On the Optimality of the ARQ-DDF Protocol

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Abstract

The performance of the automatic repeat request-dynamic decode and forward (ARQ-DDF) cooperation protocol is analyzed in two distinct scenarios. The first scenario is the multiple access relay (MAR) channel where a single relay is dedicated to simultaneously help several multiple access users. For this setup, it is shown that the ARQ-DDF protocol achieves the optimal diversity multiplexing tradeoff (DMT) of the channel. The second scenario is the cooperative vector multiple access (CVMA) channel where the users cooperate in delivering their messages to a destination equipped with multiple receiving antennas. For this setup, we develop a new variant of the ARQ-DDF protocol where the users are purposefully instructed not to cooperate in the first round of transmission. Lower and upper bounds on the achievable DMT are then derived. These bounds are shown to converge to the optimal tradeoff as the number of transmission rounds increases.

1 Background

The dynamic decode and forward (DDF) protocol was proposed in [1] as an efficient method to exploit cooperative diversity in the half-duplex relay channel (the same protocol was independently devised for other settings [2, 3]). Here, we combine the DDF protocol with the ARQ mechanism to derive new variants that are matched to the multiple access relay (MAR) and cooperative vector multiple access (CVMA) channels. These variants, some of them presented in [9], [10] and [11], are shown to achieve the optimal tradeoff between throughput and reliability, in the high signal to noise ratio (SNR) regime. For simplicity of presentation, we restrict ourselves to the two-user scenario. In principle, our results generalize to channels with more users but the mathematical development becomes involved and offers no additional insights.

Throughout the paper, all the channels are assumed to be flat Rayleigh-fading and quasi-static. The quasi-static assumption implies that the channel gains remain fixed over one coherence interval and change, independently, from one coherence interval to the next. In order to highlight the benefits of cooperation and ARQ, as opposed to temporal interleaving, we adopt the long-term static channel model of [6] where all the ARQ rounds corresponding to a certain message take place over the same coherence interval. We further assume that the channel gains are mutually independent with unit variance. The additive Gaussian noise processes at different nodes are zero-mean, mutually-independent,

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circularly-symmetric, and white. Furthermore, the variances of these noise processes are proportional to one another such that there are always *fixed* offsets between different channels' SNRs. All the nodes are assumed to operate synchronously and to have the same power constraint. We impose a short-term power constraint on the nodes ensuring that the average energy available for each symbol is fixed. Except for section 3, where the destination is assumed to have multiple receiving antennas, each node is equipped with a single antenna. We adopt the coherent transmission paradigm where only the receiving node of any link knows the channel gain. Except for the ACK/NACK feedback bits, no other channel state information (CSI) is available to the transmitting nodes. The maximum number of transmission rounds is restricted to L, where $L = 1$ corresponds to the Non-ARQ scenario. We also assume the nodes to operate in the half-duplex mode, i.e., at any point in time, a node can either transmit or receive, but not both. This constraint is motivated by the typically large difference between the incoming and outgoing power levels. Throughout the paper, we use random Gaussian code-books with asymptotically large block lengths to derive information theoretic bounds on the achievable performance. Results related to the design of practical coding/decoding schemes that approach the fundamental limits established here will be reported elsewhere (e.g., [4]). Our analysis tool is the diversity multiplexing tradeoff (DMT) introduced by Zheng and Tse in [5]. To define DMT for a symmetric multiple access channel with two users, we consider a family of codes $C(\rho) = \{C_1(\rho), C_2(\rho)\}$ labeled by the operating SNR ρ, such that the code $C_j(\rho)$ (used by user $j \in \{1, 2\}$) has a rate $R(\rho)/2$ bits per channel use (BPCU) and a maximum likelihood (ML) error probability $P_{E_j}(\rho)$. For this family, we define the multiplexing gain r and diversity gain d as

$$r \triangleq \lim_{\rho\to\infty} \frac{R(\rho)}{\log \rho}, \qquad\qquad d \triangleq \min_{j\in\{1,2\}} \{-\lim_{\rho\to\infty} \frac{\log P_{E_j}(\rho)}{\log \rho}\}. \tag{1}$$

In ARQ scenarios, i.e., $L > 1$, we replace r with the effective multiplexing gain r_e introduced in [6] to capture the variable-rate nature of ARQ schemes.

The rest of this correspondence is organized as follows. Section 2 is devoted to the MAR channel whereas our results for the CVMA channel are presented in Section 3. We finish with a few concluding remarks in Section 4. To enhance the flow of the paper, we only outline the main ideas of the proofs in the body of the paper and relegate all the technical details to the Appendix.

2 The Multiple Access Relay Channel

In the two-user multiple access relay (MAR) channel, one relay node is assigned to assist the two multiple access users. The users are not allowed to help each other (due to practical limitations, for example). The relay node is constrained by the half-duplex assumption. We proceed towards our main result in this section via a step-by-step approach. First, we prove the optimality of the ARQ-DDF protocol in the relay channel (i.e., a MAR channel with a single user). The proof for this result introduces the machinery necessary to handle the ARQ mechanism. In Lemma 2, we analyze the DDF protocol (without ARQ) in the two user MAR channel. This step elucidates the multi-user aspects of the problem. Finally, we combine these ideas in Theorem 3 to prove the optimality of the DDF-ARQ protocol in the two-user MAR channel.

Lemma 1 *The optimal DMT for the relay channel with $L \geq 2$ ARQ rounds is given by*

$$d_R(r_e, L) = 2(1 - \frac{r_e}{L}) \quad for \quad 1 > r_e \geq 0. \tag{2}$$

Furthermore, this optimal tradeoff is achieved by the proposed ARQ-DDF protocol.

