Linear Threaded Algebraic Space–Time Constellations

Mohamed Oussama Damen, *Member, IEEE*, Hesham El Gamal, *Senior Member, IEEE*, and Norman C. Beaulieu, *Fellow, IEEE*

***Abstract*—Space–time (ST) constellations that are linear over the field of complex numbers are considered. Relevant design criteria for these constellations are summarized and some fundamental limits to their achievable performances are established. The fundamental tradeoff between rate and diversity is investigated under different constraints on the peak power, receiver complexity, and rate scaling with the signal-to-noise ratio (SNR). A new family of constellations that achieve optimal or near-optimal performance with respect to the different criteria is presented. The proposed constellations belong to the threaded algebraic ST (TAST) signaling framework, and achieve the optimal minimum squared Euclidean distance and the optimal delay. For systems with one receive antenna, these constellations also achieve the optimal peak-to-average power ratio for quadrature amplitude modulation (QAM) and phase-shift keying (PSK) input constellations, as well as optimal coding gains in certain scenarios. The framework is general for any number of transmit and receive antennas and allows for realizing the optimal tradeoff between rate and diversity under different constraints. Simulation results demonstrate the performance gaines offered by the proposed designs in average power and peak power limited systems.**

***Index Terms*—Constant modulus, diversity-versus-rate tradeoff, maximum-likelihood (ML) decoding, multiple-input multiple-output (MIMO) channels, space–time constellations.**

I. Introduction

WIRELESS channels are characterized by complex physical layer effects resulting from multiple users sharing spectrum in a multipath fading environment. In such environments, reliable communication is sometimes possible only through the use of diversity techniques in which the receiver processes multiple replicas of the transmitted signal under varying channel conditions. Antenna diversity techniques have received considerable attention recently due to the significant gains promised by information-theoretic studies. While the use of multiple receive antennas is a well-explored problem, the design of space–time (ST) signals that exploit the available capacity in multitransmit antenna systems still faces many challenges.

Tarokh *et al.* [1] coined the name *space–time coding* for this two-dimensional signal design paradigm. Over the past five years, several ST coding schemes have been proposed in the literature. In this paper, we focus our attention on the class of ST signals which are linear over the field of complex numbers [2]. The different design criteria proposed for this class of signals will be summarized and some fundamental limits on their performances will be established. Furthermore, we shed more light on the tradeoffs involved in the design of linear ST signals. In particular, we investigate the different parameters that limit the achievable diversity advantage. We first show that one can simultaneously achieve full transmission rate, in terms of the number of symbols per channel use, and full diversity by this class of signals. We then demonstrate how the constraints on the peak power, complexity, and rate scaling with the signal-to-noise ratio (SNR) may limit the achievable diversity advantage. Motivated by the fundamental limits on the performance achievable with these signals, we choose to refer to them as *linear space–time constellations* in the sequel. This name also distinguishes this notion of linearity from the traditional coding theoretic linearity over finite fields and rings.

Recently, a new framework for constructing full diversity, full-rate, and polynomial complexity ST signals, i.e., the threaded algebraic ST (TAST) signaling framework, was proposed [3]. In this paper, we exploit this framework to construct ST constellations that achieve optimal or near-optimal values for multiple, different design criteria. In particular, the proposed constellations are optimized for *both* average power-limited and peak-power-limited systems. Moreover, they will be shown to achieve the optimal tradeoff between diversity and rate for systems with arbitrary numbers of transmit and receive antennas under different constraints.

The remainder of this paper is organized as follows. Section II presents the multiple-antenna signaling model adopted in our work. In Section III, we summarize the different design criteria proposed for linear ST constellations and establish some fundamental limits on their achievable performances. In Section IV, we establish the fundamental tradeoff between rate and diversity under different constraints. Section V presents the optimized full-rate and full diversity constellations for multiple-input single-output (MISO) systems. The extension of the proposed designs to multiple-input multiple-output (MIMO) channels is outlined in Section IV. In Section VII, we

Manuscript received August 28, 2002; revised June 4, 2003. The work of M. O. Damen and N. C. Beaulieu was supported in part by the Alberta Informatics Circle of Research Excellence (iCORE) and the Natural Science and Engineering Research Council of Canada (NSERC). The work of H. El Gamal was supported in part by the National Science Foundation under Grant ITR–0219892. The material in this paper was presented in part at the International Conference on Communications, Anchorage, AK, May 2003 and at the IEEE International Symposium on Information Theory, Yokohama, Japan, June/July 2003.

M. O. Damen and N. C. Beaulieu are with the Department of Electrical and Computer Engineering, University of Alberta, ECERF, Edmonton, AB, Canada T6G 2V4 (e-mail: damen@ee.ualberta.ca; beaulieu@ee.ualberta.ca).

H. El Gamal is with the Department of Electrical Engineering, The Ohio State University, Columbus, OH 43210-1272 (e-mail: helgamal@ee.eng.ohio-state.edu).

Communicated by G. Caire, Associate Editor for Communications.

Digital Object Identifier 10.1109/TIT.2003.817423

generalize the proposed constructions to realize the optimal diversity-versus-rate tradeoff under different constraints. Numerical results are presented in Section VIII for certain representative scenarios. Some concluding remarks are given in Section IX. Finally, all the proofs are detailed in the Appendix.

II. System Model

We consider signaling over an $M \times N$ MIMO channel. A $K \times 1$ information symbol vector $\boldsymbol{u} = (u_1, \ldots, u_K)^T \in \mathcal{U}^K$, where $\mathcal{U}$ denotes the input constellation, is mapped by a constellation encoder $\boldsymbol{\gamma}$ into an $MT \times 1$ output vector $\boldsymbol{\gamma}(\boldsymbol{u})$, with components from the alphabet $\prod_{i=1}^{MT} \mathcal{S}_i$ (i.e., $\boldsymbol{\gamma}: \mathcal{U}^K \to \prod_{i=1}^{MT} \mathcal{S}_i$). In general, we allow the transmitted single-dimensional constellation $\mathcal{S}_i$ to vary across time and space. All the proposed constellations, however, will have the symmetry property that $|\mathcal{S}_i| = |\mathcal{S}_j|$. In this work, we also assume that the information symbol vector is a random variable with a uniform distribution over $\mathcal{U}^K$. An ST formatter, "$\boldsymbol{F}$," then maps each encoded symbol vector $\boldsymbol{\gamma}(\boldsymbol{u})$ into an $M \times T$ ST constellation $\boldsymbol{C}_{\boldsymbol{u}} = \boldsymbol{F}(\boldsymbol{\gamma}(\boldsymbol{u}))$, where M symbols c_{mt} $(m = 1, \ldots, M)$ are transmitted simultaneously from the M transmit antennas at time t, $t = 1, \ldots, T$. When there is no confusion, we denote the ST constellation by $\boldsymbol{C}$. The transmission rate of the constellation $\boldsymbol{C}$ is, therefore, equal to K/T symbols per channel use (PCU). The throughput of the system, in bits PCU, is therefore given by $K/T \log_2 |\mathcal{U}|$.

The $N \times T$ received signal matrix $\boldsymbol{X}$, after matched filtering and sampling at the symbol rate, can be written as

$$\boldsymbol{X} = \sqrt{\rho}\boldsymbol{H}\boldsymbol{C}_{\boldsymbol{u}} + \boldsymbol{W} \tag{1}$$

where $\boldsymbol{H} = [h_{nm}]$ is the $N \times M$ channel matrix, and h_{nm} denotes the fading coefficient between the mth transmit and the nth receive antenna. These fading coefficients are assumed to be independent, and identically distributed (i.i.d.) zero-mean complex Gaussian random variables with unit variance per complex dimension. In the quasi-static, frequency nonselective fading model adopted in this paper, the fading coefficients are assumed to be fixed during one codeword (i.e., T time periods) and change independently from one codeword to the next. The entries of the $N \times T$ noise matrix $\boldsymbol{W}$, i.e., w_{nt}, are assumed to be independent samples of a zero-mean complex Gaussian random process with unit variance per complex dimension. We further impose the average power constraint that

$$\sum_{m=1}^{M} \sum_{t=1}^{T} E_{\boldsymbol{u}} |c_{mt}|^2 = T \tag{2}$$

where $E_{\boldsymbol{u}}$ refers to expectation with respect to the random data vector $\boldsymbol{u}$. The received SNR at every antenna is, therefore, independent of the number of transmit antennas and is equal to ρ. Moreover, we assume that the channel state information (CSI) is available *a priori* only at the receiver. Unless otherwise stated, we focus our attention on constellations $\mathcal{U} \subset \xi\mathbb{Z}[w_{\mathcal{M}}]$, where ξ is a normalization constant and

$$\mathbb{Z}[w_{\mathcal{M}}] \triangleq \left\{ \sum_{k=0}^{\phi(\mathcal{M})-1} a_k w_{\mathcal{M}}^k,\ a_k \in \mathbb{Z},\ k = 0, \ldots, \phi(\mathcal{M})-1 \right\} \tag{3}$$

is the ring of integers of the $\mathcal{M}$th cyclotomic number field $\mathbb{Q}(w_{\mathcal{M}})$, with $\mathbb{Z}$ the ring of integer numbers, $w_{\mathcal{M}} = e^{2i\pi/\mathcal{M}}$ the $\mathcal{M}$th primitive root of unity, and $\phi(\mathcal{M})$ denoting the Euler ϕ-function that measures the number of integers less than $\mathcal{M}$ and coprime with it. Without loss of generality, we will assume that ξ is adjusted to normalize the average power of $\mathcal{U}$ to one (i.e., $E[|u|^2] = 1$). With a slight abuse of notation, we will rely on the isomorphism between $\xi\mathbb{Z}[w_{\mathcal{M}}]$ and $\mathbb{Z}[w_{\mathcal{M}}]$ and refer to both rings as $\mathbb{Z}[w_{\mathcal{M}}]$ when there is no confusion. We also denote the minimum squared Euclidean distance of $\mathcal{U}$ as $d_{\mathcal{U}}^2$. Finally, we note that this set of constellations contains all the pulse amplitude modulation (PAM) constellations (i.e., $\mathcal{M} = 1, 2$), square quadrature amplitude modulation (QAM) constellations (i.e., $\mathcal{M} = 4$), constellations carved from the hexagonal lattice (i.e., $\mathcal{M} = 3$), and phase-shift keying (PSK) constellations (i.e., $\mathcal{M} \geq 5$).

By stacking all the columns of matrix $\boldsymbol{X}$ in one column, i.e., $\boldsymbol{x} \triangleq \mathrm{vec}(\boldsymbol{X})$, the received signal in (1) can be written in a vector form as

$$\boldsymbol{x} = \sqrt{\rho}\boldsymbol{\mathcal{H}}\boldsymbol{c}_{\boldsymbol{u}} + \boldsymbol{w} \tag{4}$$

where $\boldsymbol{\mathcal{H}} \triangleq \boldsymbol{I}_T \otimes \boldsymbol{H} \in \mathbb{C}^{TN \times TM}$, $\otimes$ denotes the Kronecker matrix product, $\boldsymbol{c}_{\boldsymbol{u}} \triangleq \mathrm{vec}(\boldsymbol{C})$, and $\boldsymbol{w} \triangleq \mathrm{vec}(\boldsymbol{W})$, with $\mathbb{C}$ the field of complex numbers. If the alphabet $\mathcal{U}$ belongs to a number ring $\mathcal{R}$ (e.g., $\mathcal{U} =$ 4-QAM $\subset \mathcal{R} = \mathbb{Z}[i]$), then one calls the constellation $\boldsymbol{C}$ linear over $\mathcal{R}$ if $\boldsymbol{C}_{\boldsymbol{u}'} + \boldsymbol{C}_{\boldsymbol{u}''} = \boldsymbol{C}_{\boldsymbol{u}'+\boldsymbol{u}''}$, for $\boldsymbol{u}', \boldsymbol{u}'' \in \mathcal{U}^K$. In this case, there exists a generator matrix $\boldsymbol{\mathcal{M}} \in \mathbb{C}^{TM \times K}$ such that $\boldsymbol{c}_{\boldsymbol{u}} = \boldsymbol{\mathcal{M}}\boldsymbol{u}$. Then, (4) is a linear system with $N \times T$ equations and K unknowns with the combining matrix $\boldsymbol{\mathcal{H}\mathcal{M}}$. The maximum-likelihood (ML) solution in this scenario can be implemented using the sphere decoder [4] whose average complexity is only polynomial in K for $K \leq N \times T$ and medium to large SNRs [5], [4], [6]. For $K > N \times T$, one can use the generalized sphere decoder [7] whose complexity is exponential in $K - N \times T$ and polynomial in $N \times T$.

The main reason for restricting the discussion in this paper to linear constellations is to benefit from the linear complexity ST encoder and the polynomial complexity ML decoding allowed by the linearity property when $K \leq N \times T$. The linearity of the constellation, however, implies some fundamental performance limits as detailed in the following two sections.

III. Design Criteria and Fundamental Limits

One of the fundamental challenges in the design of ST signals is the fact that the optimal design criteria depend largely on the system parameters (e.g., number of receive antennas) and quality of service constraints (e.g., maximum allowable delay). One of the advantages of the proposed TAST constellations is that they *nearly* optimize these different criteria *simultaneously*.

