission format for $2 \times k$ space–time codes. Let C be a linear, left-systematic $[2k, k]$ code with generator matrix

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix}$$

where the submatrices $\boldsymbol{I}$, $\boldsymbol{0}$, and $\boldsymbol{A}_{ij}$ are of dimension $k/2 \times k/2$. In the new transmission format, we split the information vector into two parts $\overline{x}_1$ and $\overline{x}_2$ which are transmitted across different antennas. Thus the corresponding space–time code $\mathcal{C}$ consists of codeword matrices of the form

$$\boldsymbol{c} = \begin{bmatrix} \overline{x}_1 & \overline{p}_1 \\ \overline{x}_2 & \overline{p}_2 \end{bmatrix}$$

where $\overline{p}_1 = \overline{x}_1 \boldsymbol{A}_{11} \oplus \overline{x}_2 \boldsymbol{A}_{21}$ and $\overline{p}_2 = \overline{x}_1 \boldsymbol{A}_{12} \oplus \overline{x}_2 \boldsymbol{A}_{22}$.

For such codes, the following theorem gives sufficient conditions on the binary connection matrices to ensure full spatial diversity of the space–time code.

Proposition 25: Let $\boldsymbol{A}_{12}$, $\boldsymbol{A}_{21}$, and

$$\boldsymbol{A} = \sum_{i=1}^{2} (\boldsymbol{A}_{i1} \oplus \boldsymbol{A}_{i2})$$

be nonsingular matrices over $\mathbb{F}$. Then the space–time code $\mathcal{C}$ achieves full $L = 2$ spatial diversity.

Proof: The conditions follow immediately from the stacking construction theorem applied to the matrices

$$\boldsymbol{M}_1 = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{A}_{11} \\ \boldsymbol{0} & \boldsymbol{A}_{21} \end{bmatrix} \qquad \boldsymbol{M}_2 = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{A}_{12} \\ \boldsymbol{I} & \boldsymbol{A}_{22} \end{bmatrix}$$

since the sum $\boldsymbol{M} = \boldsymbol{M}_1 \oplus \boldsymbol{M}_2$ may be reduced to the form

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{A}_{11} \oplus \boldsymbol{A}_{12} \\ \boldsymbol{0} & \boldsymbol{A} \end{bmatrix}$$

by Gaussian elimination. □

The conditions of the proposition are not difficult to satisfy. For example, consider the linear 2×4 space–time code $\mathcal{C}$ whose codewords

$$\boldsymbol{c} = \begin{bmatrix} x_{11} & x_{12} & p_{11} & p_{12} \\ x_{21} & x_{22} & p_{21} & p_{22} \end{bmatrix}$$

are governed by the parity-check equations

$$\begin{aligned} p_{11} &= x_{12} \oplus x_{21} \oplus x_{22} \\ p_{12} &= x_{12} \oplus x_{22} \\ p_{21} &= x_{11} \oplus x_{21} \\ p_{22} &= x_{11} \oplus x_{12} \oplus x_{21}. \end{aligned}$$

The underlying binary code C has a generator matrix with submatrices

$$\boldsymbol{A}_{11} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \qquad \boldsymbol{A}_{12} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\boldsymbol{A}_{21} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad \boldsymbol{A}_{22} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

which meet the requirements of the proposition. Hence, $\mathcal{C}$ achieves 2-level spatial diversity.

$L = 3$ Diversity: Similar derivations for $L = 3$ antennas are straightforward; but for the sake of brevity we provide only an example. This example is particularly interesting in that it provides maximum possible bandwidth efficiency (rate 1 transmission) while attaining full spatial diversity for BPSK or QPSK modulation. The space–time block codes derived from complex generalized orthogonal designs for $L > 2$, on the other hand, achieve full diversity only at a loss in bandwidth efficiency. The problem of finding generalized orthogonal designs of rates greater than $3/4$ for $L > 2$ is a difficult open problem [14]. Neither can one design rate 1 space–time block codes of short length by using the general method of delay diversity. By contrast, the following rate 1 space–time block code for $L = 3$ was derived by hand.

Let $\mathcal{C}$ consist of the codeword matrices

$$\boldsymbol{c} = \begin{bmatrix} \overline{x}\boldsymbol{M}_1 \\ \overline{x}\boldsymbol{M}_2 \\ \overline{x}\boldsymbol{M}_3 \end{bmatrix}$$

where

$$\boldsymbol{M}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{M}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\boldsymbol{M}_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

It is easily verified that $\boldsymbol{M}_1$, $\boldsymbol{M}_2$, and $\boldsymbol{M}_3$ satisfy the stacking construction constraints. Thus $\mathcal{C}$ is a 3×3 space–time code achieving full spatial diversity (shown in Section V for QPSK as well as BPSK transmission). Since $\mathcal{C}$ admits a simple maximum-likelihood decoder (code dimension is three), it can be used as a 3-diversity space–time applique for BPSK- or QPSK-modulated systems similar to the 2-diversity Alamouti scheme.

Similar examples for arbitrary $L > 3$ can also be easily derived. Of course, the complexity of the decoder for the higher dimensional codes eventually becomes an issue.

V. Theory of QPSK Space–Time Codes

Due to the *binary* rank criterion developed for QPSK codes, the rich theory developed in Section IV for BPSK-modulated space–time codes largely carries over to QPSK modulation. In fact, as we shall see, space–time codes for BPSK modulation are of fundamental importance in the theory of space–time codes for QPSK modulation.

A. $\mathbb{Z}_4$ Stacking Constructions

The binary indicant projections allow the fundamental stacking construction for BPSK-modulated space– time codes to be "lifted" to the domain of QPSK-modulated space–time codes.

Theorem 26: Let $\boldsymbol{M}_1, \boldsymbol{M}_2, \cdots, \boldsymbol{M}_L$ be $\mathbb{Z}_4$-valued $m \times n$ matrices of standard row type $1^\ell 2^{m-\ell}$ having the property that

$$\forall\, a_1, a_2, \cdots, a_L \in \mathbb{F}: \quad a_1\Xi(\boldsymbol{M}_1) \oplus a_2\Xi(\boldsymbol{M}_2) \oplus \cdots \oplus a_L\Xi(\boldsymbol{M}_L) \text{ is nonsingular unless } a_1 = a_2 = \cdots = a_L = 0.$$

Let $\mathcal{C}$ be the $L \times n$ space–time code of size $M = 2^{\ell+m}$ consisting of all matrices

$$\boldsymbol{c}(\overline{x}, \overline{y}) = \begin{bmatrix} (\overline{x}\,\overline{y})\boldsymbol{M}_1 \\ (\overline{x}\,\overline{y})\boldsymbol{M}_2 \\ \vdots \\ (\overline{x}\,\overline{y})\boldsymbol{M}_L \end{bmatrix}$$

where $(\overline{x}\,\overline{y})$ denotes an arbitrary indexing vector of information symbols $\overline{x} \in \mathbb{Z}_4^\ell$ and $\overline{y} \in \mathbb{F}^{m-\ell}$. Then, for QPSK transmission, $\mathcal{C}$ satisfies the QPSK binary rank criterion and achieves full spatial diversity L.