Proof: (Sketch) The converse is readily obtained by applying the results of [6] to the genie-aided 2×1 multiple-input multiple-output (MIMO) channel. The proof of the achievability part, on the other hand, involves two main ingredients; 1) showing that the average throughput, as a function of SNR, converges asymptotically to the throughput corresponding to one round of transmission, and 2) showing that the performance is dominated by the errors for which L rounds of transmission has been requested. By combining these two steps, it can be shown that from a probability of error perspective, the effective multiplexing gain shrinks to r_e/L. The optimality of the ARQ-DDF protocol then follows from the fact that the throughput range of interest (i.e., $r_e/L < 1/L$) falls within the optimality range of the DDF protocol (i.e., $r \leq 1/2$) [1].

In our DDF protocol for the non-ARQ MAR channel, the two sources transmit their individual messages during every symbol interval in the codeword, while the relay listens to the sources until it collects sufficient energy to decode *both* of them error-free. After decoding, the relay uses an *independent* code book to encode the two messages *jointly*. The encoded symbols are then transmitted for the rest of the codeword.

Lemma 2 *The optimal diversity gain for the symmetric two-user MAR channel is upper bounded by*

$$d_{MAR}(r) \leq \begin{cases} 2 - r & if \;\; \frac{1}{2} \geq r \geq 0 \\ 3(1-r) & if \;\; 1 \geq r \geq \frac{1}{2} \end{cases} . \tag{3}$$

Furthermore, the DMT achieved by the DDF protocol is lower bounded by

$$d_{DDF\text{-}MAR}(r) \geq \begin{cases} 2 - r & if \;\; \frac{1}{2} \geq r \geq 0 \\ 3(1-r) & if \;\; \frac{2}{3} \geq r \geq \frac{1}{2} \\ 2\frac{1-r}{r} & if \;\; 1 \geq r \geq \frac{2}{3} \end{cases} . \tag{4}$$

Proof: (Sketch) The upper bound is obtained through a min-cut max-flow argument. The proof for the achievability part follows the same lines as that of Theorem 5 in [1].

Fig. 1 compares the upper and lower bounds in Lemma 2 where the optimality of the DDF protocol for $2/3 \geq r \geq 0$ is evident. This observation is the key to establishing the following result.

Theorem 3 *The optimal DMT for the symmetric two-user MAR channel with $L \geq 2$ ARQ rounds is given by*

$$d_{MAR}(r_e, L) = 2 - \frac{r_e}{L} \quad for \quad 1 > r_e \geq 0. \tag{5}$$

Furthermore, this optimal tradeoff is achieved by the proposed ARQ-DDF protocol.

Proof: (Sketch) The proof essentially reduces to combining the results in Lemma 1 and Lemma 2. One needs, however, to be careful with a few technical details as reported in the Appendix.

We note that the optimality in Theorem 3 extends to the N-user MAR channel. Overall, the main conclusion in this section is establishing the fact that a *single* relay can be efficiently shared by *several* multiple access users such that it enhances the diversity gain achieved by *all* of them.

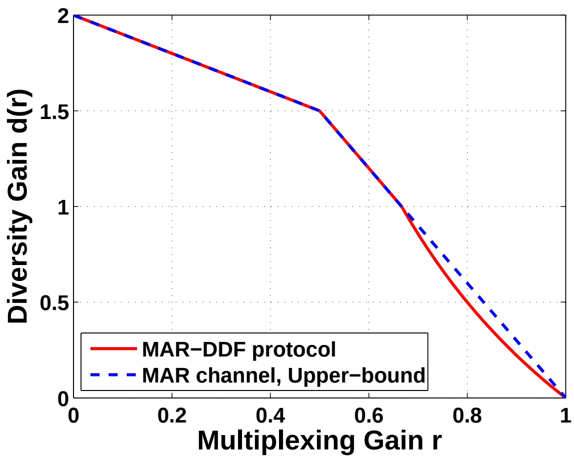

Figure 1: The DMT achieved by the DDF protocol in the MAR channel, along with an upper-bound on the achievable DMT ($L = 1$).

3 The Cooperative Vector Multiple-Access Channel

In the cooperative vector multiple access (CVMA) channel, the two single-antenna users are allowed to assist each other, as long as they do not violate the half duplex constraint. The challenge in this scenario stems from the availability of two receiving antennas at the destination, which increases the channel's degrees of freedom to two. Loosely speaking, in order to exploit these two degrees of freedom, the two users need to transmit *new* independent symbols continuously, which prevents them from cooperation under the half duplex constraint (non-ARQ case). More rigorously, it is straightforward to see that with $L = 1$, any half-duplex cooperation protocol that achieves full diversity (i.e., $d(0) = 3$), falls short of achieving full rate (i.e., $d(r) > 0$, for all $r < 2$). To get around this problem, in the case of $L \geq 2$, we purposefully instruct the users *not* to cooperate in the first round of transmission. In fact, a user continues transmitting its message while it receives NACK signals. Only when a user receives an ACK signal, it starts listening to the other user (assuming the other user has not been successfully decoded yet). Once the cooperating user decodes the message of its partner, it starts helping (i.e., the typical DDF methodology). The following result establishes lower and upper bounds on the DMT achieved by this protocol.

