On the Design of Space–Time and Space–Frequency Codes for MIMO Frequency-Selective Fading Channels

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***Abstract*—Recently, the authors introduced an algebraic design framework for space–time coding in flat-fading channels [1]–[4]. In this correspondence, we extend this framework to design algebraic codes for multiple-input multiple-output (MIMO) frequency-selective fading channels. The proposed codes strive to optimally exploit both the spatial and frequency diversity available in the channel. We consider two design approaches: The first uses space–time coding and maximum likelihood decoding to exploit the multi-path nature of the channel at the expense of increased receiver complexity. Within this time domain framework, we also propose a serially concatenated coding construction which is shown to offer a performance gain with a reasonable complexity iterative receiver in some scenarios. The second approach utilizes the orthogonal frequency division multiplexing technique to transform the MIMO multi-path channel into a MIMO flat block fading channel. The algebraic framework [1] is then used to construct space-frequency codes (SFC) that optimally exploit the diversity available in the resulting flat block fading channel. Finally, the two approaches are compared in terms of decoder complexity, maximum achievable diversity advantage, and simulated frame error rate performance in certain representative scenarios.**

***Index Terms*—Algebraic space-time codes (STC), diversity, fading channels, multiple transmit and receive antennas, stacking construction.**

I. Introduction

The first prior work on space–time coding for frequency-selective channels appeared in [5], where it was argued that space–time codes (STCs) designed to achieve a certain diversity advantage in flat-fading channels, will achieve at least the same diversity advantage in frequency-selective fading channels. Similar arguments also appear in [6]. Recent independent work in this area include a space–time block code in [7] and some orthogonal frequency-division multiplexing (OFDM)-based designs in [8], [9].

In this correspondence, we design STCs that *fully* exploit the spatial and frequency diversity available in the channel. The uniqueness of our correspondence is that we attempt to realize this goal by developing an algebraic framework for STC design in such channels. This framework benefits from our earlier work [1], [4] for nonlayered STC design in multiple-input multiple-output (MIMO) flat-fading channels. Codes designed using this framework can achieve *guaranteed* level of diversity, which includes both space and frequency diversity. The focus on diversity and coding gain design criteria, rather than other design criterion optimized for concatenated coding systems,[1] is justified here since we are mainly interested in trellis STCs. We further limit ourselves in this correspondence to standard single-dimensional quadrature amplitude modulation (QAM).[2]

We consider two approaches for system design that trade diversity advantage for receiver complexity. The time-domain approach combines algebraic space–time coding with maximum-likelihood (ML) decoding to achieve the maximum possible diversity advantage in MIMO frequency-selective channels. Within this framework, we also propose a serially concatenated coding approach that offers a performance gain with a reasonable complexity iterative decoder [11]. The time-domain approach suffers from large trellis complexity that grows exponentially with the number of resolvable paths as will be shown later.

This exponentially growing complexity motivates the frequency-domain approach where an OFDM front-end is utilized to transform the intersymbol-interference (ISI)-fading channel into a flat block-fading channel. Then, we construct algebraic *space–frequency* codes (SFC) that strive to optimally exploit the diversity available in this *space–frequency* block-fading channel.

The rest of this correspondence is organized as follows. The system model is presented in Section II. In Section III, we compute the outage probability for the MIMO frequency-selective fading model. This theoretical limit serves as a lower bound on the frame error-rate performance, and motivates the design of coding schemes that exploit the higher diversity advantage allowed by the channel frequency selectivity. In Section IV, we present an algebraic framework for the design of STCs in MIMO frequency-selective fading channels. This framework benefits from the theory developed in [1], [4]. The space–frequency coding approach is investigated in Section V. In Section VI, we present simulation results that compare the two approaches and demonstrate the full diversity performance achieved by the proposed techniques. Finally, Section VII offers some concluding remarks.

II. Signal Model

We consider a multiple-antenna communication system with L_t transmit and L_r receive antennas. In this correspondence, we are interested in the scenario where the channel state information is only available at the receiver [1], [12], [13]. The transmitter is equipped with a channel encoder that accepts input from the information source and outputs a coded stream of higher redundancy suitable for error correction processing at the receiver. The encoded output stream is modulated and distributed among the L_t antennas. The transmissions from each of the L_t transmit antennas are simultaneous and synchronous. The signal received at each antenna is, therefore, a superposition of the L_t transmitted signals corrupted by additive white Gaussian noise and the multiplicative ISI fading. At the receiver end, the signal $r_t^{(j)}$ received by antenna j at time t is given by

$$r_t^{(j)} = \sqrt{E_s} \sum_{l=0}^{L_{\rm ISI}-1} \sum_{i=1}^{L_t} \alpha_l^{(i,j)} s_{t-l}^{(i)} + n_t^{(j)} \tag{1}$$

where E_s is the energy per transmitted symbol; $\alpha_l^{(i,j)}$ is the complex Gaussian path gain from transmit antenna i to receive antenna j for the lth path; $L_{\rm ISI}$ is the length of the channel impulse response;[3] $s_t^{(i)}$ is

[1]For example, the mutual information criterion [10].

[2]The single dimension refers to one complex dimension.

[3]Here we assume that the taps are spaced at integer multiples of the symbol duration, which is the worst case in terms of designing full diversity codes [14]. A generalization to fractionally spaced taps will be the subject of future work.

Manuscript received April 26, 2001; revised February 27, 2003. This work was supported by the National Science Foundation under Grant CCR-0118859 and NCR-9706372. The material in this paper was presented in part at the Fourth International Symposium on Wireless Personal Multimedia Communications, Aalborg, Denmark, September 2001 and at the International Conference on Communications (ICC 2001), Helsinki, Finland, June 2001.

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Communicated by R. Urbanke, Associate Editor for Coding Techniques.

Digital Object Identifier 10.1109/TIT.2003.815804

the symbol transmitted from antenna i at time t; $n_t^{(j)}$ is the additive white Gaussian noise sample for receive antenna j at time t. The noise samples are independent samples of circularly symmetric zero-mean complex Gaussian random variable with a variance $N_0/2$ per dimension. The different path gains $\alpha_l^{(i,j)}$ are assumed to be statistically independent. The fading model of primary interest is that of a quasi-static Rayleigh fading process in which the complex fading gains are constant over one codeword but are independent from one codeword to the next. Unless otherwise stated, all taps are assumed to have equal power. The received signal can be rewritten in a vector form as

$$\underline{r}_t = \sqrt{E_s} \sum_{l=0}^{L_{\rm ISI}-1} \underline{s}_{t-l}\alpha_l + \underline{n}_t \tag{2}$$

where

$$\underline{r}_t = [\, r_t^{(1)} \quad r_t^{(2)} \quad \dots \quad r_t^{(L_r)} \,]$$
$$\underline{s}_t = [\, s_t^{(1)} \quad s_t^{(2)} \quad \dots \quad s_t^{(L_t)} \,]$$
$$\underline{n}_t = [\, n_t^{(1)} \quad n_t^{(2)} \quad \dots \quad n_t^{(L_r)} \,]$$

and

$$\alpha_l = \begin{bmatrix} \alpha_l^{(1,1)} & \dots & \alpha_l^{(1,L_r)} \\ \vdots & \ddots & \vdots \\ \alpha_l^{(L_t,1)} & \dots & \alpha_l^{(L_t,L_r)} \end{bmatrix}.$$

The signal-to-noise ratio at receive antenna j is defined as

$$\text{SNR}^{(j)} = \frac{E_S}{N_0} \sum_{i=1}^{L_t} \sum_{l=0}^{L_{\rm ISI}-1} E\left(|\alpha_l^{(i,j)}|^2\right) \tag{3}$$

where the assumption of equal powers in the taps implies that

$$\text{SNR}^{(1)} = \cdots = \text{SNR}^{(L_r)} = \text{SNR}.$$

III. Outage Probability

In this section, we investigate the theoretic limit, outage probability [15]–[17], of the MIMO frequency-selective fading channel model in Section II. The outage probability provides a lower bound on the frame error-rate performance when the distribution of the transmitted signals are assumed independent and identically distributed (i.i.d.) circularly symmetrical complex Gaussian, and hence, quantifies the potential gain offered by the frequency selectivity of the channel.

The outage probability $P_{\rm out}(R)$ is defined as the probability that the mutual information rate $\mathbb{I}_{|\alpha}^{\infty}$ between the transmitted signal and received signal is below a certain transmission rate R, i.e.,

$$P_{\rm out}(R) := \Pr\left\{\alpha \colon \mathbb{I}_{|\alpha}^{\infty} \le R\right\}.$$

The mutual information rate $\mathbb{I}_{|\alpha}^{\ell}$, which is a function of the channel coefficients α, is defined as

$$\mathbb{I}_{|\alpha}^{\ell} := \frac{1}{\ell}\, \mathbb{I}(\boldsymbol{s};\, \boldsymbol{r}|\boldsymbol{\alpha} = \alpha)$$

where

$$\boldsymbol{r} = \begin{bmatrix} \underline{r}_0 \\ \vdots \\ \underline{r}_{\ell-1} \end{bmatrix}$$

$$\boldsymbol{s} = \begin{bmatrix} \underline{s}_0 \\ \vdots \\ \underline{s}_{\ell-1} \end{bmatrix}$$

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{L_{\rm ISI}-1} \end{bmatrix} \tag{4}$$

and α is a realization of $\boldsymbol{\alpha}$.

In [18], Hirt and Massey proposed a technique to evaluate the mutual information using circular convolution approximation for scalar ISI channels. In this section, we extend their technique to multiple-antenna (vector) channels. First, let us modify the convolution in (2) to the circular convolution form as follows:

$$\underline{r}'_t = \sqrt{E_s} \sum_{k=0}^{L_{\rm ISI}-1} \underline{s}_{(t-k) \bmod \ell}\, \alpha_k + \underline{n}_t, \quad t = 0, 1, \dots, \ell - 1. \tag{5}$$

The corresponding mutual information rate $\mathbb{I}'^{\ell}_{|\alpha}$ is thus defined as

$$\mathbb{I}'^{\ell}_{|\alpha} := \frac{1}{\ell}\, \mathbb{I}(\boldsymbol{s};\, \boldsymbol{r}'|\boldsymbol{\alpha} = \alpha).$$

Using a similar argument to that in [18], it can be easily shown that

$$\mathbb{I}'^{\infty}_{|\alpha} = \mathbb{I}^{\infty}_{|\alpha}. \tag{6}$$

The modified mutual information rate $\mathbb{I}'^{\ell}_{|\alpha}$ can be conveniently evaluated through the discrete Fourier transform (DFT) [18]. Applying the DFT to both sides of (5), one obtains

$$\underline{\tilde{r}}'_m = \underline{\tilde{s}}_m \tilde{\alpha}_m + \underline{\tilde{n}}_m, \qquad m = 0, 1, \dots \ell - 1$$

where

$$\underline{\tilde{r}}'_m = \sum_{t=0}^{\ell-1} \underline{r}'_t \exp\left(-j\, \frac{2\pi}{\ell}\, mt\right)$$
$$\underline{\tilde{s}}_m = \sum_{t=0}^{\ell-1} \underline{s}_t \exp\left(-j\, \frac{2\pi}{\ell}\, mt\right)$$
$$\tilde{\alpha}_m = \sum_{l=0}^{L_{\rm ISI}-1} \alpha_l \exp\left(-j\, \frac{2\pi}{\ell}\, lm\right)$$
$$\underline{\tilde{n}}_m = \sum_{t=0}^{\ell-1} \underline{n}_t \exp\left(-j\, \frac{2\pi}{\ell}\, mt\right).$$

We assume the input signals $s_t^{(i)}$'s have i.i.d. circularly symmetrical complex Gaussian distribution across space and time. Due to the orthogonality of the basis functions, the signals $\tilde{s}_m^{(i)}$'s in frequency domain are also i.i.d circularly symmetric complex Gaussian distributed random variables. The same is true for the noise.