Proof: Suppose that for some $\overline{x}_0, \overline{y}_0$, not both zero, the codeword $\boldsymbol{c}(\overline{x}_0, \overline{y}_0)$ has Ξ-projection not of full rank over $\mathbb{F}$. It must be shown that the matrices $\boldsymbol{M}_i$ do not have the stated nonsingularity property.

Case i)—$\beta(\overline{x}_0) \neq 0$: If there are rows of $\boldsymbol{c}$ that are multiples of two, the failure of the $\boldsymbol{M}_i$ to satisfy the nonsingularity property is easily seen. In this case, there is some row ℓ of $\boldsymbol{c}$ for which

$$0 = \beta\left((\overline{x}_0\,\overline{y}_0)M_\ell\right) = \left(\beta(\overline{x}_0)\,\overline{0}\right)\Xi(\boldsymbol{M}_\ell).$$

Hence, $\Xi(\boldsymbol{M}_\ell)$ is singular, establishing the desired result.

Therefore, we assume that $\boldsymbol{c}$ has no rows that are multiples of two, so that $\Xi(\boldsymbol{c}) = \beta(\boldsymbol{c})$. Then there exist $a_1, a_2, \cdots, a_L \in \mathbb{F}$, not all zero, such that

$$\begin{aligned} 0 &= a_1\beta\left(\overline{x}_0\boldsymbol{M}_1\right) \oplus a_2\beta\left(\overline{x}_0\boldsymbol{M}_2\right) \oplus \cdots \oplus a_L\beta\left(\overline{x}_0\boldsymbol{M}_L\right) \\ &= \beta(\overline{x}_0)\left(a_1\Xi(\boldsymbol{M}_1) \oplus a_2\Xi(\boldsymbol{M}_2) \oplus \cdots \oplus a_L\Xi(\boldsymbol{M}_L)\right). \end{aligned}$$

Since $\beta(\overline{x}_0) \neq 0$, we have

$$a_1\Xi(\boldsymbol{M}_1) \oplus a_2\Xi(\boldsymbol{M}_2) \oplus \cdots \oplus a_L\Xi(\boldsymbol{M}_L)$$

is singular, as was to be shown.

Case ii)—$\beta(\overline{x}_0) = 0$: In this case, all of the rows of $\boldsymbol{c}$ are multiples of two. Letting

$$\boldsymbol{c} = \begin{bmatrix} (2\overline{x}_0'\,\overline{y}_0)\boldsymbol{M}_1 \\ (2\overline{x}_0'\,\overline{y}_0)\boldsymbol{M}_2 \\ \vdots \\ (2\overline{x}_0'\,\overline{y}_0)\boldsymbol{M}_L \end{bmatrix}$$

where $\overline{x}_0' \in \mathbb{F}^\ell$, we have

$$\Xi(\boldsymbol{c}) = \begin{bmatrix} (\overline{x}_0'\,\overline{y}_0)\Xi(\boldsymbol{M}_1) \\ (\overline{x}_0'\,\overline{y}_0)\Xi(\boldsymbol{M}_2) \\ \vdots \\ (\overline{x}_0'\,\overline{y}_0)\Xi(\boldsymbol{M}_L) \end{bmatrix}.$$

By hypothesis, there exist $a_1, a_2, \cdots, a_L \in \mathbb{F}$, not all zero, such that

$$\begin{aligned} &a_1 \cdot (\overline{x}_0'\,\overline{y}_0)\Xi(\boldsymbol{M}_1) \oplus a_2 \cdot (\overline{x}_0'\,\overline{y}_0)\Xi(\boldsymbol{M}_2) \\ &\quad \oplus \cdots \oplus a_L \cdot (\overline{x}_0'\,\overline{y}_0)\Xi(\boldsymbol{M}_L) = 0. \end{aligned}$$

Then

$$a_1\Xi(\boldsymbol{M}_1) \oplus a_2\Xi(\boldsymbol{M}_2) \oplus \cdots \oplus a_L\Xi(\boldsymbol{M}_L)$$

is singular as was to be shown. □

In summary, the stacking of $\mathbb{Z}_4$-valued matrices will produce a QPSK-modulated space–time code achieving full spatial diversity if the stacking of their Ξ-projections produces a BPSK-modulated space–time code achieving full diversity. Thus the binary constructions lift in a natural way. Analogs of the transmit delay diversity construction, rate $1/L$ convolutional code construction, $|\mathcal{A}|\mathcal{A} \oplus \mathcal{B}|$ construction, and multistacking construction all follow as immediate consequences of the QPSK stacking construction and the corresponding results for BPSK-modulated space–time codes.

Theorem 27: Let $\mathcal{C}$ be the $\mathbb{Z}_4$-valued, $L \times (n + L - 1)$ space–time code produced by applying the stacking construction to the matrices

$$\begin{aligned} \boldsymbol{M}_1 &= [\boldsymbol{G} \quad \mathbf{0}_{k\times(L-1)}] \\ \boldsymbol{M}_2 &= [\mathbf{0}_{k\times 1} \quad \boldsymbol{G} \quad \mathbf{0}_{k\times(L-2)}], \cdots \\ \boldsymbol{M}_L &= [\mathbf{0}_{k\times(L-1)} \quad \boldsymbol{G}] \end{aligned}$$

where $\mathbf{0}_{i\times j}$ denotes the all-zero matrix consisting of i rows and j columns and $\boldsymbol{G}$ is the generator matrix of a linear $\mathbb{Z}_4$-valued code of length n. If $\Xi(\boldsymbol{G})$ is of full rank over $\mathbb{F}$, then the QPSK-modulated code $\mathcal{C}$ achieves full spatial diversity L.

