On the Design of Algebraic Space–Time Codes for MIMO Block-Fading Channels

Hesham El Gamal, *Member, IEEE,* and A. Roger Hammons, Jr., *Member, IEEE*

***Abstract*—The availability of multiple transmit antennas allows for two-dimensional channel codes that exploit the spatial transmit diversity. These codes were referred to as space–time codes by Tarokh *et al.* Most prior works on space–time code design have considered quasi-static fading channels. In this paper, we extend our earlier work on algebraic space–time coding to block-fading channels. First, we present baseband design criteria for space–time codes in multi-input multi-output (MIMO) block-fading channels that encompass as special cases the quasi-static and fast fading design rules. The diversity advantage baseband criterion is then translated into *binary* rank criteria for phase shift keying (PSK) modulated codes. Based on these *binary* criteria, we construct algebraic space–time codes that exploit the spatial and temporal diversity available in MIMO block-fading channels. We also introduce the notion of universal space–time codes as a generalization of the smart–greedy design rule. As a part of this work, we establish another result that is important in its own right: we generalize the full diversity space–time code constructions for quasi--static channels to allow for higher rate codes at the expense of minimal reductions in the diversity advantage. Finally, we present simulation results that demonstrate the excellent performance of the proposed codes.**

***Index Terms*—Block-fading channels, multiple transmit and receive antennas, space–time codes, wireless communication.**

I. Introduction

RECENT works have investigated the design of channel codes that exploit the spatial transmit diversity available in coherent multi-input multi-output (MIMO) channels (e.g., [1]–[10]). These works have primarily focused on the quasi-static fading model in which the path gains remain fixed throughout the codeword. In this paper, we extend our earlier work on quasi-static fading channels to the more general block-fading model. In this model, the codeword is composed of multiple blocks. The fading coefficients are constant over one fading block but are independent from block to block. The number of fading blocks per codeword can be regarded as a measure of the interleaving delay allowed in the system, so that systems subject to a strict delay constraint are usually characterized by a small number of independent blocks.

The information-theoretic capacity of MIMO block-fading channels has been studied by Biglieri *et al.* assuming the availability of channel state information (CSI) at both the transmitter and receiver [11]. One of their key conclusions was that antenna diversity can be a substitute for temporal diversity. This conclusion does not necessarily hold for the case of interest in this paper where CSI is only available at the receiver. The code design objective in this scenario is to exploit both spatial and temporal diversity.

In this paper, we present baseband design criteria that determine the diversity and coding advantages achieved in MIMO block-fading channels. We show that the design criteria in [12], [13] for quasi-static and fast-fading MIMO channels are special cases of the new criteria. Then, we follow the approach in [14] to translate the diversity advantage baseband design criterion into binary design criteria for phase shift keying (PSK) modulated codes. The binary design criteria facilitate the development of an algebraic framework for space–time code design in this scenario. This algebraic framework opens the door for constructing codes that realize the optimum tradeoff between diversity advantage and transmission rate for systems with arbitrary numbers of transmit antennas and fading blocks per codeword. We also introduce the notion of *universal space–time coding* which aims at constructing codes that exploit the additional temporal diversity *whenever available*. This notion is a generalization of the smart greedy design principle [12] for the present scenario.

In addition to its importance for space–time code design in time-varying channels, the MIMO block-fading model has proved to be beneficial in the frequency-selective scenario. In [15], we utilize the algebraic framework developed here to construct *space–frequency* codes that exploit the frequency diversity available in MIMO frequency-selective fading channels.

As a part of this work, we generalize the quasi-static full diversity space–time code constructions in [14] to allow for *higher rate* codes with arbitrary diversity advantages. This generalization offers more flexibility in the tradeoff between diversity advantage and transmission rate, especially in systems with large numbers of transmit antennas.

The rest of this paper is organized as follows. Section II introduces the system model and briefly reviews the design rules in MIMO quasi-static and fast-fading channels. The design criteria for space–time codes in MIMO block-fading channels are presented in Section III. Section IV is devoted to the development of the algebraic design framework. Simulation results for codes constructed based on the new framework are provided in Section V for some representative scenarios. Finally, we offer some concluding remarks in Section VI.

Manuscript received August 10, 2001; revised June 25, 2002. The material in this paper was presented in part at the IEEE International Symposium on Information Theory, Washington, DC, June 2001.

H. El Gamal is with the Electrical Engineering Department, the Ohio State University, Columbus, OH 43210 USA (e-mail: helgamal@ee.eng.ohio-state.edu).

A. R. Hammons, Jr. is with Corvis Corporation, Columbia, MD 21046-3403 USA (e-mail: rhammons@corvis.com).

Communicated by G. Caire, Associate Editor for Communications.

Digital Object Identifier 10.1109/TIT.2002.806116

II. Space–Time Coding Concepts

In this section, we review the main concepts of space–time code design and introduce our notation. The system model is presented in Section II-A and we review briefly the design criteria for quasi-static and fast-fading channels in Sections II-B and II-C, respectively.

A. System Model

In the system under consideration, the source generates k information symbols from the discrete alphabet $\mathcal{X}$, which are encoded by the error control code C to produce codewords of length $N = nL_t$ over the symbol alphabet $\mathcal{Y}$. The encoded symbols are parsed among L_t transmit antennas and then mapped by the modulator into constellation points from the discrete complex-valued signaling constellation Ω for transmission across the channel. The modulated streams for all antennas are transmitted simultaneously. At the receiver, there are L_r receive antennas to collect the incoming transmissions. The received baseband signals are subsequently decoded by the space–time decoder.

Following [14], we formally define a space–time code to consist of an underlying error control code together with a spatial parser.

Definition 1: An $L_t \times n$ space–time code $\mathcal{C}$ of size S consists of an $(L_t n, S)$ error control code C and a spatial parser σ that maps each codeword vector $\underline{c} \in C$ to an $L_t \times n$ matrix $\boldsymbol{c}$ whose entries are a rearrangement of those of $\underline{c}$.

Except as noted to the contrary, we will assume that the standard parser maps

$$\underline{c} = (c_1^1, c_1^2, \dots, c_1^{L_t}, c_2^1, c_2^2, \dots, c_2^{L_t}, \dots, c_n^1, c_n^2, \dots, c_n^{L_t}) \in C$$

to the matrix

$$\boldsymbol{c} = \begin{bmatrix} c_1^1 & c_2^1 & \cdots & c_n^1 \\ c_1^2 & c_2^2 & \cdots & c_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ c_1^{L_t} & c_2^{L_t} & \cdots & c_n^{L_t} \end{bmatrix}.$$

In this notation, it is understood that c_t^i is the code symbol assigned to transmit antenna i at time t.

Let f: $\mathcal{Y} \to \Omega$ be the modulator mapping function. Then $\boldsymbol{s} = f(\boldsymbol{c})$ is the baseband version of the codeword as transmitted across the channel. For this system, we have the following baseband model of the received signal:

$$y_t^j = \sqrt{E_s} \sum_{i=1}^{L} \alpha_t^{ij} s_t^i + n_t^j \tag{1}$$

where E_s is the energy per transmitted symbol; α_t^{ij} is the complex path gain from transmit antenna i to receive antenna j at time t; $s_t^i = f(c_t^i)$ is the transmitted constellation point from antenna i at time t; n_t^j is the additive white Gaussian noise sample for receive antenna j at time t. The noise samples are independent samples of zero-mean circularly symmetric complex Gaussian random variable with variance $N_0/2$ per dimension. The different path gains α_t^{ij} are assumed to be spatially white samples of zero-mean complex Gaussian random variable with unit variance. The CSI is assumed to be known *a priori* only at the receiver.

The fading model of primary interest is that of a block flat Rayleigh-fading process in which the codeword encompasses M fading blocks. The complex fading gains are constant over one fading block but are independent from block to block. The quasi-static and fast-fading models are special cases of the block-fading model in which $M = 1$ and $M = n$, respectively, [13], [12], [14]. For simplicity, it is also assumed here that M divides n.

B. Quasi-Static Fading Channel

For the quasi-static flat Rayleigh-fading channel the pairwise error probability that the decoder will prefer the alternate codeword $\boldsymbol{e}$ to $\boldsymbol{c}$ is upper-bounded by [13], [12]:

$$P(\boldsymbol{c} \to \boldsymbol{e}) \le \left(\frac{1}{\prod_{i=1}^{d} (1 + \lambda_i E_s / 4N_0)} \right)^{L_r} \tag{2}$$

$$\le \left(\frac{\mu E_s}{4N_0} \right)^{-dL_r} \tag{3}$$

where $d = \mathrm{rank}(f(\boldsymbol{c}) - f(\boldsymbol{e}))$ and $\mu = (\lambda_1 \lambda_2 \cdots \lambda_d)^{1/d}$ is the geometric mean of the nonzero eigenvalues of

$$\boldsymbol{A} = (f(\boldsymbol{c}) - f(\boldsymbol{e}))(f(\boldsymbol{c}) - f(\boldsymbol{e}))^{\mathrm{H}}.$$

The exponent dL_r is sometimes called the "diversity advantage," while the multiplicative factor μ is sometimes called the "coding advantage." The parameter d is the diversity provided by the multiple transmit antennas. The pairwise error probability leads to the following rank and equivalent product distance criteria for space–time code design in quasi-static channels [13], [12].

• *Rank Criterion*: Maximize the transmit diversity advantage $d = \mathrm{rank}(f(\boldsymbol{c}) - f(\boldsymbol{e}))$ over all pairs of distinct codewords $\boldsymbol{c}$, $\boldsymbol{e} \in \mathcal{C}$.