Since the DFT is an invertible operation, the mutual information rate between random variables in time domain is equal to that in frequency domain. Therefore,

$$\mathbb{I}'^{\ell}_{|\alpha} = \frac{1}{\ell} \sum_{m=0}^{\ell-1} \log_2\left(\det\left(\boldsymbol{I}_{L_r} + \frac{\text{SNR}}{L_t} \tilde{\alpha}_m^T \tilde{\alpha}_m^*\right)\right) \tag{7}$$

where T denotes transpose, $*$ denotes complex conjugate. By the definition of Riemann integral and (6), we obtain

$$\mathbb{I}^{\infty}_{|\alpha} = \int_0^1 \log_2\left(\det\left(\boldsymbol{I}_{L_r} + \frac{\text{SNR}}{L_t}\, \tilde{\alpha}_f^T \tilde{\alpha}_f^*\right)\right) df$$

where

$$\tilde{\alpha}_f = \sum_{k=0}^{L_{\rm ISI}-1} \alpha_k \exp(-j2\pi f k).$$

It is often more convenient to use (7) to evaluate the outage probability by Monte Carlo simulation. In this section, we follow the Monte

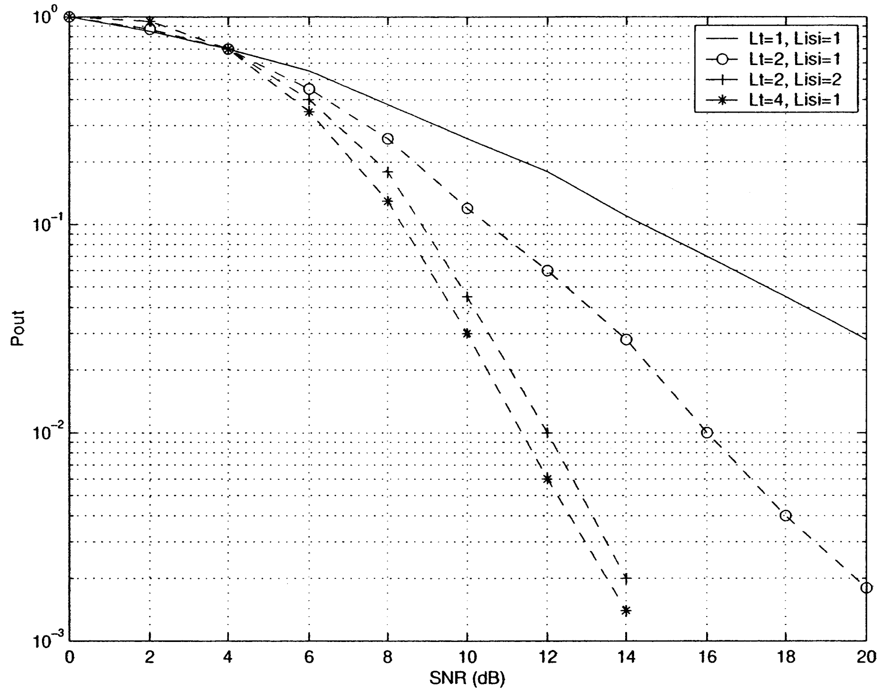

Fig. 1. The outage probability for systems with different number of transmit antennas, and different number of taps at transmission rate 2 bits/symbol.

Carlo simulation approach with a frame length $\ell = 64$. The use of larger frame sizes does not significantly change the results.

Fig. 1 reports the outage probability for systems with different number of transmit antennas, one receive antenna, and one or two taps at a transmission rate $R = 2$ bits/symbol. It is shown that the outage probability for the case of two transmit antennas and two taps is close to that obtained with four transmit antenna and one tap. This observation and the fact that non-ISI channels with four transmit antenna and one receive antenna are known to offer four levels of diversity[4] suggest that the frequency selectivity of the channel can be exploited to achieve a higher diversity advantage. The significance of the higher diversity advantage can be easily seen when comparing with the case of two transmit antennas and one tap. At $P_{\rm out}(R) = 0.01$, the ISI offers a potential gain of 4 dB. This potential performance gain motivates the construction of efficient coding schemes in the following sections.

IV. Space–Time Coding Approach

In this section, we consider the single-carrier time-domain approach where algebraic STCs that efficiently exploit the diversity available in the MIMO frequency-selective fading channel are proposed. An algebraic framework for the design of trellis STCs is first presented. This framework assumes ML decoding that accounts for the ISI nature of the channel at the receiver. Then, we present a serially concatenated coding scheme that allows for further performance gains without undue complexity.

A. Trellis STC

As in [1], we formally define an STC to consist of an underlying error control code together with a spatial parsing formatter.

Definition 1: An $L_t \times \ell$ STC $\mathcal{C}$ of size M consists of an $(L_t\ell, M)$ error-control code C and a spatial parser σ that maps each codeword vector $\overline{c} \in C$ to an $L_t \times \ell$ matrix $\boldsymbol{c}$ whose entries are a rearrangement of those of $\overline{c}$. The STC $\mathcal{C}$ is said to be linear if both C and σ are linear in the domain where C is defined.

Except as noted to the contrary, we will assume that the standard parser maps

$$\overline{c} = \left(c_1^{(1)}, c_1^{(2)}, \ldots, c_1^{(L_t)}, c_2^{(1)}, c_2^{(2)}, \ldots, c_2^{(L_t)}, \ldots, c_\ell^{(1)}, c_\ell^{(2)}, \ldots, c_\ell^{(L_t)}\right) \in C$$

to the matrix

$$\boldsymbol{c} = \begin{bmatrix} c_1^{(1)} & c_2^{(1)} & \cdots & c_\ell^{(1)} \\ c_1^{(2)} & c_2^{(2)} & \cdots & c_\ell^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ c_1^{(L_t)} & c_2^{(L_t)} & \cdots & c_\ell^{(L_t)} \end{bmatrix}_{L_t \times \ell}. \tag{8}$$

The baseband codeword $\boldsymbol{f(c)}$ is obtained by applying the modulation operator f on the components of $\boldsymbol{c}$. This modulation operator maps the entries of $\boldsymbol{c}$ into constellation points from the discrete complex-valued signaling constellation Ω for transmission across the channel. In this notation, it is understood that $c_t^{(i)}$ is the code symbol assigned to transmit antenna i at time t and $s_t^{(i)} = f(c_t^{(i)})$.

The diversity advantage of an STC is defined as the minimum absolute value of the slope of any pairwise probability of error versus signal-to-noise ratio curve on a log–log scale. The following rank criterion was proposed in [12], [13] to maximize the spatial transmit diversity advantage in quasi-static flat-fading MIMO channels

- *Baseband Rank Criterion*: Maximize $d = \operatorname{rank}(f(\boldsymbol{c}) - f(\boldsymbol{e}))$ over all pairs of distinct codewords $\boldsymbol{c}, \boldsymbol{e} \in \mathcal{C}$.

Therefore, full spatial transmit diversity is achieved if and only if $\operatorname{rank}(f(\boldsymbol{c}) - f(\boldsymbol{e})) = L_t$ for all pairs of distinct codewords $\boldsymbol{c}, \boldsymbol{e} \in \mathcal{C}$. Note that in the presence of L_r receive antennas, the total diversity advantage achieved by a full diversity code is $L_t L_r$.

[4]That is, with proper code design, the slope of the pairwise probability of error curve approaches four on a log–log scale [12].

STC constructions for frequency-selective fading channels is based on the idea of “virtual transmit antennas”: In an ISI environment with $L_{\rm ISI}$ resolvable paths, a space–time system with L_t transmit antennas is equivalent to a space–time system operating in flat-fading channel with $L_t L_{\rm ISI}$ transmit antennas. In the equivalent model, however, the codeword matrices are restricted to a certain special structure. This structure is captured in the following definition for the baseband codeword matrix in the ISI environment:

$$\boldsymbol{f}(\boldsymbol{c})_{\rm ISI} = \begin{bmatrix} f_0(\boldsymbol{c}) \\ f_1(\boldsymbol{c}) \\ \vdots \\ f_{L_{\rm ISI}-1}(\boldsymbol{c}) \end{bmatrix}_{L_t L_{\rm ISI} \times (\ell + L_{\rm ISI} - 1)}$$

where $\boldsymbol{c}$ is the codeword matrix as defined in (8) and $f_m(\boldsymbol{c})$ is a time-delayed copy of $f(\boldsymbol{c})$, i.e.,

$$f_m(\boldsymbol{c}) = [\mathbf{0}_{L_t \times m}\ f(\boldsymbol{c})\ \mathbf{0}_{L_t \times (L_{\rm ISI}-m-1)}].$$

Now, the received signal matrix $\boldsymbol{r}$ can be written as

$$\boldsymbol{r} = \boldsymbol{f}(\boldsymbol{c})_{\rm ISI}^T \boldsymbol{\alpha} + \boldsymbol{n} \tag{9}$$

where $\boldsymbol{r}$ and $\boldsymbol{\alpha}$ are defined in (4). From this equivalent model, it is clear that in frequency-selective fading channels, the maximum transmit diversity order is $L_t L_{\rm ISI}$. Not all codes that achieve full space diversity in flat-fading channels can achieve this maximum diversity. However, a carefully designed code can achieve it. In particular, codes designed using our algebraic framework can achieve the maximum diversity order. This achievability is established later in a comment after Proposition 6. Following in the footsteps of [13] and [12], we have the following baseband design criterion for STCs in the ISI channel under consideration.

- *ISI Baseband Rank Criterion*: Maximize

  $$d = \operatorname{rank}(\boldsymbol{f}(\boldsymbol{c})_{\rm ISI} - \boldsymbol{f}(\boldsymbol{e})_{\rm ISI})$$

  over all pairs of distinct codewords $\boldsymbol{c}, \boldsymbol{e} \in \mathcal{C}$.

Full transmit diversity in this scenario is equal to $L_t L_{\rm ISI}$, and is achieved if and only if $\operatorname{rank}(\boldsymbol{f}(\boldsymbol{c})_{\rm ISI} - \boldsymbol{f}(\boldsymbol{e})_{\rm ISI}) = L_t L_{\rm ISI}$ for all pairs of distinct codewords $\boldsymbol{c}, \boldsymbol{e} \in \mathcal{C}$.

Next we follow the approaches in [1], [4] to develop binary and Σ_o-rank criteria that facilitate the construction of algebraic binary STCs for binary phase-shift keying (BPSK) systems, $\mathbb{Z}_4$ STCs for quaternary phase-shift keying (QPSK) systems, and $\mathbb{Z}_{2^k}(j)$ STCs for square shaped 2^{2k}-QAM systems with arbitrary number of transmit antennas and channel impulse response lengths.

Let us define a new codeword matrix $\boldsymbol{c}_{\rm ISI}$ that captures the ISI nature of the channel as

$$\boldsymbol{c}_{\rm ISI} = \begin{bmatrix} \boldsymbol{c}_0 \\ \boldsymbol{c}_1 \\ \vdots \\ \boldsymbol{c}_{L_{\rm ISI}-1} \end{bmatrix}_{L_t L_{\rm ISI} \times (\ell + L_{\rm ISI} - 1)}$$

where $\boldsymbol{c}_m$ is a time-delayed copy of $\boldsymbol{c}$, i.e.,

$$\boldsymbol{c}_m = [\mathbf{0}_{L_t \times m}\ \boldsymbol{c}\ \mathbf{0}_{L_t \times (L_{\rm ISI}-m-1)}].$$

First, we observe that in general

$$\boldsymbol{f}(\boldsymbol{c}_{\rm ISI}) \neq \boldsymbol{f}(\boldsymbol{c})_{\rm ISI} \tag{10}$$

since

$$f(\underline{\mathbf{0}}) \neq \underline{\mathbf{0}}. \tag{11}$$

However, we note that the diversity advantage only depends on the differences between codewords rather than the codewords themselves, and we have

$$\boldsymbol{f}(\boldsymbol{c}_{\rm ISI}) - \boldsymbol{f}(\boldsymbol{e}_{\rm ISI}) = \boldsymbol{f}(\boldsymbol{c})_{\rm ISI} - \boldsymbol{f}(\boldsymbol{e})_{\rm ISI} \tag{12}$$

for any signaling constellation. This observation is the key to the algebraic space–time constructions developed in this section.

In the following, we focus our discussion on binary codes for BPSK modulation. Extensions to $\mathbb{Z}_4$ QPSK codes and $\mathbb{Z}_{2^k}(j)$ QAM codes will be briefly outlined at the end of the section.