Theorem 28: The natural space–time code $\mathcal{C}$ associated with the rate $1/L$ convolutional code C over $\mathbb{Z}_4$ achieves full spatial diversity L for QPSK transmission if the transfer function matrix $\boldsymbol{G}(D)$ of C has Ξ-projection of full rank L as a matrix of coefficients over $\mathbb{F}$.

Theorem 29: The $\mathbb{Z}_4$-valued space–time codes $\mathcal{C}_1 = |\mathcal{A}|\mathcal{B}|$ and $\mathcal{C}_2 = |\mathcal{A}|\mathcal{A} \oplus \mathcal{B}|$ satisfy the QPSK binary rank criterion if and only if the $\mathbb{Z}_4$-valued space–time codes $\mathcal{A}$ and $\mathcal{B}$ do.

Theorem 30: Let $\mathcal{C}$ be the $L \times n$ space–time code of size $M = 2^{u+m}$ consisting of the codeword matrices

$$\boldsymbol{c} = \begin{bmatrix} (\overline{x}\,\overline{y})\boldsymbol{M}_1 \\ (\overline{x}\,\overline{y})\boldsymbol{M}_2 \\ \vdots \\ (\overline{x}\,\overline{y})\boldsymbol{M}_L \end{bmatrix}$$

where $\overline{x} \in \mathbb{Z}_4^u$, $\overline{y} \in \mathbb{F}^{m-u}$, and the $\mathbb{Z}_4$-valued $\boldsymbol{M}_1, \boldsymbol{M}_2, \cdots, \boldsymbol{M}_L$ of standard row type $1^u 2^{m-u}$ satisfy the stacking construction constraints for QPSK-modulated codes. For $i = 1, 2, \cdots, m$, let $(\boldsymbol{M}_{1i}, \boldsymbol{M}_{2i}, \cdots, \boldsymbol{M}_{\ell i})$ be an ℓ-tuple of distinct matrices from the set $\{\boldsymbol{M}_1, \boldsymbol{M}_2, \cdots, \boldsymbol{M}_L\}$. Then, the space–time code $\mathcal{C}_{\ell,m}$ consisting of the codewords

$$\boldsymbol{c} = \begin{bmatrix} (\overline{x}_1\,\overline{y}_1)\boldsymbol{M}_{11} & (\overline{x}_2\,\overline{y}_2)\boldsymbol{M}_{12} & \cdots & (\overline{x}_m\,\overline{y}_m)\boldsymbol{M}_{1m} \\ (\overline{x}_1\,\overline{y}_1)\boldsymbol{M}_{21} & (\overline{x}_2\,\overline{y}_2)\boldsymbol{M}_{22} & \cdots & (\overline{x}_m\,\overline{y}_m)\boldsymbol{M}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ (\overline{x}_1\,\overline{y}_1)\boldsymbol{M}_{\ell 1} & (\overline{x}_2\,\overline{y}_2)\boldsymbol{M}_{\ell 2} & \cdots & (\overline{x}_m\,\overline{y}_m)\boldsymbol{M}_{\ell m} \end{bmatrix}$$

is an $\ell \times mn$ space–time code of size M^m that achieves full diversity ℓ. Setting $(\overline{x}_1\,\overline{y}_1) = (\overline{x}_2\,\overline{y}_2) = \cdots = (\overline{x}_m\,\overline{y}_m) = (\overline{x}\,\overline{y})$ produces an $\ell \times mn$ space–time code of size M that achieves full diversity.

As a consequence of these results, for example, one sees that the binary connection polynomials of Table I can be used to generate linear, $\mathbb{Z}_4$-valued, rate $1/L$ convolutional codes whose natural space–time formatting achieves full spatial diversity L. More generally, one may use any set of $\mathbb{Z}_4$-valued connection polynomials whose modulo 2 projections appear in the table.

The transformation theorem also extends to QPSK-modulated space–time codes in a straightforward manner.

Theorem 31: Let $\mathcal{C}$ be a $\mathbb{Z}_4$-valued, $L \times m$ space–time code satisfying the QPSK binary rank criterion with respect to Ξ-indicants, and let $\boldsymbol{M}$ be an $m \times n$ $\mathbb{Z}_4$-valued matrix whose binary projection $\beta(\boldsymbol{M})$ is nonsingular over $\mathbb{F}$. Consider the $L \times n$ space–time code $\boldsymbol{M}(\mathcal{C})$ consisting of all codeword matrices

$$\boldsymbol{M}(\boldsymbol{c}) = \begin{bmatrix} \overline{c}_1\boldsymbol{M} \\ \overline{c}_2\boldsymbol{M} \\ \vdots \\ \overline{c}_L\boldsymbol{M} \end{bmatrix}$$

where $\boldsymbol{c} = [\overline{c}_1 \; \overline{c}_2 \; \cdots \; \overline{c}_L]^T \in \mathcal{C}$. Then, $\boldsymbol{M}(\mathcal{C})$ satisfies the QPSK binary rank criterion and thus, for QPSK transmission, achieves full spatial diversity L.

Proof: Let $\boldsymbol{c}$, $\boldsymbol{c}'$ be distinct codewords in $\mathcal{C}$, and let $\Delta\boldsymbol{c} = \boldsymbol{c} \ominus_4 \boldsymbol{c}'$. Without loss of generality, we may assume that $\Delta\boldsymbol{c}$ is of standard row type $1^\ell 2^{L-\ell}$. Since $\beta(\boldsymbol{M})$ is nonsingular, we have $\beta(\overline{x})\beta(\boldsymbol{M}) = 0$ if and only if $\beta(\overline{x}) = 0$. Hence, $\boldsymbol{M}(\Delta\boldsymbol{c})$ is also of standard row type $1^\ell 2^{L-\ell}$. It is to be shown that $\Xi(\boldsymbol{M}(\Delta\boldsymbol{c}))$ is of rank L.