Theorem 4 *The optimal diversity gain for the symmetric two-user CVMA channel with L ARQ rounds is upper bounded by*

$$d_{CVMA}(r_e, L) \leq \min\left\{3(1 - \frac{r_e}{2L}), 4 - \frac{3r_e}{L}\right\}, \quad \textit{for} \ \ 2 > r_e \geq 0. \tag{6}$$

For $L = 2$, the diversity gain achieved by the proposed ARQ-DDF protocol satisfies

$$d_{DDF-CVMA}(r_e, 2) \geq \left\{ \begin{array}{ll} 3 - r_e, & 1 > r_e \geq 0 \\ 4 - 2r_e, & \frac{4}{3} > r_e \geq 1 \\ 2 - \frac{r_e}{2}, & 2 > r_e \geq \frac{4}{3} \end{array} \right. . \tag{7}$$

Furthermore, as L increases, the ARQ-DDF diversity gain converges to the optimal value, i.e.,

$$\lim_{L \to \infty} d_{DDF-CVMA}(r_e, L) = 3, \quad \textit{for} \ \ 2 > r_e \geq 0. \tag{8}$$

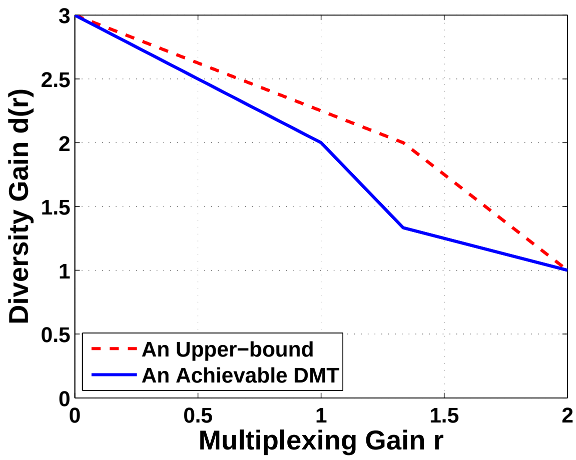

Figure 2: The DMT achieved by the ARQ-DDF protocol in the CVMA channel, along with an upper-bound on the achievable DMT ($L = 2$).

Proof: (Sketch) The upper bound is obtained through a min-cut max-flow argument. The lower bound, on the other hand, results from analyzing a sub-optimal decoder which ignores parts of the received signal to simplify the mathematical development. The key observation for proving the asymptotic optimality result in this theorem is that the dominant error event corresponds to the case when one of the users is decoded successfully, while the other one remains in error, even after L rounds of transmission. In fact, the proposed cooperation protocol is devised with this observation in mind, i.e., the asymptotic optimality of the ARQ-DDF protocol follows from the fact that, as L increases, every user gets a better chance of being helped by the other one.

Fig. 2 compares the upper and lower bounds in (6) and (7) for $L = 2$. Clearly, this figure shows the full diversity and full rate properties of the proposed ARQ-DDF protocol. Finally, we observe that the analysis of the ARQ-DDF protocol in Theorem 4 can be repeated for $L > 2$. We have, however, chosen to omit this analysis since it is rather tedious and not necessarily inspiring.

4 Conclusions

In this correspondence, we combined the DDF protocol with the ARQ mechanism to develop efficient cooperation schemes for the MAR and CVMA channels. The proposed ARQ-DDF protocol was shown to achieve the optimal DMT for the MAR channel. In the CVMA scenario, we argued that the ARQ-DDF protocol achieves significant cooperative diversity gains while exploiting all of the channel's degrees of freedom and despite the half duplex constraint. Furthermore, the diversity gain achieved by the ARQ-DDF protocol in this scenario was shown to converge to the optimal value, as the number of ARQ rounds increases.

5 Appendix

We start with a few notations that are needed throughout the proofs. The SNR of a link, ρ, is defined as

$$\rho \triangleq \frac{E}{\sigma^2}, \tag{9}$$

where E denotes the average energy available for transmission of a symbol across the link and σ^2 denotes the variance of the noise observed at the receiving end of the link. We say that $f(\rho)$ is *exponentially* equal to ρ^b, denoted by $f(\rho) \doteq \rho^b$, when

$$\lim_{\rho\to\infty} \frac{\log f(\rho)}{\log \rho} = b. \tag{10}$$

In (10), b is called the *exponential order* of $f(\rho)$. $\dot{\leq}$, $\dot{\geq}$ are defined similarly.

We denote the maximum allowable number of ARQ rounds by L ($L = 1$ corresponds to the non-ARQ scenario), where each round consists of T consecutive symbol intervals. We denote the first-round rate of transmission by R_1 and the average throughput by η. These two quantities are related through [6]

$$\eta = \frac{R_1}{1 + \sum_{\ell=1}^{L-1} p(\ell)}, \tag{11}$$

where $p(\ell)$ is the probability that the destination requests for the $(\ell + 1)$th round of transmission. We define the first-round multiplexing gain, r_1, and effective multiplexing gain, r_e, as

$$r_1 \triangleq \lim_{\rho\to\infty} \frac{R_1}{\log \rho}, \qquad r_e \triangleq \lim_{\rho\to\infty} \frac{\eta}{\log \rho}. \tag{12}$$

From (11), we note that if $\{p(\ell)\}_{\ell=1}^{L-1}$ decay polynomially with ρ, then $\eta \doteq R_1$ and thus $r_e = r_1$.