For binary codes with BPSK modulation, elements in $\boldsymbol{c}$ are drawn from the field $\mathbb{F} = \{0, 1\}$ of integers modulo 2. The modulation operator f maps the symbol $c_t^{(i)} \in \mathbb{F}$ to the constellation point

$$s_t^{(i)} = f(c_t^{(i)}) \in \{-1, 1\}$$

according to the rule $f(c_t^{(i)}) = (-1)^{c_t^{(i)}}$.

The binary rank criterion for full diversity STCs in ISI channels can be stated as follows.

Proposition 2 (The ISI Binary Rank Criterion): Let $\mathcal{C}$ be a linear $L_t \times \ell$ STC in binary field $\mathbb{F}$ with underlying binary code C of length $N = L_t \ell$ operating in an ISI channel with $L_{\rm ISI}$ paths, where $\ell \geq L_t L_{\rm ISI}$. Suppose that every nonzero codeword $\boldsymbol{c}$ corresponds to a matrix $\boldsymbol{c}_{\rm ISI}$, which is of full rank $L_t L_{\rm ISI}$ over the binary field $\mathbb{F}$. Then, for BPSK transmission over the frequency-selective quasi-static fading channel, the STC $\mathcal{C}$ achieves full transmit diversity $L_t L_{\rm ISI}$.

Proof: This proposition can be proven by replacing $\boldsymbol{c}$ with $\boldsymbol{c}_{\rm ISI}$ in the proof of the BPSK binary rank criterion for flat-fading channels in [1]. □

While the previous result was stated for full transmit diversity codes, it readily generalizes to any order of transmit diversity less than or equal to $L_t L_{\rm ISI}$. The ISI channel binary rank criterion opens the door for the following stacking construction which establishes an algebraic framework for STC design in MIMO ISI fading channels.

Proposition 3 (The ISI Stacking Construction for BPSK Modulation): Let $\boldsymbol{M}_1, \boldsymbol{M}_2, \ldots, \boldsymbol{M}_{L_t}$ be binary matrices of dimension $k \times \ell$, $\ell \geq k$, and let $\mathcal{C}$ be the $L_t \times \ell$ STC of dimension k consisting of the codeword matrices

$$\boldsymbol{c} = \begin{bmatrix} \underline{x}\,\boldsymbol{M}_1 \\ \underline{x}\,\boldsymbol{M}_2 \\ \vdots \\ \underline{x}\,\boldsymbol{M}_{L_t} \end{bmatrix}$$

where $\underline{x}$ denotes an arbitrary k-tuple of information bits and $L_t \leq \ell$. Denote

$$\boldsymbol{M}_{n,m} = \left[\mathbf{0}_{k \times (m-1)} \boldsymbol{M}_n \mathbf{0}_{k \times (L_{\rm ISI}-m)}\right] \tag{13}$$

where $\mathbf{0}_{\boldsymbol{k} \times (\boldsymbol{m}-\mathbf{1})}$ is the $k \times (m-1)$ all-zero matrix. Then, $\mathcal{C}$ satisfies the ISI binary rank criterion, and thus, for BPSK transmission over the frequency-selective quasi-static fading channel, achieves full transmit diversity $L_t L_{\rm ISI}$, if and only if $\boldsymbol{M}_{1,1}, \boldsymbol{M}_{2,1}, \ldots, \boldsymbol{M}_{L_t, L_{\rm ISI}}$ have the property that

$$\forall a_1, a_2, \ldots, a_{L_t L_{\rm ISI}} \in \mathbb{F}:$$

$$\boldsymbol{M} = a_1 \boldsymbol{M}_{1,1} \oplus a_2 \boldsymbol{M}_{2,1} \oplus \cdots \oplus a_{L_t L_{\rm ISI}} \boldsymbol{M}_{L_t, L_{\rm ISI}}$$

is of full rank k unless $a_1 = \cdots = a_{L_t L_{\rm ISI}} = 0$.

TABLE I
THE DIVERSITY ADVANTAGE FOR BPSK ALGEBRAIC SPACE–TIME CODES WITH OPTIMAL FREE DISTANCE FOR MIMO FREQUENCY-SELECTIVE FADING CHANNELS

L_t	ν	**Connection Polynomials**	d for $L_{\rm ISI}=1$	$L_{\rm ISI}=2$	$L_{\rm ISI}=3$	$L_{\rm ISI}=4$
2	2	5, 7	2	4	5	6
	3	64, 74	2	4	6	7
	4	46, 72	2	4	6	8
	5	65, 57	2	4	6	8
	6	554, 744	2	4	6	8
3	3	54, 64, 74	3	5	6	7
	4	52, 66, 76	3	6	7	8
	5	47, 53, 75	3	6	8	9
	6	554, 624, 764	3	6	9	10
4	4	52, 56, 66, 76	4	6	7	8
	5	53, 67, 71, 75	4	7	8	9
5	5	75, 71, 73, 65, 57	5	7	8	9

Proof: First we note that

$$\boldsymbol{c}_{\rm ISI} = \begin{bmatrix} \underline{x}\,\boldsymbol{M}_{1,1} \\ \underline{x}\,\boldsymbol{M}_{2,1} \\ \vdots \\ \underline{x}\,\boldsymbol{M}_{L_t, L_{\rm ISI}} \end{bmatrix}$$

then the proof follows the same argument used for the stacking construction in flat-fading channels [1]. □

The stacking construction is general and applies for block as well as trellis codes. Trellis codes, however, enjoy more practical importance because they allow for a reasonable complexity ML decoder. Therefore, we will content ourselves only with trellis codes in the remainder of this section.

Let C be the binary, rate $1/L_t$, convolutional code having transfer function matrix [19]

$$\boldsymbol{G}(D) = [\, g_1(D) \quad g_2(D) \quad \cdots \quad g_{L_t}(D)\,].$$

The *natural BPSK STC $\mathcal{C}$ associated with C* is defined to consist of the codeword matrices $\boldsymbol{c}(D) = \boldsymbol{G}^{\rm T}(D)x(D)$, where the polynomial $x(D)$ represents the input information bit stream. In other words, for the natural STC, we adopt the natural transmission format in which the output coded bits generated by $g_i(D)$ are transmitted via antenna i. As in [12], we assume the trellis codes are terminated by tail bits. Thus, if $x(D)$ is restricted to a block of N information bits, then $\mathcal{C}$ is an $L_t \times (N+\nu+L_{\rm ISI}-1)$ STC, where $\nu = \max_{1\le i\le L_t}\{\deg g_i(x)\}$ is the maximal memory order of the convolutional code C. Denote

$$\boldsymbol{G}_{\rm ISI}(D) = [g_{1,1}(D)\; g_{2,1}(D)\; \cdots\; g_{L_t,1}(D)\; \cdots\; g_{L_t, L_{\rm ISI}}(D)]$$

where $g_{n,m} = D^{(m-1)} g_n$. Then, we have the following result that characterizes the performance of natural space–time convolutional codes in MIMO ISI channels.

Proposition 4: The natural STC $\mathcal{C}$ associated with the rate $1/L_t$ convolutional code C satisfies the binary rank criterion, and thus achieves full transmit diversity for BPSK transmission in an ISI channel with $L_{\rm ISI}$ paths, if and only if the transfer function matrix $\boldsymbol{G}_{\rm ISI}(D)$ of C has full rank $L_t L_{\rm ISI}$ as a matrix of coefficients over $\mathbb{F}$.

Proof: One can easily prove Proposition 4 by observing that

$$\sum_{1\le i\le L_t,\, 1\le j\le L_{\rm ISI}} a_{i,j}\, g_{i,j}(D) x(D) = 0$$

for some $x(D) \neq 0$ iff

$$\sum_{1\le i\le L_t,\, 1\le j\le L_{\rm ISI}} a_{i,j}\, g_{i,j}(D) = 0.$$

This proof readily generalizes to recursive convolutional codes. □

Proposition 4 extends in a straightforward fashion to convolutional codes with arbitrary rates and arbitrary diversity orders. Since the coefficients of $\boldsymbol{G}_{\rm ISI}(D)$ form a binary matrix of dimension $(\nu + L_{\rm ISI}) \times L_t L_{\rm ISI}$ and the column rank must be equal to the row rank, the proposition provides a simple bound as to how complex the convolutional code must be in order to satisfy the full diversity ISI channel binary rank criterion.

Corollary 5: The maximum diversity order achieved by an STC based on an underlying rate $1/L_t$ convolutional code C with a maximal memory order ν in an $L_{\rm ISI}$ paths ISI channel is $\nu + L_{\rm ISI}$.

This bound shows that, for a fixed trellis complexity, increasing the number of antennas beyond $L_t = \frac{\nu + L_{\rm ISI}}{L_{\rm ISI}}$ will not increase the diversity advantage. This fact is validated by the results reported in Table I. Since the number of paths is not known *a priori* at the transmitter, it is desirable to construct STCs that achieve the maximum diversity order for an arbitrary number of paths. This leads to the notion of *universal STCs* that combine the maximum spatial diversity with the frequency diversity, *whenever available*. Within the class of universal STCs with maximum diversity advantage, one would ideally pick the code with the maximum product distance [12], [13]. Unfortunately, to the best of our knowledge, no systematic framework for optimizing the code product distance has been proposed in the literature. In the absence of such a framework, we have resorted in [1] to optimizing the code free distance. The simulated performance of those optimal free-distance STCs achieving full diversity was later shown to compare favorably with other proposed STCs obtained by exhaustive computer search methods [20]. Reference [1, Table I] reports full-diversity convolutional codes with maximum free distance for flat-fading channels covering a wide range of constraint lengths and numbers of antennas. In Table I, we report the diversity advantage achieved by these codes in ISI channels with different $L_{\rm ISI}$. From the table, one can see that these codes are *universal* in the sense that they achieve the maximum possible diversity order given by Corollary 5 in all considered cases. In addition, the good performance of these codes is validated by the simulation results in Section VI.

The ISI binary rank criterion and stacking construction for BPSK modulation can be extended to obtain similar results for QPSK

TABLE II
LINEAR $\mathbb{Z}_4$ SPACE–TIME CODES FOR QPSK MODULATION IN MIMO FREQUENCY-SELECTIVE FADING CHANNELS

L	ν	**Connection Polynomials**
2	1	$1+2D, 2+D$.
	2	$1+2D+D^2, 1+D+D^2$.
	3	$1+D+2D^2+D^3, 1+D+D^2+D^3$.
	4	$1+2D+2D^2+D^3+D^4, 1+D+D^2+2D^3+D^4$.
	5	$1+D+2D^2+D^3+2D^4+D^5, 1+2D+D^2+D^3+D^4+D^5$.
3	2	$1+2D+2D^2, 2+D+2D^2, 1+D+2D^2$.
	3	$1+2D+D^2+D^3, 1+D+2D^2+D^3, 1+D+D^2+D^3$.
	4	$1+2D+D^2+2D^3+D^4, 1+D+2D^2+D^3+D^4, 1+D+D^2+D^3+D^4$.
	5	$1+2D+2D^2+D^3+D^4+D^5, 1+2D+D^2+2D^3+D^4+D^5,$ $1+D+D^2+D^3+2D^4+D^5$.
4	3	$1+2D+2D^2+2D^3, 2+D+2D^2+2^3, 2+2D+D^2+2D^3, 2+2D+2D^2+D^3$
	4	$1+2D+D^2+2D^3+D^4, 1+2D+D^2+D^3+D^4, 1+D+2D^2+D^3+D^4,$ $1+D+D^2+D^3+D^4$.
	5	$1+2D+D^2+2D^3+D^4+D^5, 1+D+2D^2+D^3+D^4+D^5,$ $1+D+D^2+2D^3+2D^4+D^5, 1+D+D^2+D^3+2D^4+D^5$.
5	4	$1+2D+2D^2+2D^3+2D^4, 2+D+2D^2+2D^3+2D^4, 2+2D+D^2+2D^3+2D^4,$ $2+2D+2D^2+D^3+2D^4, 2+2D+2D^2+2D^3+D^4$.
	5	$1+D+D^2+D^3+2D^4+D^5, 1+D+D^2+2D^3+2D^4+D^5,$ $1+D+D^2+2D^3+D^4+D^5, 1+D+2D^2+D^3+2D^4+D^5,$ $1+2D+D^2+D^3+D^4+D^5$.

modulation using the machinery introduced in [1] (those results were omitted here for brevity). As a consequence of the QPSK ISI binary rank criterion and stacking construction, one sees that the binary connection polynomials of Table I can be used to generate linear, $\mathbb{Z}_4$-valued, rate $1/L_t$ space–time trellis codes for QPSK modulation. More generally, one may use any set of $\mathbb{Z}_4$-valued connection polynomials whose modulo 2 projections appear in Table I. In most of the cases considered, the best performance was obtained from the $\mathbb{Z}_4$ codes constructed by replacing the zero coefficients by twos in Table I. This lifting produces the codes reported in Table II. The simulation results reported in Section VI will therefore focus on these codes.