Note that

$$\Xi(\boldsymbol{M}(\Delta\boldsymbol{c})) = \begin{bmatrix} \beta(\Delta\overline{c}'_1)\beta(\boldsymbol{M}) \\ \vdots \\ \beta(\Delta\overline{c}'_\ell)\beta(\boldsymbol{M}) \\ \beta(\Delta\overline{c}'_{\ell+1})\beta(\boldsymbol{M}) \\ \vdots \\ \beta(\Delta\overline{c}'_L)\beta(\boldsymbol{M}) \end{bmatrix}$$

where $\Delta\overline{c}_i = 2\Delta\overline{c}'_i$ for $i > \ell$. Suppose there are coefficients $a_1, a_2, \cdots, a_L \in \mathbb{F}$ such that

$$\begin{aligned} 0 &= a_1 \cdot \beta(\Delta\overline{c}_1)\beta(\boldsymbol{M}) \oplus \cdots \oplus a_\ell \cdot \beta(\Delta\overline{c}_\ell)\beta(\boldsymbol{M}) \\ &\quad \oplus a_{\ell+1} \cdot \beta(\Delta\overline{c}'_{\ell+1})\beta(\boldsymbol{M}) \oplus \cdots \oplus a_L \cdot \beta(\Delta\overline{c}'_L)\beta(\boldsymbol{M}) \\ &= [a_1\beta(\Delta\overline{c}_1) \oplus \cdots \oplus a_\ell\beta(\Delta\overline{c}_\ell) \oplus a_{\ell+1}\beta(\Delta\overline{c}'_{\ell+1}) \\ &\quad \oplus \cdots \oplus a_L\beta(\Delta\overline{c}'_L)]\beta(\boldsymbol{M}). \end{aligned}$$

Then, since $\beta(\boldsymbol{M})$ is nonsingular, we have

$$a_1\beta(\Delta\overline{c}_1) \oplus \cdots \oplus a_\ell\beta(\Delta\overline{c}_\ell) \oplus a_{\ell+1}\beta(\Delta\overline{c}'_{\ell+1}) \oplus \cdots \oplus a_L\beta(\Delta\overline{c}'_L) = 0.$$

But, by hypothesis, $\Xi(\Delta\boldsymbol{c})$ is of full rank. Hence, $a_1 = a_2 = \cdots = a_L = 0$, and therefore $\Xi(\boldsymbol{M}(\Delta\boldsymbol{c}))$ is also of full rank L as required. □

As in the binary case, the transformation theorem implies that certain concatenated coding schemes preserve the full spatial diversity of a space–time code. Finally, we note that the results in Section IV-D regarding the special cases of $L = 2$ and 3 for BPSK codes also lift to full diversity space–time codes for QPSK modulation.

B. Dyadic Construction

Two BPSK space–time codes can be directly combined as in a dyadic expansion to produce a $\mathbb{Z}_4$-valued space–time code for QPSK modulation. If the component codes satisfy the BPSK binary rank criterion, the composite code will satisfy the QPSK binary rank criterion. Such codes are also of interest because they admit low-complexity multistage decoders based on the underlying binary codes.

Theorem 32: Let $\mathcal{A}$ and $\mathcal{B}$ be binary $L \times n$ space–time codes satisfying the BPSK binary rank criterion. Then the $\mathbb{Z}_4$-valued space–time code $\mathcal{C} = \mathcal{A} + 2\mathcal{B}$ is an $L \times n$ space–time code that satisfies the QPSK binary rank criterion and thus, for QPSK modulation, achieves full spatial diversity L.

Proof: Let $\boldsymbol{z}_1 = \boldsymbol{a}_1 + 2\boldsymbol{b}_1$ and $\boldsymbol{z}_2 = \boldsymbol{a}_2 + 2\boldsymbol{b}_2$ be codewords in $\mathcal{C}$, with $\boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathcal{A}$ and $\boldsymbol{b}_1, \boldsymbol{b}_2 \in \mathcal{B}$. Then the $\mathbb{Z}_4$ difference between the two codewords is

$$\Delta\boldsymbol{z} = \Delta\boldsymbol{a} + 2\nabla\boldsymbol{a} + 2\Delta\boldsymbol{b}$$

where $\Delta\boldsymbol{a} = \boldsymbol{a}_1 \oplus \boldsymbol{a}_2$, $\Delta\boldsymbol{b} = \boldsymbol{b}_1 \oplus \boldsymbol{b}_2$, and $\nabla\boldsymbol{a} = (\mathbf{1} \oplus \boldsymbol{a}_1) \odot \boldsymbol{a}_2$. In the latter expression, $\mathbf{1}$ denotes the all-one matrix and $\odot$ denotes componentwise multiplication. The modulo 2 projection is $\beta(\Delta\boldsymbol{z}) = \Delta\boldsymbol{a}$, which is nonsingular and equal to $\Xi(\Delta\boldsymbol{z})$ unless $\Delta\boldsymbol{a} = 0$. In the latter case, $\nabla\boldsymbol{a} = \mathbf{0}$, so that $\Delta\boldsymbol{z} = 2\Delta\boldsymbol{b}$. Then $\Xi(\Delta\boldsymbol{z}) = \Delta\boldsymbol{b}$, which is nonsingular unless $\Delta\boldsymbol{b} = 0$. □

From the proof, it is clear that, in the dyadic construction, one can also take the constituent codes $\mathcal{A}$ and $\mathcal{B}$ to be $\mathbb{Z}_4$-valued codes whose binary projections $\beta(\mathcal{A})$ and $\beta(\mathcal{B})$ are full-diversity BPSK space-time codes.

C. Mapping Codes to Space–Time Codes

Let C be a linear code of length n over $\mathbb{Z}_4$. For any codeword $\overline{c}$, let $w_i(\overline{c})$ denote the number of times the symbol $i \in \mathbb{Z}_4$ appears in $\overline{c}$. Furthermore, let $w_i(C)$ denote the maximum number of times the symbol $i \in \mathbb{Z}_4$ appears in any nonzero codeword of C.

The spatial diversity achievable by a space–time code $\mathcal{C}$ is at most the spatial diversity achievable by any of its subcodes. For a linear $\mathbb{Z}_4$-valued code C, the code $2C$ is a subcode whose minimum and maximum Hamming weights among nonzero codewords are given by

$$d_{\min}(2C) = \min_{\overline{c} \in C} \{w_1(\overline{c}) + w_{-1}(\overline{c})\}$$
$$d_{\max}(2C) = \max_{\overline{c} \in C} \{w_1(\overline{c}) + w_{-1}(\overline{c})\}.$$

Thus we have the following result.

Proposition 34: Let C be a linear code of length n over $\mathbb{Z}_4$. Then, for any QPSK transmission format, the corresponding space–time code $\mathcal{C}$ achieves spatial diversity at most

$$L \le \min\{d_{\min}(2C),\, n - d_{\max}(2C) + 1\}.$$

VI. Analysis of Existing Space–Time Codes

A. TSC Space–Time Trellis Codes

In [13], Tarokh, Seshadri, and Calderbank (TSC) provide a detailed investigation of the baseband rank and product distance criteria for a variety of channel conditions and present a small number of handcrafted codes for low levels of spatial diversity to illustrate the utility of these space–time coding ideas. They do not present any general space–time code designs or design rules of wide applicability.