1) Diversity Order and Coding Gain: Under the quasi-static assumption, the Chernoff upper bound on the pairwise error probability of the ML detection of $\boldsymbol{u}''$ given that $\boldsymbol{u}' \neq \boldsymbol{u}''$ was transmitted is given by [12], [1]

$$\begin{aligned}\Pr\{\boldsymbol{u}' \longrightarrow \boldsymbol{u}''\} &\leq \left(\det\left(\boldsymbol{I}_M + \frac{\rho}{4}\boldsymbol{A}(\boldsymbol{u}', \boldsymbol{u}'')\right)\right)^{-N} \\ &= \left(\prod_{\ell=1}^{r(\boldsymbol{A}(\boldsymbol{u}', \boldsymbol{u}''))} \left(1 + \frac{\rho}{4}\lambda_\ell(\boldsymbol{u}', \boldsymbol{u}'')\right)\right)^{-N}\end{aligned} \tag{5}$$

where $\boldsymbol{I}_M$ is the $M \times M$ identity matrix

$$\boldsymbol{A}(\boldsymbol{u}', \boldsymbol{u}'') \triangleq (\boldsymbol{C}_{\boldsymbol{u}'} - \boldsymbol{C}_{\boldsymbol{u}''})(\boldsymbol{C}_{\boldsymbol{u}'} - \boldsymbol{C}_{\boldsymbol{u}''})^H$$

the superscript ^{H} denotes the conjugate transpose operator, $r(\boldsymbol{A}(\boldsymbol{u}', \boldsymbol{u}''))$ is the rank of $\boldsymbol{A}(\boldsymbol{u}', \boldsymbol{u}'')$, and $\lambda_1(\boldsymbol{u}', \boldsymbol{u}''), \ldots, \lambda_{r(\boldsymbol{A}(\boldsymbol{u}', \boldsymbol{u}''))}(\boldsymbol{u}', \boldsymbol{u}'')$ are the nonzero eigenvalues of $\boldsymbol{A}(\boldsymbol{u}', \boldsymbol{u}'')$. One can easily see that the largest power of the inverse of the SNR (i.e., diversity order) in (5) is equal to $N \times r(\boldsymbol{A}(\boldsymbol{u}', \boldsymbol{u}''))$, and the dominant term in (5) at large SNR is

$$\prod_{\ell=1}^{r(\boldsymbol{A}(\boldsymbol{u}', \boldsymbol{u}''))} \lambda_\ell(\boldsymbol{u}', \boldsymbol{u}'').$$

This observation gives rise to the well-known determinant and rank criteria [1]. Therefore, a *full diversity* ST constellation $\boldsymbol{C}$ achieves the maximum diversity order of MN. In addition, one refers to the term

$$\delta_{\boldsymbol{C}} \triangleq \min_{\boldsymbol{u}' \neq \boldsymbol{u}'' \in \mathcal{U}^K} \left(\prod_{\ell=1}^{M} \lambda_\ell(\boldsymbol{u}', \boldsymbol{u}'') \right)^{1/M} \tag{6}$$

as the coding gain of the full diversity constellation $\boldsymbol{C}$.

Using the linearity of the constellation, the average power constraint, and the geometric mean/arithmetic mean inequality, one can see that the maximum achievable coding gain for a linear ST constellation that supports L symbols PCU from an input constellation $\mathcal{U}$ is given by

$$\delta_{\boldsymbol{C}} \le \frac{d_{\mathcal{U}}^2}{ML}. \tag{7}$$

2) Squared Euclidean Distance: For small SNR and/or large numbers of receive antennas, one can see that the dominant term in (5) is the squared Euclidean distance of ST constellation $\boldsymbol{C}$ given by

$$d_{\boldsymbol{C}}^2 \triangleq \min_{\boldsymbol{u}', \boldsymbol{u}'' \in \mathcal{U}^K} \sum_{\ell=1}^{M} \lambda_\ell(\boldsymbol{u}', \boldsymbol{u}''). \tag{8}$$

Again, using the linearity and average power constraints, one can show that

$$d_{\boldsymbol{C}}^2 \le \frac{d_{\mathcal{U}}^2}{L} \tag{9}$$

for a linear ST constellation that supports L symbols PCU.

One can use (7) and (9) to extract a useful design guideline. The **nonlinear** shrinking of $d_{\mathcal{U}}^2$ with the size of the constellation implies that the upper bounds in (7) and (9) are maximized by maximizing the number of symbols PCU for a fixed throughput. The only exception to this rule is when one moves from a binary phase-shift keying (BPSK) to a quaternary phase-shift keyng (QPSK) constellation where the two choices are equivalent. This exception can be attributed to the *wasteful* nature of the BPSK constellation. The maximum value of L is, however, limited to $\min(N, M)$ to facilitate polynomial complexity ML decoding [2], [4], [13], [14]. This argument implies that the choice $L = \min(N, M)$ strikes a very favorable tradeoff between performance and complexity. Therefore, all the proposed constellations will be constructed to achieve $\min(N, M)$ symbols PCU. Moreover, we will show that the proposed constellations achieve the upper bound on the squared Euclidean distance with equality in all cases.

3) Peak-to-Average Power Ratio (PAR): The PAR of the ST constellation plays an important role in peak-power-limited systems because a high value of the PAR will shift the operating point to the nonlinear region of the power amplifier which may cause power clipping and/or distortion. Therefore, it is desirable to construct ST constellations with low PAR values. We define the baseband PAR [15] for a given constellation $\mathcal{U}$ as

$$\text{PAR}_{\mathcal{U}} \triangleq \frac{\max |u|^2}{E[|u|^2]}, \qquad u \in \mathcal{U}. \tag{10}$$

For example, the PAR for a square $\mathcal{M}$-QAM constellation equals $3\frac{\sqrt{\mathcal{M}}-1}{\sqrt{\mathcal{M}}+1}$. The PAR of the ST constellation $\boldsymbol{C}$ is given by

$$\text{PAR}_{\boldsymbol{C}} \triangleq \frac{\max |c_{mt}|^2}{E[|c_{mt}|^2]}, \qquad c_{mt} \in \mathcal{S}_i, \; i = 1, \ldots, MT \tag{11}$$

where $\mathcal{S}_i$ is the single-dimension alphabet at the output of the encoder. For symmetric ST constellations, with the same average power transmitted from all the antennas and the same PAR for all the $\mathcal{S}_i$, the average power constraint can be used to simplify (11) to

$$\text{PAR}_{\boldsymbol{C}} \triangleq M \max |c_{11}|^2, \qquad c_{11} \in \mathcal{S}_1. \tag{12}$$

The linearity of the ST encoder and the independence of the inputs imply the following lower bound on the PAR of the constellation:

$$\text{PAR}_{\boldsymbol{C}} \ge \text{PAR}_{\mathcal{U}} \tag{13}$$

where $\mathcal{U}$ is the input constellation. Guided by the single-antenna scenario, one can see that there is a fundamental tradeoff between optimizing the performance of the constellation in average-power-limited systems and minimizing the PAR. For example, it is well known that QAM constellations outperform PSK constellations in terms of average power performance while PSK constellations offer the optimum PAR. To quantify and utilize this tradeoff, we define the normalized coding gain and squared Euclidean distance, respectively, as

$$\eta_{\boldsymbol{C}} \triangleq \frac{\delta_{\boldsymbol{C}}}{\text{PAR}_{\boldsymbol{C}}} \tag{14}$$

$$\chi_{\boldsymbol{C}}^2 \triangleq \frac{d_{\boldsymbol{C}}^2}{\text{PAR}_{\boldsymbol{C}}}. \tag{15}$$

These metrics play the same role as the coding gain and squared Euclidean distance, defined earlier, in the case of peak-power-limited systems. They will be used in the sequel to guide the design and measure the optimality of the proposed ST constellations. Combining (7), (9), (10), and (13), we obtain the following upper bounds on the normalized coding gain and squared Euclidean distance:

$$\eta_{\boldsymbol{C}} \le \frac{d_{\mathcal{U}}^2}{M \times L \times \text{PAR}_{\mathcal{U}}} \tag{16}$$

$$\chi_{\boldsymbol{C}}^2 \le \frac{d_{\mathcal{U}}^2}{L \times \text{PAR}_{\mathcal{U}}}. \tag{17}$$

We will show later that these bounds are achievable for constellations that support one symbol PCU.

4) Delay: One can easily see that a nonzero coding gain, and hence, full diversity, can only be achieved if $T \geq M$ (i.e., so that $\boldsymbol{A}$ can have full row rank). Therefore, the ST constellation $\boldsymbol{C}$ will be called delay optimal if $T = M$. All the constellations considered in this paper are delay optimal by construction, and hence, we will always assume that $T = M$ unless otherwise stated. The optimality of the delay is also desirable from a complexity point of view since it minimizes the dimension of the sphere decoder (we will elaborate on this issue in the numerical results section).

5) Mutual Information: Assuming that the ST constellation will be concatenated with a Gaussian outer codebook, Hassibi and Hochwald [2] proposed the *average* mutual information between the input of the ST constellation and the received signal as the design metric. They further presented a numerical optimization technique for constructing constellations with **near**-optimal average mutual information. It is straightforward to see that the optimal constellation is the one that *preserves* the capacity of the channel, and hence, we will refer to it as an *information lossless* constellation. The prime example of an information lossless constellation is the *identity* parser which distributes the output symbols of the outer code across the M transmit antennas periodically. As noted in [2], however, optimizing the mutual information only is not sufficient to guarantee good performance. Furthermore, imposing the constraint that $L \leq \min(N, M)$ generally entails a loss in the mutual information when $N < M$ (the only known exception for this observation is the Alamouti scheme with one receive antenna). In the sequel, we will show that the average mutual information achieved by the **full diversity** constellations proposed here is optimal (i.e., information lossless) when $N \geq M$ and very close to the optimized values in [2] when $N < M$. We, however, observe that there is no guarantee that the constellations obtained using the optimization technique in [2] will achieve the optimized coding gains and squared Euclidean distances of the proposed constellations.

IV. The Diversity-Versus-Rate Tradeoff

In multiple-antenna systems, one can increase the transmission rate at the expense of a certain loss in the diversity advantage. Earlier attempts to characterize this tradeoff have defined the transmission rate as the number of transmitted symbols PCU (e.g., in [16]). Such tradeoff is obviously unnecessary. The TAST constellations presented here (and earlier in [3]) offer a constructive proof that one can *simultaneously* achieve full diversity while transmitting at the full-rate of $\min(M, N)$ symbols PCU. The tradeoff between rate and diversity becomes only **necessary** if one imposes further requirements on the system. Three scenarios are considered in the following subsections. In Section IV-A, we follow the approach proposed in [17] and allow the transmission rate to increase with the SNR. Then, we characterize the diversity-versus-rate tradeoff under peak power and complexity constraints in Sections IV-B and IV-C, respectively.

A. Rate Scaling With the SNR

In [17], the transmission rate, in bits PCU, is allowed to grow with the SNR as

$$R = r \log_2 \rho \tag{18}$$

where r is defined as the multiplexing gain. The authors further characterize the optimal tradeoff between the achievable diversity gain d, $0 \leq d \leq MN$, and the achievable multiplexing gain r, $0 \leq r \leq \min(M, N)$, for an $M \times N$ MIMO system as

$$d = (M - r)(N - r). \tag{19}$$

This characterization has an elegant interpretation for MIMO systems with fixed transmission rates.

Proposition 1: Let $\mathcal{C}(R)$ be an ST signaling scheme that supports an arbitrary rate R in bits PCU. Then, $\mathcal{C}$ achieves the optimal diversity-versus-multiplexing tradeoff if

$$P_e(\mathcal{C}(R), \rho) = \alpha P_{\text{out}}(R, \rho), \qquad \forall R, \rho \tag{20}$$

where ρ is the SNR, $P_{\text{out}}(R, \rho)$ is the outage probability at R and ρ, $P_e(\mathcal{C}(R), \rho)$ is the probability of error at this particular rate and SNR, and α is an arbitrary constant.

Proposition 1 means that the *gap* between the performance of the optimal transmission scheme and the outage probability should be *independent* of the transmission rate and the SNR. Proposition 1 also highlights the fact that this tradeoff characterization **does not** capture the coding gain of the constellation (i.e., the optimal tradeoff curve is achieved for any constant α). One can, therefore, augment this tradeoff characterization by requiring that $\alpha = 1$ for the optimal scheme.

In the sequel, we will argue that the proposed constellations achieve the optimal tradeoff between the diversity and rate, for $N = 1$ and $N \geq M$, when concatenated with an outer Gaussian codebook under the ML decoding assumption. We will further present simulation results which indicate that the proposed constellations achieve the optimal tradeoff curve even when the inputs are drawn from uncoded QAM constellations, where the constellation size increases with the SNR.

B. Tradeoff Under Peak Power Constraints

In order to simultaneously achieve full diversity and full transmission rate in an unconstrained system, the TAST constellations induce an *expansion* of the output constellations $\mathcal{S}_i$ [3]. In fact, this constellation expansion is a characteristic of most ST signals that are linear over the field of complex numbers (e.g., [2]). The constellation expansion, however, results in an increase in the peak transmitted power. In order to avoid the increase of the peak power, one can limit the **output** constellations (i.e., $\mathcal{S}_i$) to be *standard*, but possibly different, QAM or PSK constellations. This constraint, however, imposes the following fundamental limit on the tradeoff between transmission rate and diversity advantage. This bound is obtained from the Singleton bound and assumes a symmetric ST constellation with $|\mathcal{S}_i| = |\mathcal{S}_j|$ [1], [9]

$$R \le \log_2 |\mathcal{S}_i| \left(M - \frac{d}{N} + 1\right). \tag{21}$$

For example, to achieve full diversity (i.e., $d = MN$), the maximum transmission rate is one symbol, drawn from $\mathcal{S}_i$, PCU which corresponds to $\log_2 |\mathcal{S}_i|$ bits PCU *irrespective of the number of receive antennas* [1], [9].

By combining L symbols from $\mathcal{U}$ to obtain a symbol from $\mathcal{S}_i$, a linear ST constellation can achieve full diversity only if

$$|\mathcal{S}_i| \ge |\mathcal{U}|^L, \qquad \forall\, i \tag{22}$$

as predicted by the Singleton bound (21). All the constellations proposed here satisfy the lower bound in (22) with equality. Now, by imposing the constraint that $|\mathcal{S}_i| = |\mathcal{U}|$ (i.e., no increase in the PAR), equality in the Singleton bound (21) can be satisfied with linear ST constellations *only* in the full diversity scenario (i.e., $d = MN$). In general, the suboptimality of linear ST constellations in peak-power-limited ST systems is formalized in the following result.