• *Product Distance Criterion*: Maximize the coding advantage $\mu = (\lambda_1 \lambda_2 \cdots \lambda_d)^{1/d}$ over all pairs of distinct codewords $\boldsymbol{c}$, $\boldsymbol{e} \in \mathcal{C}$.

The rank criterion is the more important of the two as it determines the asymptotic slope of the performance curve as a function of E_s/N_0 [12].

In [14], we developed a binary rank criterion that facilitates the design of algebraic binary space–time codes for binary phase shift keying (BPSK) modulation. Using the binary rank criterion, we also proposed the following general construction for full diversity space–time codes [14].

Theorem 2 **(Stacking Construction)***:* Let $\boldsymbol{M}_1, \boldsymbol{M}_2, \dots, \boldsymbol{M}_{L_t}$ be binary matrices of dimension $k \times n$, $n \ge k$, and let $\mathcal{C}$

be the $L_t \times n$ space–time code of dimension k consisting of the codeword matrices

$$\boldsymbol{c} = \begin{bmatrix} \underline{x}\boldsymbol{M}_1 \\ \underline{x}\boldsymbol{M}_2 \\ \vdots \\ \underline{x}\boldsymbol{M}_{L_t} \end{bmatrix}$$

where $\underline{x}$ denotes an arbitrary k-tuple of information bits and $L_t \leq n$. Then $\mathcal{C}$ satisfies the binary rank criterion, and thus, for BPSK transmission over the quasi-static fading channel, achieves full spatial transmit diversity L_t, if and only if $\boldsymbol{M}_1, \boldsymbol{M}_2, \ldots, \boldsymbol{M}_{L_t}$ have the property that

$$\forall\, a_1, a_2, \ldots, a_{L_t} \in \mathbb{F}\colon \boldsymbol{M} = a_1\boldsymbol{M}_1 \oplus a_2\boldsymbol{M}_2 \oplus \cdots \oplus a_{L_t}\boldsymbol{M}_{L_t}$$

is of full rank k unless $a_1 = a_2 = \cdots = a_{L_t} = 0$.

The binary rank criterion and stacking construction lift to the $\mathbb{Z}_4$ domain where they describe full diversity quaternary phase shift keying (QPSK)-modulated space–time codes (see [14] for details). In this paper, we generalize the stacking construction in two different ways. First, we present a generalization that allows for increasing the transmission rate at the expense of a minimal reduction in the diversity advantage. Second, we extend this construction to MIMO block-fading channels where the constructed codes exploit the spatial and temporal diversity available in the channel without compromising the transmission rate.

C. Fast-Fading Channel

For the fast-fading channel, the pairwise error probability can be upper-bounded by [12]

$$P(\boldsymbol{c} \to \boldsymbol{e}) \leq \left(\frac{1}{\prod_{t=1}^{n} \left(1 + \left|f\left(\overline{c}_t\right) - f\left(\overline{e}_t\right)\right|^2 E_s/4N_0\right)} \right)^{L_r} \leq \left(\frac{\mu E_s}{4N_0}\right)^{-dL_r}$$

where $\overline{c}_t$ is the tth column of $\boldsymbol{c}$, $\overline{e}_t$ is the tth column of $\boldsymbol{e}$, d is the number of columns $\overline{c}_t$ that are different from $\overline{e}_t$, and

$$\mu = \left(\prod_{\overline{c}_t \neq \overline{e}_t} \left|f\left(\overline{c}_t\right) - f\left(\overline{e}_t\right)\right|^2 \right)^{1/d}.$$

The diversity advantage is now dL_r, and the coding advantage is μ.

Thus, the fundamental design criteria for space–time codes over fast-fading channels are the following [12]:

• *Columnwise Hamming Distance Criterion*: Maximize the number of column differences $d = |\{t\colon \overline{c}_t \neq \overline{e}_t\}|$ over all pairs of distinct codewords $\boldsymbol{c}, \boldsymbol{e} \in \mathcal{C}$.

• *Product Criterion*: Maximize the coding advantage

$$\mu = \left(\prod_{\overline{c}_t \neq \overline{e}_t} \left|f\left(\overline{c}_t\right) - f\left(\overline{e}_t\right)\right|^2 \right)^{1/d}$$

over all pairs of distinct codewords $\boldsymbol{c}, \boldsymbol{e} \in \mathcal{C}$.

Since real fading channels will be neither quasi-static nor fast fading but something in between, Tarokh *et al.* [12] suggested the *ad hoc* strategy of designing space–time codes based on a combination of the quasi-static and fast-fading design criteria. They refer to space–time codes designed according to the hybrid criteria as “smart greedy codes,” meaning that the codes seek to exploit both spatial and temporal diversity whenever available. The fact that the smart greedy approach only considers the two *extreme* scenarios of quasi-static and fast fading motivates the design of *universal space–time codes* that strive to achieve the same goal in the parameterized MIMO block-fading channels.

III. Space–Time Design Criteria for Block Fading Channels

Now, we consider the MIMO block-fading model and develop a systematic approach for designing space–time codes that achieve the maximum diversity advantage for any coding rate, number of transmit antennas, and temporal interleaving depth. We start in this section by developing the design criteria that govern the code performance in such channels.

A. Baseband Design Criterion

Under the block-fading assumption, the path gains are constant over n/M consecutive symbol durations. We will use the notation $(.)[m]$ to denote the single or two-dimensional vector of values of parameter $(.)$ for the mth fading block. Accordingly, we have[1]

$$\begin{aligned}
\alpha^{ij}[m] &= \alpha^{ij}_{(m-1)n/M+1} = \cdots = \alpha^{ij}_{mn/M} \\
\underline{Y}[m] &= \left[y^1_{((m-1)n/M)+1}, \ldots, y^1_{mn/M}, \ldots, y^{L_r}_{mn/M}\right]_{1\times\frac{nL_r}{M}} \\
\underline{N}[m] &= \left[n^1_{((m-1)n/M)+1}, \ldots, n^1_{mn/M}, \ldots, n^{L_r}_{mn/M}\right]_{1\times\frac{nL_r}{M}} \\
\underline{A}[m] &= \left[\alpha^{11}[m], \ldots, \alpha^{L_t1}[m], \ldots, \alpha^{L_tL_r}[m]\right]_{1\times L_tL_r} \\
\boldsymbol{c}[m] &= \left[\overline{c}_{((m-1)n/M)+1}, \ldots, \overline{c}_{mn/M}\right]_{L_t\times\frac{n}{M}}
\end{aligned}$$

$$\boldsymbol{D}_{\boldsymbol{c}}[m] = \begin{bmatrix} f(\boldsymbol{c}[m]) & \mathbf{0}_{L_t\times\frac{n}{M}} & \cdots & \mathbf{0}_{L_t\times\frac{n}{M}} \\ \mathbf{0}_{L_t\times\frac{n}{M}} & f(\boldsymbol{c}[m]) & \cdots & \mathbf{0}_{L_t\times\frac{n}{M}} \\ \vdots & \ddots & \ddots & \mathbf{0}_{L_t\times\frac{n}{M}} \\ \mathbf{0}_{L_t\times\frac{n}{M}} & \mathbf{0}_{L_t\times\frac{n}{M}} & \cdots & f(\boldsymbol{c}[m]) \end{bmatrix}_{L_t\times n}.$$

Then, for $1 \leq m \leq M$

$$\underline{Y}[m] = \sqrt{E_s}\,\underline{A}[m]\boldsymbol{D}_c[m] + \underline{N}[m]. \tag{4}$$

Assuming that codeword $\boldsymbol{c}$ is transmitted, the conditional pairwise error probability that the decoder will prefer an alternate codeword $\boldsymbol{e}$ to $\boldsymbol{c}$ is now given by

$$P(\boldsymbol{c} \to \boldsymbol{e} | \{\alpha^{ij}\}) = P(V < 0 | \{\alpha^{ij}\})$$

where

$$V = \sum_{m=1}^{M} \left[\left\|\underline{A}[m](\boldsymbol{D}_c[m] - \boldsymbol{D}_e[m]) + \underline{N}[m]\right\|^2 - \left\|\underline{N}[m]\right\|^2 \right]$$

[1]The dimensions of the arrays and matrices are reported as subscripts.

is a Gaussian random variable with mean

$$\mathrm{E}\{V\} = \sum_{m=1}^{M} \|\underline{A}[m](\boldsymbol{D}_c[m] - \boldsymbol{D}_e[m])\|^2$$

and variance

$$\mathrm{Var}\{V\} = 2N_0\mathrm{E}\{V\}.$$

Thus,

$$\begin{aligned} &P(V < 0|\{\alpha^{ij}\}) \\ &\quad = Q\left(\frac{\sum_{m=1}^{M} \|\underline{A}[m](\boldsymbol{D}_c[m] - \boldsymbol{D}_e[m])\|}{\sqrt{2N_0}}\right) \\ &\quad \le \frac{1}{2}\exp\left\{-\frac{1}{4N_0}\sum_{m=1}^{M}\|\underline{A}[m](\boldsymbol{D}_c[m] - \boldsymbol{D}_e[m])\|^2\right\}. \end{aligned}$$

Following the approach in [12], [13], the pairwise probability of error can be manipulated to yield the fundamental bound

$$P(\boldsymbol{c} \to \boldsymbol{e}) \le \prod_{m=1}^{M} \left(\frac{\mu_m E_s}{4N_0}\right)^{-d_m L_r} \tag{5}$$

where

$$\begin{aligned} d_m &= \mathrm{rank}(f(\boldsymbol{c}[m]) - f(\boldsymbol{e}[m])) \\ \mu_m &= \left(\prod_{m=1}^{M} \lambda_1[m]\lambda_2[m] \cdots \lambda_{d_m}[m]\right)^{1/d_m} \end{aligned}$$

and $\lambda_1[m]$, $\lambda_2[m]$, $\ldots$, $\lambda_{d_m}[m]$ are the nonzero eigenvalues of

$$\boldsymbol{A}[m] = (f(\boldsymbol{c}[m]) - f(\boldsymbol{e}[m]))(f(\boldsymbol{c}[m]) - f(\boldsymbol{e}[m]))^{\mathrm{H}}.$$

Hence, the generalized diversity and product distance criteria for space–time codes over MIMO block-fading channels are as follows.