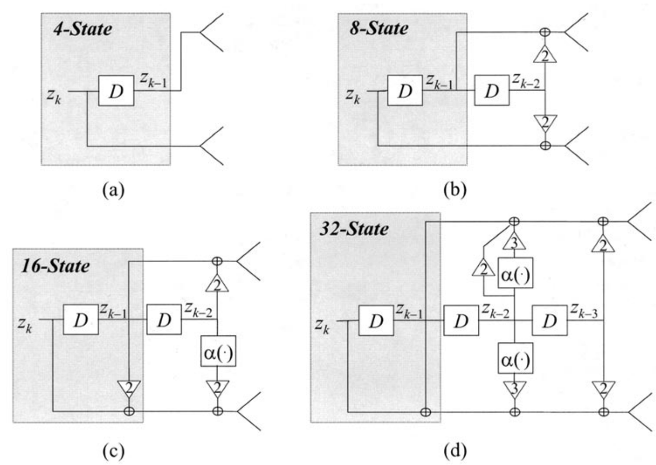

Fig. 3. $\mathbb{Z}_4$ encoders for TSC space–time trellis codes for QPSK modulation.

For $L = 2$ transmit antennas, they present four handcrafted $\mathbb{Z}_4$ space–time trellis codes, containing 4, 8, 16, and 32 states, respectively, that achieve full spatial diversity. They note that the 4-state code satisfies their simple design rules regarding diverging and merging trellis branches, and, therefore, is of full rank, but that the other codes do not and require a "trickier" bit of analysis exploiting geometric uniformity in order to confirm that full spatial diversity is achieved. The binary rank criterion for QPSK-modulated space–time codes, however, allows this determination to be done in a straightforward manner. In fact, the binary analysis shows that all of the handcrafted codes employ a simple common device to ensure that full spatial diversity is achieved.

Convolutional encoder block diagrams for the $\mathbb{Z}_4$ codes of [13] are shown in Fig. 3. The 4-state and 8-state codes are both linear over $\mathbb{Z}_4$, with transfer function matrices $\boldsymbol{G}_4(D) = [D \;\; 1]$ and $\boldsymbol{G}_8(D) = [D + 2D^2 \;\; 1 + 2D^2]$, respectively. By inspection, both satisfy the QPSK binary rank criterion and, therefore, achieve $L = 2$ spatial diversity.

The 16-state and 32-state codes are nonlinear over $\mathbb{Z}_4$. In this case, the binary rank criterion must be applied to all differences between codewords. For the 16-state code, the codeword matrices are of the following form:

$$\boldsymbol{c} = \begin{bmatrix} x_1(D) \\ x_2(D) \end{bmatrix} = \begin{bmatrix} D + 2D^2 & 0 \\ 1 + 2D & 2D^2 \end{bmatrix} \begin{bmatrix} z(D) \\ \alpha(z(D)) \end{bmatrix}.$$

For the 32-state code, the codewords are given by

$$\boldsymbol{c} = \begin{bmatrix} x_1(D) \\ x_2(D) \end{bmatrix} = \begin{bmatrix} D + 2D^2 + 2D^3 & 3D^2 \\ 1 + D + 2D^3 & 3D^2 \end{bmatrix} \begin{bmatrix} z(D) \\ \alpha(z(D)) \end{bmatrix}.$$

Due to the initial delay structure (enclosed by dashed box in Fig. 3) that is common to all four code designs, the first unit input $z_t = \pm 1$—or first nonzero input $z_t = 2$ if $z(D)$ consists only of multiples of 2—results in two consecutive columns that are multiples of $[0 \;\; 1]^T$ and $[1 \;\; x]^T$, where $x \in \mathbb{Z}_4$ is arbitrary. The only exception occurs in the case of the 32-state code when the first ± 1 is immediately preceded by a 2. In this case, the last nonzero entry results in a column that is a multiple of $[1 \;\; \pm 1]^T$.

Hence, the Ψ-projection of the codeword differences is always of full rank. By the QPSK binary rank criterion, all four codes achieve full $L = 2$ spatial diversity.

For $L = 4$ transmit antennas, Tarokh *et al.* exhibit the full-diversity space–time code corresponding to the linear $\mathbb{Z}_4$-valued convolutional code with transfer function $\boldsymbol{G}(D) = [1 \quad D \quad D^2 \quad D^3]$, a simple form of repetition delay diversity. As noted in Theorem 13 and also in [8], this design readily generalizes to spatial diversity levels $L > 4$. The stacking and related constructions presented in this paper, however, provide more general full-diversity space–time codes for $L \geq 2$.

B. GFK Space–Time Trellis Codes

In [8], Grimm, Fitz, and Krogmeier (GFK) proved that, for all $L \geq 2$, trellis-coded delay diversity schemes achieve full spatial diversity with the fewest possible number of trellis states. As a generalization of the TSC simple design rules for $L = 2$ diversity, they also introduced the notion of zeros symmetry to guarantee full spatial diversity for $L \geq 2$.

The results of a computer search undertaken to identify space–time trellis codes of full diversity and good coding advantage are also presented in [8]. Their table of best known codes for BPSK modulation covers the cases of $L = 2, 3,$ and 5 antennas. For QPSK codes, their table covers only $L = 2$. The 4-state and 8-state QPSK codes provide 1.5- and 0.62-dB additional coding advantage, respectively, compared to the corresponding TSC trellis codes.

All of the BPSK codes satisfy the zeros symmetry criterion. Since zeros symmetry for BPSK codes is a very special case of satisfying the binary rank criterion, all of the BPSK codes are special cases of our more general stacking construction.

The QPSK space–time codes are different. Some of the QPSK codes satisfy the zeros symmetry criterion; some do not. Except for the trivial delay diversity code (constraint length $\nu = 2$ with zeros symmetry), all of them are nonlinear codes over $\mathbb{Z}_4$ that do not fall under any of our general constructions. With one exception, they all satisfy the QPSK binary rank criterion. We consider these interesting codes individually.