For QAM codes, we utilize the concept of Σ_o-rank introduced in [4][5] where the elements in $\boldsymbol{c}$ are drawn from the complex integer ring $\mathbb{Z}_{2^k}(j)$. The modulation operator f maps the symbol $c_t^{(i)} \in \mathbb{Z}_{2^k}(j)$ to the constellation point $s_t^{(i)} = f(c_t^{(i)})$ according to the rule

$$f(c_t^{(i)}) = c_t^{(i)} - ((2^k-1)/2 + j(2^k-1)/2)$$

which is called translation mapping. This machinery facilitates the design of full diversity QAM STCs as formalized in the following proposition.

Proposition 6 (The ISI Stacking Construction for QAM): Let $\boldsymbol{M}_1, \boldsymbol{M}_2, \cdots, \boldsymbol{M}_{L_t}$ be $\mathbb{Z}_{2^k}(j)$ matrices of dimension $k \times \ell$, $\ell \geq k$, and let $\mathcal{C}$ be the $L_t \times \ell$ STC consisting of the codeword matrices

$$\boldsymbol{c} = \begin{bmatrix} \underline{x}\boldsymbol{M}_1 \\ \underline{x}\boldsymbol{M}_2 \\ \vdots \\ \underline{x}\boldsymbol{M}_{L_t} \end{bmatrix}$$

[5]The Σ_o-rank definitions necessary for the development in this correspondence are included in the Appendix.

where $\underline{x}$ denotes an k-tuple information sequence defined on integer ring $\mathbb{Z}_{2^k}$ and $L_t \leq \ell$. Denote

$$\boldsymbol{M}_{n,m} = \left[\mathbf{0}_{L_t\times(m-1)}\boldsymbol{M}_n\mathbf{0}_{L_t\times(L_{\mathrm{ISI}}-m)}\right] \tag{14}$$

where $\mathbf{0}_{L_t\times(m-1)}$ is the $L_t \times (m-1)$ all-zero matrix. Then, $\mathcal{C}$ achieves full transmit diversity $L_t L_{\mathrm{ISI}}$ for QAM transmission over the frequency selective quasi-static fading channel, if $\boldsymbol{M}_{1,1}, \boldsymbol{M}_{2,1}, \cdots, \boldsymbol{M}_{L_t, L_{\mathrm{ISI}}}$ have the property that

$\forall\, \Sigma_o-$coefficient set $\{a_1, a_2, \ldots, a_{L_t L_{\mathrm{ISI}}}\}$ defined on $\mathbb{Z}_{2^k}(j)$:
$\boldsymbol{M} = a_1\boldsymbol{M}_{1,1} \oplus a_2\boldsymbol{M}_{2,1} \oplus \cdots \oplus a_{L_tL_{\mathrm{ISI}}}\boldsymbol{M}_{L_t, L_{\mathrm{ISI}}}$
is not singular, i.e.,

$\underline{x}\boldsymbol{M} \neq \underline{0}$ for all $\underline{x} \neq \underline{0}$

where $\oplus$ is modulo 2^k addition.

Proof: By replacing the generator matrices with $M_{n,m}$, the proof follows from the Σ_o-rank conditions on generator matrices in [4]. □

This sufficient condition of achieving full diversity is a set of finite number of linear constraints. When the unknowns in the generator matrices, or equivalently the memory size for trellis-type codes, increase, one can always find generator matrices that satisfy all of the constraints. Since the BPSK binary rank criterion is a special case of this Σ_o-rank criterion, the above establishes that using the algebraic framework in this correspondence we can always design codes that achieve full space and frequency diversity.

Similar to the BPSK case, the ISI stacking construction for QAM constellations can be used to develop a systematic approach for designing full diversity trellis codes. Let C be a $\mathbb{Z}_{2^k}$, rate $1/(4L_t)$ code with the transfer function matrix

$$\boldsymbol{G}(D) = [\,g_1(D) \quad g_2(D) \quad \cdots \quad g_{4L_t}(D)\,].$$

The generator polynomials of the corresponding $\mathbb{Z}_{2^k}(j)$ code is

$$\overline{\boldsymbol{G}}(D) = \begin{bmatrix} \overline{g}_1^{(1)}(D) & \cdots & \overline{g}_{L_t}^{(1)}(D) \\ \overline{g}_1^{(2)}(D) & \cdots & \overline{g}_{L_t}^{(2)}(D) \end{bmatrix} = \begin{bmatrix} g_1(D) \oplus jg_2(D) & \cdots & g_{4L_t-3}(D) \oplus jg_{4L_t-2}(D) \\ g_3(D) \oplus jg_4(D) & \cdots & g_{4L_t-1}(D) \oplus jg_{4L_t}(D) \end{bmatrix}.$$

The *natural* $2k$*-bits/symbol QAM STC* $\mathcal{C}$ *associated with* C is defined to consist of the codeword matrices $\boldsymbol{c}^{\mathrm{T}}(D) = [x^{(1)}(D)x^{(2)}(D)]\overline{\boldsymbol{G}}(D)$, where the polynomials $x^{(1)}(D)$ and $x^{(2)}(D)$ represent the $\mathbb{Z}_{2^k}$ input information streams. Define $\overline{g}_{n,m}^{(k)} = D^{(m-1)}\overline{g}_n^{(k)}$, then the following proposition establishes conditions on full diversity ISI trellis codes for QAM constellations.

Proposition 7: The natural STC $\mathcal{C}$ associated with the rate $1/4L_t$ code C satisfies the Σ_o-rank criterion, and thus achieves full transmit diversity for QAM transmission in an ISI channel with L_{ISI} paths, if the real part $g_I^{(k)}(D)$ and imaginary part $g_Q^{(k)}(D)$ of

$$g^{(k)}(D) = g_I^{(k)}(D) \oplus jg_Q^{(k)}(D) = \sum_{1\le i\le L_t,\, 1\le j\le L_{\mathrm{ISI}}} a_{i,j}\overline{g}_{i,j}^{(k)}(D)$$

satisfy the condition that $(g_I^{(1)}(x)g_Q^{(2)}(x) \ominus g_Q^{(1)}(x)g_I^{(2)}(x))$ has at least one odd coefficient for all possible Σ_o-coefficient sets

$$\{a_{i,j},\, 1 \le i \le L_t,\, 1 \le j \le L_{\mathrm{ISI}}\}.$$

Proof: This result is a straightforward application of Proposition 6 in this correspondence, and [4, Proposition 10]. □

Proposition 7 also offers an alternative approach to construct QPSK STCs from binary convolutional codes. In this case, we simply set $k = 1$ in Proposition 7, and let C be a binary, rate $1/(4L_t)$ convolutional code.

B. Concatenated Coding Approach

In this construction, the ISI channel is considered the inner code and the rank criteria are used to design an outer code such that the concatenated code achieves full diversity, assuming ML decoding. The outer code is composed of a trellis code followed by a symbol channel interleaver. For full diversity, the generator matrices $\boldsymbol{M}_i$ corresponding to the concatenation of the symbol interleaver and trellis encoders must satisfy the conditions in the stacking construction(s). The construction procedure is to select a trellis code first, then randomly select the interleaver. If the code and interleaver pair does not satisfy the rank criterion, another interleaver is randomly chosen. Experience shows that a full diversity pair can be found after a few trials.

Regarding the complexity of the rank check, if Gaussian elimination is used, the complexity is of $O(\ell^3)$. For large frame size, it is a complex process but it is only performed in the code design phase.

This coding construction is similar to the serially concatenated convolutional codes proposed in [21], and hence, allows for the application of the iterative soft-input soft-output (SISO) decoding architecture [22]. Fig. 2 shows this architecture where the maximum *a posteriori* (MAP) principle [23] is used to design the SISO decoder and equalizer (the equalizer refers to the decoder of the inner code). The soft information $P(\boldsymbol{r}, prior|s_t^{(i)})$ and $P(s_t^{(i)}, evidence)$ have exact probability meaning in the first iteration where "prior" refers to *the a priori* probability of $s_m^{(l)}$ $(1 \le l < i,\, i < l \le L_t,\, 1 \le m < t,\, t < m \le \ell)$ and "evidence" refers to the likelihood of $s_m^{(l)}$ $(1 \le l < i,\, i < l \le L_t,$ $1 \le m < t,\, t < m \le \ell)$. The equalizer and decoder exchange soft information for N_{iter} iterations before the decoder makes hard decisions on information bits. Although the iterative decoder is suboptimal, it was shown to yield near-ML decoding performance in various scenarios [24]. The simulation results in Section VI demonstrate that this suboptimal algorithm is effective and does not incur diversity loss.

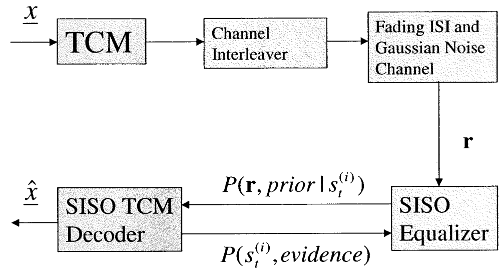

Fig. 2. Block diagram of the iterative receiver for the serially concatenated code.

The main drawback of the time-domain design approach is the complexity of the decoder. For the trellis coding approach, the ML decoder that accounts for the ISI nature of the channel has a trellis complexity proportional to $|\Omega|^{(L_{\mathrm{ISI}}+\nu)}$, where $|\Omega|$ is the size of the constellation alphabet and ν is the maximal memory order of the underlying convolutional code. The iterative approach also suffers from the exponentially growing complexity since the number of states in the SISO MAP equalizer is $|\Omega|^{L_t(L_{\mathrm{ISI}}-1)}$.

This prohibitive exponential complexity, especially when the number of transmit antennas and/or the number of paths is large, motivates the investigation of the frequency-domain approach that permits more flexibility in the tradeoff between the diversity advantage and receiver complexity.