- *Block-Fading Sum of Ranks Criterion*: Maximize the transmit diversity advantage

$$d = \sum_{m=1}^{M} d_m = \sum_{m=1}^{M} \mathrm{rank}(f(\boldsymbol{c}[m]) - f(\boldsymbol{e}[m]))$$

over all pairs of distinct codewords $\boldsymbol{c}$, $\boldsymbol{e} \in \mathcal{C}$.

- *Block-Fading Product Distance Criterion*: Maximize the coding advantage

$$\mu = \left(\prod_{m=1}^{M} \lambda_1[m]\lambda_2[m]\cdots\lambda_{d_m}[m]\right)^{1/d}$$

over all pairs of distinct codewords $\boldsymbol{c}$, $\boldsymbol{e} \in \mathcal{C}$.

It is straightforward to see that the design criteria for quasi-static and fast-fading channels can be obtained from the block fading criteria by simply letting $M = 1$ and $M = n$, respectively.

Similar to the quasi-static fading scenario, the fact that the diversity advantage baseband rank criterion applies to the complex-valued differences between baseband codewords represents a major obstacle to the systematic design of maximal diversity space–time codes. Following in the footsteps of [14], we will translate the baseband criterion into an equivalent binary design criteria for BPSK- and QPSK-modulated space–time codes. These general binary criteria are sufficient to ensure that a space–time code achieves a certain level of diversity over block-fading channels. In our approach, a certain equivalence relation is imposed upon potential baseband difference matrices, where each equivalence class contains a special representative that is seen to be a binary projection of a codeword whose rank over *the binary field* is a lower bound on the rank of any of the *complex* matrices in the equivalence class.

B. BPSK Binary Design Criterion

For BPSK modulation, the natural discrete alphabet is the field $\mathbb{F} = \{0, 1\}$ of integers modulo 2. Modulation is performed by mapping the symbol $x \in \mathbb{F}$ to the constellation point $s = f(x) \in \{-1, 1\}$ according to the rule $s = (-1)^x$. Note that it is possible for the modulation format to include an arbitrary phase offset $e^{i\phi}$, since a uniform rotation of the BPSK constellation will not affect the rank of the matrices $f(\boldsymbol{c}[m]) - f(\boldsymbol{e}[m])$ nor the eigenvalues of the matrices

$$\boldsymbol{A}[m] = (f(\boldsymbol{c}[m]) - f(\boldsymbol{e}[m]))(f(\boldsymbol{c}[m]) - f(\boldsymbol{e}[m]))^{\mathrm{H}}.$$

Notationally, the circled operator $\oplus$ will be used to distinguish modulo 2 addition from real- or complex-valued $(+, -)$ operations. Based on this, we have the following result for BPSK-modulated codes over MIMO block-fading (BF) channels.

Proposition 3 **(BPSK-BF Binary Design Criterion)***:* Let $\mathcal{C}$ be a linear $L_t \times n$ space–time code with $n \ge L_t$ used in a communication system with L_t transmit antennas and operating over a block-fading channel with M independent blocks per codeword. Let

$$d = \min_{\boldsymbol{c}\in\mathcal{C}} \sum_{m=1}^{M} \mathrm{rank}(\boldsymbol{c}[m])$$

where the rank is over the binary field $\mathbb{F}$. Then, for BPSK transmission, the space–time code $\mathcal{C}$ achieves a transmit diversity level at least as large as d.

Proof: From [14], we know that for any two codewords $\boldsymbol{c}$, $\boldsymbol{e} \in \mathcal{C}$

$$\mathrm{rank}(f(\boldsymbol{c}[m]) - f(\boldsymbol{e}[m])) \ge \mathrm{rank}(\boldsymbol{c}[m] \oplus \boldsymbol{e}[m])$$

where the rank on the left-hand side is over the complex field and on the right-hand side is over the binary field. Hence,

$$\sum_{m=1}^{M} \mathrm{rank}(f(\boldsymbol{c}[m]) - f(\boldsymbol{e}[m])) \ge \sum_{m=1}^{M} \mathrm{rank}(\boldsymbol{c}[m] \oplus \boldsymbol{e}[m]).$$

Since $\mathcal{C}$ is linear, then $\boldsymbol{c} \oplus \boldsymbol{e} \in \mathcal{C}$ and

$$\min_{\boldsymbol{c},\,\boldsymbol{e}\in\mathcal{C}} \sum_{m=1}^{M} \mathrm{rank}(f(\boldsymbol{c}[m]) - f(\boldsymbol{e}[m])) \ge \min_{\boldsymbol{c}\in\mathcal{C}} \sum_{m=1}^{M} \mathrm{rank}(\boldsymbol{c}[m]),$$

as was to be shown. □

C. QPSK Binary Design Criterion

For QPSK modulation, the natural discrete alphabet is the ring $\mathbb{Z}_4 = \{0, \pm 1, 2\}$ of integers modulo 4. Modulation is performed by mapping the symbol $x \in \mathbb{Z}_4$ to the constellation point $s \in \{\pm 1, \pm i\}$ according to the rule $s = i^x$, where $i = \sqrt{-1}$. Again, the absolute phase reference of the QPSK constellation could have been chosen arbitrarily without affecting the diversity or coding advantages of a $\mathbb{Z}_4$-valued space–time code.

For the $\mathbb{Z}_4$-valued matrix $\boldsymbol{c}[m]$, we define the binary-valued row and column indicant $\Xi(\boldsymbol{c}[m])$ and $\Psi(\boldsymbol{c}[m])$, respectively, as in [14]. These indicant projections serve to indicate certain aspects of the binary structure of the $\mathbb{Z}_4$ matrix in which multiples of two are ignored. For example, let $\boldsymbol{c}[m]$ be a $\mathbb{Z}_4$-valued matrix with exactly ℓ rows that are not multiples of two. After suitable row permutations if necessary, it has the following row structure:

$$\boldsymbol{c}[m] = \begin{bmatrix} \underline{c}_1 \quad [m] \\ \vdots \\ \underline{c}_\ell \quad [m] \\ 2\underline{c}'_{\ell+1}[m] \\ \vdots \\ 2\underline{c}'_{L_t} \quad [m] \end{bmatrix}.$$

The row-based indicant projection (Ξ-projection) is then defined as

$$\Xi(\boldsymbol{c}[m]) = \begin{bmatrix} \beta\left(\underline{c}_1[m]\right) \\ \vdots \\ \beta\left(\underline{c}_\ell[m]\right) \\ \beta\left(\underline{c}'_{\ell+1}[m]\right) \\ \vdots \\ \beta\left(\underline{c}'_{L_t}[m]\right) \end{bmatrix}$$

where $\beta(\underline{c}_\ell[m])$ is the binary projection of $\underline{c}_\ell[m]$ (i.e., modulo 2 reduction of the entries [14]). Similarly, the column-based indicant projection (Ψ-projection) is defined as

$$[\Psi(\boldsymbol{c}[m])]^{\mathrm{T}} = \Xi(\boldsymbol{c}[m]^{\mathrm{T}}). \tag{6}$$

Using these binary indicants, we have the following result that translates the baseband rank criterion into a binary design criterion for QPSK-modulated codes in a MIMO block-fading environment.

Proposition 4 **(QPSK-BF Binary Design Criterion)***:* Let $\mathcal{C}$ be a linear $L_t \times n$ $\mathbb{Z}_4$ space–time code used in a communication system with L_t transmit antennas and operating over a block-fading channel with M blocks per codeword. Let

$$d = \min_{\boldsymbol{c}\in\mathcal{C}}\left(\min\left(\sum_{m=1}^{M}\operatorname{rank}\{\Xi(\boldsymbol{c}[m])\},\ \sum_{m=1}^{M}\operatorname{rank}\{\Psi(\boldsymbol{c}[m])\}\right)\right)$$

where the rank is over the binary field $\mathbb{F}$. Then, for QPSK transmission, the space–time code $\mathcal{C}$ achieves a transmit diversity level at least as large as d.

Proof: The proof follows the technique in [14] to relate the binary rank of the indicants to the baseband rank, then it is straightforward to obtain the result using the same argument as in the BPSK scenario. □

IV. Algebraic Code Design

The MIMO block-fading channel binary design criteria permit the construction of space–time codes that realize the optimum tradeoff between transmission rate and diversity advantage. In this section, we show that the design of full diversity space–time codes is straightforward using the stacking construction in Theorem 2. However, we will also show that using full diversity codes in this scenario entails a significant loss in the transmission rate compared to the quasi-static case. This motivates the design of space–time codes that exploit the *additional* temporal diversity without compromising the transmission rate. We show that carefully constructed codes can achieve significant increases in the diversity advantage with a relatively small number of fading blocks per codeword (i.e., a small penalty in terms of interleaving delay), and without any rate reduction. The main results are developed first in Section IV-A for BPSK-modulated codes and the extension to QPSK modulation is then briefly outlined in Section IV-C.