The QPSK code of constraint length $\nu = 2$ without zeros symmetry consists of the codewords $\boldsymbol{c}$ satisfying

$$\boldsymbol{c}(D)^T = (a(D)\, b(D)) \begin{bmatrix} 1 & 2D \\ 2D & 1+2D \end{bmatrix}$$

where $a(D)$ and $b(D)$ are binary information sequences and for simplicity $+$ is used instead of $\oplus_4$ to denote modulo 4 addition. The $\mathbb{Z}_4$-difference between two codewords $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$, corresponding to input sequences $(a_1(D)\, b_1(D))$ and $(a_2(D)\, b_2(D))$, is given by

$$\Delta\boldsymbol{c} = \begin{bmatrix} \Delta a(D) + 2\nabla a(D) + 2D\Delta b(D) \\ 2D\Delta a(D) + \Delta b(D) + 2\nabla b(D) + 2D\Delta b(D) \end{bmatrix}$$

where

$$\Delta a(D) = a_1(D) \oplus a_2(D)$$
$$\nabla a(D) = (1 \oplus a_1(D)) \odot a_2(D)$$

and so forth. Here $\odot$ denotes componentwise multiplication (coefficient by coefficient).

Note that the $\mathbb{Z}_4$-difference $\Delta\boldsymbol{c}$ is not a function of the binary differences $\Delta a(D)$ and $\Delta b(D)$ alone but depends on the individual input sequences $a(D)$ and $b(D)$ through the terms $\nabla a(D)$ and $\nabla b(D)$. For reference, we note that if

$$\Delta a(D) = a_0 + a_1 D + a_2 D^2 + \cdots + a_N D^N \qquad (a_i \in \mathbb{F})$$

then

$$\Delta a(D) + 2\nabla a(D) = \pm a_0 \pm a_1 D \pm a_2 D^2 \pm \cdots \pm a_N D^N$$

for some suitable choice of sign at each coefficient.

Projecting the codeword difference $\Delta\boldsymbol{c}$ modulo 2 gives

$$\beta(\Delta\boldsymbol{c}) = \begin{bmatrix} \Delta a(D) \\ \Delta b(D) \end{bmatrix}$$

which is nonsingular unless either: i) $\Delta a(D) = 0$, $\Delta b(D) \neq 0$; ii) $\Delta b(D) = 0$, $\Delta a(D) \neq 0$; or iii) $\Delta a(D) = \Delta b(D) \neq 0$. For Case i), one finds that

$$\Xi(\Delta\boldsymbol{c}) = \Delta b(D) \begin{bmatrix} D \\ 1 \end{bmatrix}$$

which is nonsingular. For Case ii), we have

$$\Xi(\Delta\boldsymbol{c}) = \Delta a(D) \begin{bmatrix} 1 \\ D \end{bmatrix}$$

which is also nonsingular. Finally, in Case iii), we have

$$\Delta\boldsymbol{c} = \begin{bmatrix} \Delta a(D) + 2\nabla a(D) + 2D\Delta a(D) \\ \Delta a(D) + 2\nabla a(D) \end{bmatrix}. \qquad (11)$$

Thus the tth column of $\Delta\boldsymbol{c}$ is given by

$$\overline{h}_t = \begin{bmatrix} \Delta a_t + 2\nabla a_t + 2\Delta a_{t-1} \\ \Delta a_t + 2\nabla a_t \end{bmatrix}.$$

Consider the first k for which $\Delta a_k = 1$ and $\Delta a_{k+1} = 0$ (guaranteed to exist since the trellis is terminated). Then the kth and $(k+1)$th columns of $\Delta\boldsymbol{c}$ are $\overline{h}_k = [\pm 1 \quad \pm 1]^T$ and $\overline{h}_{k+1} = [2 \ 0]^T$, respectively. Thus $\Psi(\Delta\boldsymbol{c})$ is nonsingular, and the QPSK binary rank criterion is satisfied. Note that it is the extra delay term in the upper expression of (11) that serves to guarantee full spatial diversity.

The QPSK code of constraint length $\nu = 3$ with zeros symmetry consists of the codewords $\boldsymbol{c}$ satisfying

$$\boldsymbol{c}(D)^T = (a(D)\, b(D)) \begin{bmatrix} 1+3D & D+D^2 \\ 2 & 2D \end{bmatrix}.$$

For this code, the binary rank analysis is even simpler. The projection modulo 2 of the $\mathbb{Z}_4$ difference $\Delta\boldsymbol{c}$ between two codewords is given by

$$\beta\left(\Delta\boldsymbol{c}(D)\right) = \Delta a(D) \begin{bmatrix} 1 \oplus D \\ D \oplus D^2 \end{bmatrix}$$

which is nonsingular unless $\Delta a(D) = 0$ and $\Delta b(D) \neq 0$. In the latter case

$$\Delta \boldsymbol{c} = \Delta b(D) \begin{bmatrix} 2 \\ 2D \end{bmatrix}$$

whose Ξ- and Ψ-indicants are nonsingular.

The QPSK code of constraint length $\nu = 3$ without zeros symmetry, consisting of the codewords

$$\boldsymbol{c}(D)^T = (a(D)\, b(D)) \begin{bmatrix} 1+2D^2 & D+2D^2 \\ 1+2D & 2 \end{bmatrix}$$

does not satisfy the QPSK binary rank criterion. When $\Delta a(D) = 0$ but $\Delta b(D) \neq 0$, the codeword difference is

$$\Delta \boldsymbol{c} = \begin{bmatrix} \Delta b(D) + 2\nabla b(D) + 2D\Delta b(D) \\ 2\Delta b(D) \end{bmatrix}$$

for which $\Xi(\Delta \boldsymbol{c})$ and $\Psi(\Delta \boldsymbol{c})$ are both singular. The latter can be easily seen from the fact that the second row of $\Delta \boldsymbol{c}$ is two times the first row.

C. BBH Space–Time Trellis Codes

In [2], Baro, Bauch, and Hansmann (BHH) report on a recent computer search, similar to the earlier one of Grimm *et al.*, for $L = 2$ QPSK trellis codes with 4, 8, and 16 states. Their results agree with Grimm *et al.*regarding the optimal product distances; but, interestingly, the codes they give have different generators. This indicates that, at least for $L = 2$ spatial diversity, there is a multiplicity of optimal codes.

All of the BBH codes are nonlinear over $\mathbb{Z}_4$. The 4-state and 16-state codes consist of the following codeword matrices:

4-state:

$$\boldsymbol{c}(D)^T = (a(D)\, b(D)) \begin{bmatrix} 2+D & 2 \\ 3D & 2+D \end{bmatrix}$$

16-state:

$$\boldsymbol{c}(D)^T = (a(D)\, b(D)) \begin{bmatrix} 1+2D & 2+D+2D^2 \\ 2+2D^2 & 2D \end{bmatrix}.$$

The analysis showing that these two codes satisfy the QPSK binary rank criterion is straightforward and similar to that given for the GFK codes.