Proposition 2: The diversity advantage of a linear ST constellation that supports L symbols PCU using output constellations $\mathcal{S}_i$ with $|\mathcal{U}| = |\mathcal{S}_i|$, is governed by

$$d \le \left\lfloor \frac{M}{L} \right\rfloor N. \tag{23}$$

In Section VII-A, we present variants of the proposed constellations that realize this optimal tradeoff. Interestingly, Proposition 2 indicates that the vertical Bell Labs layerd space–time (V-BLAST) architecture [18] achieves the optimal diversity advantage for full-rate symmetric systems (i.e., $L = M = N$) with strict peak power constraints (i.e., $|\mathcal{S}_i| = |\mathcal{U}|$).

C. Tradeoff Under Complexity Constraints

Although ML decoding for the full-rate and full diversity linear constellations only requires polynomial complexity in $M \min(M, N)$ [4], this complexity can be prohibitive for systems with large numbers of transmit and receive antennas. This motivates the following question: *what are the achievable diversity–rate pairs (d, L) for a MIMO system under the constraint that the dimension of the sphere decoder is L_c?* The answer to this question is given in the following proposition.

Proposition 3: In an $M \times N$ MIMO system with $1 \le L_c \le M \min(M, N)$ complex dimensions in the polynomial complexity sphere decoder and a diversity advantage $1 \le d \le MN$, the number of transmitted symbols PCU satisfies

$$L \le \min\left(\frac{L_c N}{d}, M, N\right) \tag{24}$$

where we require $\frac{d}{N}$ to be an integer.

Proposition 3 means that, with complexity constraints, the choice of number of symbols PCU implies a tradeoff between the diversity advantage and the squared Euclidean distance (since a large number of symbols results in a large squared Euclidean distance as evident in (9)). One can use this observation, along with the fact that the squared Euclidean distance is the dominant factor for small SNR and/or large numbers of antennas, to conclude that the optimal choice of the number of symbols PCU depends on the available complexity, number of antennas, and SNR. Variants of the proposed constellations that achieve the optimal diversity–rate tradeoff in (24) will be presented in Section VII.

Similarly, one can investigate the *complexity constrained* tradeoff for other receiver architectures. For example, we have the following conjecture for the nulling and cancellation receiver [18], where the diversity advantage is upper-bounded by the number of excess degrees of freedom in the system of linear equations.

Conjecture 1: In an $M \times N$ MIMO system with $1 \le L_c \le M \min(M, N)$ complex dimensions in the polynomial complexity nulling and cancellation algorithm supported by the receiver, one has

$$\frac{1}{M} \le L \le \min(M, N, L_c) \tag{25}$$

$$d \le \min\left(M, \frac{L_c}{L}\right)(N - L) + 1. \tag{26}$$

Numerical results that demonstrate significant gains for constellations optimized according to this conjecture will be presented in Section VIII.

In the following, we first present the new full-rate, in terms of the number of symbols PCU, and full diversity constellations for the MISO scenario in Section V. The extension to MIMO systems is then described in Section VI. Section VII offers generalizations of the proposed constellations that realize the optimal tradeoff between diversity and rate in various scenarios.

V. Constellations for MISO Channels

The designs proposed in this paper belong to the TAST signaling framework [3]. The main idea behind this framework is to assign an algebraic code in each thread that will achieve full diversity in the absence of the other threads. One should then project the threads into different algebraic subspaces by multiplying each one with a properly chosen scaling factor[1] to ensure that the threads are *transparent* to each other. Here, we utilize this framework to construct constellations with optimal (or near-optimal) PARs, minimum squared Euclidean distances, and coding gains. Compared to the *exemplary* constructions in [3], the proposed constellations are different in two major ways. First, we impose the constraint that the number of threads is equal to the number of transmit antennas M, rather than the number of symbols PCU; this avoids sending zeros from some of the transmit antennas when $L < M$. Second, we replace the rate-one algebraic rotations used as component codes in the different threads with simple repetition codes of length M. The resulting constellation, therefore, still supports one symbol PCU while avoiding the increase of PAR incurred by the rotation and the periods of no transmission.

[1]The scaling factor will be referred to as a Diophantine number in the sequel.

TABLE I
CODING GAINS FOR $\mathcal{M}$-PSK CONSTELLATIONS WITH $\phi = e^{2i\pi/\mathcal{M}}$ FOR $\mathcal{M}$ EVEN AND $\phi = e^{2i\pi/2\mathcal{M}}$ FOR $\mathcal{M}$ ODD

$\mathcal{M}$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\delta_{\mathbf{C}}$ (dB)	3	1.8	0	−3.7	−3	−7.8	−6.5	−10.9	−9.3	−13.5	−11.6	−15.6	−13.6	−17.4	−15.3

Mathematically, over M transmit antennas and M symbol periods one sends M information symbols $u_1, \ldots, u_M$ in a circulant $M \times M$ matrix as follows:

$$\boldsymbol{\mathcal{D}}_{M,M,1}(\boldsymbol{u}) = \frac{1}{\sqrt{M}} \begin{pmatrix} u_1 & \cdots & \phi^{\frac{1}{M}} u_2 \\ \phi^{\frac{1}{M}} u_2 & \cdots & \phi^{\frac{2}{M}} u_3 \\ \vdots & \ddots & \vdots \\ \phi^{\frac{M-1}{M}} u_M & \cdots & u_1 \end{pmatrix} \tag{27}$$

where $\boldsymbol{\mathcal{D}}_{M,M,1}$ refers to the new TAST constellation with M transmit antennas, M threads, and one symbol PCU. The Diophantine number ϕ is chosen to guarantee full diversity and optimize the coding gain as formalized in the following two theorems.

Theorem 1: If the Diophantine number ϕ, with $|\phi| = 1$, is chosen such that $\{1, \phi, \ldots, \phi^{M-1}\}$ are algebraically independent over $\mathbb{Q}(w_{\mathcal{M}})$, the $\mathcal{M}$-cyclotomic number fields, then $\boldsymbol{\mathcal{D}}_{M,M,1}(\boldsymbol{u})$ in (27) achieves full diversity over all constellations carved from $\mathbb{Z}[w_{\mathcal{M}}]$. This can be achieved if ϕ is chosen such that

1) $\phi = e^{i\lambda}$ with $\lambda \neq 0 \in \mathbb{R}$ algebraic (ϕ transcendental);
2) ϕ algebraic such that $\mathbb{Q}(\phi)$ is an extension of degree greater than or equal to M over $\mathbb{Q}(w_{\mathcal{M}})$ with $\{1, \phi, \ldots, \phi^{M-1}\}$ a basis, or part of a basis of $\mathbb{Q}(\phi)$ over $\mathbb{Q}(w_{\mathcal{M}})$.

Furthermore, $\boldsymbol{\mathcal{D}}_{M,M,1}(\boldsymbol{u})$ achieves the optimal Euclidean distance of $d_{\mathcal{U}}^2$ and the optimal normalized Euclidean distance of $\frac{d_{\mathcal{U}}^2}{\mathrm{PAR}_{\mathcal{U}}}$ (the constraint $|\phi| = 1$ is imposed to ensure this property).

Theorem 2: For $M = 2^r$, $r \geq 1$, the optimal coding gain, i.e., $\delta_{\boldsymbol{C}} = \frac{d_{\mathcal{U}}^2}{M}$, can be obtained with $\boldsymbol{\mathcal{D}}_{M,M,1}$ by choosing the Diophantine number $\phi = i$ and constellations carved from $\mathbb{Z}[i]$, and for $M = 2^{r_0}3^{r_1}$, $r_0, r_1 \geq 0$ by choosing $\phi = e^{2i\pi/6}$ and constellations carved from $\mathbb{Z}[j]$.

When $M \neq 2^r$ or $M \neq 2^{r_0}3^{r_1}$, $r_0, r_1 \geq 0$, one can only guarantee local optima for the coding gains by using exhaustive computer search or by choosing the Diophantine number as an algebraic integer with the smallest degree that guarantees full diversity (as in Theorem 1). It is also interesting to note that the optimal choice of Diophantine numbers in Theorem 2 does not depend on the size of the constellation, and hence, the proposed ST constellations are universal for any constellation size in these cases.

Theorems 1 and 2 allow for constructing *constant modulus* full diversity ST constellations with polynomial complexity ML decoding for any number of transmit antennas M. This can be achieved by using $\mathcal{M}$-PSK input modulations, i.e.,

$$\mathcal{U} \triangleq \{e^{2i\pi k/\mathcal{M}}, k = 0, \ldots, \mathcal{M}-1\} \subset \mathbb{Z}[w_{\mathcal{M}}]$$

and choosing ϕ to be a root of unity which satisfies the constraint that $\{1, \phi, \ldots, \phi^{M-1}\}$ are algebraically independent over $\mathbb{Z}[w_{\mathcal{M}}]$. For a given PSK constellation, one can use algebraic methods combined with computer search, as in [14], to find ϕ that maximizes the coding gain of the system considered. For example, for $M = 2$ with the $\mathcal{M}$-PSK constellation, one has the following relation for the coding gain:

$$\delta_{\boldsymbol{\mathcal{D}}_{2,2,1}}(\phi) \triangleq \frac{1}{M} \min_{(u_1, u_2) \neq (0,0) \in \mathbb{Z}[w_{\mathcal{M}}]^2, |u_k| \leq 2, k=1,2} |u_1^2 - \phi u_2^2| \tag{28}$$

where the condition $|u_k| \leq 2$, $k = 1, 2$, ensures that u_k is a difference of two points in the $\mathcal{M}$-PSK constellation. Thus, it suffices to choose ϕ not to be a quadratic residue in $\mathbb{Z}[w_{\mathcal{M}}]$ (i.e., $\phi^{1/2} \notin \mathbb{Z}[w_{\mathcal{M}}]$) in order to guarantee a nonzero determinant. For even values of $\mathcal{M}$, the only roots of unity in $\mathbb{Q}(w_{\mathcal{M}})$ are the $\mathcal{M}$th roots of unity, and hence, one can choose $\phi = e^{2i\pi/\mathcal{M}}$ in these cases. For odd values of $\mathcal{M}$, the only roots of unity in $\mathbb{Q}(w_{\mathcal{M}})$ are the $2\mathcal{M}$th roots of unity, and hence, it suffices to one choose $\phi = e^{2i\pi/2\mathcal{M}}$ to guarantee that ϕ is not a quadratic residue in $\mathbb{Z}[w_{\mathcal{M}}]$ in these cases. This way, we can also guarantee that the determinant value in (28) is a nonzero integer from $\mathbb{Z}[w_{\mathcal{M}}]$. Furthermore, it can be shown that these values of ϕ maximize the coding gain for constant modulus transmission with two transmit antennas (Table I reports the optimized coding gains for $\mathcal{M} = 2, \ldots, 16$). For an arbitrary number of transmit antennas and arbitrary $\mathcal{M}$-PSK constellations, one can construct full diversity TAST constellations with optimal PARs and optimized coding gains by setting the Diophantine number ϕ according to the rules in [24]. Moreover, Theorem 1 is general for constellations over any number ring $\mathcal{R}$. In this case, the Diophantine number ϕ has to be chosen such that $\{1, \phi, \ldots, \phi^{M-1}\}$ are algebraically independent over the number ring considered (see the Appendix and [24] for more details). Such a generalization can be useful for including some constellations of particular interest. For example, the most energy-efficient 8-QAM constellation is given by [30]

$$\left\{1+i, -1+i, 1-i, -1-i, 1+\sqrt{3}, -\left(1+\sqrt{3}\right), \left(1+\sqrt{3}\right)i, -\left(1+\sqrt{3}\right)i\right\} \subset \mathbb{Z}[i] \cup \mathbb{Z}[j].$$

Thus, choosing ϕ such that $\{1, \phi, \ldots, \phi^{M-1}\}$ are independent over $\mathbb{Z}[i] \cup \mathbb{Z}[j]$ gives full diversity TAST constellations over the 8-QAM constellation [24].

One can also use the new constellations to gain further insight into the tradeoff between performance and complexity for the orthogonal designs. Recently, a framework for the construction of delay-optimal orthogonal ST signals was presented in [19]. It is easy to see that these signals can be obtained from the construction in (27) if we allow for a slightly more general version of repetition coding where conjugation and/or multiplication by a constant is allowed for any number of entries. For example, for $M = 4$, the delay-optimal orthogonal constellation is given by [19]

$$\mathcal{O}_4 \triangleq \begin{pmatrix} u_1 & u_2 & u_3 & 0 \\ -u_2^* & u_1^* & 0 & -u_3 \\ -u_3^* & 0 & u_1^* & u_2 \\ 0 & u_3^* & -u_2^* & u_1 \end{pmatrix} \tag{29}$$

where u_1, u_2, u_3 belong to the constellation considered. One can simply identify the threaded structure in (29) where a full diversity *generalized* repetition code is assigned to each thread. In order to ensure orthogonality, however, the fourth thread is left empty. The empty thread results in a reduced transmission rate and increased PAR. For a fixed throughput, the reduced rate of the orthogonal constellation translates into a loss in the coding gain. For example, at a rate of 3 bits PCU, the constellation $\mathcal{O}_4$ uses a 16-QAM constellation whereas the constellation $\mathcal{D}_{4,4,1}$ uses an 8-QAM constellation. This results in a coding gain of 2.2185 dB in favor of the $\mathcal{D}_{4,4,1}$ code. In addition, the constellation $\mathcal{O}_4$ has a PAR of 12/5, whereas the constellation $\mathcal{D}_{4,4,1}$ has a PAR of $5/3$ (a gain of 1.5836 dB) in this same scenario. This example illustrates the loss in performance needed to facilitate linear complexity ML decoding (with the exception of the 2×1 MISO channel, where the Alamouti scheme is optimal [20]).