A. Algebraic Framework for BPSK Space–Time Codes

In a MIMO block-fading channel with L_t transmit antennas and M fading blocks per codeword, the maximum transmit diversity is L_tM. Under this model, the space–time code $\mathcal{C}$ is defined to consist of codewords

$$\boldsymbol{c} = \begin{bmatrix} \underline{x}\boldsymbol{M}_{11} & \underline{x}\boldsymbol{M}_{12} & \cdots & \underline{x}\boldsymbol{M}_{1M} \\ \underline{x}\boldsymbol{M}_{21} & \underline{x}\boldsymbol{M}_{22} & \cdots & \underline{x}\boldsymbol{M}_{2M} \\ \vdots & \vdots & \ddots & \\ \underline{x}\boldsymbol{M}_{L_t1} & \underline{x}\boldsymbol{M}_{L_t2} & \cdots & \underline{x}\boldsymbol{M}_{L_tM} \end{bmatrix} \tag{7}$$

where $\underline{x} \in \mathbb{F}^k$, $M_{ij} \in \mathbb{F}^{k\times n/M}$. Full diversity is realized in this scenario through the multistacking code construction [14].

Corollary 5 **(Multistacking Construction)***:* The space–time code $\mathcal{C}$ will achieve full transmit diversity L_tM if for every $m, 1 \le m \le M$, the set of matrices $\{\boldsymbol{M}_{1m}, \boldsymbol{M}_{2m}, \ldots, \boldsymbol{M}_{L_tm}\}$ satisfies the stacking construction conditions in Theorem 2.

Proof: According to the binary design criterion for BPSK codes, full diversity will be achieved if for every nonzero codeword we have

$$\operatorname{rank}(\boldsymbol{c}[m]) = L_t, \qquad 1 \le m \le M.$$

The conditions in Theorem 2 ensure that $\operatorname{rank}(\boldsymbol{c}[m]) = L_t$ for every nonzero codeword, and hence, the resulting code achieves full diversity in the block fading scenario. □

Corollary 5 offers a simple construction for full diversity space–time codes in MIMO block-fading channels with arbitrary numbers of transmit antennas and fading blocks per codeword. Starting with a full diversity rate $1/L_t$ space–time code for quasi-static channels, one can *repeat* this code in every fading block to obtain a rate $1/L_tM$ code with L_tM diversity advantage. More generally, one can multiplex any "M" full diversity quasi-static codes in the different fading blocks to obtain a full diversity block-fading code. For example, the binary convolutional codes in [14, Table I] offer a rich class of quasi-static codes that can be tailored for this scenario.

TABLE I
MAXIMUM ACHIEVABLE DIVERSITY ADVANTAGE FOR FULL RATE TRANSMISSION (THAT IS, 1-BIT/TRANSMISSION INTERVAL) BPSK SPACE–TIME CODES

L_t	d_{max} for $M=1$	$M=2$	$M=4$	$M=8$
2	2	3	5	9
3	3	5	9	17
4	4	7	13	25

It is clear that the transmission rate of full diversity codes constructed according to Corollary 5 is $\frac{1}{M}$ bits per transmission interval. Comparing this rate to full diversity codes in quasi-static fading channels that achieve 1-bit/transmission interval, one observes the significant loss in throughput. This observation is formalized in the following result that establishes a fundamental limit on the tradeoff between transmission rate and diversity advantage in MIMO block-fading channels, independent of the used code.

Lemma 6: The maximum transmission rate for BPSK modulation in a communication system with L_t transmit antenna, operating over a block-fading channel with M blocks, and using a space–time code that achieves d levels of transmit diversity is $\frac{ML_t-d+1}{M}$ bits per transmission interval.

Proof: Follows directly from the Singleton bound. □

The previous result argues that the loss in rate observed in the multi-stacking construction is inherent to any full diversity code in this scenario. In Table I, we report the maximum diversity advantage for BPSK-modulated codes supporting 1-bit/transmission interval (i.e., the same rate as full diversity quasi-static codes). This table shows that carefully constructed codes can achieve a significant increase in the diversity advantage in MIMO block-fading channels without penalizing the transmission rate. At this point, we note that the limit on the transmission rate in Lemma 6 can be relaxed by using rotated constellations instead of the standard PSK constellations [16]. This line of research, however, will not be pursued further in this paper.

In the following, we investigate the design of space–time codes that realize the optimum tradeoff between the transmission rate and diversity advantage in Corollary 6 for arbitrary numbers of transmit antennas, fading blocks per codeword, and transmission rates. Our development is divided into two parts. First, we present a generalization of the stacking construction in quasi-static fading channels that allows for increasing the transmission rate at the expense of a minimal reduction in the diversity advantage. In addition to being an important result in its own right, this generalization introduces the technical machinery necessary for the second part where space–time codes are constructed to optimally exploit the diversity available in MIMO block-fading channels.

In the first step, we wish to generalize Theorem 2 to handle codes that achieve $d < L_t$ levels of spatial transmit diversity. These codes are capable of supporting higher transmission rates than full diversity codes since, according to Lemma 6, a d-diversity BPSK code can support $L_t - d + 1$ bits per transmission interval in quasi-static fading channels. This result is particularly important for systems with large numbers of transmit antennas where it may be advantageous to increase the transmission rate beyond that possible with full diversity codes while maintaining a reasonable diversity advantage.

Before proceeding further, some definitions are needed to simplify the presentation. The BPSK space–time code $\mathcal{C}$ is now defined as in the stacking construction (i.e., Theorem 2) with the minor modification that n can be smaller than k to allow for transmission rates higher than 1-bit/transmission interval. Let $GL_{L_t}(\mathbb{F})$ be the general linear group over the binary field [17].[2] The number of distinct matrices, ignoring permutations,[3] in this group is [17]

$$|GL_{L_t}(\mathbb{F})| = \frac{\prod_{i=0}^{L_t-1}\left(2^{L_t}-2^i\right)}{L_t!}. \tag{8}$$

For every matrix $\boldsymbol{G} \in GL_{L_t}(\mathbb{F})$ and every k-tuple information vector $\underline{x} \in \mathbb{F}^k$, we define

$$\mathcal{Q}(\underline{x}, \boldsymbol{G}) = \boldsymbol{G}\begin{bmatrix} \underline{x}\boldsymbol{M}_1 \\ \underline{x}\boldsymbol{M}_2 \\ \vdots \\ \underline{x}\boldsymbol{M}_{L_t} \end{bmatrix} = \begin{bmatrix} q_1(\underline{x}, \boldsymbol{G}) \\ q_2(\underline{x}, \boldsymbol{G}) \\ \vdots \\ q_{L_t}(\underline{x}, \boldsymbol{G}) \end{bmatrix} \tag{9}$$

where $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_{L_t}$ are the binary matrices used to generate the space–time code $\mathcal{C}$

$$\begin{aligned} q_i(\underline{x}, \boldsymbol{G}) &= \underline{x}\left(g(i,1)\boldsymbol{M}_1 \oplus g(i,2)\boldsymbol{M}_2 \oplus \cdots \oplus g(i,L_t)\boldsymbol{M}_{L_t}\right) \\ &= \underline{x}\mathcal{M}_i(\boldsymbol{G}) \end{aligned} \tag{10}$$

and $g(i, j)$ is the (i, j)th element of $\boldsymbol{G}$. Now, we have the following result that generalizes the stacking construction to allow for higher transmission rates in quasi-static channels.

*Theorem 7 (**Generalized Stacking Construction**):* Let $\mathcal{C}$ be a linear $L_t \times n$ space–time code as defined in Theorem 2 with the exception that $n \le k$ is allowed. Then, for BPSK transmission over the quasi-static fading channel, $\mathcal{C}$ achieves at least d levels of transmit diversity if d is the largest integer such that $\forall\, \boldsymbol{G} \in GL_{L_t}(\mathbb{F})$

$$\boldsymbol{\mathcal{M}}(\boldsymbol{G}) = [\mathcal{M}_1(\boldsymbol{G}), \mathcal{M}_2(\boldsymbol{G}), \ldots, \mathcal{M}_{L_t-d+1}(\boldsymbol{G})]$$

has full rank k over the binary field $\mathbb{F}$.

Proof: Applying the BPSK-BF binary design criterion to the quasi-static scenario, one immediately sees that $\mathcal{C}$ will achieve d levels of diversity if the binary rank for every nonzero codeword is larger than or equal to d. Now, a codeword matrix $\boldsymbol{c}$ will have a binary rank equal to d, if and only if all matrices resulting from applying any number of simple row operations to $\boldsymbol{c}$ have at least d nonzero rows—i.e., the number of zero rows is less than $L_t - d + 1$. Noting that $\{\mathcal{Q}(\underline{x}, \boldsymbol{G})\}$ is the set of matrices resulting from applying all possible combinations of simple row operations to the codeword matrix corresponding

[2] $GL_{L_t}(\mathbb{F})$ is the set of $L_t \times L_t$ full-rank binary matrices over the binary field.