5.1 Proof of Lemma 1

A simple min-cut max-flow examination reveals that the optimal diversity gain for this channel is upper bounded by those of the 2×1 and 1×2 ARQ MIMO channels, thus

$$d_{\mathrm{R}}(r_e, L) \leq 2(1 - \frac{r_e}{L}) \ \text{ for } \ 1 > r_e \geq 0. \tag{13}$$

Next we prove the achievability of this upper bound, by characterizing the DMT for the proposed ARQ-DDF relay protocol. We do this in two steps. First, we construct an ensemble of random Gaussian codes and characterize its average error probability P_E, and throughput η. We then show, through a simple expurgation argument, that there are codes in the ensemble that perform at least as well as these two averages, therefore achieving P_E and η *simultaneously.*

Let $C(\rho) = \{C_s(\rho), C_r(\rho)\}$ denote the random codes used by the source and the relay, respectively. These are codes of length LT symbols, rate R_1/L BPCU and generated by an i.i.d complex Gaussian random process of mean zero and variance E. Let us denote the message to be sent by m_0. This message consists of R_1T information bits. We denote the source and relay codewords corresponding to m_0 by $\mathbf{x}_s(m_0)$ and $\mathbf{x}_r(m_0)$, respectively. We also denote the signature of message m_0 at the destination by $\mathbf{s}(m_0)$, i.e.,

$$\mathbf{y} = \mathbf{s}(m_0) + \mathbf{n}, \tag{14}$$

where $\mathbf{y}$ and $\mathbf{n}$ denote the destination received signal and additive noise, respectively. It is important to realize that $\mathbf{s}(m_0)$, not only depends on the message m_0, but also on the channel realization and the relay noise. Also, notice that $\mathbf{x}_r(m_0)$ is only partially transmitted. This is because the half-duplex relay, itself, needs first to listen to the source to be able to decode the message. Finally, we use superscript ℓ to denote the portion of the signal that corresponds to the first ℓ rounds of transmission. The decoder $\{\varphi, \psi\}$, consists of two functions, $\varphi = \{\varphi^\ell\}_{\ell=1}^{L}$ and $\psi = \{\psi^\ell\}_{\ell=1}^{L-1}$.

- At round ℓ $(L \geq \ell \geq 1)$, φ^ℓ outputs the message that minimizes $|\mathbf{y}^\ell - \mathbf{s}^\ell|^2$, i.e.

$$\varphi^\ell(\mathbf{y}^\ell) = \arg\min_m |\mathbf{y}^\ell - \mathbf{s}^\ell(m)|^2, \quad \text{for} \quad L \geq \ell \geq 1. \tag{15}$$

  We denote the event that $\varphi^\ell(\mathbf{y}^\ell)$ differs from m_0, with m_0 denoting the transmitted message, by E^ℓ.

- At round ℓ $(L-1 \geq \ell \geq 1)$, ψ^ℓ outputs a one, if m is the *unique* message for which

$$|\mathbf{y}^\ell - \mathbf{s}^\ell(m)|^2 \leq \ell T(1+\delta)\sigma^2, \tag{16}$$

  where σ^2 denotes the destination noise variance and δ is some positive value. In any other case, ψ^ℓ outputs a zero. We denote the event that ψ^ℓ outputs a one, by A^ℓ.

The decoder uses φ and ψ to decode the message as follows:

1. At the end of round ℓ, $(L-1 \geq \ell \geq 1)$, the decoder computes both, $\varphi^\ell(\mathbf{y}^\ell)$ and $\psi^\ell(\mathbf{y}^\ell)$. If $\psi^\ell(\mathbf{y}^\ell) = 1$, then the decoder declares $\varphi^\ell(\mathbf{y}^\ell)$ as the received message and sends back an ACK. Otherwise, it requests for another round of transmission by sending back a NACK signal.
2. At the end of the Lth round, though, the decoder outputs $\varphi^L(\mathbf{y}^L)$ as the received message.

To characterize the average error probability, P_E, we first use the Bayes' rule to write

$$P_E \leq P_{E|\overline{E}_r} + P_{E_r},$$

where E_r and $\overline{E}_r$ denote the events that the relay makes an error in decoding the message, and its complement, respectively. Since the relay starts transmission only after the mutual information between its received signal and the source signal exceeds $R_1 T$, we have (e.g., refer to Theorem 10.1.1 in [8])

$$P_{E_r} \leq \epsilon, \quad \text{for any} \quad \epsilon > 0.$$

This means that

$$P_E \dot{\leq} P_{E|\overline{E}_r}.$$

For the sake of notational simplicity, in the sequel, we denote $P_{E|\overline{E}_r}$ by P_E. In characterizing P_E, we take the approach of El Gamal, Caire and Damon in [6], i.e., we upper bound P_E by

$$P_E \leq \sum_{\ell=1}^{L-1} P_{E^\ell, A^\ell} + P_{E^L}. \tag{17}$$

Notice that P_{E^ℓ, A^ℓ} upper bounds the probability of *undetected* errors with $\ell < L$ rounds of transmission, while P_{E^L} upper bounds the probability of *decoding* errors at the end of L transmission rounds. The next step in characterizing P_E is to show that, for the decoder of interest, the undetected errors do not dominate the overall error event, i.e.,

$$P_E \dot{\leq} P_{E^L}. \tag{18}$$

Toward this end, we note that

$$P_{E^\ell, A^\ell} \leq \Pr\{|\mathbf{n}^\ell|^2 > \ell T(1+\delta)\sigma^2\}. \tag{19}$$

To understand (19), let us assume that $|\mathbf{n}^\ell|^2 \leq \ell T(1+\delta)\sigma^2$. One can then use (14) to conclude that $|\mathbf{y}^\ell - \mathbf{s}^\ell(m_0)|^2 \leq \ell T(1+\delta)\sigma^2$, where m_0 denotes the transmitted message. This, however, is in contradiction with $E^\ell \cap A^\ell$. This is because the latter event implies that some message m_1, other than m_0, is the *unique* message for which $|\mathbf{y}^\ell - \mathbf{s}^\ell(m_1)|^2 \leq \ell T(1+\delta)\sigma^2$. Thus $E^\ell \cap A^\ell \subseteq \{|\mathbf{n}^\ell|^2 > \ell T(1+\delta)\sigma^2\}$, which means that (19) is indeed true. Now, $|\mathbf{n}^\ell|^2$ has a central Chi-squared distribution with $2\ell T$ degrees of freedom. One can use the Chernoff bound to upper bound the tail of this distribution to get

$$\Pr\{|\mathbf{n}^\ell|^2 > \ell T(1+\delta)\sigma^2\} \leq (1+\delta)^{\ell T} e^{-\ell T \delta}. \tag{20}$$