V. Space–Frequency Coding Approach

In the frequency-domain approach, an OFDM front-end is utilized to transform the ISI channel into flat, however, selective fading channel. The baseband signal assigned to each antenna is passed through an inverse fast Fourier transform (IFFT) before transmission. Adopting the notation in (1), the transmitted signal from antenna i at the nth interval is now given by

$$x_n^{(i)} = \sqrt{E_s}\sum_{k=0}^{N-1} s_k^{(i)} \exp\left(-j\frac{2\pi kn}{N}\right) \tag{15}$$

where N is block length. A cyclic prefix of length $L_{\mathrm{ISI}} - 1$ is added to eliminate the ISI between consecutive OFDM symbols. At the receiver end, the signal $y_n^{(j)}$ received by antenna j at time n is given by

$$y_n^{(j)} = \sum_{l=0}^{L_{\mathrm{ISI}}-1}\sum_{i=1}^{L_t} \alpha_l^{(i,j)} x_{t-l}^{(i)} + n_n^{(j)} = \sqrt{E_s}\sum_{l=0}^{L_{\mathrm{ISI}}-1}\sum_{i=1}^{L_t}\sum_{k=0}^{N-1} \alpha_l^{(i,j)} s_k^{(i)} \exp\left(-j\frac{2\pi k(n-l)}{N}\right) + n_n^{(j)}. \tag{16}$$

The fast Fourier transform (FFT) operator is then applied to the received signal to yield

$$r_t^{(j)} = \sum_{n=0}^{N-1} y_n^{(i)} \exp\left(j\frac{2\pi nt}{N}\right) = \sqrt{E_s}\sum_{i=1}^{L_t}\left(\sum_{l=0}^{L_{\mathrm{ISI}}-1} \alpha_l^{(i,j)} \exp\left(-j\frac{2\pi lt}{N}\right)\right) s_t^{(i)} + N_t^{(j)} = \sqrt{E_s}\sum_{i=1}^{L_t} H_t^{(i,j)} s_t^{(i)} + N_t^{(j)} \tag{17}$$

where $N_t^{(j)}$ are independent noise samples of circularly symmetric zero-mean complex Gaussian random variable with variance $N_0/2$ per dimension. The complex fading coefficients of the equivalent channel model $H_t^{(ij)}$ have the following autocorrelation function:

$$\begin{aligned} R(i_1-i_2,\, j_1-j_2,\, t_1-t_2) &= E\left(H_{t_1}^{(i_1j_1)} H_{t_2}^{(i_2j_2)*}\right) \\ &= \delta(i_1-i_2,\, j_1-j_2) \sum_{l=0}^{L_{\rm ISI}-1} \exp\left(-j\frac{2\pi l(t_1-t_2)}{N}\right) \end{aligned} \tag{18}$$

where $\delta(i,\, j)$ is the Dirac-delta function. From (18), it is clear that the fading coefficients of the equivalent channel are spatially independent and that

$$R\left(0,\, 0,\, \frac{kN}{L_{\rm ISI}}\right) = 0, \qquad \text{for } k = 1,\, 2,\, \ldots,\, L_{\rm ISI}-1.$$

This observation suggests that the equivalent *space–frequency* flat fading channel can be approximated by the piecewise constant MIMO block-fading model. In this model, the codeword encompasses $L_{\rm ISI}$ fading blocks. The complex fading gains are constant over one fading block but are independent from block to block. This model implicitly assumes that the number of frequency "bins" is much larger than the length of dominant error events, which is reasonable in most practical systems.

Now we consider the design of *space–frequency* codes (SFC) for this frequency-domain approach. Here we benefit from the algebraic framework introduced in [2] for constructing codes in MIMO block-fading channels. Similar to the previous section, we focus our attention on trellis-based codes because of the availability of reasonable complexity ML decoders. Also, we limit our discussion to BPSK-modulated systems relying on the fact that QPSK codes can be obtained from the BPSK codes as described in the previous section. Consider the general case where C is a binary convolutional code of rate $k/L_tL_{\rm ISI}$. The encoder processes k binary input sequences $x_1(t), x_2(t), \ldots, x_k(t)$ and produces $L_tL_{\rm ISI}$ coded output sequences $y_1(t), y_2(t), \ldots, y_{L_tL_{\rm ISI}}(t)$ which are multiplexed together to form the output codeword. The encoder action is summarized by the matrix equation

$$\boldsymbol{Y}(D) = \boldsymbol{X}(D)\boldsymbol{G}(D)$$

where

$$\begin{aligned} \boldsymbol{Y}(D) &= [Y_1(D)\, Y_2(D)\, \cdots\, Y_{L_tL_{\rm ISI}}(D)] \\ \boldsymbol{X}(D) &= [X_1(D)\, X_2(D)\, \cdots\, X_k(D)] \end{aligned}$$

and

$$\boldsymbol{G}(D) = \begin{bmatrix} G_{1,1}(D) & G_{1,2}(D) & \cdots & G_{1,L_tL_{\rm ISI}}(D) \\ G_{2,1}(D) & G_{2,2}(D) & \cdots & G_{2,L_tL_{\rm ISI}}(D) \\ \vdots & \vdots & \ddots & \vdots \\ G_{k,1}(D) & G_{k,2}(D) & \cdots & G_{k,L_tL_{\rm ISI}}(D) \end{bmatrix}. \tag{19}$$

We consider the natural space–time formatting of C in which the output sequence corresponding to $Y_{(m-1)L_t+l}(D)$ is assigned to the lth transmit antenna in the mth fading block and wish to characterize the diversity that can be achieved by this scheme. Our algebraic analysis technique considers the rank of matrices formed by concatenating linear combinations of the column vectors

$$\boldsymbol{F}_\ell(D) = \begin{bmatrix} G_{1,\ell}(D) \\ G_{2,\ell}(D) \\ \vdots \\ G_{k,\ell}(D) \end{bmatrix}. \tag{20}$$

Using the same approach as in [2], we define $\mathcal{G}$ to be the set of binary full-rank matrices $\{\boldsymbol{G}\colon \boldsymbol{G} = [g_{i,j}]_{L_t\times L_t}\}$ resulting from applying any number of simple row operations to the identity matrix I_{L_t}, and $\forall \boldsymbol{G}_m \in \mathcal{G}$, $1 \le i \le L_t$, $1 \le m \le L_{\rm ISI}$ we define

$$R_i^{(\boldsymbol{G}_m,\, m)}(D) = [g_{i,1}(m)I_k,\, g_{i,2}(m)I_k,\, \ldots,\, g_{i,L_t}(m)I_k] \cdot \begin{bmatrix} \boldsymbol{F}_{(m-1)L_t+1}(D) \\ \boldsymbol{F}_{(m-1)L_t+2}(D) \\ \vdots \\ \boldsymbol{F}_{mL_t}(D) \end{bmatrix}. \tag{21}$$

Then we have the following algebraic construction for BPSK space–frequency convolutional codes.

Proposition 8: In a MIMO OFDM-based communication system with L_t transmit antennas operating over a frequency-selective block-fading channel with $L_{\rm ISI}$ blocks, let $\mathcal{C}$ denote the SFC consisting of the binary convolutional code C, whose $k \times L_tL_{\rm ISI}$ transfer function matrix is

$$\boldsymbol{G}(D) = [\boldsymbol{F}_1(D)\, \cdots\, \boldsymbol{F}_{L_{\rm ISI}L_t}(D)]$$

and the spatial parser σ in which the output

$$Y_{(m-1)L_t+l}(D) = \boldsymbol{X}(D)\boldsymbol{F}_{(m-1)L_t+l}(D)$$

is assigned to antenna l in fading block m. Then, with BPSK transmission, $\mathcal{C}$ achieves d levels of transmit diversity if d is the largest integer such that for every $\boldsymbol{G}_1 \in \mathcal{G}, \ldots, \boldsymbol{G}_{L_{\rm ISI}} \in \mathcal{G}$

$$0 \le m_1 \le \min(L_t,\, L_{\rm ISI}L_t - d + 1),\, \ldots,\, 0 \le m_{L_{\rm ISI}} \le \min(L_t,\, L_{\rm ISI}L_t - d + 1)$$

and

$$\sum_{i=1}^{L_{\rm ISI}} m_i = L_{\rm ISI}L_t - d + 1.$$

$$\boldsymbol{R}_{m_1,\ldots,m_{L_{\rm ISI}}}^{(\boldsymbol{G}_1,\ldots,\boldsymbol{G}_{L_{\rm ISI}})}(D) = [R_0^{(\boldsymbol{G}_1,1)}(D),\, \ldots,\, R_{m_1}^{(\boldsymbol{G}_1,1)}(D),\, R_0^{(\boldsymbol{G}_2,2)}(D),\, \ldots,\, R_{m_2}^{(\boldsymbol{G}_2,2)}(D),\, \ldots,\, R_{m_{L_{\rm ISI}}}^{(\boldsymbol{G}_{L_{\rm ISI}},L_{\rm ISI})}(D)]$$

has a full rank k over the space of all formal series.

Proof: Please refer to [2]. □

This result allows for constructing convolutional SFC that realize the optimum tradeoff between transmission rate and diversity order for BPSK modulation with an arbitrary coding rate, number of transmit antenna, and number of fading blocks. Similar to [3], one can utilize this framework to construct rate $1/n'$ bit interleaved space–frequency convolutional codes.

We now focus on rate $1/L_t$ codes in order to achieve the same transmission rate as that in the space–time coding approach. The output sequence $Y_i(D)$ is assigned to the ith antenna. The stream assigned to each antenna is then distributed across the different fading blocks using a periodic bit interleaver. Assuming that the number of resolvable paths is known at the transmitter, the interleaver mapping function π is defined as

$$\pi(i) = \left\lfloor \frac{i}{L_{\rm ISI}} \right\rfloor + \frac{N}{L_{\rm ISI}} (i)_{L_{\rm ISI}} \tag{22}$$

where $(\,)_m$ refers to the modulo m operation, $0 \le i \le N-1$, and N is the codeword length which is assumed to be a multiple of $L_{\rm ISI}$.

In the absence of prior information on the number of resolvable paths, it seems difficult to develop an interleaving scheme capable of exploiting all the frequency diversity, *whenever available*, for an arbitrary *unknown* number of paths. In the special case where the number of paths is restricted to be a power of two (i.e., $L_{\rm ISI} = 2^r$ for any arbitrary integer r), and we assume that the maximum possible number of paths $L_{\rm ISI}^{(\max)}$ is known at the transmitter, we have the following construction for the *universal* interleaving map:

$$\pi(i) = \sum_{k=0}^{\log_2(L_{\rm ISI}^{(\max)})} a_k \frac{N}{2^{k+1}} + \left\lfloor \frac{i}{L_{\rm ISI}^{(\max)}} \right\rfloor \tag{23}$$

where

$$a_k = \left(\frac{(i)_{L_{\rm ISI}^{(\max)}} - \sum_{j=0}^{k-1} a_j 2^j}{2^k} \right)_2. \tag{24}$$

This interleaving scheme distributes the input sequence periodically among the $L_{\rm ISI}$ fading blocks for any $L_{\rm ISI} = 2^r$ and $L_{\rm ISI} \le L_{\rm ISI}^{(\max)}$. In practice, $L_{\rm ISI}^{(\max)}$ can be chosen to be larger than the maximum number of resolvable paths expected in this particular scenario, and hence, no feedback is required from the receiver. If the number of paths is not a power of two, the diversity advantage is lower-bounded by that achieved with the number of paths equal to $L_{\rm ISI}^{(approx)} = 2^r$ where r is the largest integer such that $L_{\rm ISI}^{(approx)} < L_{\rm ISI}$.

In Table III, we report the diversity advantage achieved by the optimal free-distance binary codes when used as space–frequency codes in this scenario. It is clear from the table that *carefully* constructed codes can achieve significant increases in the diversity advantage by exploiting the frequency diversity without any reduction in the transmission rate. While the codes in the table do not realize the maximum possible diversity advantage in all scenarios, they strike a reasonable balance between the diversity advantage and coding gain. This fact will be validated by an example in Section VI. For more exhaustive results on algebraic code constructions that achieve the maximum diversity advantage in MIMO block-fading channels, the reader is referred to [2].

The tradeoff between the two design approaches can now be assessed. The OFDM-based approach was motivated by the need for lower complexity ML receiver. This is emphasized in the fact that the ML decoder complexity in the OFDM approach *does not* increase exponentially with the number of resolvable paths in contrast to the space–time coding approach. This does not mean, however, that the complexity of the decoder does not depend on the number of paths. As shown in Table III, as the number of paths increases, codes with larger constraint lengths are needed to efficiently exploit the diversity available in the channel. Unlike the space–time coding approach, however, one can trade diversity advantage for a reduction in complexity by choosing codes with small constraint lengths. This trade is not possible in the space–time coding approach because, irrespective of the constraint length of the code, the complexity of the (ML) decoder grows exponentially with the number of resolvable paths. The main disadvantage of the OFDM-based approach is the loss in diversity advantage compared to the space–time coding approach. This fact is formalized in the following result.

TABLE III
THE DIVERSITY ADVANTAGE FOR BPSK ALGEBRAIC SPACE–FREQUENCY CODES WITH OPTIMAL FREE DISTANCE FOR MIMO FREQUENCY-SELECTIVE FADING CHANNELS

L_t	ν	**Connection Polynomials**	d for $L_{\rm ISI}=1$	$L_{\rm ISI}=2$	$L_{\rm ISI}=3$
2	2	5, 7	2	2	3
	3	64, 74	2	3	3
	4	46, 72	2	3	4
	5	65, 57	2	3	4
	6	554, 744	2	3	4
3	3	54, 64, 74	3	4	4
	4	52, 66, 76	3	3	5
	5	47, 53, 75	3	4	6
	6	554, 624, 764	3	5	5
4	4	52, 56, 66, 76	4	4	5

Lemma 9: The maximum transmit diversity advantage achieved in a BPSK OFDM MIMO wireless system with L_t transmit antennas and $L_{\rm ISI}$ resolvable paths/antenna supporting a throughput of 1 bit/s/Hz is $L_{\rm ISI}(L_t - 1) + 1$.