[3] All the matrices that can be obtained from one matrix through row permutations are counted only once. The reason is that all these matrices result in the same test in our approach.

to the input stream $\underline{x}$, one can easily see that $\mathcal{C}$ will achieve d levels of diversity if

$$\forall \boldsymbol{G} \in GL_{L_t}(\mathbb{F}),\ \underline{x} \in \mathbb{F}^K$$
$$[q_1(\underline{x}, \boldsymbol{G}),\ q_2(\underline{x}, \boldsymbol{G}),\ \ldots,\ q_{L_t-d+1}(\underline{x}, \boldsymbol{G})] \neq \underline{0}. \quad (11)$$

Using the fact that

$$q_i(\underline{x}, \boldsymbol{G}) = \underline{x}\mathcal{M}_i(\boldsymbol{G})$$

we can see that the condition in (11) will be satisfied for every nonzero input stream $\underline{x}$ if and only if

$$\boldsymbol{\mathcal{M}}(\boldsymbol{G}) = [\mathcal{M}_1(\boldsymbol{G}),\ \mathcal{M}_2(\boldsymbol{G}),\ \ldots,\ \mathcal{M}_{L_t-d+1}(\boldsymbol{G})]$$

has full rank k over the binary field for every $\boldsymbol{G} \in GL_{L_t}(\mathbb{F})$. □

It is clear that for $d = L_t$, the condition in Theorem 7 reduces to that in the stacking construction (i.e., Theorem 2). The generalized stacking construction allows for significant gains in throughput in certain scenarios. For example, with n transmit antennas, the rate can be increased from 1-bit/transmission interval to 2-bit/transmission interval at the expense of a reduction in diversity advantage from n to $n-1$.

The next step is to extend the generalized stacking construction to MIMO block-fading channels. First, some more notation is needed to handle this scenario: for every fading block $m \in \{1, \ldots, M\}$ and input information sequence $\underline{x} \in \mathbb{F}^k$, we define $\boldsymbol{G}_m = [g_m(i, j)] \in GL_{L_t}(\mathbb{F})$ and

$$\mathcal{Q}(\underline{x}, \boldsymbol{G}_m, m) = \boldsymbol{G}_m \begin{bmatrix} \underline{x}\boldsymbol{M}_{1m} \\ \underline{x}\boldsymbol{M}_{2m} \\ \vdots \\ \underline{x}\boldsymbol{M}_{L_t m} \end{bmatrix} = \begin{bmatrix} q_1(\underline{x}, \boldsymbol{G}_m, m) \\ q_2(\underline{x}, \boldsymbol{G}_m, m) \\ \vdots \\ q_{L_t}(\underline{x}, \boldsymbol{G}_m, m) \end{bmatrix} \quad (12)$$

where

$$\begin{aligned} q_i(\underline{x}, \boldsymbol{G}_m, m) &= \underline{x}(g_m(i, 1)\boldsymbol{M}_{1m} \oplus g_m(i, 2)\boldsymbol{M}_{2m} \\ &\quad \oplus \cdots \oplus g_m(i, L_t)\boldsymbol{M}_{L_t m}) \\ &= \underline{x}\mathcal{M}_{i,m}(\boldsymbol{G}_m). \end{aligned} \quad (13)$$

$g_m(i, j)$ is the (i, j)th element of $\boldsymbol{G}_m$, and $q_0(\underline{x}, \boldsymbol{G}_m, m)$, $\mathcal{M}_{0,m}(\boldsymbol{G}_m)$ are empty. Using this notation, we are ready to generalize the quasi-static generalized stacking construction to block-fading channels with arbitrary numbers of transmit antennas and fading blocks per codeword.

*Theorem 8 (***Block Fading Generalized Stacking Construction***):* Let $\mathcal{C}$ be a linear $L_t \times n$ space–time code as defined in (7). Then, for BPSK transmission in a MIMO block channel with L_t transmit antennas and M fading blocks per codeword, $\mathcal{C}$ achieves at least d levels of transmit diversity if d is the largest integer such that

$$\forall \boldsymbol{G}_1 \in GL_{L_t}(\mathbb{F}), \ldots, \boldsymbol{G}_M \in GL_{L_t}(\mathbb{F}),\ 0 \le m_i \le L_t,$$
$$\text{and } \sum_{i=1}^{M} m_i = ML_t - d + 1$$

$$\begin{aligned} &\boldsymbol{\mathcal{M}}_{m_1,\ldots,m_M}(\boldsymbol{G}_1, \ldots, \boldsymbol{G}_M) \\ &\quad = [\mathcal{M}_{0,1}(\boldsymbol{G}_1), \ldots, \mathcal{M}_{m_1,1}(\boldsymbol{G}_1), \mathcal{M}_{0,2}(\boldsymbol{G}_2), \ldots, \\ &\qquad \mathcal{M}_{m_2,2}(\boldsymbol{G}_2), \ldots, \mathcal{M}_{m_M,M}(\boldsymbol{G}_M)] \end{aligned}$$

has a full rank k over the binary field $\mathbb{F}$.

Proof: From the BPSK-BF binary design criterion, we know that the diversity advantage of the space–time code $\mathcal{C}$ is given by

$$d \ge \min_{\boldsymbol{c} \in \mathcal{C}} \sum_{m=1}^{M} \operatorname{rank}(\boldsymbol{c}[m]). \quad (14)$$

Let $\boldsymbol{H}[m] = \boldsymbol{G}_m \boldsymbol{c}[m]$, then

$$d \ge ML_t - \max_{\underline{x} \in \mathbb{F}^k, \boldsymbol{G}_1, \ldots, \boldsymbol{G}_m \in GL_{L_t}(\mathbb{F})} \sum_{m=1}^{M} N_r(\boldsymbol{H}[m]) \quad (15)$$

where $N_r(.)$ is the number of zero rows in the matrix $(.)$. It is straightforward to see that

$$\max_{\underline{x} \in \mathbb{F}^k} \sum_{m=1}^{M} N_r(\boldsymbol{H}[m]) = N_{\max} \quad (16)$$

if and only if $N_{\max}$ is the smallest integer such that any $N_{\max}+1$ rows chosen from $\{\boldsymbol{H}[1], \ldots, \boldsymbol{H}[M]\}$ have at least one nonzero row. This condition is satisfied if $N_{\max}$ is the smallest integer such that $\boldsymbol{\mathcal{M}}_{m_1,\ldots,m_M}(\boldsymbol{G}_1, \ldots, \boldsymbol{G}_M)$ has full rank k over the binary field $\mathbb{F}$ for any M-tuple $\{m_1, \ldots, m_M\}$ with $0 \le m_i \le L_t$ and $\sum_{i=1}^{M} m_i = N_{\max} + 1$. This means that if v is the smallest integer such that $\boldsymbol{\mathcal{M}}_{m_1,\ldots,m_M}(\boldsymbol{G}_1, \ldots, \boldsymbol{G}_M)$ has full rank k over the binary field $\mathbb{F}$ for any M-tuple $\{m_1, \ldots, m_M\}$ with $0 \le m_i \le L_t$, $\sum_{i=1}^{M} m_i = v + 1$, and

$$\forall \boldsymbol{G}_1 \in GL_{L_t}(\mathbb{F}), \ldots, \boldsymbol{G}_M \in GL_{L_t}(\mathbb{F})$$

then

$$\max_{\underline{x} \in \mathbb{F}^k, \boldsymbol{G}_1, \ldots, \boldsymbol{G}_m \in GL_{L_t}(\mathbb{F})} \sum_{m=1}^{M} N_r(\boldsymbol{H}[m]) = v \quad (17)$$

and, hence,

$$d \ge ML_t - v \quad (18)$$

which was to be proven. □

One notes that this proposition only establishes a lower bound on the diversity advantage due to the fact that the BPSK-BF binary design criterion provides only sufficient conditions. In most cases considered in this paper, however, this lower bound was found to correspond to the *true* diversity advantage as validated by simulation results. It is straightforward to see that the multistacking construction can be obtained by setting the diversity advantage $d = ML_t$ in Theorem 8.

Theorem 8 enables the construction of space–time convolutional and block codes that optimally exploit the diversity available in MIMO block-fading channels with arbitrary numbers of transmit antennas and fading blocks per codeword. In the following, we consider in some detail the design of space–time convolutional codes. The main advantage of such codes is the availability of computationally efficient maximum-likelihood decoders.

Following the approach in [18], let C be a binary convolutional code of rate k/ML_t. The encoder processes k binary input sequences $x_1(t), x_2(t), \ldots, x_k(t)$ and produces ML_t coded output sequences $y_1(t), y_2(t), \ldots, y_{ML_t}(t)$ which are multiplexed together to form the output codeword. A sequence $\{x(t)\}$ is often represented by the formal series

$$X(D) = x(0) + x(1)D + x(2)D^2 + \cdots.$$

We refer to $\{x(t)\} \leftrightarrow X(D)$ as a D-transform pair. The action of the binary convolutional encoder is linear and is characterized by the so-called impulse responses $g_{i,j}(t) \leftrightarrow G_{i,j}(D)$ associating output $y_j(t)$ with input $x_i(t)$. Thus, the encoder action is summarized by the matrix equation

$$\boldsymbol{Y}(D) = \boldsymbol{X}(D)\boldsymbol{G}(D)$$

where

$$\boldsymbol{Y}(D) = [Y_1(D) \quad Y_2(D) \quad \cdots \quad Y_{ML_t}(D)]$$
$$\boldsymbol{X}(D) = [X_1(D) \quad X_2(D) \quad \cdots \quad X_k(D)]$$

and

$$\boldsymbol{G}(D) = \begin{bmatrix} G_{1,1}(D) & G_{1,2}(D) & \cdots & G_{1,ML_t}(D) \\ G_{2,1}(D) & G_{2,2}(D) & \cdots & G_{2,ML_t}(D) \\ \vdots & \vdots & \ddots & \vdots \\ G_{k,1}(D) & G_{k,2}(D) & \cdots & G_{k,ML_t}(D) \end{bmatrix}. \quad (19)$$