This, however, in conjunction with (19) means that, for any $\delta > 0$, it is possible to choose T large enough, such that

$$P_{E^\ell, A^\ell} \leq \epsilon, \quad \text{for any} \quad \epsilon > 0. \tag{21}$$

Now, (18) follows from (21), together with (17). Examination of E^L reveals that P_{E^L} is the probability of error for the DDF relay protocol at a multiplexing gain of r_1/L, i.e.

$$P_{E^L} \doteq \rho^{-d_{\text{DDF-R}}(\frac{r_1}{L})}, \tag{22}$$

where $d_{\text{DDF-R}}(\cdot)$ denotes the diversity gain achieved by the DDF relay protocol [1]. Using (18), we conclude

$$P_E \dot{\leq} \rho^{-d_{\text{DDF-R}}(\frac{r_1}{L})}. \tag{23}$$

The final step is to show that for $1 > r_1 \geq 0$, we have $r_1 = r_e$. Towards this end, we use (11) and notice that for the scenario of interest,

$$p(\ell) \triangleq P_{\overline{A^1}, \cdots, \overline{A^\ell}}, \quad \text{for} \quad L > \ell > 0, \tag{24}$$

where $\overline{A^\ell}$ denotes the complement of A^ℓ. To characterize $p(\ell)$, we first upper bound it by

$$p(\ell) \leq P_{\overline{A^\ell}}. \tag{25}$$

Careful examination of $\overline{A^\ell}$ reveals that

$$\overline{A^\ell} = R_0^\ell \cup R_1^\ell, \tag{26}$$

where R_0^ℓ denotes the subset of destination signals, $\mathbf{y}^\ell$, not included in any of the spheres of squared radius $\ell T(1+\delta)\sigma^2$, centered at the signatures of all possible messages $\{\mathbf{s}^\ell(m)\}$, i.e.,

$$R_0^\ell \triangleq \cap_m \{|\mathbf{y}^\ell - \mathbf{s}^\ell(m)|^2 > \ell T(1+\delta)\sigma^2\}. \tag{27}$$

R_1^ℓ, on the other hand, represents the subset of destination signals, $\mathbf{y}^\ell$, included in more than one such spheres, i.e.,

$$R_1^\ell \triangleq \cup_m R_1^\ell(m), \tag{28}$$

where

$$R_1^\ell(m) \triangleq \cup_{m_1 \neq m} \{|\mathbf{y}^\ell - \mathbf{s}^\ell(m)|^2 \leq \ell T(1+\delta)\sigma^2, |\mathbf{y}^\ell - \mathbf{s}^\ell(m_1)|^2 \leq \ell T(1+\delta)\sigma^2\}. \tag{29}$$

Note that $R_1^\ell(m)$ consists of the intersections of the sphere corresponding to message m, with those of the other messages. Next, we assume that message m_0 is transmitted and characterize $P_{\overline{A^\ell}|m_0}$. Toward this end, we write

$$\begin{aligned} \overline{A^\ell} &= R_0^\ell \cup R_1^\ell, \quad \text{or} \\ \overline{A^\ell} &= R_0^\ell \cup \left(R_1^\ell \cap \overline{R_1^\ell(m_0)}\right) \cup R_1^\ell(m_0), \end{aligned} \tag{30}$$

where the last step follows from (28). Now, examining (27), we have

$$R_0^\ell \subseteq \{|\mathbf{y}^\ell - \mathbf{s}^\ell(m_0)|^2 > \ell T(1+\delta)\sigma^2\}.$$

On the other hand, realizing that $R_1^\ell \cap \overline{R_1^\ell(m_0)}$ consists of the intersections of all spheres excluding the one corresponding to m_0, gives

$$R_1^\ell \cap \overline{R_1^\ell(m_0)} \subseteq \{|\mathbf{y}^\ell - \mathbf{s}^\ell(m_0)|^2 > \ell T(1+\delta)\sigma^2\}.$$

Thus

$$\begin{aligned} R_0^\ell \cup \left(R_1^\ell \cap \overline{R_1^\ell(m_0)}\right) &\subseteq \{|\mathbf{y}^\ell - \mathbf{s}^\ell(m_0)|^2 > \ell T(1+\delta)\sigma^2\}, \\ &= \{|\mathbf{n}^\ell|^2 > \ell T(1+\delta)\sigma^2\}, \end{aligned}$$

where the last step follows from (14) and the assumption that m_0 is the transmitted message. Recalling (20), we conclude

$$\Pr\{R_0^\ell \cup \left(R_1^\ell \cap \overline{R_1^\ell(m_0)}\right)|m_0\} \leq \epsilon, \quad \text{for any} \quad \epsilon > 0.$$

This, together with (30), means that

$$P_{\overline{A^\ell}|m_0} \dot{\leq} \Pr\{R_1^\ell(m_0)|m_0\}. \tag{31}$$

Characterization of $\Pr\{R_1^\ell(m_0)|m_0\}$ is a little bit more involved. In particular, if we let $a \triangleq \mathbf{s}^\ell(m_0) - \mathbf{s}^\ell(m_1)$, $b \triangleq \mathbf{y}^\ell - \mathbf{s}^\ell(m_0)$ and $\Delta \triangleq \ell T(1+\delta)\sigma^2$, then

$$\begin{aligned} \{|b|^2 \leq \Delta, |a+b|^2 \leq \Delta\} =& \{|a|^2 \leq 4\Delta, |b|^2 \leq \Delta, |a+b|^2 \leq \Delta\} \cup \\ & \{|a|^2 > 4\Delta, |b|^2 \leq \Delta, |a+b|^2 \leq \Delta\}. \end{aligned}$$