One can also generalize this argument to the case of the non-delay-optimal orthogonal signals of rates $1/2$ in [21] by considering them as a concatenation of two delay-optimal threaded constellations. This generalization, however, does not contribute more insights, and hence, the corresponding details will be omitted for brevity.

VI. Extension to MIMO Channels

Now, we extend our approach to MIMO channels $(N > 1)$. In this case, sending $L = \min(M, N)$ symbols PCU gives the maximum possible rate with polynomial complexity ML decoding [4]; therefore, the number of information symbols to be sent over M transmit antennas and M symbol periods (i.e., optimal delay) should be $M \min(M, N)$. In our approach, we partition the input information symbols into M streams of L symbols (i.e., $\boldsymbol{u}_j \triangleq (u_{j1}, \ldots, u_{jL})^T$, $j = 1, \ldots, M$). Each stream $\boldsymbol{u}_j$ is then fed to a component encoder γ_j, where the number of coded symbols at the output of the encoder is M. The output stream from each encoder will be assigned to a different thread. The component encoders should be constructed to ensure full diversity in the absence of other threads and guarantee that the threads are *transparent* to each other [3]. Without loss of generality, we will consider the following assignment of ST *cells* to the jth thread (with the convention that time indexes span $[0, M-1]$):

$$\ell_j = \{(\lfloor t + j - 1 \rfloor_M + 1, t): 0 \le t < T\}, \quad \text{for } 1 \le j \le M \tag{30}$$

where $\lfloor \cdot \rfloor_M$ denotes the $\text{mod-}M$ operation [3]. Note that since the number of threads is *always* equal to M, we avoid having periods of no transmission as incurred in the constructions in [3]. The component *linear* encoders, i.e., γ_j, $j = 1, \ldots, M$, are given by

$$\gamma_j(\boldsymbol{u}_j) = \phi_{j-1}\boldsymbol{s}_j = \phi_{j-1}\tilde{\boldsymbol{M}}\boldsymbol{u}_j \tag{31}$$

where ϕ_{j-1}, $j = 1, \ldots, M$ are the Diophantine numbers that separate the different threads, and $\tilde{\boldsymbol{M}}$ is an $M \times L$ matrix containing the normalized first L columns of the $M \times M$ full diversity rotation matrix $\boldsymbol{M}$ [3], [13], [11], [10]. For the special case, when M is divisible by L, the matrix $\tilde{\boldsymbol{M}}$ can be obtained in a slightly different way. Rather than deleting the last columns of the $M \times M$ full diversity matrix, one can obtain $\tilde{\boldsymbol{M}}$ by stacking M/L full diversity matrices of dimension $L \times L$. In this way, we decrease the algebraic degrees of the rotation matrix elements, and hence, reduce the degrees of the algebraic Diophantine numbers that achieve full diversity (see Theorem 3). In the sequal, $\mathbb{Q}(\theta)$ will always denote the algebraic number field that contains the input alphabet $\mathcal{U}$ and the rotation entries.

The following examples illustrate the proposed construction.

Examples:

1) M **divisible by** $L = \min(N, M)$.

For $L = N = 1$, the proposed constellation reduces to that given by (27). In this case, $\tilde{\boldsymbol{M}} = (1, \ldots, 1)^T$, and $\gamma_j(u_j) = \phi^{(j-1)/M}\tilde{\boldsymbol{M}}u_j$ are the full diversity component encoders. For $M = 4$, $L = N = 2$, we have

$$\mathcal{D}_{4,4,2}(\boldsymbol{u}) \triangleq \frac{1}{\sqrt{4}} \begin{pmatrix} s_{11} & \phi^{3/4}s_{42} & \phi^{2/4}s_{31} & \phi^{1/4}s_{22} \\ \phi^{1/4}s_{21} & s_{12} & \phi^{3/4}s_{41} & \phi^{2/4}s_{32} \\ \phi^{2/4}s_{31} & \phi^{1/4}s_{22} & s_{11} & \phi^{3/4}s_{42} \\ \phi^{3/4}s_{41} & \phi^{2/4}s_{32} & \phi^{1/4}s_{21} & s_{12} \end{pmatrix} \tag{32}$$

where $(s_{k1}, s_{k2})^T = \boldsymbol{M}(u_{k1}, u_{k2})^T$, $k = 1, \ldots, 4$, with $\boldsymbol{M}$ the optimal 2×2 complex or real full diversity rotation [3], and $u_{k1}, u_{k2} \in \mathbb{Z}[i]$, $k = 1, \ldots, 4$. One proves that $\phi = e^{i\pi/16}$ (of degree 4 over $\mathbb{Q}(\theta)$ when using the 2×2 optimal complex rotation) achieves full diversity over all QAM constellations. Moreover, we have found the Diophantine number $\phi = e^{2i\pi/7}$ achieves a local optimum of the coding gain for the 4-QAM constellation in this configuration. Note that the benefit of using a repetition code when L divides M is the small degree of $\mathbb{Q}(\theta)$, which has a degree of 4 here as opposed to 8 when using the optimal 4×4 complex rotation matrix [3]. This implies a smaller degree of the Diophantine number ϕ that separates the different threads, giving in turn a better coding gain [3].

2) M **is not divisible by** $L = N$.

For $M = 3$ and $L = N = 2$, we have

$$\mathcal{D}_{3,3,2}(\boldsymbol{u}) \triangleq \frac{1}{\sqrt{3}} \begin{pmatrix} s_{11} & \phi^{2/3}s_{32} & \phi^{1/3}s_{23} \\ \phi^{1/3}s_{21} & s_{12} & \phi^{2/3}s_{33} \\ \phi^{2/3}s_{31} & \phi^{1/3}s_{22} & s_{13} \end{pmatrix} \tag{33}$$

where $(s_{k2}, s_{k2}, s_{k3})^T = \boldsymbol{M}(u_{k1}, u_{k2}, 0)^T$, $k = 1, 2, 3$, $\boldsymbol{M}$ is the optimal 3×3 complex or real rotation [3], u_{k1}, u_{k2},

$k = 1, 2, 3$, belong to the considered constellation, and ϕ is chosen to ensure full diversity (as formalized in Theorem 3). For example, we have found $\phi = e^{2i\pi/9}$ to give a local optima of the coding gain for the 4-QAM constellation when using the optimal 3×3 real rotation [3]

The desirable properties of the proposed constellations are formalized in the following theorem.

Theorem 3: If the Diophantine numbers

$$\left\{\phi_0 = 1,\, \phi_1 = \phi^{1/M},\, \ldots,\, \phi_{M-1} = \phi^{(M-1)/M}\right\}$$

are selected to be transcendental or algebraic such that $\left\{1, \phi, \phi^2, \ldots, \phi^{M-1}\right\}$ are independent over $\mathbb{Q}(\theta)$, then the new ST constellation will achieve full diversity. The PAR of the proposed constellations increases only linearly with the number of symbols L (i.e., $\text{PAR}_{\mathcal{D}_{M,M,L}} \le L \times \text{PAR}_{\mathcal{U}}$). Moreover, the proposed constellations achieve the optimal Euclidean distance of $\frac{d^2_{\mathcal{U}}}{L}$.

Interestingly, if we allow for the use of generalized repetition codes, then we can obtain $\mathcal{D}_{M,M,L}$ as the sum of L different *variants* of $\mathcal{D}_{M,M,1}$. For example, with $M = N = L = 2$, we have

$$\mathcal{D}_{2,2,2}(\boldsymbol{u}) = \frac{1}{\sqrt{4}} \times \left\{ \begin{pmatrix} u_1 & \phi^{1/2} u_2 \\ \phi^{1/2} u_2 & u_1 \end{pmatrix} + \theta \begin{pmatrix} u_3 & -\phi^{1/2} u_4 \\ \phi^{1/2} u_4 & -u_3 \end{pmatrix} \right\}. \quad (34)$$

We also note that in order to use the proposed constructions with PSK input constellations, one needs to construct full diversity algebraic rotations for these constellations (i.e., to construct full diversity rotations over $\mathbb{Z}[w_{\mathcal{M}}]$). Although the techniques in [11], [3] are optimized for constellations carved from $\mathbb{Z}[i]$, one can still utilize them to construct full diversity rotations over $\mathcal{M}$-PSK by considering the Galois *extension* of degree M over $\mathbb{Q}(w_{\mathcal{M}})$. For example, for $M = 2$ the rotation

$$\boldsymbol{M} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \phi \\ 1 & -\phi \end{pmatrix}$$

guarantees full diversity for two-dimensional $\mathcal{M}$-PSK constellations if ϕ is not a quadratic residue in $\mathbb{Z}[w_{\mathcal{M}}]$ as reported in Table I. The construction of optimal algebraic rotations for constellations carved from $\mathbb{Z}[w_{\mathcal{M}}]$ is investigated in [24].

VII. Tradeoffs

The constellations presented thus far achieve full-rate and full diversity, simultaneously, for arbitrary numbers of transmit and receive antennas. Now, we consider generalizations of these constellations that allow for realizing the optimal tradeoffs, established in Section IV, when further constraints are imposed on the system.

In the case when the transmission rate is allowed to grow with the SNR, we have the following result.

Propostion 4: The proposed constellations achieve the optimal diversity-versus-multiplexing tradeoff when concatenated with a Gaussian codebook under an ML decoding assumption for $N = 1$ and $N \ge M$.

Here, we would like to warn the reader that Proposition 4 is limited by the need for an outer Gaussian codebook. This limitation does not allow for an explicit design of reduced complexity decoding algorithms. In Section VIII, however, we present simulation results which indicate that the proposed constellations achieve the optimal multiplexing-versus-diversity tradeoff with QAM inputs and using the sphere decoder at the receiver. Unfortunately, we do not have a proof for this observation at the moment.

A. Trading Diversity for Reduced PAR

Here, we impose the constraint that the PAR of the linear ST constellation is equal to that of the input constellation $\mathcal{U}$. Suppose that the transmitter wants to send $L \le \min(M, N)$ symbols, drawn from $\mathcal{U}$, PCU without increasing the size of $\mathcal{U}$. Then, as predicted by Proposition 2, the maximum achievable transmit diversity in this case equals $d = N\lfloor \frac{M}{L} \rfloor$. Therefore, it suffices to consider only signaling schemes with an integer L such that M is divisible by L. In the proposed scheme, we only send the first M/L columns of the constellation matrix $\mathcal{D}_{M,M,1}(\boldsymbol{u})$ in (27). It is easily seen that this constellation supports $L = MN/d$ symbols PCU, and achieves a diversity advantage d, while preserving the PAR of the input constellation.

B. Trading Diversity for Reduced Complexity

We consider the scenario where a sphere decoder with L_c complex dimensions is used at the receiver. Given a diversity order of d (divisible by N), let $T \triangleq \frac{d}{N} \le M$. Then, we construct an $M \times T$ TAST constellation with M threads of length T each. Consider the threading in (30), where we assign *scaled* full diversity diagonal algebraic ST (DAST) constellations [3] of length T to the different threads. To prove that this TAST constellation achieves full diversity (with the correct choice of the Diophantine number), we distinguish between the following two cases.

1) If $T = M$, then one has a square $M \times M$ TAST constellation that achieves full diversity when the Diophantine number ϕ is chosen to be algebraic or transcendental and $\{1, \phi, \ldots, \phi^{M-1}\}$ are algebraically independent over $\mathbb{Q}(\theta)$.

2) To prove that this $M \times T$ matrix is full rank when $T < M$, we augment the constellation matrix to a square matrix by adding to thread ℓ_j, the numbers

$$\phi^{(j-1)/M}\alpha_{j,1},\, \phi^{(j-1)/M}\alpha_{j,2},\, \ldots,\, \phi^{(j-1)/M}\alpha_{j,M-T}$$

with

$$\alpha_{j,k} \ne 0 \in \mathbb{Q}(\theta), \qquad k = 1 \ldots M-T,\; j = 1, \ldots, M.$$

One can prove that the resulting square matrix satisfies the full rank condition with the appropriate choice of the Diophantine numbers as in Case 1 (for a detailed proof, the interested reader is referred to the proof of Theorem 1 and [3, Theorems 1 and 4]). It follows that the first T columns of this matrix are linearly independent, and therefore, the considered constellation achieves a diversity of $d = TN$.

One can also see that a rate of M symbols PCU is realized by the proposed constellations if a full-rate (i.e., one symbol PCU) DAST constellation is used in each thread. Limiting the dimensionality of the sphere decoder to L_c complex dimensions is, however, achieved by *zero-setting* some of the symbols. So if $L_c/T \leq \min(M, N)$ one can write $TM = L_c + n_1T + n_2$, with $n_1 \geq 0$ and $0 \leq n_2 < T$. This suggests that if one deletes the last n_1 threads in our $M \times T$ TAST constellation, and sets

$$u_{M-n_1+1, T-n_2+1} = \cdots = u_{M-n_1+1, T} = 0$$

one obtains a transmission rate of L_c/T symbols PCU while allowing for the poynomial complexity sphere decoder with L_c dimensions. Finally, we note that deleting some threads and zero-setting some symbols in a thread does not affect the diversity gain.

The following examples illustrate the proposed scheme. Let $M = 3, N = 3$, and consider the following choices of L_c.

1) $L_c = \min(M, N) = 3$ allows for the following choices of (d, L):

a) $(d = 3, L = 3)$: The truncated $M \times d$ TAST constellation reduces to the well-known V-BLAST system.

b) $(d = 6, L = 3/2)$: One sends the following TAST constellation:

$$\frac{1}{\sqrt{3}} \begin{pmatrix} s_{11} & \phi^{2/3}s_{32} \\ \phi^{1/3}s_{21} & s_{12} \\ \phi^{2/3}s_{31} & \phi^{1/3}s_{22} \end{pmatrix} \tag{35}$$

where $(s_{j1}, s_{j2})^T = \boldsymbol{M}(u_{j1}, u_{j2})^T$, $u_{j1}, u_{j2} \in \mathcal{U}$, $j = 1, 2, 3$, with $\boldsymbol{M}$ the 2×2 optimal rotation matrix, and ϕ chosen such that $\{1, \phi, \phi^2\}$ are independent over $\mathbb{Q}(\theta)$ (e.g., when $\mathcal{U}$ is a QAM constellation, and $\boldsymbol{M}$ is complex [3], then $\theta = e^{i\pi/4}$ and one can choose $\phi = e^{2i\pi/5}$). To obtain a rate of $3/2$ compatible with the complexity $L_c = 3$, one sets $u_{31} = u_{32} = u_{21} = 0$.

c) $(d = 9, L = 1)$: This is achieved by the TAST constellation in (27).