We consider the natural space–time formatting of C in which the output sequence corresponding to $Y_{(m-1)L_t+l}(D)$ is assigned to the lth transmit antenna in the mth fading block, and wish to characterize the diversity that can be achieved by this scheme. Similar to Theorem 8, the algebraic analysis technique considers the rank of matrices formed by concatenating linear combinations of the column vectors

$$\boldsymbol{F}_\ell(D) = \begin{bmatrix} G_{1,\ell}(D) \\ G_{2,\ell}(D) \\ \vdots \\ G_{k,\ell}(D) \end{bmatrix}. \quad (20)$$

Following in the footsteps of Theorem 8, we define $\forall\, \boldsymbol{G}_m \in GL_{L_t}(\mathbb{F})$, $1 \le i \le L_t$, $1 \le m \le M$

$$\begin{aligned} \mathcal{M}_i^{(\boldsymbol{G}_m, m)}(D) = & g_m(i, 1)\boldsymbol{F}_{(m-1)L_t+1}(D) \\ & \oplus g_m(i, 2)\boldsymbol{F}_{(m-1)L_t+2}(D) \\ & \oplus \cdots \oplus g_m(i, L_t)\boldsymbol{F}_{mL_t}(D) \end{aligned} \quad (21)$$

where $\mathcal{M}_0^{(\boldsymbol{G}_m, m)}(D)$ is empty. Then, the following result characterizes the diversity advantage achieved by BPSK space–time convolutional codes in block-fading channels.

Proposition 9: In a communication system with L_t transmit antennas operating over a block-fading channel with M independent blocks per codeword, let $\mathcal{C}$ denote the space–time code consisting of the binary convolutional code C, whose $k \times ML_t$ transfer function matrix is $\boldsymbol{G}(D) = [\boldsymbol{F}_1(D) \quad \cdots \quad \boldsymbol{F}_{ML_t}(D)]$, and the spatial parser σ in which the output $Y_{(m-1)L_t+l}(D) = \boldsymbol{X}(D)\boldsymbol{F}_{(m-1)L_t+l}(D)$ is assigned to the lth antenna in the mth fading block. Then, for BPSK transmission, $\mathcal{C}$ achieves at least d levels of transmit diversity if d is the largest integer such that

$$\forall\, \boldsymbol{G}_1 \in GL_{L_t}(\mathbb{F}), \ldots, \boldsymbol{G}_M \in GL_{L_t}(\mathbb{F}),\ \ 0 \le m_i \le L_t, \quad \text{and } \sum_{i=1}^{M} m_i = ML_t - d + 1$$

TABLE II
RATE $1/2$ UNIVERSAL BINARY SPACE–TIME CODES FOR TWO TRANSMIT ANTENNAS

ν	**Connection Polynomials**	d for $M=1$	$M=2$	$M=3$	$M=4$
2	6, 7	**2**	**3**	3	3
3	64, 34	**2**	**3**	**4**	4
4	72, 46	**2**	**3**	**4**	4
5	55, 37	**2**	**3**	**4**	**5**

TABLE III
RATE $1/3$ UNIVERSAL BINARY SPACE–TIME CODES FOR THREE TRANSMIT ANTENNAS

ν	**Connection Polynomials**	d for $M=1$	$M=2$	$M=3$
3	54, 64, 74	**3**	4	4
4	62, 76, 26	**3**	**5**	5
5	55, 65, 62	**3**	**5**	6

$$\begin{aligned} &\boldsymbol{\mathcal{M}}_{m_1,\ldots,m_M}^{(\boldsymbol{G}_1,\ldots,\boldsymbol{G}_M)}(D) \\ &= \Big[\mathcal{M}_0^{(\boldsymbol{G}_1,1)}(D), \ldots, \mathcal{M}_{m_1}^{(\boldsymbol{G}_1,1)}(D), \mathcal{M}_0^{(\boldsymbol{G}_2,2)}(D), \ldots, \\ &\quad \mathcal{M}_{m_2}^{(\boldsymbol{G}_2,2)}(D), \ldots, \mathcal{M}_{m_M}^{(\boldsymbol{G}_M,M)}(D)\Big] \end{aligned}$$

has a rank k over the space of all formal series.

Proof: Straightforward by specializing the technique used in the proof of Theorem 8 to the convolutional codes scenario as in [18]. □

We note that, similar to [18], this framework encompasses as a special case rate $1/n'$ convolutional codes with bit or symbol interleaving across the transmit antennas and fading blocks.

Theorem 8 and Proposition 9 allow for generalizing the *smart and greedy* design rule to the present scenario. Since in many practical applications the number of independent fading blocks per codeword is not known *a priori* at the transmitter, it is desirable to construct codes that exploit the temporal diversity *whenever* available. This leads to the notion of *universal* space–time codes that achieve the maximum diversity advantage, allowed by their transmission rate according to Lemma 6, with different numbers of fading blocks per codeword (i.e., different M). More specifically, we have the following definition.

Definition 10: A rate r binary space–time code $\mathcal{C}$ is said be a universal BPSK space–time code for a $(L_t, M_{\max})$ MIMO block-fading channel if it achieves the maximum transmit diversity advantage, allowed by the transmission rate and decoder complexity, with L_t transmit antennas and arbitrary $M \le M_{\max}$ fading blocks per codeword.

In this definition, it is assumed that the output symbols from the baseline code C will be periodically distributed across the transmit antennas and fading blocks for every M. Rate $1/L_t$ *universal* space–time codes are particularly important because they achieve *full* transmit diversity in quasi-static fading channels. In Tables II and III, we present the best rate $1/L_t$ universal codes found through an exhaustive search for two and three transmit antennas with different numbers of fading blocks per codeword.[4] These codes also maximize the free distance within the codes with the same diversity advantage profile. In our search, we assumed bit multiplexing across the different transmit antennas and fading blocks.

B. Suboptimality of the Concatenated Coding Approach

Several works have considered the design of concatenated coding schemes for time varying MIMO channels—e.g., [19]. The basic idea is to construct an outer code to exploit the *temporal* diversity along with an inner code that exploits the *spatial* diversity. In this approach, the outer code is composed of a baseline code and a multiplexer that distributes the coded symbols across the different fading blocks. The input assigned to each fading block is *independently* coded with an inner *spatial* code.[5]

The following result establishes an upper bound on the diversity advantage that demonstrates the suboptimality of this coding scheme. The bound also offers some guidelines on the optimal rate allocation for the inner and outer codes.

Proposition 11: Let $\mathcal{C}$ be a linear concatenated space–time code using a rate r_o outer code that achieves d_t levels of temporal transmit diversity and rate r_i inner code that achieve d_s levels of spatial transmit diversity used in a communication system with L_t transmit antennas and M independent blocks per codeword. Then, $\mathcal{C}$ achieves at least $d_c = d_t d_s$ levels of transmit diversity for BPSK transmission with maximum-likelihood decoding. Furthermore, for a fixed transmission rate $\eta = r_o r_i L_t$ bits per transmission interval, we have

$$d_c \leq (\lfloor M - \sqrt{M\eta} \rfloor + 1)(\lfloor L_t - \sqrt{M\eta} \rfloor + 1)$$

where equality can be achieved if and only if the rates are allocated as $r_o = \sqrt{\frac{\eta}{M}}$, $r_i = \frac{\sqrt{\eta M}}{L_t}$.

Proof: Let $c_o \in C_o$ denote a codeword in the outer code C_o. In the concatenated coding approach, the codeword c_o is partitioned into M substreams (i.e., $c_o = c_o[1]|c_o[2]|\cdots|c_o[M]$), where each substream is independently encoded by the inner spatial code $\mathcal{C}_i$. Let $\mathrm{wt}\,(c_o[i])$ denote the binary weight of $c_o[i]$, and

$$\begin{aligned} I(\mathrm{wt}\,(c_o[i])) &= 1, \qquad \text{if } \mathrm{wt}\,(c_o[i]) \geq 1 \\ &= 0, \qquad \text{otherwise.} \end{aligned} \tag{22}$$

Using the linearity of the code C_o, it is easy to see that

$$d_t = \min_{c_o \in C_o,\, c_o \neq \underline{0}} \sum_{i=1}^{M} I(\mathrm{wt}\,(c_o[i])). \tag{23}$$

From the BPSK-BF rank criterion, we know that the diversity advantage is lower-bounded by

$$d \geq \min_{\boldsymbol{c}_i \neq \mathbf{0}} \sum_{k=1}^{M} \mathrm{rank}(\boldsymbol{c}_i[k]). \tag{24}$$

[4]Bold letters are used to identify the cases where the maximum diversity advantage is achieved.

[5]In most proposed approaches, this inner code is derived from the orthogonal designs.