Proof: Follows directly from the Singleton bound and can be easily generalized to arbitrary single-dimensional constellations. □

It is clear that the maximum diversity advantage in this approach is lower than that in the space–time coding approach. Tables I and III compare the diversity advantage achieved by STCs and SFCs for different values of L_t and $L_{\rm ISI}$. We hasten to stress that the loss in diversity advantage in the space-frequency coding approach is a direct result of the use of standard single-dimensional constellation. One can alleviate this loss, at the cost of an additional increase in the peak-to-average power ratio and complexity, by using multidimensional rotated constellations. Finally, as it will be shown in the following section, the diversity advantage loss in the space–frequency coding approach may not result in a performance loss for the frame error rate range of interest.

VI. NUMERICAL RESULTS

In this section, we present simulated frame error-rate performance results for the two approaches presented in Sections IV and V. Our main focus will be the codes presented in Tables I–III. Except for the serially concatenated coding scenario, joint ML decoding and equalization that accounts for the ISI nature of the channel is assumed at the receiver. In most cases, we restrict the simulated frame error rates to be $\ge$1% because of the practical significance of this range, and to limit the simulation time. Unless otherwise stated, the frame length corresponds to 100 simultaneous transmissions from all antennas. We used a periodic bit level interleaver in the space–frequency coding simulations.

A. BPSK Codes

Figs. 3–8 report the performance of the space–time and space–frequency coding approaches in BPSK systems with different numbers of transmit antennas L_t, receive antennas L_r, resolvable paths $L_{\rm ISI}$, and receiver trellis complexity. The number of states in the figures represents the ML decoder trellis complexity. For the OFDM approach, this number is equal to the number of states in the underlying convolutional code, however, for the space-time coding approach, this number accounts for the additional complexity dictated by the ISI nature of the channel. In Figs. 3 and 4, we show the gain in performance with increasing numbers of resolvable paths. In the

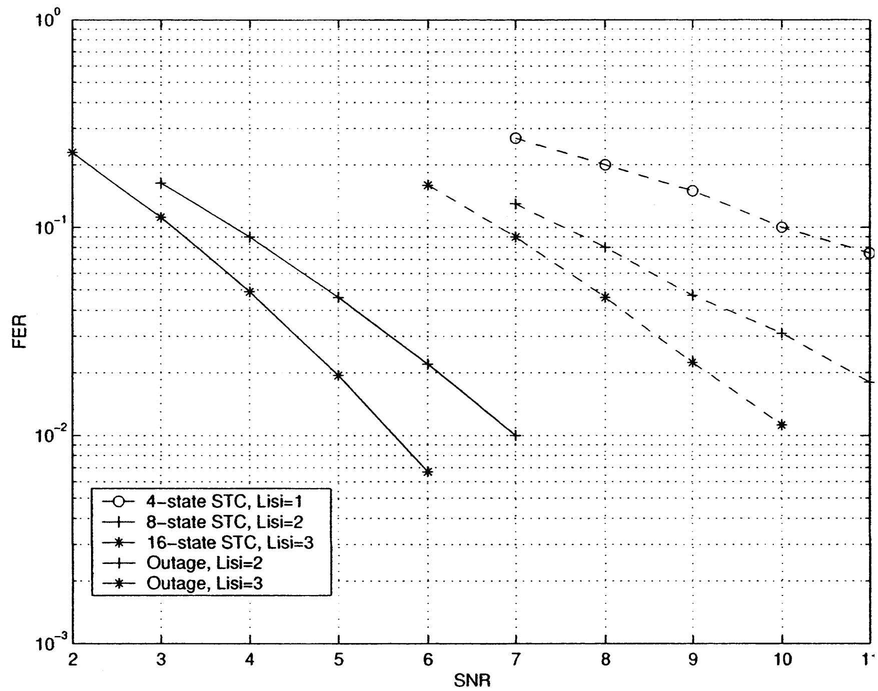

Fig. 3. Performance of four-state BPSK STC with $L_t = 2$, and $L_r = 1$.

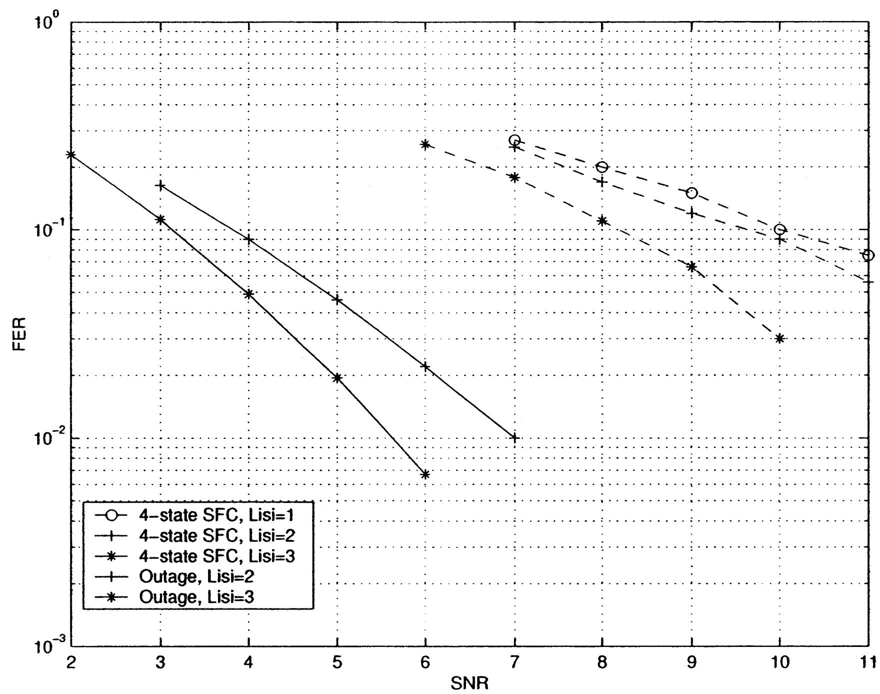

Fig. 4. Performance of four-state BPSK SFC with $L_t = 2$, and $L_r = 1$.

single-carrier approach, this improvement comes at the expense of increased receiver complexity—the number of states in the ML receiver grows exponentially with the number of resolvable paths. On the other hand, for the space–frequency coding approach, the performance improvement did not entail any increase in complexity. It is also worth noting that the improvement in performance in the space–frequency coding approach was marginal when $L_{\rm ISI}$ was increased from one to two because, as reported in Table I, the diversity advantage of the four-state code is the same in both scenarios. In Figs. 5–8, we compare the space–time and space–frequency coding approaches. It is shown that when the same code is used in both schemes, the space–time coding approach always provides a gain in performance, however, at the expense of higher receiver complexity. Whereas, if the receiver complexity is fixed in both approaches, the space–frequency coding approach sometimes offers better performance. This may seem to contrast the intuition based on the superiority of the space–time coding approach in terms of diversity advantage. This seeming contradiction can be attributed to two reasons. First, the same receiver complexity allows the space–frequency coding approach to utilize more sophisticated codes that offer larger coding gains. Second, the

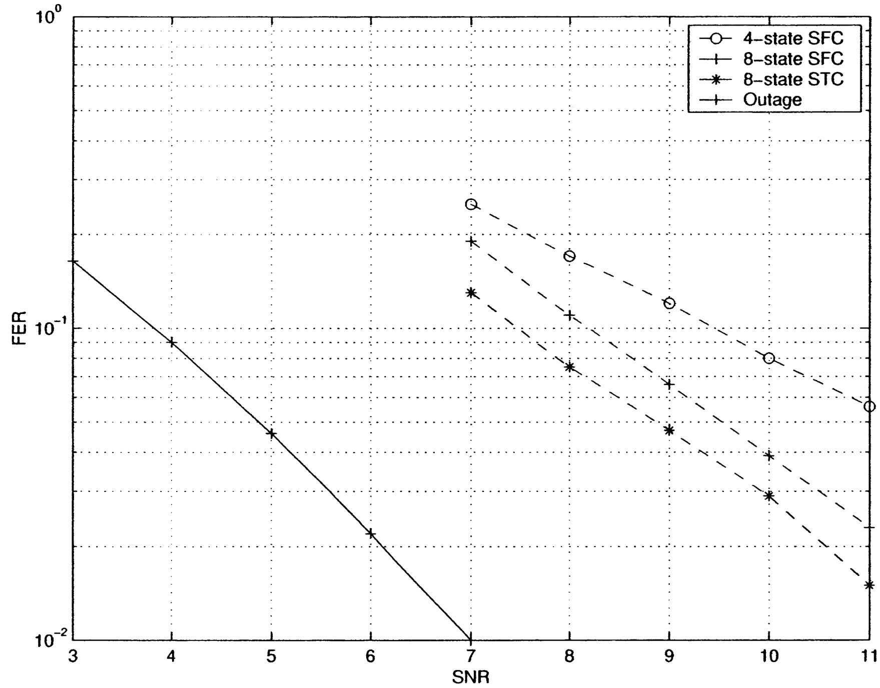

Fig. 5. Performance of BPSK STC and SFC with $L_t = 2$, $L_r = 1$, and $L_{\rm ISI} = 2$.

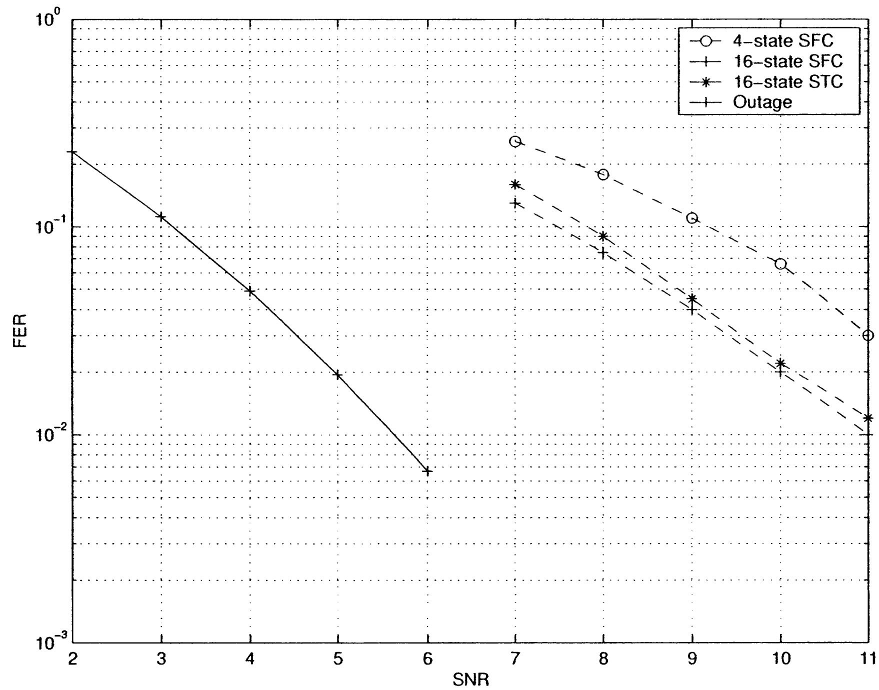

Fig. 6. Performance of BPSK STC and SFC with $L_t = 2$, $L_r = 1$, and $L_{\rm ISI} = 3$.

effect of the space–time coding superior diversity advantage may only become apparent at significantly larger signal-to-noise ratios. This observation, however, indicates that the space–frequency coding approach may yield superior performance in some practical applications. Fig. 9 highlights the importance of careful design to optimize the diversity advantage. In the figure, we compare the four-state $(5, 7)$ optimal free-distance SFC with the four-state $(6, 7)$ in a system with $L_t = 2$, $L_r = 1$, and $L_{\rm ISI} = 2, 3$. As reported in Table I, the $(5, 7)$ code achieves $d = 2, 3$ for $L_{\rm ISI} = 2, 3$, respectively. Whereas the $(6, 7)$ code achieves $d = 3$ in both cases (for $L_{\rm ISI} = 2$, $d = 3$ is the maximum possible diversity advantage for this throughput). As shown in the figure, for the $L_{\rm ISI} = 2$ case, the superior diversity advantage of the $(6, 7)$ is apparent in the steeper slope of the frame error-rate curve. This results in a gain of about 1 dB at 0.01 frame error rate. On the other hand, for the $L_{\rm ISI} = 3$ case, it is shown that the $(5, 7)$ code exhibits a superior coding gain compared with the $(6, 7)$ code.