2) $L_c = 6$ allows the following two possibilities for (d, L) (24):

a) $(d = 6, L = 3)$: This is achieved by the constellation in (33) without zero-setting any information symbols.

b) $(d = 9, L = 2)$: This is achieved by the constellation in (33) where $(s_{j1}, s_{j2}, s_{j3})^T = \boldsymbol{M}(u_{j1}, u_{j2}, u_{j3})^T$, $u_{j1}, u_{j2}, u_{j3} \in \mathcal{U}$, $j = 1, 2$, the third thread is deleted, and ϕ is chosen such that $\{1, \phi\}$ are independent over $\mathbb{Q}(\theta)$.

3) $L_c = 9$ allows for transmitting at full-rate and full diversity by using the TAST constellation

$$\frac{1}{\sqrt{3}} \begin{pmatrix} s_{11} & \phi^{2/3}s_{32} & \phi^{1/3}s_{23} \\ \phi^{1/3}s_{21} & s_{12} & \phi^{2/3}s_{33} \\ \phi^{2/3}s_{31} & \phi^{1/3}s_{22} & s_{31} \end{pmatrix} \tag{36}$$

where $(s_{j1}, s_{j2}, s_{j3})^T = \boldsymbol{M}(u_{j1}, u_{j2}, u_{j3})^T$, $u_{j1}, u_{j2}, u_{j3} \in \mathcal{U}$, $j = 1, 2, 3$, with $\boldsymbol{M}$ the 3×3 optimal rotation matrix, and ϕ is chosen such that $\{1, \phi, \phi^2\}$ are independent over $\mathbb{Q}(\theta)$.

One can use the same technique for optimizing the performance of the proposed constellations with the nulling and cancellation receiver (i.e., by finding the optimal pair of diversity and number of symbols PCU). Although this approach is motivated by a conjecture, the numerical results in the following section demonstrate the significant gains, compared to the V-BLAST with the same complexity, for example, in various scenarios.

TABLE II
COMPARISONS OF THE MUTUAL INFORMATION OF THE TAST CONSTELLATIONS AND THE LINEAR DISPERSION CODES AT AN SNR OF 20 dB

(M, N)	TAST const., $T = M$	LD const., T [2, Table I]	Channel capacity
$(2, 1)$	5.893	$(6.28, T = 2)$	6.28
$(3, 1)$	5.893	$(6.25, T = 4), (6.28, T = 6)$	6.41
$(3, 2)$	11.27	$(11.63, T = 4)$	12.14
$(4, 1)$	5.893	$(6.34, T = 4)$	6.47
$(4, 2)$	11.27	$(11.84, T = 6)$	12.49
$(8, 4)$	22.14	$(23.10, T = 8)$	24.94

VIII. Numerical Results

In this section, we report numerical results that illustrate the near optimal performance of the proposed constellations. Table II compares the average mutual information achieved by our constellations with the constellations optimized according to the criterion in [2]. As a benchmark, we also report the ergodic channel capacity in these scenarios [2, Table I]. One can see that, although our constellations are not specifically optimized to maximize this criterion, their average mutual information is very close to that of the optimized constellations in [2]. The proposed constellations also achieve the maximum average mutual information, i.e., ergodic channel capacity, whenever $N \geq M$. In addition, we observe that the constellations in [2] are obtained through a numerical optimization approach for these specific system parameters (i.e., N, M, SNR), are not delay optimal, and do not *necessarily* achieve full diversity.

For $N = 1$, we note that the proposed constellations give the same performance as the DAST constellations [13] in average-power-limited systems. One can easily show that these constellations are equivalent in this scenario by using the discrete Fourier transform (DFT) in order to diagonalize the circulant matrix (27). In peak-power-limited systems, however, the proposed constellations offer a gain of

$$10 \log_{10} \frac{1}{2M \sin^2(\frac{\pi}{4M})} \approx 10 \log_{10}(M) - 0.9121 \text{ dB} \quad (M \geq 2)$$

with respect to the DAST constellations.

In Fig. 1, we compare the rate $3/4$ orthogonal design, the linear dispersion (LD) constellation [2], and the proposed TAST constellation with $M = 3$ and $N = 1$ at a rate of $R = 6$ bits PCU in a peak-power-limited scenario. The TAST constellation proposed is delay optimal whereas the other two schemes require more delay $(T = 4)$. The smaller delay of the proposed scheme implies a reduction in complexity, compared to the LD constellation, due to the smaller dimension of the sphere decoder. We also note that, although the slope of the bit error rate-versus-SNR curve is almost the same for the three constellations, only the orthogonal design and the proposed TAST constellation can be proved to achieve full diversity. Fig. 2 shows

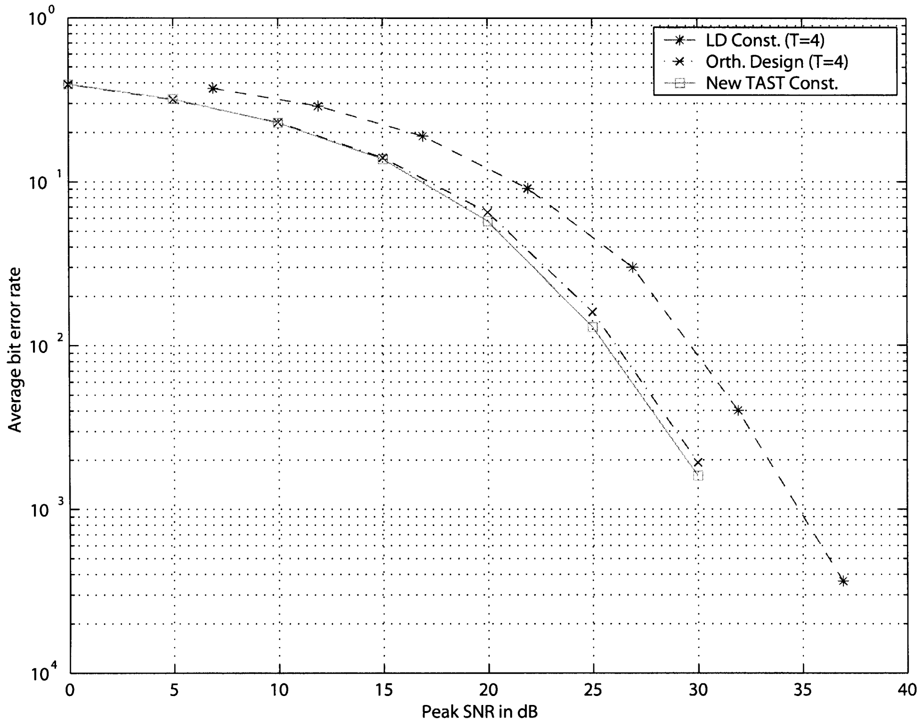

Fig. 1. The performances of the linear dispersion (LD), orthogonal design, and $\mathcal{D}_{3,3,1}$ constellations, for $M = 3$, $N = 1$, and $R = 6$ bits PCU.

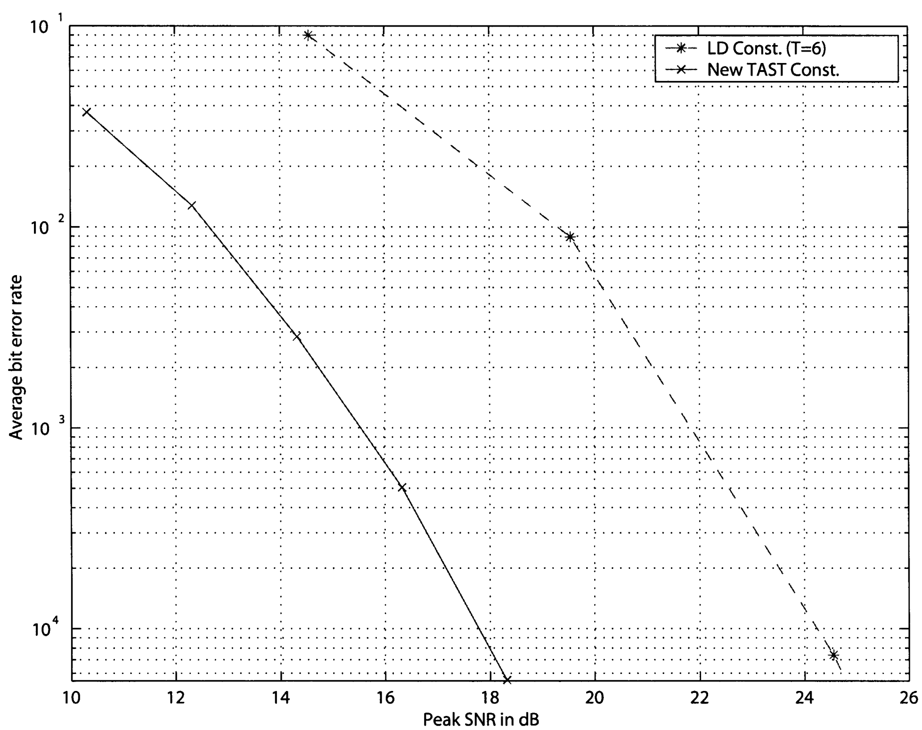

Fig. 2. The performances of the linear dispersion and $\mathcal{D}_{4,4,2}$ constellations for $M = 4$, $N = 2$, and $R = 4$ bits PCU.

the performance of the TAST constellation proposed and a linear dispersion constellation with $M = 4$ and $N = 2$ at a transmission rate of $R = 4$ bits PCU in a peak-power-limited system. Again, the TAST constellation offers the advantages of delay optimality and lower decoding complexity in addition to superior performance.

In Table III, we report the different performance metrics for ST constellations constructed for the 2×2 scenario with 4

TABLE III
PERFORMANCES OF 2×2 ST CONSTELLATIONS AT 4 BITS PCU

$\mathbf{C} =$	Alamouti scheme[21]	$\mathbf{\Theta}_2$[4]	$\mathbf{C}_2$[14]	Tirkk.-Hott.[24]	LD[2]	$\mathcal{D}_{2,2,2}$
$\delta_{\mathbf{C}}$ (dB)	-10	-8.0107	-9.2665	-7.2354	$-\infty$	-8.8807
$d^2_{\mathbf{C}}$ (dB)	-3.9794	0	0	0	0	0
$\mathrm{PAR}_{\mathbf{C}}$ (dB)	2.5527	5.0974	2.7360	4.0866	3.0103	2.3226
$\eta_{\mathbf{C}}$ (dB)	-12.5527	-13.1077	-12.0007	-11.3220	$-\infty$	-11.2032
$\chi^2_{\mathbf{C}}$ (dB)	-6.5321	-5.0974	-2.7360	-4.0866	-3.0103	-2.3226

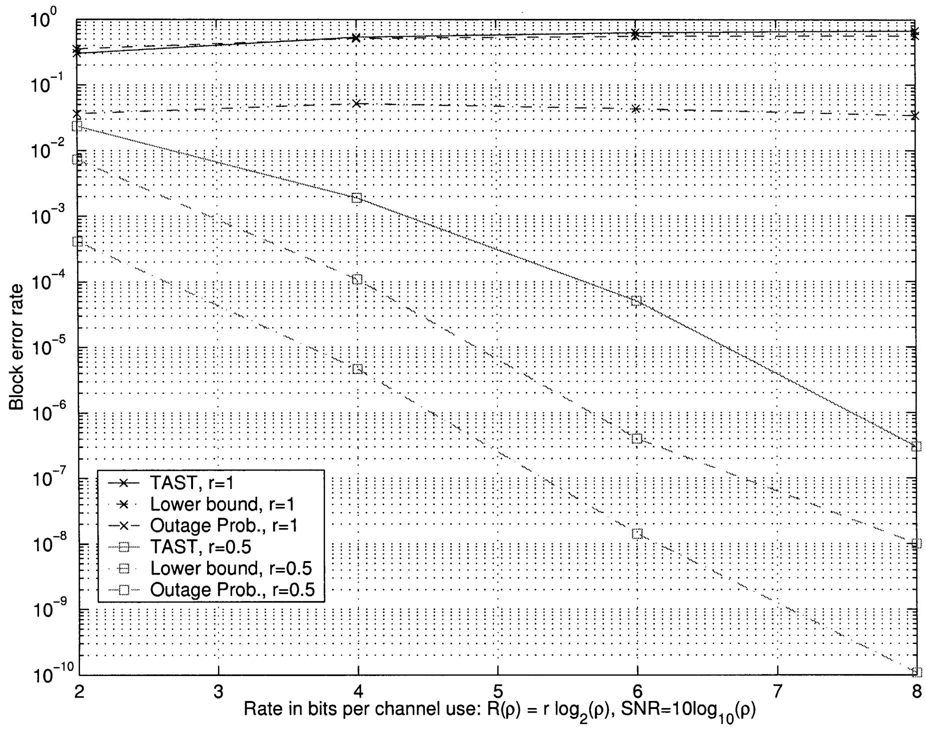

Fig. 3. The diversity-versus-multiplexing tradeoffs of the new TAST constellations $\mathcal{D}_{4,4,1}$.

bits PCU (i.e., $\mathbf{\Theta}_2$ [4], $\boldsymbol{B}_2$ [14], the constellation LD_2 [2], the Tirkonnen–Hottinen scheme [23], the Alamouti scheme [20], and the proposed constellation $\mathcal{D}_{2,2,2}$). We note that among these schemes, the proposed constellation $\mathcal{D}_{2,2,2}$ achieves the best PAR, normalized coding gain $\eta_{\boldsymbol{C}}$, normalized squared Euclidean distance $\chi^2_{\boldsymbol{C}}$, and squared Euclidean distance $d^2_{\boldsymbol{C}}$. With respect to coding gain, the proposed constellation is near-optimal. One can also see that the Alamouti scheme has the worst squared Euclidean distance. This inferior squared Euclidean distance of the Alamouti code is, in fact, the cause of its bad performance at low SNR. Finally, unlike other constellations, which are optimized for this particular rate, the $\mathcal{D}_{2,2,2}$ and $\boldsymbol{B}_2$ constellations achieve full diversity for arbitrary inputs carved from $\mathbb{Z}[w_{\mathcal{M}}]$ for $\mathcal{M} \geq 1$.