The 8-state BBH code consists of the codeword matrices

$$\boldsymbol{c}(D)^T = (a(t)\, b(t)) \begin{bmatrix} D & 1 \\ 2+2D+2D^2 & 2+2D^2 \end{bmatrix}$$

which expression can be rearranged to give

$$\boldsymbol{c}(D) = a(D) \begin{bmatrix} D \\ 1 \end{bmatrix} + 2b(D) \begin{bmatrix} 1+D+D^2 \\ 1+D^2 \end{bmatrix}.$$

Whereas the GFK 8-state code does not satisfy the QPSK binary rank criterion, the BBH 8-state code does and is in fact an example of our dyadic construction $\mathcal{C} = \mathcal{A} + 2\mathcal{B}$. By inspection, the two component space–time codes $\mathcal{A}$ and $\mathcal{B}$, with transfer functions

$$\boldsymbol{G}_{\mathcal{A}}(D) = \begin{bmatrix} D \\ 1 \end{bmatrix} \qquad \boldsymbol{G}_{\mathcal{B}}(D) = \begin{bmatrix} 1+D+D^2 \\ 1+D^2 \end{bmatrix}$$

respectively, both have β-projections satisfying the BPSK binary rank criterion.

These results, together with those of Grimm *et al.*show that the class of space–time codes satisfying the binary rank criteria is indeed rich and includes, for every case searched thus far, optimal codes with respect to coding advantage.

D. Space–Time Block Codes from Orthogonal Designs

The orthogonal designs of Alamouti [1] and Tarokh *et al.*[14], [15] can give rise to nonlinear space–time codes of very short block length provided the PSK modulation format is chosen so that the constellation is closed under complex conjugation. These derived codes are interesting in the context of this paper because they settle the question as to whether or not all full diversity BPSK space–time codes satisfy the binary rank criteria.

Consider the original Alamouti design in which the modulated codewords are of the form

$$\begin{bmatrix} x_1 & x_2^* \\ x_2 & -x_1^* \end{bmatrix}$$

where x_1, x_2 are BPSK constellation points. Assuming the on-axis BPSK constellation, the corresponding space–time block code $\mathcal{C}$ consists of all binary matrices of the form

$$\boldsymbol{c} = \begin{bmatrix} a & b \\ b & 1 \oplus a \end{bmatrix}.$$

As pointed out by Calderbank [5], this simple code provides $L = 2$ diversity gain but no coding gain. The difference between two modulated codewords has determinant

$$\det \begin{vmatrix} (-1)^{a_1} - (-1)^{a_2} & (-1)^{b_1} - (-1)^{b_2} \\ (-1)^{b_1} - (-1)^{b_2} & -[(-1)^{a_1} - (-1)^{a_2}] \end{vmatrix} = -\left([(-1)^{a_1} - (-1)^{a_2}]^2 + [(-1)^{b_1} - (-1)^{b_2}]^2\right)$$

which is zero if and only if the two codewords are identical ($a_1 = a_2$ and $b_1 = b_2$). On the other hand, the corresponding binary difference of the unmodulated codewords is given by

$$\Delta \boldsymbol{c} = \begin{bmatrix} a_1 \oplus a_2 & b_1 \oplus b_2 \\ b_1 \oplus b_2 & a_1 \oplus a_2 \end{bmatrix}.$$

But, if $a_1 \oplus a_2 = b_1 \oplus b_2 = 1$, for example, the difference is

$$\Delta \boldsymbol{c} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

a matrix that is singular over $\mathbb{F}$. Hence, $\mathcal{C}$ achieves full spatial diversity but does not satisfy the BPSK binary rank criterion.

VII. Extensions to Non-Quasi–Static Fading Channels

For the fast fading channel [13], the baseband model differs from (1) in that the complex path gains now vary independently from symbol to symbol

$$y_t^j = \sum_{i=1}^{n} \alpha_{ij}(t) s_t^i \sqrt{E_s} + n_t^j. \tag{12}$$

Let codeword $\boldsymbol{c}$ be transmitted. In this case, the pairwise error probability that the decoder will prefer the alternate codeword $\boldsymbol{e}$ to $\boldsymbol{c}$ can be upper-bounded by

$$P(\boldsymbol{c} \to \boldsymbol{e}) \leq \left(\frac{1}{\prod_{t=1}^{n} (1 + |\, f(\overline{c}_t) - f(\overline{e}_t)\,|^2 E_s/4N_0)} \right)^{L_r} \leq \left(\frac{\mu E_s}{4N_0} \right)^{-dL_r},$$

where $\overline{c}_t$ is the tth column of $\boldsymbol{c}$, $\overline{e}_t$ is the tth column of $\boldsymbol{e}$, d is the number of columns $\overline{c}_t$ that are different from $\overline{e}_t$, and

$$\mu = \left(\prod_{\overline{c}_t \neq \overline{e}_t} |f(\overline{c}_t) - f(\overline{e}_t)|^2 \right)^{1/d}.$$

The diversity advantage is now dL_r, and the coding advantage is μ.

Thus the fundamental design criteria [13] for space–time codes over fast fading channels are the following:

- *Distance Criterion*: Maximize the number of column differences $d = |\{t : \overline{c}_t \neq \overline{e}_t\}|$ over all pairs of distinct codewords $\boldsymbol{c}$, $\boldsymbol{e} \in \mathcal{C}$.
- *Product Criterion*: Maximize the coding advantage

$$\mu = \left(\prod_{\overline{c}_t \neq \overline{e}_t} |f(\overline{c}_t) - f(\overline{e}_t)|^2 \right)^{1/d}.$$

over all pairs of distinct codewords $\boldsymbol{c}$, $\boldsymbol{e} \in \mathcal{C}$.

Since real fading channels will be neither quasi-static nor fast-fading but something in between, Tarokh *et al.* [13] suggest designing space–time codes based on a combination of the quasi-static and fast-fading design criteria. They refer to space–time codes designed according to the hybrid criteria as "smart greedy codes," meaning that the codes seek to exploit both spatial and temporal diversity whenever available.