Since the second set on the right-hand side of this expression is empty, we get

$$\{|b|^2 \leq \Delta, |a+b|^2 \leq \Delta\} = \{|a|^2 \leq 4\Delta, |b|^2 \leq \Delta, |a+b|^2 \leq \Delta\}$$
$$\subseteq \{|a|^2 \leq 4\Delta\}.$$

This, along with (29), results in

$$R_1^{\ell}(m_0) \subseteq \cup_{m_1 \neq m_0} \{|\frac{\mathbf{s}^{\ell}(m_0) - \mathbf{s}^{\ell}(m_1)}{2}|^2 \leq \ell T(1+\delta)\sigma^2\},$$

which gives

$$\Pr\{R_1^{\ell}(m_0)|m_0\} \leq \sum_{m_1 \neq m_0} \Pr\{|\frac{\mathbf{s}^{\ell}(m_0) - \mathbf{s}^{\ell}(m_1)}{2}|^2 \leq \ell T(1+\delta)\sigma^2\}. \quad (32)$$

We identify the right hand side of (32), as the union bound on the ML error probability, conditioned on transmission of m_0 (refer to equation (17) in [5], for a very similar expression). Here, the noise variance is $(1+\delta)\sigma^2$, the code length is ℓT and the rate is R_1/ℓ BPCU. As a result

$$\Pr\{R_1^{\ell}(m_0)\} \dot{\leq} \rho^{-d_{\text{DDF-R}}(\frac{r_1}{\ell})}, \quad \text{for} \quad L > \ell > 0, \quad (33)$$

where $d_{\text{DDF-R}}(.)$ is the diversity gain achieved by the DDF relay protocol [1]. Now (33), along with (31) and (25) gives

$$p(\ell) \dot{\leq} \rho^{-d_{\text{DDF-R}}(\frac{r_1}{\ell})}, \quad \text{for} \quad L > \ell > 0. \quad (34)$$

This means that over the range of $1 > r_1 \geq 0$, the probabilities $\{p(\ell)\}_{\ell=1}^{L-1}$, decay polynomially with ρ. As a result, over this range, r_e equals r_1, i.e.

$$r_e = r_1, \quad \text{for} \quad 1 > r_1 \geq 0. \quad (35)$$

Now (35), together with (23), and the fact that for $1 > r_e \geq 0$, $d_{\text{DDF-R}}(\frac{r_e}{L}) = 2(1 - \frac{r_e}{L})$, gives

$$P_E \dot{\leq} \rho^{-2(1-\frac{r_e}{L})}, \quad \text{for} \quad 1 > r_e \geq 0. \quad (36)$$

Note that (36) only characterizes the relation between the *average* error probability P_E and *average* throughput η. To complete the proof, we need to show that there exists a code in the ensemble, that *simultaneously* achieves P_E and η, as characterized by (36). Toward this end, we use Lemma 11 of [6]. The application of this lemma to the case of interest is immediate and thus the proof is complete.

5.2 Proof of Lemma 2

A simple max-cut min-flow examination reveals that the optimal diversity gain for this channel is upper bounded by

$$d_{\text{MAR}}(r) \leq \min\{d_{3\times1}(r), d_{2\times2}(r), d_{2\times1}(\frac{r}{2}), d_{1\times2}(\frac{r}{2})\}, \quad (37)$$

where $d_{m\times n}(\cdot)$ denotes the optimal diversity gain for an $m \times n$ MIMO channel. Now, (37) results in (3) and the proof of the converse part is complete.

In order to derive a lower bound on the diversity gain achieved by the DDF MAR protocol, we upper bound the *source-specific* ML error probabilities, with that of the *joint* ML decoder. Furthermore, instead of characterizing the latter probability for specific codes, in the sequel, we characterize its average, P_E, over the ensemble of random Gaussian codes. It is then straightforward to see that there exists a code in the ensemble, whose error probability is better than P_E. To characterize P_E, we use the Bayes' rule to derive the following upper bound

$$P_E \leq P_{E|\overline{E}_r} + P_{E_r},$$

where E_r and $\overline{E}_r$ denote the events that the relay makes errors in decoding the messages, and its complement, respectively. Next, we note that if we denote the signals transmitted by the two sources and the relay by $\{x_{j,k}\}_{k=1}^{T}$ and $\{x_{r,k}\}_{k=T'+1}^{T}$, respectively, and the signals received by the relay and the destination by $\{y_{r,k}\}_{k=1}^{T'}$ and $\{y_k\}_{k=1}^{T}$, then the number of symbol intervals T' that the relay waits before decoding the messages satisfies

$$\frac{TR}{2} \leq I(\{x_{1,k}\}_{k=1}^{T'}; \{y_{r,k}\}_{k=1}^{T'}|\{x_{2,k}\}_{k=1}^{T'}), \tag{38}$$

$$\frac{TR}{2} \leq I(\{x_{2,k}\}_{k=1}^{T'}; \{y_{r,k}\}_{k=1}^{T'}|\{x_{1,k}\}_{k=1}^{T'}), \tag{39}$$

$$TR \leq I(\{x_{1,k}\}_{k=1}^{T'}, \{x_{2,k}\}_{k=1}^{T'}; \{y_{r,k}\}_{k=1}^{T'}). \tag{40}$$