We now investigate the performance of the proposed constellations in terms of the diversity-versus-multiplexing tradeoff as defined in [17]. In the study, we use the proposed constellations with L symbols PCU where the size of the QAM constellation increases with the SNR according to $|\mathcal{U}| = \rho^{r/L}$. The throughput, in bits PCU, is therefore given by $R(\rho) = r \log_2 \rho$ [17]. Figs. 3 and 4 show the block error rate performance curves as functions of the transmission rates PCU $R(\rho)$ for the proposed constellations with different parameters. In the figures, we also plot the outage probability curves for the given values of r. It is noted that for short-length constellations $(T = M)$ the outage probability curve is not always a lower bound on the block error rate. This is because the outage is only a lower bound on the block error probability when $T \longrightarrow \infty$ [17]. For this reason, we derive the following lower bound on the block error rate for finite length T (see the Appendix for the proof)

$$\Pr\{\text{block error}\} \geq E_{\boldsymbol{H}}\left[\max\left\{0, 1 - \frac{\boldsymbol{I}(\boldsymbol{X};\boldsymbol{Y}|\boldsymbol{H}=H)}{RT} - \frac{1}{RT}\right\}\right] \quad (37)$$

where R is the transmission rate in bits PCU, $\boldsymbol{I}(\boldsymbol{X};\boldsymbol{Y}|\boldsymbol{H}=H)$ is the mutual information between the input $\boldsymbol{X}$ and the output $\boldsymbol{Y}$ conditioned on the channel realization $\boldsymbol{H} = H$ [17]. The difference between the lower bound in (37) and the outage probability is that in the latter case one assumes $\Pr\{\text{block error}\} = 1$ for

$$\boldsymbol{I}(\boldsymbol{X};\boldsymbol{Y}|\boldsymbol{H}=H) < RT$$

(i.e., when an outage occurs). The lower bound (37) is also shown in Figs. 3 and 4. One can see from the figures that the curve slope of the proposed constellations almost coincides with the slope of the outage probability and the lower bound curves in all the considered scenarios [17]. It is also worth noting that

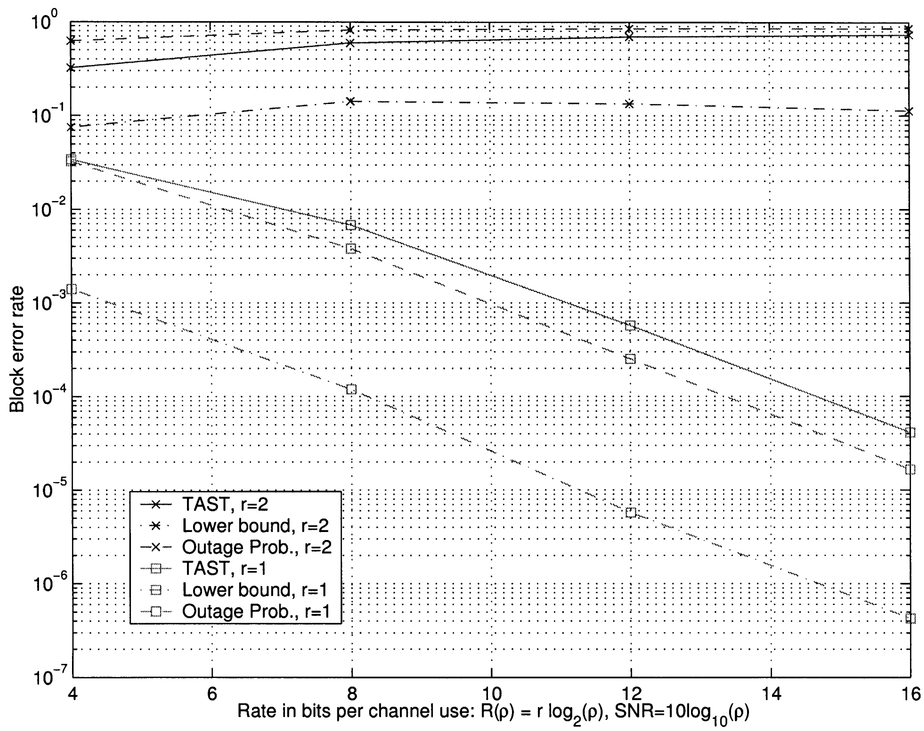

Fig. 4. The diversity-versus-multiplexing tradeoffs of the TAST constellations $\mathcal{D}_{2,2,2}$.

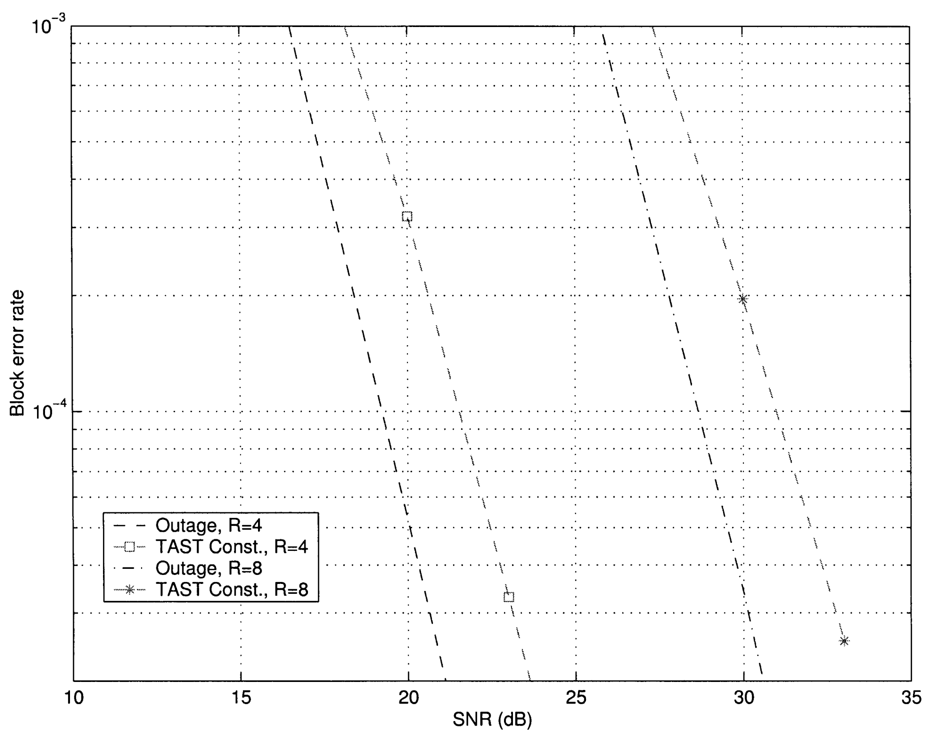

Fig. 5. Scaling the error probability with the outage of the TAST constellation $\mathcal{D}_{2,2,2}$.

this scenario is *pessimistic* in the sense that an uncoded QAM constellation is used instead of the outer Gaussian code used in [17]. Note that in Fig. 4, with $r = 1$, the block error rate is only 3 dB away from the outage probability.

More results relevant to this point are presented in Figs. 5 and 6, where we compare the Alamouti scheme, the LD, and the TAST constellations ($M = N = L = 2$), with outage probability at 4 and 8 bits PCU. Note that the LD constella-

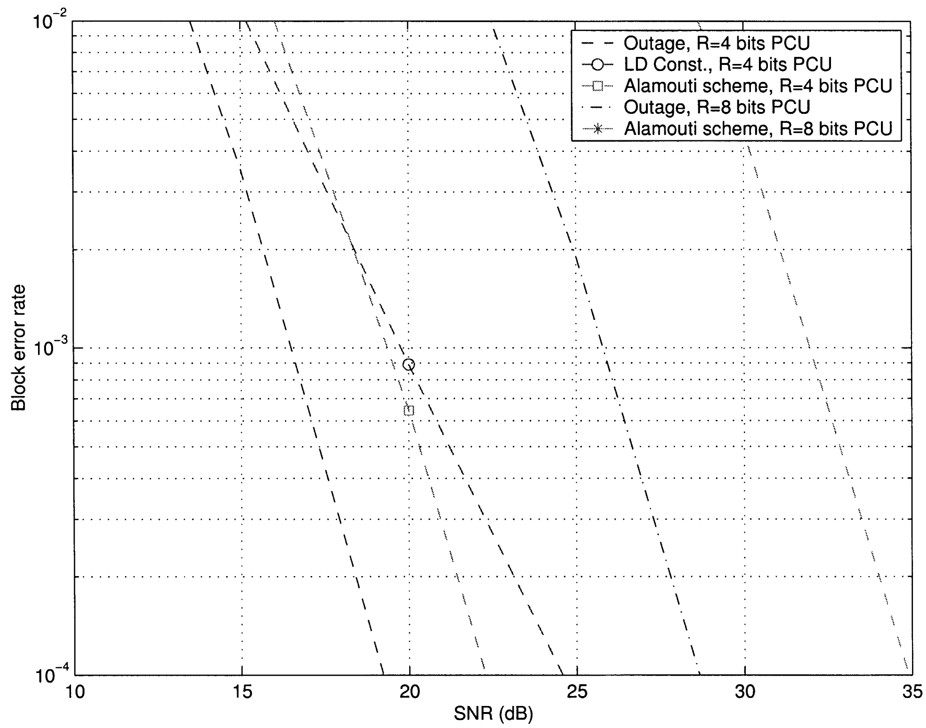

Fig. 6. Scaling the error probability with the outage of the LD constellation and the Alamouti scheme.

tion performance curve flattens out as compared to the outage curve, whereas both the Alamouti scheme curve and the TAST constellation curve follow the outage (in slope). However, the differ- ence (i.e., α in (20)) is almost fixed at about 2.5 dB for the TAST constellation with both rates, whereas this difference increases for the Alamouti scheme from about 3 to 6 dB with the rate increase. All these results support our claim that thepro-posed constellations achieve the optimal diversity-versus-multiplexing curve even with uncoded inputs.

Finally, Fig. 7 illustrates the benefit of matching the TAST constellation to the nulling and cancellation receiver as proposed in Section VII-B (Conjecture 1).[2] The MIMO system parameters are $M = N = 4$, and $R = 8$ bits PCU. Fig. 7 shows the performances of the V-BLAST constellation with $L = 4$ and 4-QAM, the TAST constellation with $L = 4$ and 4-QAM, the TAST constellation with $L=2$ and 16-QAM, and the V-BLAST with $L=2$ and 16-QAM (in this case we only use two transmit antennas). The two full-rate constellations achieve only one or-der of diversity under nulling and cancellation, the half rate V-BLAST scheme achieves $d = 3$, and Conjecture 1 predicts that the half rate TAST scheme achieves $d = 9$. One can see that the half rate TAST constellation achieves the best performance in this scenario. This is an example of scenarios where reducing the transmission rate, in terms of the number of symbols, will lead to improved performance. We hasten to stress that this observation is a direct result of using the reduced complexity nulling and cancellation receiver. We also observe that the full-rate TAST constellation is worse than the V-BLAST under nulling and cancellation. The reason is that, while both constellations achieve the same diversity order under nulling and cancellation, the high correlation between the TAST constellation symbols results in degraded performance. In summary, this example highlights the importance of optimizing the ST constellation based on the receiver characteristics.

[2]In the simulations, we used the zero-forcing decision feedback equalization (ZE-DFE) algorithmin accordance with Conjuncture 1. However, better performance can be obtained using the minimum-mean square error (MMSE) DFE, as well known.

IX. Concluding Remarks

In this paper, we have reviewed the relevant design criteria for linear ST constellations and established some fundamental limits on the performance achievable by this class of signals. We have characterized the fundamental tradeoff between diversity and rate under different constraints. We further presented a new family of constellations within the TAST signaling framework that achieve optimal or near-optimal performance with respect to these criteria. In particular, the proposed constellations were shown to achieve the optimal tradeoff between diversity and multiplexing, the optimal squared Euclidean distance, and the optimal delay. For systems with one receive antenna, the proposed constellations also achieve the optimal PAR for QAM and PSK input constellations. With respect to the average mutual information and coding gain criteria, the proposed constellations were shown to outperform or rival the best designs proposed in the literature in all the considered scenarios. The proposed framework is general for any number of transmit and receive antennas and allows for exploiting the polynomial complexity ML sphere decoder. Variants of the proposed constellations for reducing the complexity of the receiver and optimizing the PAR were also presented. Numerical results that demonstrate the excellent performance of the proposed designs in average-power- and peak-power-limited systems were also presented.