The fact that the inner code is designed to achieve d_s levels of spatial diversity means that

$$\begin{aligned} \mathrm{rank}(\boldsymbol{c}_i[k]) &\geq d_s, \qquad \text{if } I(\mathrm{wt}\,(c_o[k])) = 1 \\ &= 0, \qquad \text{if } I(\mathrm{wt}\,(c_o[k])) = 0. \end{aligned} \tag{25}$$

Combining (23), (24), and (25), we obtain the first part of the proposition

$$d \geq d_c = d_t d_s. \tag{26}$$

From Lemma 6, we know that

$$d_t \leq \lfloor M(1 - r_o) \rfloor + 1 \tag{27}$$

$$d_s \leq \lfloor L_t(1 - r_i) \rfloor + 1 \tag{28}$$

and hence,

$$d_c \leq (\lfloor M(1 - r_o) \rfloor + 1)(\lfloor L_t(1 - r_i) \rfloor + 1). \tag{29}$$

Subject to the constraint of a transmission rate equal to η bits per transmission interval, the right-hand side of (29) is maximized by assigning $r_o = \sqrt{\frac{\eta}{M}}$, $r_i = \frac{\sqrt{\eta M}}{L_t}$ (this result can be obtained using the standard Lagrange multiplier optimization approach). This allocation results in

$$d_c \leq \left(\left\lfloor M - \sqrt{M\eta} \right\rfloor + 1\right)\left(\left\lfloor L_t - \sqrt{M\eta} \right\rfloor + 1\right). \tag{30}$$

□

Component convolutional codes for this approach can be designed using Proposition 9 after setting $L_t = 1$, $M = 1$, respectively. Compared with the achievable diversity advantage suggested by Lemma 6, Proposition 11 implies that the concatenated coding approach is usually suboptimal. For example, assuming that $M = L_t = 3$, $\eta = 1$-bit/transmission interval, the achievable diversity advantage is $d_{\max} = 7$. The possible transmit diversity for the concatenated coding approach is, however, $d_c = 4$.[6] Furthermore, the optimal rate allocation strategy suggested by Theorem 11 argues that using a *full* spatial diversity inner code is not always the optimal solution since full spatial diversity is only possible with rate $1/L_t$ inner codes which may not agree with the optimal rate-allocation strategy. In the previous example, using full spatial diversity inner code will result in $d_c = 3$. The inner code uses all the available redundancy, and hence, the temporal diversity is not exploited.

C. QPSK Modulation

As a consequence of the QPSK-BF binary design criterion, one sees that the binary constructions proposed for BPSK modulation in Section IV-A can also be used to construct QPSK modulated codes. For example, the binary connection polynomials of Tables II and III can be used to generate linear, $\mathbb{Z}_4$-valued, trellis QPSK codes that achieve the same diversity advantage as the *baseline* binary code [14]. More generally, one may use any set of $\mathbb{Z}_4$-valued connection polynomials whose modulo 2 projections [14] appear in the tables. Ideally, it is desirable to find the code that offers the best coding gain (i.e., product distance)

[6]Again, there is some looseness in this argument since d_c may be lower than the true diversity advantage.

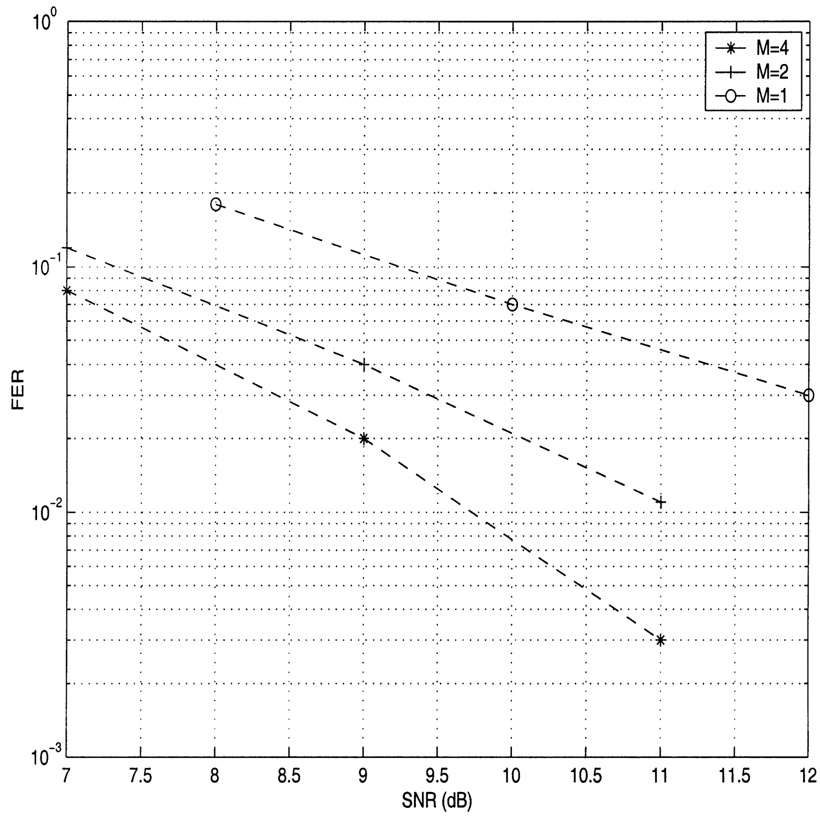

Fig. 1. Performance of 16-state BPSK space–time code with $L_t = 2$, $L_r = 1$, and different numbers of fading blocks per codeword.

within this class of codes. However, in the absence of a systematic approach for optimizing the coding gain, one has to restore to empirical searching techniques. Experimentally, we found that codes obtained by replacing each zero in the binary generator matrix by a two yield the best simulated frame error rate performance in most cases. Another way for designing codes for QPSK modulation is to combine two binary codes $\mathcal{A}, \mathcal{B}$ with the same diversity advantage in a dyadic format $\mathcal{C} = \mathcal{A} + 2\mathcal{B}$. The resulting $\mathbb{Z}_4$ code achieves the same diversity advantage as the binary codes for QPSK transmission [14]. It is worth noting that these *lifting* techniques are general for arbitrary numbers of transmit antennas and fading blocks per codeword.

V. Numerical Results

Fig. 1 highlights the performance gains possible with increasing the number of fading blocks per codeword. In the figure, we report the performance of the universal 16-state BPSK code in a system with $L_t = 2$ and $L_r = 1$. It is clear from the figure that the diversity advantage increases as the number of fading blocks increases, and hence, the performance curve becomes steeper. It is worth noting that the gain in performance in this scenario does not entail any additional receiver complexity. Fig. 2 highlights the importance of careful design to optimize the diversity advantage. In the figure, we compare the four-state $(5, 7)$ optimal free distance space–time code with the four-state $(6, 7)$ code in a BPSK system with $L_t = 2$, $L_r = 1$, and $M = 2, 3$. Using Proposition 9, one can see that the $(5, 7)$ code achieves $d = 2, 3$ for $M = 2, 3$, respectively. The $(6, 7)$ code, on the other hand, achieves $d = 3$ in both cases (note that for $M = 2$, $d = 3$ is the maximum possible diversity advantage for this throughput). As shown in the figure, for the $M = 2$ case, the superior diversity advantage of the $(6, 7)$ code is apparent in the steeper slope of the frame error rate curve. This results in a gain of about 1 dB at 0.01 frame error rate. For the $M = 3$ case, however, it is clear that the $(5, 7)$ code exhibits a superior coding gain compared with the $(6, 7)$ code.

Fig. 3 compares the performance of Tarokh, Seshadri, and Calderbank (TSC) 16-state code [12], and a new linear $\mathbb{Z}_4$ code obtained by "*lifting*" the binary $(6, 7)$ code[7] with QPSK modulation in the same scenario (i.e., $L_t = M = 2$). By examining the error events of the TSC code, one can easily see that this code only achieves $d = 2$, whereas the new code achieves $d = 3$ as argued in Section IV-C. It is clear from the figure that the superior diversity advantage of the new code allows for significant gains especially at high signal-to-noise ratios. Fig. 4 compares the performance of the recently proposed turbo space–time code [5] with the new 64-state linear $\mathbb{Z}_4$ code with generator polynomial $(1 + D + D^2 + D^3, 1 + D + 2D^2 + D^3)$[8] in a system with $L_t = M = 2$ and $L_r = 1$. One observes the significant performance gain and reduction in receiver complexity offered

[7]The $\mathbb{Z}_4$ generator polynomial of the new code is $(1+D+D^2, 1+D+2D^2)$.

[8]This code is obtained by lifting the 16-state universal code in Table I.

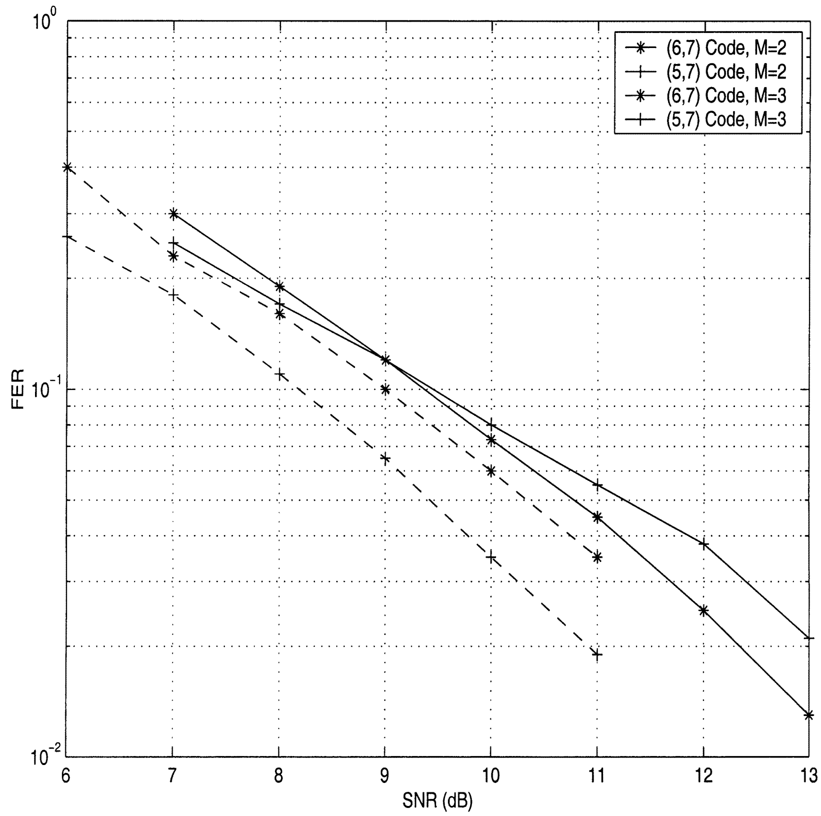

Fig. 2. Performance comparison of two 4-state BPSK space–time codes with $L_t = 2$, $L_r = 1$, and different numbers of fading blocks per codeword.