In these expressions, R is the *total* data rate (in BPCU) at the destination and $I(.;.)$ denotes the mutual information function. Now, observing that (38), (39) and (40) guarantee that (e.g., refer to section 14.3.1 of [8])

$$P_{E_r} \leq \epsilon, \quad \text{for any} \quad \epsilon > 0,$$

we conclude

$$P_E \dot{\leq} P_{E|\overline{E}_r}.$$

For the sake of notational simplicity, in the sequel, $P_{E|\overline{E}_r}$ is denoted by P_E. In characterizing P_E, we follow the approach of Tse, Viswanath and Zheng [7], by partitioning the error event E into the set of partial error events E_I, i.e.,

$$E = \bigcup_I E_I.$$

where I denotes any *nonempty* subset of $\{1, 2\}$ and E_I (referred to as type-I error) is the event that the joint ML decoder incorrectly decodes the messages from sources whose indices belong to I while correctly decoding all other messages. Because the partial error events are mutually exclusive, we have

$$P_E = \sum_I P_{E_I}. \tag{41}$$

To characterize P_{E_I}, we use the Bayes' rule to derive the following upper-bound

$$P_{E_I} = P_{O_I} P_{E_I|O_I} + P_{E_I,\overline{O}_I}$$
$$P_{E_I} \leq P_{O_I} + P_{E_I,\overline{O}_I},$$

where O_I and $\overline{O}_I$ denote the type-I outage event and its complement, respectively. The type-I outage event is defined such that P_{O_I} dominates $P_{E_I,\overline{O}_I}$, i.e.

$$P_{E_I,\overline{O}_I} \dot{\leq} P_{O_I}. \tag{42}$$

Thus,

$$P_{E_I} \dot{\leq} P_{O_I}. \tag{43}$$

Characterization of O_I, however, requires derivation of $P_{PE_I|g,h}$, i.e., the joint ML decoder's type-I pairwise error probability (PEP), conditioned on a particular channel realization and averaged over the ensemble of random Gaussian codes. For this purpose, let us denote the gain of the channels connecting the two sources to the relay by h_1 and h_2 and those of the channels connecting the two sources and the relay to the destination by g_1, g_2 and g_r, respectively. It is then straightforward to see that (refer to [1])

$$P_{PE_{\{1\}}|g,h} \leq (1 + \frac{1}{2}\rho|g_1|^2)^{-T'}(1 + \frac{1}{2}\rho(|g_1|^2 + |g_r|^2))^{-(T-T')},$$
$$P_{PE_{\{1,2\}}|g,h} \leq (1 + \frac{1}{2}\rho(|g_1|^2 + |g_2|^2))^{-T'}(1 + \frac{1}{2}\rho(|g_1|^2 + |g_2|^2 + |g_r|^2))^{-(T-T')}.$$

Notice that since the channel is symmetric, $P_{E_{\{1\}}} \doteq P_{E_{\{2\}}}$, i.e., we do not need to characterize $P_{PE_{\{2\}}|g,h}$. Let us denote the exponential orders of $\{1/|g_j|^2\}_{j=1}^2$ and $1/|g_r|^2$ by $\{v_j\}_{j=1}^2$ and v_r, respectively, and those of $\{1/|h_j|^2\}_{j=1}^2$ by $\{u_j\}_{j=1}^2$. Realizing that, at a rate of $R = r\log\rho$ BPCU and a codeword length of T, there are a total of ρ^{Tr} codewords in the code, we get

$$P_{E_{\{1\}}|v,u} \dot{\leq} \rho^{-T[f(1-v_1)^+ + (1-f)(1-\min\{v_1,v_r\})^+ - \frac{r}{2}]}, \tag{44}$$
$$P_{E_{\{1,2\}}|v,u} \dot{\leq} \rho^{-T[f(1-\min\{v_1,v_2\})^+ + (1-f)(1-\min\{v_1,v_2,v_r\})^+ - r]}, \tag{45}$$

where $f \triangleq T'/T$. Careful examination of (44) reveals that defining $O_{\{1\}}^+$ as

$$O_{\{1\}}^+ \triangleq \{(v_1, \cdots, u_2) \in \mathbb{R}^{5+} | f(1-v_1)^+ + (1-f)(1-\min\{v_1, v_r\})^+ \leq \frac{r}{2}\}, \tag{46}$$

satisfies (42). In this expression, $O_{\{1\}}^+$ denotes $O_{\{1\}} \cap \mathbb{R}^{5+}$, where $\mathbb{R}^{n+}$ represents the set of all nonnegative real n-tuples (for an explanation on why we are only concerned about O_I^+, please refer to [1]). This is because, if (46) is satisfied, then through choosing a large enough T, $P_{E_{\{1\}},\overline{O}_{\{1\}}}$ can be made arbitrarily small. Likewise, defining $O_{\{1,2\}}^+$ as

$$O_{\{1,2\}}^+ \triangleq \{(v_1, \cdots, u_2) \in \mathbb{R}^{5+} | f(1-\min\{v_1, v_2\})^+ + (1-f)(1-\min\{v_1, v_2, v_r\})^+ \leq r\}, \tag{47}$$

satisfies (42). With the type-I outage events specified, the only thing left is to characterize P_{O_I} (please refer to [1]), i.e.

$$P_{O_I} \doteq \rho^{-d_I(r)} \quad \text{where} \quad d_I(r) \triangleq \inf_{O_I^+}\{v_1 + v_2 + v_r + u_1 + u_2\}. \tag{48}$$

Toward this end, we use (46) to derive $\inf_{(v_1,v_2,v_r)\in O_{\{1\}}^+}\{v_1 + v_2 + v_r\}$, as a *function* of f, i.e.