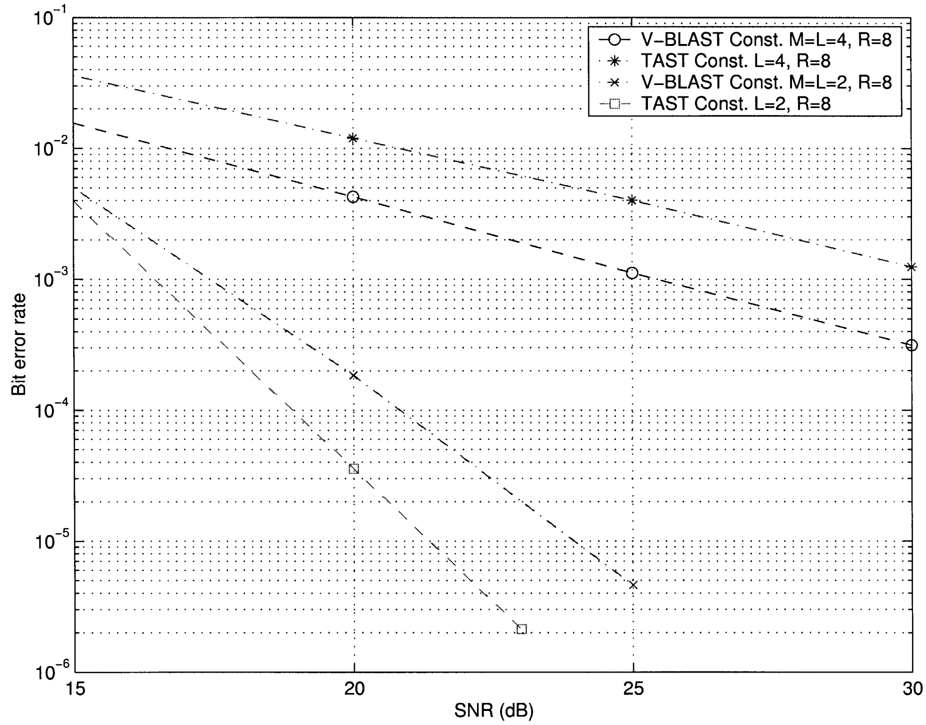

Fig. 7. Optimizing the transmitter diversity for nulling and cancellation algorithms when $M = N = 4,\ R = 8$ bits PCU.

As a final remark, we observe that the proposed constellations were optimized assuming ML decoding. We, however, realize that the use of an outer code may be necessary in certain applications. In this scenario, the ST constellation decoder must be optimized to provide soft outputs for the outer decoder. Several approaches have already been proposed in the literature for modifying the decoder to provide soft outputs. More generally, we believe that one should investigate the *joint optimization* of the ST constellation, the outer code, and the decoding algorithm in this scenario.

APPENDIX

Proof of Proposition 1: Assume that $P_e = \alpha P_{\rm out}(R,\, \rho)$. By [17], at high SNR, one has

$$P_{\rm out}(R,\, \rho) = c\rho^{-(M-\frac{R}{\log\rho})(N-\frac{R}{\log\rho})} \tag{38}$$

where c is a constant. This gives

$$\log P_{\rm out}(R,\, \rho) = -(M\log\rho - R)\left(N - \frac{R}{\log\rho}\right) + \log c. \tag{39}$$

Let $R_j = r\log\rho_j,\ j = 1,\, 2$, then one has

$$\frac{\log(P_e(\mathcal{C}(R_1),\, \rho_1)) - \log(P_e(\mathcal{C}(R_2),\, \rho_2))}{\log\rho_2 - \log\rho_1} = (M-r)(N-r) \tag{40}$$

which is the diversity order achieved by $\mathcal{C}(R)$ and equals the optimal tradeoff curve [17].

Note that the condition that α is a constant independent of the SNR is only sufficient. The proof of Proposition 1 still holds if α varies with the SNR such that $\alpha/\log\rho \longrightarrow 0$ when ρ increases.

Proof of Proposition 2: For simplicity's sake, one assumes that the constellations considered are real with a real generator matrix $\boldsymbol{\mathcal{M}}$. Generalization to complex constellations is straightforward, and can be done, for example, by treating the real and imaginary parts as two independent symbols from real constellations. The condition $|\mathcal{S}_j| = |\mathcal{U}|\ \forall j$ implies that no combination of symbols is allowed by the linear ST constellation. Therefore, in order to achieve a diversity d at a transmission rate of L symbols PCU, each of the TL symbols has to appear d/N times in the $M \times T$ ST matrix. Since the total number of positions is MT, it follows that $MT \geq LTd/N$, and hence $d \leq MN/L$.

Proof of Proposition 3: First, noting that L_c is the number of independent complex symbols transmitted during a codeword of length T gives a limit on the transmission rate of $L = L_c/T$ symbols PCU. Second, observing that the maximum possible transmit diversity equals $d = N\min(T,\, M)$ under quasi-static fading assumptions, gives, for each value of d, a maximum possible value of L which is equal to $L = \min(\frac{NL_c}{d},\, M,\, N)$, where $\frac{NL_c}{d}$ guarantees that one has L_c symbols in an $N \times d/N$ matrix (d/N being an integer) and the condition $L \leq \min(M,\, N)$ is required by the linear system obtained at the receiver (d equations with a total of Ld/N unknowns where each equation contains at most M unknowns (1)).

We will employ techniques and results from algebraic number theory in the proofs of Theorems 1, 2, and 3. The interested reader is referred to [25]–[29] for comprehensive treatments of algebraic number theory; for abridged and concise summaries, the reader is referred to papers that use these algebraic tools in the wireless communications context such as [3], [11], [13].

Before stating the Proofs of Theorems 1, 2, and 3, we recall the following lemma from [3].

Lemma 1: Let

$$\left\{\phi_0 = 1,\, \phi_1 = \phi^{1/M},\, \ldots,\, \phi_{L-1} = \phi^{(L-1)/M}\right\}$$

be the Diophantine numbers used in the TAST constellation $\mathcal{T}_{M,L,L}(\boldsymbol{u})$, with

$$\boldsymbol{u} \triangleq (\boldsymbol{u}_1^T, \ldots, \boldsymbol{u}_L^T)^T \quad \text{and} \quad \boldsymbol{u}_j \triangleq (u_{j1}, \ldots, u_{jM})^T$$

denoting the information symbols assigned to the jth layer which is rotated by matrix $\boldsymbol{M}$ and then multiplied by the Diophantine number ϕ_{j-1}. Then the determinant of matrix $\mathcal{T}_{M,L,L}(\boldsymbol{u})$ can be written as

$$\Delta_{M,L,L}(\boldsymbol{u}) \triangleq \det \mathcal{T}_{M,L,L}(\boldsymbol{u}) = \epsilon_1 X_1 + \epsilon_2 X_2 \phi + \cdots + \epsilon_L X_L \phi^{L-1} \tag{41}$$

where

$$X_1 = \prod_{k=1}^{M} s_{1k} \quad \text{and} \quad X_L = \prod_{k=1}^{M} s_{Lk}$$

The terms $X_j \in \mathbb{Q}(\theta)$ contain the cross terms of $\prod_{k=1}^{M} s_{j_k k}$, $j_k \in \{1, \ldots, M\}$ and $\epsilon_j = \pm 1$, $j = 1, \ldots, L$, depends on the positions of layer ℓ_j in the matrix $\mathcal{T}_{M,L,L}(\boldsymbol{u})$, and is called the signature of the layer ℓ_j. Furthermore, the cross terms X_j, $j = 1, \ldots, L$ are algebraic integers in $\mathbb{Q}(\theta)$.

For the proof of Lemma 1 please refer to [3].

Proof of Theorem 1: Using Lemma 1, the determinant of matrix $\sqrt{M}\mathcal{D}_{M,M,1}(\boldsymbol{u})$ can be written as (for convenience, we multiply matrix $\mathcal{D}_{M,M,1}(\boldsymbol{u})$ (27) by $\sqrt{M}$)

$$\Delta_{M,M,1}(\boldsymbol{u}) \triangleq \det \sqrt{M}\mathcal{D}_{M,M,1}(\boldsymbol{u}) = \epsilon_1 X_1 + \epsilon_2 X_2 \phi + \cdots + \epsilon_M X_M \phi^{M-1} \tag{42}$$

where $X_1 = u_1^M$, $X_M = u_M^M \in \mathbb{Z}[w_{\mathcal{M}}]$, the terms X_j contain the cross terms of $\prod_{k=1}^{M} u_{j_k} \in \mathbb{Z}[w_{\mathcal{M}}]$, $j_k \in \{1, \ldots, M\}$, and $\epsilon_j = \pm 1$, $j = 1, \ldots, M$, is the signature of layer ℓ_j and depends on its positions in the matrix $\mathcal{D}_{M,M,1}(\boldsymbol{u})$. Now, let ϕ be chosen algebraic or transcendental ($\phi = e^{i\lambda}$, with $\lambda \neq 0 \in \mathbb{R}$ algebraic [27]) such that $\{1, \phi, \ldots, \phi^{M-1}\}$ are algebraically independent over $\mathbb{Q}(w_{\mathcal{M}})$, and suppose that $\Delta_{M,M,1}(\boldsymbol{u}) = 0$. Equation (42), implies that $u_1 = u_M = 0$; therefore, matrix $\mathcal{D}_{M,M,1}(\boldsymbol{u})$ now has only $M-2$ threads. Using Lemma 1 gives

$$\Delta_{M,M,1}(\boldsymbol{u}) = \epsilon_2 \overline{X}_2 \phi + \cdots + \epsilon_{M-1} \overline{X}_{M-1} \phi^{M-2} \tag{43}$$

with $\overline{X}_2 = u_2^M$ and $\overline{X}_{M-1} = u_{M-1}^M$. This implies that $u_2 = u_{M-1} = 0$ since $\{\phi, \ldots, \phi^{M-2}\}$ is a free set over $\mathbb{Q}(w_{\mathcal{M}})$. Continuing this process, one concludes that $u_1 = \cdots = u_M = 0$, implying that $\Delta_{M,M,1}(\boldsymbol{u}) \neq 0$ for $\boldsymbol{u} \neq 0$, and the new TAST constellation achieves full diversity over all constellations carved from $\mathbb{Z}[w_{\mathcal{M}}]$.

Regarding the PARs of the new TAST constellations, it is clear from (27) and the constraint that $|\phi| = 1$ that no increase occurs in the PAR of the original constellation, giving, thus, optimal PARs for these constellations.

The fact that the new TAST constellations conserve the minimum squared Euclidean distance of the input modulation $\mathcal{U}$ follows from observing that they are obtained by unitary transformations on symbols from $\mathcal{U}$ [30].

One needs the following Proposition [25] in order to prove Theorem 2.

Proposition 5: Every conjugate of w_n is an nth root of unity and not an mth root of unity for any $m < n$ (i.e., primitive nth root). Further, all w_n^k, $1 \le k \le n$, $(k|n) = 1$, are conjugates of w_n.

It follows that $\mathbb{Q}(w_n)$ contains all the conjugates of w_n, thus, $\mathbb{Q}(w_n)$ is a Galois *extension* of $\mathbb{Q}$ (i.e., $\mathbb{Q}(w_n^k) = \mathbb{Q}(w_n)$ for $1 \le k \le n$, $(k|n) = 1$). Furthermore

$$\left\{1,\, (w_n^k),\, (w_n^k)^2,\, \ldots,\, (w_n^k)^{\phi(n)-1}\right\}$$

is a basis of $\mathbb{Q}(w_n)$ over $\mathbb{Q}$, for $1 \le k \le n$, $(k|n) = 1$.

Proof of Theorem 2: Since matrix $\mathcal{D}_{M,M,1}(\boldsymbol{u})^T$ is circulant (27), one has [33]

$$\Delta_{M,M,1}(\boldsymbol{u}) = \prod_{k=1}^{M} \Big(u_1 + \phi^{\frac{1}{M}} u_2 (w_M^k) + \cdots + \phi^{\frac{M-1}{M}} u_M (w_M^k)^{M-1} \Big) \tag{44}$$

where $w_M^k = e^{2i\pi k/M}$, $k = 1, \ldots, M$, is the Mth root of unity. Let $M = 2^r$, $r \ge 1$, $\mathcal{M} = 4$, and choose $\boldsymbol{u} \in \mathbb{Z}[i]^M$, then all the cross terms in (42) belong to $\mathbb{Z}[i]$. Furthermore, by letting $\phi = i$, one ensures that the determinant in (42) belongs to $\mathbb{Z}[i]$. Thus, it suffices to prove that this choice of ϕ makes the determinant nonzero in order to conclude that the minimal absolute value of the determinant equals 1 (after proper scaling with the normalization constant). To this end, one examines the terms $\phi^{1/M} w_M^k$, for $k = 1, \ldots, M$, in (44). First, note that $\phi^{1/M} w_M^k$, $k = 1, \ldots M$, are all roots of the minimal polynomial $\mu(X) = X^M - i$ over $\mathbb{Q}(i)$. Since $\mathbb{Q}(w_{4M})$ is an extension of degree M over $\mathbb{Q}(i)$ [25], one only needs to prove that $\phi^{1/M} w_M^k$, $k = 1, \ldots M$ are conjugates, and then Proposition 5 implies that

$$\left\{1,\, \phi^{1/M} w_M^k,\, \ldots,\, \phi^{(M-1)/M} w_M^{(M-1)k}\right\}, \quad k = 1, \ldots, M$$

is a basis of $\mathbb{Q}(w_{4M})$ over $\mathbb{Q}(i)$. One has

$$\phi^{1/M} w_M^k = e^{2i\pi/4M} e^{2i\pi k/M} = e^{2i\pi(4k+1)/4M} = w_{4M}^{4k+1}.$$

For $M = 2^r$, one easily proves that $(4k+1|4M) = 1$, for $k = 1, \ldots, M$, since $4k+1$ is odd and $4M$ is only divisible by powers of 2. Thus, using Proposition 5, it follows that $\phi^{1/M} w_M^k$ is a conjugate of w_{4M}, with

$$\left\{1,\, \phi^{1/M} w_M^k,\, \phi^{2/M} w_M^{2k},\, \ldots,\, \phi^{(M-1)/M} w_M^{k(M-1)}\right\}$$

as a free set over $\mathbb{Q}(i)$. It follows that each term in the product in (44) is nonzero for $\boldsymbol{u} \neq \boldsymbol{0}$ and the new TAST constellation has a maximum coding gain of 1 for $M = 2^r$ when choosing $\phi = i$ and constellations carved from $\mathbb{Z}[i]$.