Fig. 10 compares the performance of the serially concatenated coding approach with that of the trellis-coding approach. In both cases, we assume $L_t = 2$, $L_r = 1$, and $L_{\rm ISI} = 3$. The same four-state code is used in both scenarios. This entails a 16-state ML decoder for

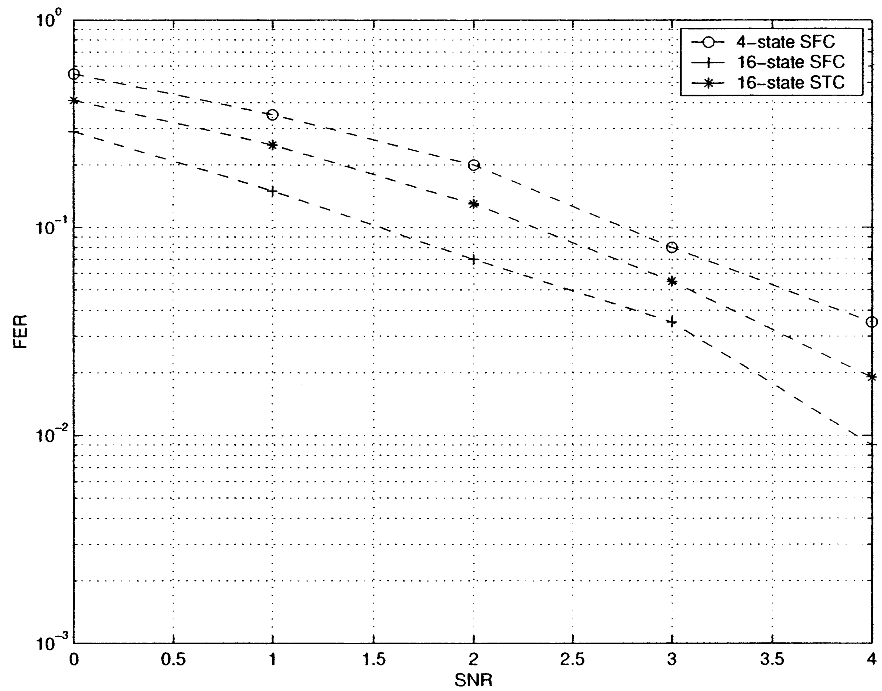

Fig. 7. Performance of BPSK STC and SFC with $L_t = 2$, $L_r = 2$, and $L_{\rm ISI} = 3$.

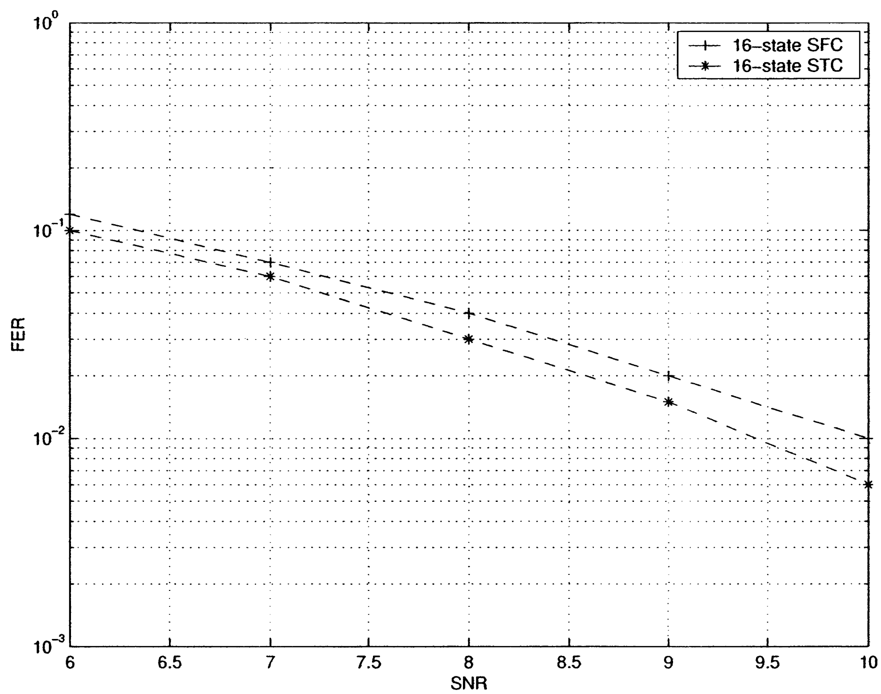

Fig. 8. Performance of BPSK STC and SFC with $L_t = 3$, $L_r = 1$, and $L_{\rm ISI} = 2$.

the trellis-coding approach, whereas the iterative decoder uses two SISO decoders with five decoding iterations (the inner decoder has 16 states and the outer decoder has four states). It is shown in the figure that the serially concatenated code offers a performance gain of 1 dB, however, at the expense of increased receiver complexity.

B. QPSK Codes

The frame size in this scenario corresponds to 130 simultaneous transmissions for all antennas. In Fig. 11, we compare the four-state code of Tarokh, Seshadri, and Calderbank (TSC) [12] and the new linear $\mathbb{Z}_4$ four-state code when used in the single carrier space–time coding paradigm. The same comparison is repeated in Fig. 12 for the space-frequency approach with 16-state codes. In the two cases considered, it is shown that the new codes are uniformly better than the TSC codes. However, the significance of the achieved gain depends largely on the system parameters. For example, this gain seems to increase with the number of receive antennas.

Fig. 13 compares the performance of the serially concatenated coding approach with that of the trellis-coding approach. In both cases,

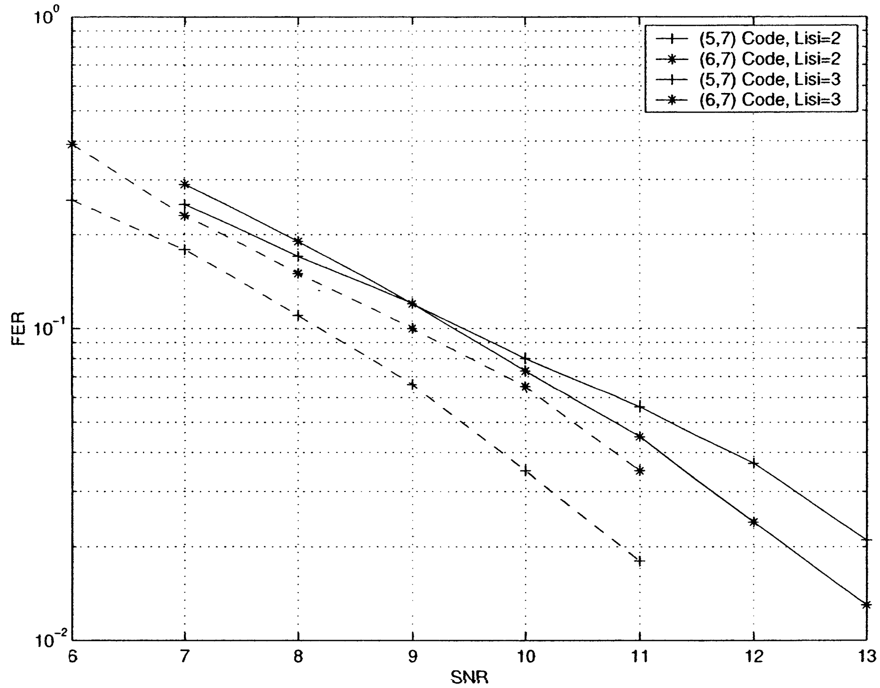

Fig. 9. Performance comparison of two four-state BPSK SFCs for $L_t = 2$, $L_r = 1$.

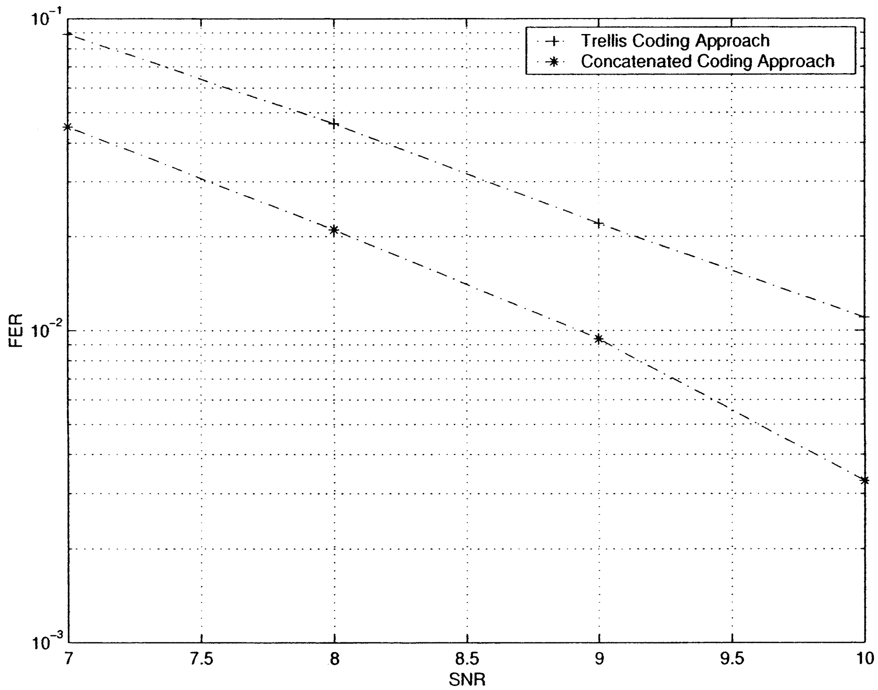

Fig. 10. Performance comparison of the serially concatenated coding approach and the trellis coding approach (BPSK).

we assume $L_t = 2$, $L_r = 1$, and $L_{\mathrm{ISI}} = 2$. The same 16-state code is used in both scenarios. Similar to the BPSK scenario, it is shown in the figure that the serially concatenated code offers a performance gain of 1 dB, however, at the expense of increased receiver complexity.

Finally, the utility of the OFDM SFC approach in reducing the receiver complexity can be observed in the following simple example. In a system with $L_t = 3$ and $L_{\mathrm{ISI}} = 2$, a full diversity QPSK space–time coding would entail a 1024-state (ML) decoder, whereas with only 64-state receiver the space–frequency coding approach can achieve four levels of diversity and very reasonable coding gain.