A handcrafted example of a two-state smart–greedy space–time trellis code for $L = 2$ antennas and BPSK modulation was given. This code is a special case of our multistacking construction applied to the two binary rate $1/2$ convolutional codes having respective transfer function matrices

$$\boldsymbol{G}_1(D) = \begin{bmatrix} 1 \oplus D \\ D \end{bmatrix} \qquad \boldsymbol{G}_2(D) = \begin{bmatrix} 1 \\ 1 \oplus D \end{bmatrix}.$$

The M-TCM example of [13] can also be analyzed using the binary rank criteria. The other smart–greedy examples are based on traditional concatenated coding schemes with space–time trellis codes as inner codes.

We note that our general $|\mathcal{A}|\mathcal{B}|$, $|\mathcal{A}|\mathcal{A} \oplus \mathcal{B}|$, destacking, multistacking, and concatenated code constructions provide a large class of space–time codes that are "smart–greedy." Furthermore, the common practice in wireless communications of interleaving within codewords to randomize burst errors on such channels is a special case of the transformation theorem. Specific examples of new, more sophisticated smart–greedy codes can be easily obtained, for example, by destacking or multistacking the space–time trellis codes of Table I. These latter designs may make possible the design of space–time overlays for existing wireless communication systems whose forward error-correction schemes are based on standard convolutional codes. The extra diversity of the spatial overlay would then serve to augment the protection provided by the traditional temporal coding.

VIII. Extensions to Higher Order Constellations

Direct extension of the binary rank analysis to general $L \times n$ space–time codes over the alphabet $\mathbb{Z}_{2^r}$ for 2^r-PSK modulation with $r \geq 3$ is difficult. Special cases such as 8-PSK codes with $L = 2$, however, are tractable. Thus the 8–PSK-modulated space–time codes of [13] are covered by the binary rank criteria.

For general constellations, Tarokh *et al.* [13] note that multilevel coding techniques [4], [10] can produce powerful space–time codes for high bit rate applications while admitting a simpler multilevel decoder. Our results regarding multilevel PSK constructions will be presented separately [6]. Since at each level binary decisions are made, the binary rank criteria can be used to design space–time codes that provide guaranteed levels of diversity at each bit decision.

IX. Conclusions

We have developed general design criteria for PSK-modulated space–time codes, based on the binary rank of the unmodulated codewords, to ensure that full spatial diversity is achieved. For BPSK modulation, we have shown that the binary rank criterion provides a complete characterization of space–time codes achieving full spatial diversity when no knowledge is available regarding the distribution of $\pm$ signs among the baseband differences. For QPSK modulation, the binary rank criterion is also broadly applicable. We have shown that the binary design criteria significantly simplify the problem of designing space–time codes to achieve full spatial diversity and have furthermore shown that much of what is currently known about PSK-modulated space–time codes is covered by these design criteria. Finally, we have introduced several new construction methods that are quite general and open the door to more sophisticated and more powerful codes for both quasi-static and time-varying fading channels.

Acknowledgment

A. R. Hammons, Jr. would like to thank former colleague, M. P. Fitz, for useful discussions while at the Ohio State University, especially with regard to the material of Section II.

The authors would also like to express their appreciation to the reviewers for their help in improving the paper, thanking reviewer A for the especially careful and thoughtful critique and reviewer B for the reference to recent search results by Baro *et al.* [2].

References

[1] S. M. Alamouti, "A simple transmitter diversity scheme for wireless communications," *IEEE J. Select. Areas Commun.*, vol. 16, pp. 1451–1458, Oct. 1998.

[2] S. Baro, G. Bauch, and A. Hansmann, "Improved codes for space–time trellis-coded modulation," *IEEE Commun. Lett.*, vol. 4, pp. 20–22, Jan. 2000.

[3] E. Biglieri, J. Proakis, and S. Shimai (Shitz), "Fading channels: Information-theoretic and communication aspects," *IEEE Trans. Inform. Theory*, vol. 44, pp. 2619–2693, Oct. 1998.

[4] A. R. Calderbank, "Multilevel codes and multistage decoding," *IEEE Trans. Commun.*, vol. 37, pp. 222–229, Mar. 1989.

[5] A. R. Calderbank, "The art of signaling: Fifty years of coding theory," *IEEE Trans. Inform. Theory*, vol. 44, pp. 2561–2596, Oct. 1998.

[6] H. El Gamal and A. R. Hammons Jr., "On the application of binary rank criteria to the design of multilevel space-time codes," in preparation.

[7] G. J. Foschini and M. Gans, "On the limits of wireless communication in a fading environment when using multiple antennas," *Wireless Personal Commun.*, Mar. 1998.

[8] J. Grimm, M. P. Fitz, and J. V. Krogmeier, "Further results in space-time coding for Rayleigh fading," in *Proc. 1998 Allerton Conf.*.

[9] J.-C. Guey, M. P. Fitz, M. R. Bell, and W.-Y. Kuo, "Signal design for transmitter diversity wireless communication systems over Rayleigh fading channels," in *1996 IEEE Vehicular Technology Conf.*, Atlanta, GA, 1996, pp. 136–140.

[10] H. Imai and S. Hirakawa, "A new multilevel coding method using error-correcting codes," *IEEE Trans. Inform. Theory*, vol. IT-23, pp. 371–377, May 1977.

[11] S. Lin and D. J. Costello Jr., *Error Control Coding: Fundamentals and Applications*. Englewood Cliffs, NJ: Prentice-Hall, 1983.

[12] J. G. Proakis, *Digital Communications*, 3rd ed. New York: McGraw-Hill, 1995.

[13] V. Tarokh, N. Seshadri, and A. R. Calderbank, "Space-time codes for high data rate wireless communication: Performance criterion and code construction," *IEEE Trans. Inform. Theory*, vol. 44, pp. 744–765, Mar. 1998.

[14] V. Tarokh, H. Jafarkhani, and A. R. Calderbank, "Space-time block coding for wireless communications: Theory of generalized orthogonal designs," *IEEE Trans. Inform. Theory*, vol. 45, pp. 1456–1467, July 1999.

[15] ——, "Space-time block coding for wireless communications: Performance results," *IEEE J. Select. Areas Commun.*, vol. 17, pp. 451–460, Mar. 1999.

[16] E. Teletar, "Capacity of multi-antenna Gaussian channels," AT&T Bell Labs., Tech. Rep., June 1995.

[17] S. B. Wicker, *Error Control Systems for Digital Communication and Storage*. Englewood Cliffs, NJ: Prentice-Hall, 1995.