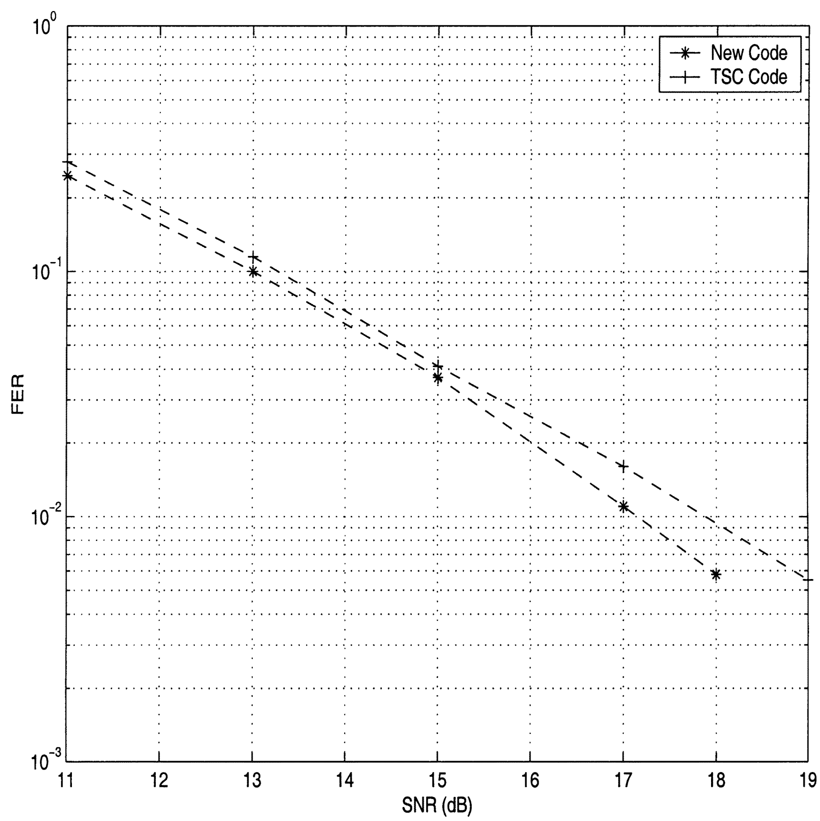

Fig. 3. Performance comparison of the new 16-state QPSK space–time code and the TSC space–time code with $L_t = M = 2$, and $L_r = 1$.

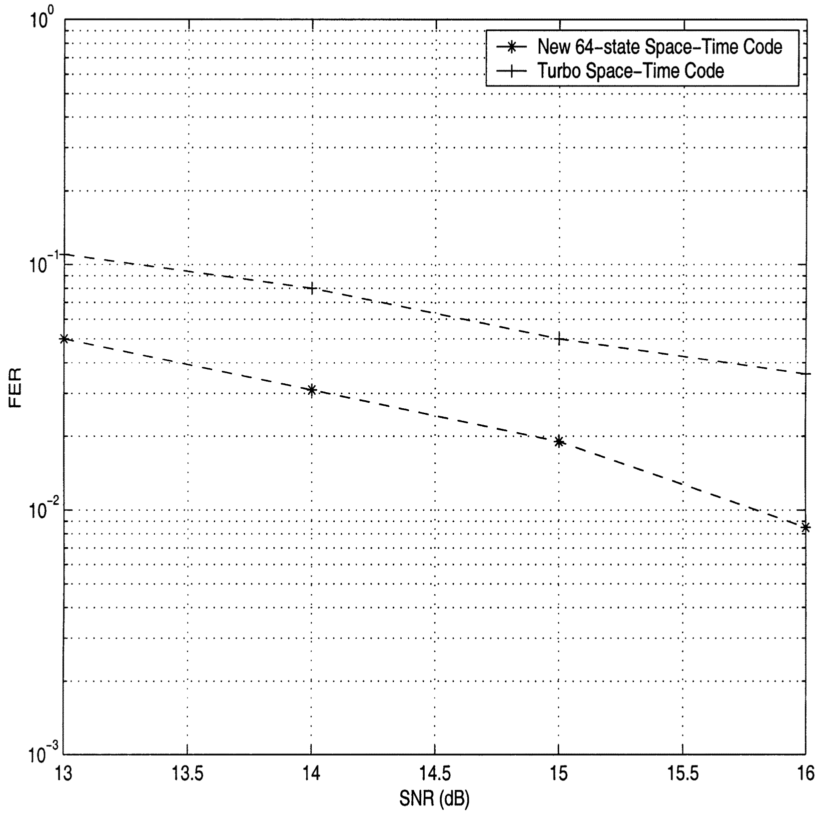

Fig. 4. Performance comparison of the new 64-state QPSK space–time code and the new Turbo space–time code with $L_t = M = 2$, and $L_r = 1$.

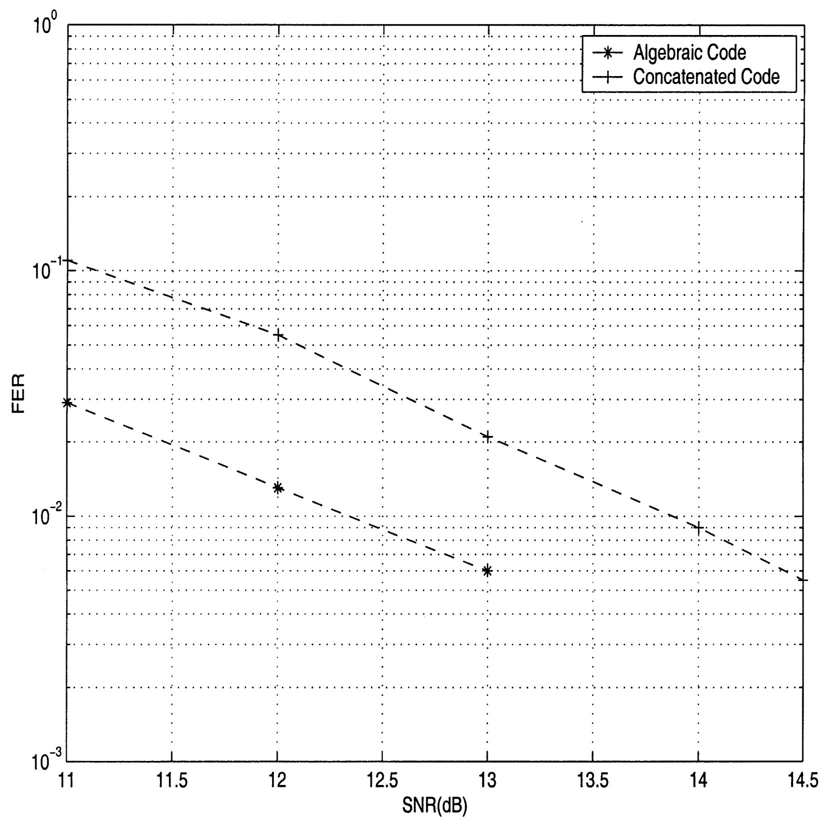

Fig. 5. Performance comparison of the algebraic QPSK space–time code and the concatenated space–time code in a system with $L_t = 2$, $L_r = 1$, and $M = 4$.

by the new code compared to the turbo code in this scenario with *limited degrees of diversity*.[9]

In Fig. 5, we compare the proposed algebraic QPSK space–time codes with the concatenated coding approach in a systems with $L_t = 2$, $L_r = 1$, and $M = 4$. The QPSK code is obtained by lifting the eight-state binary code in Table II to the $\mathbb{Z}_4$ ring. Alamouti orthogonal design is used as an inner code in the concatenated coding approach. The outer code is the eight-state binary code proposed for the block-fading channel in [20]. To achieve a 2-bits/channel use, similar to the algebraic QPSK code, we used 16-QAM modulation in the concatenated coding approach. As shown in the figure, the algebraic QPSK code *matched to the MIMO-BF channel* offers a gain of 1.5 dB. It is also worth noting that further gain may be achieved if the 16-QAM modulation along with a lower rate *algebraically constructed* code are used. This, however, requires extending our results to higher order constellations which is beyond the scope of this paper.

VI. CONCLUSION

In this paper, we considered the design of space–time codes for MIMO block-fading channels. We derived the baseband design criteria that determine the diversity and coding advantage achieved by space–time codes in MIMO block-fading channels. For BPSK and QPSK modulated codes, we presented binary design criteria that allow for designing space–time codes that exhibit the optimum diversity-versus-throughput tradeoff. These binary criteria were then utilized to develop an algebraic framework for space–time code design in MIMO block-fading channels. Several design examples for trellis codes suitable for BPSK and QPSK transmissions were also presented together with simulation results that demonstrate the efficacy of the proposed framework.

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