VII. Conclusion

In this correspondence, we considered the design of coding schemes for MIMO frequency-selective fading channels. We investigated two design approaches for constructing algebraic codes that exploit the diversity available in such channels. The time-domain approach is based on space–time coding with ML decoding at the receiver. Using the algebraic framework established in [1], [4] for flat-fading channels, we proposed general constructions for space–time codes that achieve full diversity for systems with arbitrary number of transmit antennas and resolvable paths. The exponentially growing complexity of the receiver

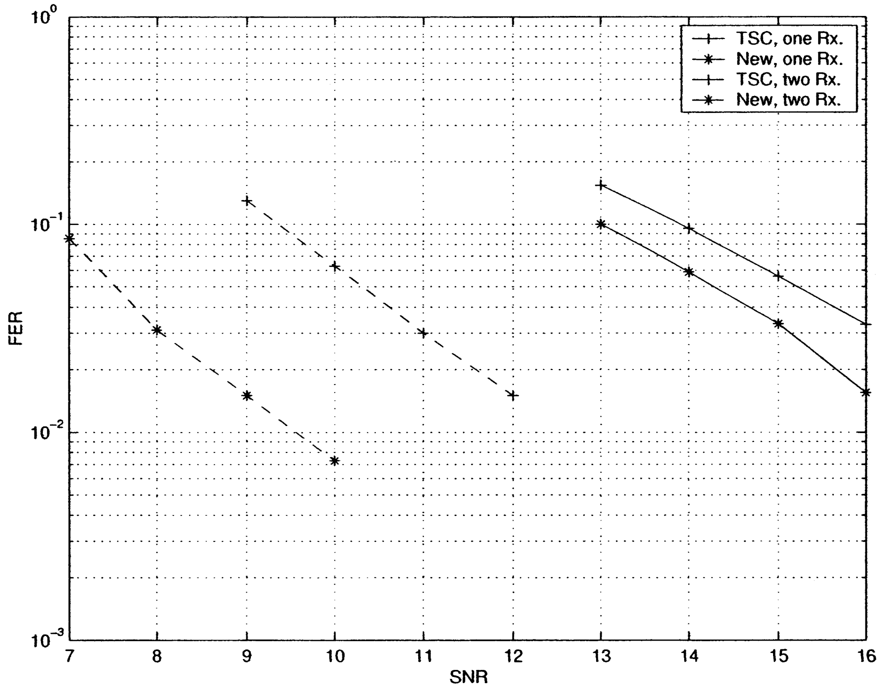

Fig. 11. Performance of two four-state QPSK STCs for $L_t = 2$ and $L_{\rm ISI=2}$.

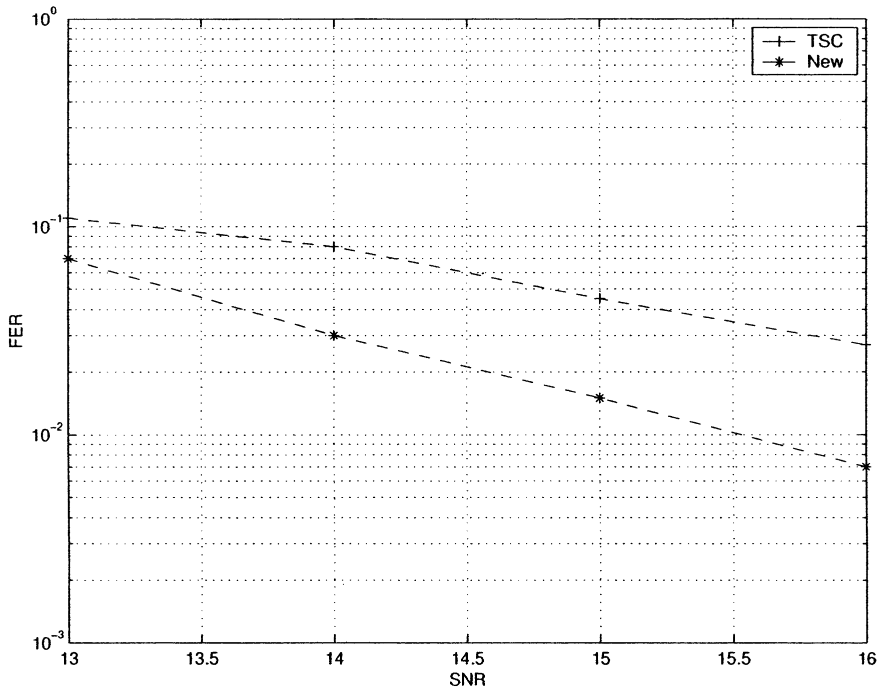

Fig. 12. Performance of two 16-state QPSK SFCs for $L_t = 2$, $L_r = 1$, and $L_{\rm ISI} = 3$.

in this approach motivated the consideration of the frequency-domain paradigm. In this scenario, we utilized an OFDM front-end to transform the ISI channel into a MIMO flat block-fading channel. We then construct *space–frequency* codes that exploit the diversity available in the MIMO block-fading channel. The frequency-domain approach was shown to trade the diversity advantage for a significant reduction in the receiver complexity. Finally, we presented simulation results that compare the performance of the two approaches and demonstrate the excellent performance of the proposed algebraic codes in MIMO frequency-selective fading channels.

APPENDIX
Σ_o-RANK

In this appendix, we provide necessary definitions related to Σ_o-rank theory for the convenience of readers. The Σ_o-rank theory developed in [4] is for codes defined on the ring $\mathbb{Z}_{2^k}(j)$. In the sequel, $\oplus_n$ is used to denote the modulo n addition and subtraction, and the subscript n is dropped if the context is clear.

The definition of ring $\mathbb{Z}_{2^k}(j)$ is given first.

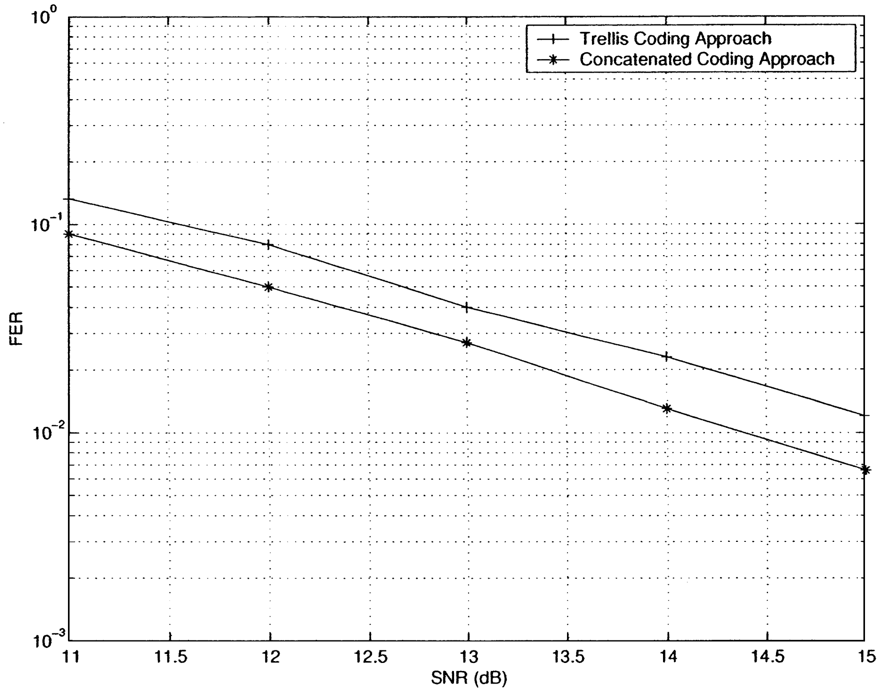

Fig. 13. Performance comparison of the serially concatenated coding approach and the trellis coding approach (QPSK).

Definition 10 (Ring $\mathbb{Z}_{2^k}(j)$): The ring $\mathbb{Z}_{2^k}(j)$ is a finite set and k is a positive integer. Each element V has the form

$$V = V_I \oplus_{2^k} jV_Q$$

where the real part V_I and imaginary part V_Q are integers in $\mathbb{Z}_{2^k}$ and $j^2 = -1 = -1 + 2^k$. In this correspondence, nonnegative integers in $\{0, 1, \ldots, 2^k - 1\}$ are used to label elements in $\mathbb{Z}_{2^k}$. The addition and multiplication in this ring are the addition and multiplication in complex number field followed by modulo 2^k operation on the real part and the imaginary part.

We use translation mapping as constellation mapping for codes defined on $\mathbb{Z}_{2^k}(j)$.

Definition 11 (Translation Mapping): Translation mapping $f(\cdot)$ maps a $\mathbb{Z}_{2^k}(j)$ element c to a complex number s by

$$s = c - ((2^k - 1)/2 + j(2^k - 1)/2).$$

It results in a square 2^{2k} QAM constellation.

The concept of Σ_o-coefficients is used to define Σ_o-rank.

Definition 12 (Σ_o-Coefficient Set): A group of coefficients $\{\alpha_1, \alpha_2, \ldots, \alpha_L\}$, defined on $\mathbb{Z}_{2^k}(j)$ is said to be Σ_o-coefficient set if there exists at least one i^* $(1 \le i^* \le L)$ such that $a_{i^*} + b_{i^*}$ is odd, where $a_{i^*} \oplus jb_{i^*} = \alpha_{i^*}$.

Σ_o-rank is the core definition for Σ_o-rank theory.

Definition 13 (Column Σ_o-Rank): A matrix $\boldsymbol{V}$ defined on the ring $\mathbb{Z}_{2^k}(j)$ has column Σ_o-rank L if L is the maximum number of column vectors of $\boldsymbol{V}$, such that

$$\exists\, \mathcal{V} = \left\{\vec{V}_{i_1}, \ldots, \vec{V}_{i_L}\right\}, \quad \bigoplus_{l=1}^{L} \alpha_l \vec{V}_{i_l} \neq \vec{0}$$

for all possible Σ_o-coefficient set $\{\alpha_1, \alpha_2, \ldots, \alpha_L\}$.

The row Σ_o-rank can be similarly defined. Since column Σ_o-rank is equal to row Σ_o-rank [4], the common value is denoted as Σ_o-rank.

Definition 14 (Full Σ_o-Rank): An m by n matrix $\boldsymbol{V}$ defined on ring $\mathbb{Z}_{2^k}(j)$ is said to be of full Σ_o-rank if it has Σ_o-rank equal to the minimum of m and n.

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Bandwidth-Efficient Linear Modulations for Multiple-Antenna Transmission

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***Abstract*—Some $M \times T$ modulation matrices for M transmit antennas and T symbol periods, with $M = 2, 3, 4$ and $T = 2$, and $M = T = 4$ are studied. A transmission rate of M symbols per channel use and a transmit diversity order of $\min(M, T)$ are achieved over a quasi-static fading channel when using rotated versions of a multidimensional quadratic amplitude modulation with spectral efficiency 2 bits/symbol. Extension to input constellations with higher spectral efficiencies is then considered. The modulations are then generalized to any number of transmit antennas M and any number of symbol periods T, such that a transmission rate of M symbols per channel use, and a transmit diversity of T are achieved under fast fading (ergodic scenario). By means of signal space diversity, the proposed modulations exploit the degrees of freedom of multiantenna channels and have moderate detection complexity at moderate and large signal-to-noise ratios (SNRs).**

***Index Terms*—Constellations, diversity methods, maximum-likelihood (ML) detection, modulation, multiple-input multiple-output (MIMO) systems.**

I. Introduction

The field of space–time (ST) coding and modulation has gained much interest due to the increasing need to transmit reliable information at high rates over wireless channels [1]–[10]. Theoretical investigations of M-transmit and N-receive antennas systems showed that the capacity of such systems increases linearly with the minimum of M and N [2], [3]. High data rates are obtained by simultaneously sending signals from several transmit antennas. To protect the integrity of the transmitted information, transmit diversity is obtained by introducing redundancy among the transmitted signals over M transmit antennas (space) and T time periods (time). Under quasi-static fading, the maximum combined transmit–receive diversity order equals MN [4], [5].

In [6], the Vertical Bell Labs Layered Space–Time Architecture (V-BLAST) multiantenna prototype was proposed in order to achieve very high data rates through spatial multiplexing. The detection algorithm considered in [6] does not fully exploit the receive diversity as is done when applying a maximum-likelihood (ML) detection algorithm such as the sphere decoder [7]. Alamouti proposed a modulation scheme over $M = 2$ transmit and N receive antennas where a rate of one symbol per channel use (PCU) with $2N$ diversity was achieved [8]. The ML detection of the Alamouti scheme can be implemented by a linear complexity decorrelator. The latter scheme was generalized to $M \geq 2$ by Tarokh *et al.* [9], where complex ST modulations were proposed with rate $1/2$ and $3/4$ symbol PCU. In [10], a different approach is proposed to exploit the transmit–receive diversity using rotated constellations. The so-called diagonal algebraic space–time (DAST) modulations [10] achieve diversity MN and have a rate of one symbol PCU using real or complex input constellations. Hassibi

Manuscript received August 8, 2001; revised August 9, 2002. The material in this correspondence was presented in part at the IEEE Global Telecommunications Conference, San Antonio, TX, November 2001.

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Communicated by G. Caire, Associate Editor for Communications.
Digital Object Identifier 10.1109/TIT.2003.815779