Similarly, let now $M = 2^{r_0} 3^{r_1}$, $r_0, r_1 \ge 0$, $\mathcal{M} = 3$, $\phi = e^{2i\pi/6} = -j^2$, and $\boldsymbol{u} \in \mathbb{Z}[j]^M$. Then, from (42), one has

$$\Delta_{\mathcal{D}_{M,M,1}(\boldsymbol{u})} \in \mathbb{Z}[j].$$

Now, examine the terms

$$\phi^{1/M} w_M^k = e^{2i\pi(6k+1)/6M}, \qquad \text{for } k = 1, \ldots, M$$

in (44). Again, $\phi^{1/M} w_M^k$, $k = 1, \ldots, M$ are all roots of the minimal polynomial $\mu(X) = X^M + j^2$ with coefficients in $\mathbb{Z}[j]$. Since $\mathbb{Q}(w_{6M})$ is an extension of degree M over $\mathbb{Q}(j)$ [25], and since $(6k+1|6M) = 1$ for $M = 2^{r_0} 3^{r_1}$ and $k = 1, \ldots, M$ (the only factors of $6M$ are 3 and powers of 2, which are not written in the form $6k+1$), then, using Proposition 5 and (44), one proves that the new TAST constellation has a maximum coding gain of 1 for $M = 2^{r_0}3^{r_1}$, $r_0, r_1 \geq 0$, $\phi = -j^2$, and constellations carved from $\mathbb{Z}[j]$.

Proof of Theorem 3: We only give an outline of the proof since it is similar to Theorem 1 and [3, Theorem 1]. The proof is done by contradiction where one supposes that $\Delta_{M,M,L}(\boldsymbol{u})=0$ for certain $\boldsymbol{u} \neq 0$. Then, one uses (41) and the independence of the set of Diophantine numbers $\{1, \phi, \ldots, \phi^{M-1}\}$ over $\mathbb{Q}(\theta)$ in order to prove that $\boldsymbol{u} = 0$ by induction over the number of information streams L.

Regarding the PAR of the new TAST constellations, one notes that in both constructions, the rotation matrix combines only L constellation points at a time regardless of whether M is divisible by L. Therefore, the PAR increases at most as L, i.e., $\text{PAR}_{\boldsymbol{\mathcal{D}}_{M,M,L}} \leq \text{PAR}_{\mathcal{U}} \times L$. When using $L \times L$ complex or real rotation matrices built over cyclotomic number fields of degree $2L$ [11] [13], one has

$$\text{PAR}_{\boldsymbol{\mathcal{D}}_{M,M,L}} = \text{PAR}_{\mathcal{U}} \times \frac{1}{2L\sin^2\frac{\pi}{4L}}. \tag{45}$$

Note that the advantage of the construction when M is divisible by L is to reduce the degree of $\mathbb{Q}(\theta)$, which is useful for enhancing the coding gain.

Now, the fact that the new TAST constellations are obtained by means of unitary transformations (rotations and repetition codes) over symbols from $\mathcal{U}$, proves that $d^2_{\boldsymbol{\mathcal{D}}_{M,M,L}} = d^2_{\mathcal{U}}/L$.

Proof of Proposition 4: For the diversity-versus-multiplexing tradeoff when $N = 1$, it suffices to prove that $d = M$ when $r = 0$, and $r = 1$ when $d = 0$. The first point is readily proved by our argument regarding the full diversity of the proposed constellation. For the second point, we use the circulant nature of the transmitted matrix to write the received signal as

$$\boldsymbol{x} = \alpha\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{M}}\boldsymbol{u} + \boldsymbol{w} \tag{46}$$

where

$$\boldsymbol{\mathcal{M}} \triangleq \text{diag}\left(1, \phi^{1/M}, \ldots, \phi^{(M-1)/M}\right)$$

$\alpha \triangleq \sqrt{\frac{\rho}{M}}$ with ρ the SNR, and

$$\boldsymbol{\mathcal{H}} \triangleq \begin{pmatrix} h_1 & h_2 & h_3 & \cdots & h_M \\ h_2 & h_3 & h_4 & \cdots & h_1 \\ h_3 & h_4 & h_5 & \cdots & h_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_M & h_1 & h_2 & \cdots & h_{M-1} \end{pmatrix} \tag{47}$$

with $h_1, \ldots, h_M$ the channel coefficients. It follows that

$$\boldsymbol{x} = \alpha\boldsymbol{F}_M^H\boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{F}_M\boldsymbol{\mathcal{M}}\boldsymbol{u} + \boldsymbol{w} \tag{48}$$

where $\boldsymbol{F}_M$ is the $M \times M$ DFT matrix, $\boldsymbol{P}$ is an $M \times M$ permutation matrix (i.e., the entries of $\boldsymbol{P}$ are 0 or 1 and $\boldsymbol{P}\boldsymbol{P}^T = \boldsymbol{I}$), and $\boldsymbol{\Lambda}$ is a diagonal matrix with i.i.d. Gaussian entries

$$\lambda_k = h_1 + h_2(w_M^k) + \cdots + h_M(w_M^k)^{M-1}, \quad k = 1, \ldots, M$$

with $w_M = e^{2i\pi/M}$ [31], [32]. We multiply both sides in (48) by $\boldsymbol{P}^T\boldsymbol{F}_M$ to obtain

$$\boldsymbol{r} \triangleq \boldsymbol{P}^T\boldsymbol{F}_M\boldsymbol{x} = \alpha\boldsymbol{\Lambda}\boldsymbol{F}_M\boldsymbol{\mathcal{M}}\boldsymbol{u} + \boldsymbol{n} \tag{49}$$

where $\boldsymbol{n} \triangleq \boldsymbol{P}^T\boldsymbol{F}_M\boldsymbol{w}$ is a white Gaussian $M \times 1$ column vector. Observing that $\boldsymbol{F}_M\boldsymbol{\mathcal{M}}\boldsymbol{u}$ is a white Gaussian vector when $\boldsymbol{u}$ is white Gaussian, one can see that the model in (49) is equivalent to a single-antenna system with M independent fading blocks. Now, one can invoke the argument in [17] to show that $d = M(1 - r)$ in this case which proves that $r = 1, d = 0$ is achievable by the proposed constellation.

For $N \geq M > 1$, the unitary nature of the transformation ensures that, when $N \geq M$ with white Gaussian input, the transmitted symbols are i.i.d. Gaussian random variables. This fact, along with the argument in [17], ensure that the proposed constellations will achieve the optimal tradeoff between diversity and multiplexing with ML decoding. The simulation results further indicate that these constellations achieve the optimal tradeoff with uncoded QAM constellations.

Proof of the Lower Bound on the Block Error Rate (37): We use the notations of [17]. Let $\Pr\{\text{error}|\boldsymbol{H} = H\}$ denote the block error rate when the channel $\boldsymbol{H} = H$. Then, by Fano's inequality, one has [17]

$$\Pr\{\text{error}|\boldsymbol{H} = H\} \geq 1 - \frac{\boldsymbol{I}(\boldsymbol{X};\boldsymbol{Y}|\boldsymbol{H} = H)}{RT} - \frac{1}{RT}. \tag{50}$$

By taking the expectation with respect to $\boldsymbol{H}$ of both sides in (50), and by substituting 0 for the right side when it has a negative value one obtains

$$\Pr\{\text{block error}\} \geq E\left[\max\left\{0, 1 - \frac{\boldsymbol{I}(\boldsymbol{X};\boldsymbol{Y}|\boldsymbol{H} = H)}{RT} - \frac{1}{RT}\right\}\right]. \tag{51}$$

REFERENCES

[1] V. Tarokh, N. Seshadri, and A. R. Calderbank, "Space-time codes for high data rate wireless communications: Performance criterion and code construction," *IEEE Trans. Inform. Theory*, vol. 44, pp. 744–765, Mar. 1998.

[2] B. Hassibi and B. Hochwald, "High-rate codes that are linear in space and time," *IEEE Trans. Inform. Theory*, vol. 48, pp. 1804–1824, July 2002.

[3] H. El Gamal and M. O. Damen, "Universal space-time coding," *IEEE Trans. Inform. Theory*, vol. 49, pp. 1097–1119, May 2003.

[4] M. O. Damen, A. Chkeif, and J.-C. Belfiore, "Lattice codes decoder for space-time codes," *IEEE Commun. Lett.*, vol. 4, pp. 161–163, May 2000.

[5] U. Fincke and M. Pohst, "Improved methods for calculating vectors of short length in a lattice, including a complexity analysis," *Math. Computation*, vol. 44, pp. 463–471, Apr. 1985.

[6] H. Vikalo and B. Hassibi, "The expected complexity of sphere decoding, Part I: Theory, Part II: Applications," *IEEE Trans. Signal Processing*, submitted for publication.

[7] M. O. Damen, K. Abed-Meraim, and J.-C. Belfiore, "A generalized sphere decoder for asymmetrical space-time communication architecture," *Electron. Lett.*, vol. 36, no. 2, pp. 166–167, Jan. 2000.

[8] H. El Gamal and A. R. Hammons, Jr., "A new approach to layered space–time coding and signal processing," *IEEE Trans. Inform. Theory*, vol. 47, pp. 2321–2334, Sept. 2001.

[9] ——, "Algebraic space-time codes for block fading channels," in *Proc. IEEE Int. Symp. Information Theory*, Washington, DC, June 2001, p. 152.

[10] J. Boutros and E. Viterbo, "Signal space diversity: A power and bandwidth efficient diversity technique for the Rayleigh fading channel," *IEEE Trans. Inform. Theory*, vol. 44, pp. 1453–1467, July 1998.

[11] X. Giraud, E. Boutillon, and J.-C. Belfiore, "Algebraic tools to build modulation schemes for fading channels," *IEEE Trans. Inform. Theory*, vol. 43, pp. 938–952, May 1997.

[12] J.-C. Guey, M. P. Fitz, M. R. Bell, and W.-Y. Kuo, "Signal design for transmitter diversity wireless communication systems over Rayleigh fading channels," in *Proc. IEEE Vehicular Technology Conf.*, Atlanta, GA, Apr. 1996, pp. 136–140.

[13] M. O. Damen, K. Abed-Meraim, and J.-C. Belfiore, "Diagonal algebraic space-time block codes," *IEEE Trans. Inform. Theory*, vol. 48, pp. 628–636, Mar. 2002.

[14] M. O. Damen, A. Tewfik, and J.-C. Belfiore, "A construction of a space-time code based on number theory," *IEEE Trans. Inform. Theory*, vol. 48, pp. 753–761, Mar. 2002.

[15] H. Minn, C. Tellambura, and V. K. Bhargava, "On the peak factors of sampled and continuous signals," *IEEE Commun. Lett.*, vol. 5, pp. 129–132, Apr. 2001.

[16] Y. Xin, Z. Liu, and G. B. Giannakis, "High-rate layered space-time coding based on constellation precoding," in *Proc. Wireless Communications and Networking Conf.*, vol. 1, Orlando, FL, Mar. 17–21, 2002, pp. 471–476.

[17] L. Zheng and D. N. C. Tse, "Diversity and multiplexing: A fundamental tradeoff in multiple antenna channels," *IEEE Trans. Inform. Theory*, vol. 49, pp. 1073–1096, May 2003.

[18] G. Golden, C. Foschini, R. Valenzuela, and P. Wolniasky, "Detection algorithm and initial laboratory results using V-BLAST space-time communication architecture," *Electron. Lett.*, vol. 35, pp. 14–16, Jan. 1999.

[19] O. Tirkkonen and A. Hottinen, "Square-matrix embeddable space-time block codes for complex signal constellations," *IEEE Trans. Inform. Theory*, vol. 48, pp. 384–396, Feb. 2002.

[20] S. M. Alamouti, "A simple transmit diversity technique for wireless communications," *IEEE J. Select. Areas Commun.*, vol. 16, pp. 1451–1458, Oct. 1998.

[21] V. Tarokh, H. Jafarkhani, and A. R. Calderbank, "Space-time block codes from orthogonal designs," *IEEE Trans. Inform. Theory*, vol. 45, pp. 1456–1466, July 1999.

[22] O. Tirkkonen, A. Boariu, and A. Hottinen, "Minimal nonorthogonality rate 1 space-time block code for $3+$ Tx antennas," in *Proc. IEEE Int. Symp.. Spread Spectrum Techniques and Applications*, Parsippany, NJ, Sept. 2000, pp. 429–432.

[23] O. Tirkkonen and A. Hottinen, "Improved MIMO performance with nonorthogonal space-time block codes," in *Proc. IEEE Global Communications Conf.*, San Antonio, TX, Nov. 2001, pp. 1122–1126.

[24] M. O. Damen, H. El Gamal, and N. C. Beaulieu, "Systematic construction of full diversity algebraic constellations," *IEEE Trans. Inform. Theory*, to be published.

[25] D. A. Marcus, *Number Fields*. New York: Springer-Verlag, 1977.

[26] S. Lang, *Algebraic Number Theory*. New York: Springer-Verlag, 1970.

[27] A. B. Shidlovskii, *Transcendental Numbers*, W. de Gruyter, Ed. New York, 1989.

[28] H. Cohen, *A Course in Computational Algebraic Number Theory*. Berlin, Germany: Springer-Verlag, 1993.

[29] W. M. Schmidt, *Diophantine Approximation*. Berlin, Germany: Springer-Verlag, 1980.

[30] J. G. Proakis, *Digital Communications*, 4th ed. New York: McGraw-Hill, 2000.

[31] C. F. Van Loan, *Computational Frameworks for the Fast Fourier Transform*. Philadelphia, PA: SIAM, 1992.

[32] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 1993.

[33] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, 6th ed. San Diego, CA: Academic, 2000.