$$\inf_{(v_1,v_2,v_r)\in O_{\{1\}}^+}\{v_1 + v_2 + v_r\}(f) = \lambda_{\{1\}}(f),$$

where

$$\lambda_{\{1\}}(f) \triangleq \begin{cases} 2-r, & \frac{1}{2} > f \geq 0 \\ 2-\frac{r}{2(1-f)}, & 1-\frac{r}{2} > f \geq \frac{1}{2} \\ \frac{2-r}{2f}, & 1 \geq f \geq 1-\frac{r}{2} \end{cases} \tag{49}$$

Likewise, one can use (47) to derive

$$\inf_{(v_1,v_2,v_r)\in O^+_{\{1,2\}}} \{v_1+v_2+v_r\}(f) = \lambda_{\{1,2\}}(f),$$

where

$$\lambda_{\{1,2\}}(f) \triangleq \begin{cases} 3(1-r), & \frac{2}{3} > f \geq 0 \\ 3-\frac{r}{1-f}, & 1-r > f \geq \frac{2}{3} \\ 2\frac{1-r}{f}, & 1 \geq f \geq 1-r \end{cases}, \quad \text{if} \quad \frac{1}{3} > r \geq 0 \tag{50}$$

or

$$\lambda_{\{1,2\}}(f) \triangleq \begin{cases} 3(1-r), & \frac{2}{3} > f \geq 0 \\ 2\frac{1-r}{f}, & 1 \geq f \geq \frac{2}{3} \end{cases}, \quad \text{if} \quad 1 \geq r \geq \frac{1}{3} \tag{51}$$

Now, to complete the derivation of $d_I(r)$, we need to characterize $\inf_{(u_1,u_2)\in\mathbb{R}^{2+}}\{u_1 + u_2\}(f)$. For this purpose, we use (38), (39) and (40) to derive

$$T' = \min\{T, \max\{\lceil \frac{TR}{2\log_2(1+\min\{|h_1|^2,|h_2|^2\}c\rho)}\rceil, \lceil \frac{TR}{\log_2(1+(|h_1|^2+|h_2|^2)c\rho)}\rceil\}\},$$

where c is the ratio of destination noise variance to that of the relay and $\lceil x \rceil$ denotes the closest integer to x towards plus infinity. In terms of channel exponential orders, this last expression can be rewritten as

$$f = \min\{1, \max\{\frac{r}{2(1-\max\{u_1,u_2\})^+}, \frac{r}{(1-\min\{u_1,u_2\})^+}\}\}, \quad (u_1,u_2)\in\mathbb{R}^{2+}. \tag{52}$$

Now, using (52), one can show that

$$\inf_{(u_1,u_2)\in\mathbb{R}^{2+}} \{u_1+u_2\}(f) = \lambda(f),$$

where

$$\lambda(f) \triangleq \begin{cases} 2(1-\frac{r}{f}), & \frac{3r}{2} > f \geq r \\ 1-\frac{r}{2f}, & 1 \geq f \geq \frac{3r}{2} \end{cases}. \tag{53}$$

Using (48) we conclude

$$d_{\{1\}}(r) = \inf_f \{\lambda(f) + \lambda_{\{1\}}(f)\},$$

which can be characterized using (49) and (53),

$$d_{\{1\}}(r) = \begin{cases} 2-r, & \frac{1}{2} > r \geq 0 \\ \frac{4-5r}{2(1-r)}, & \frac{2}{3} \geq r \geq \frac{1}{2} \\ \frac{2-r}{2r}, & 1 \geq r \geq \frac{2}{3} \end{cases}. \tag{54}$$

Similarly,

$$d_{\{1,2\}}(r) = \inf_f \{\lambda(f) + \lambda_{\{1,2\}}(f)\},$$

which can be derived using (50), (51) and (53),

$$d_{\{1,2\}}(r) = \begin{cases} 3(1-r), & \frac{2}{3} > r \geq 0 \\ 2\frac{1-r}{r}, & 1 \geq r \geq \frac{2}{3} \end{cases}. \tag{55}$$

Now, (54) and (55), together with (48), (43) and (41), result in (4) and thus complete the proof of the achievability part.

5.3 Proof of Theorem 3

A simple max-cut min-flow examination reveals that the optimal diversity gain for this channel is upper bounded by

$$d_{\mathrm{MAR}}(r_e, L) \leq \min\{d_{3\times1}(r_e, L), d_{2\times2}(r_e, L), d_{2\times1}(\frac{r_e}{2}, L), d_{1\times2}(\frac{r_e}{2}, L)\}, \tag{56}$$

where $d_{m\times n}(\cdot,\cdot)$ denotes the optimal diversity gain for an $m \times n$ ARQ MIMO channel (refer to [6]). Now, (56) results in

$$d_{\mathrm{MAR}}(r_e, L) \leq 2 - \frac{r_e}{L}, \quad \text{for} \quad 1 > r_e \geq 0, \tag{57}$$

which completes the proof of the converse.

Next, we prove that the proposed protocol achieves this upper bound. To do this, we only need to describe the encoder and the decoder. The rest of the proof then follows that of Lemma 1, line by line. Toward this end, let $C(\rho) = \{C_1(\rho), C_2(\rho), C_r(\rho)\}$ denote the random codes used by the two sources and the relay, respectively. These are codes of length LT, generated by an i.i.d complex Gaussian random process of mean zero and variance E. The rates of these codes are different, though. While $C_1(\rho)$ and $C_2(\rho)$ are of rate $R_1/2L$ BPCU, $C_r(\rho)$ is of rate R_1/L BPCU. In other words, the relay code has twice the rate of the source codes. This means that, corresponding to each pair of source codewords $(\mathbf{x}_1(m_1), \mathbf{x}_2(m_2)) \in C_1(\rho) \times C_2(\rho)$, there exists a codeword $\mathbf{x}_r(\mathbf{m}) \in C_r(\rho)$, where $\mathbf{m} \triangleq (m_1, m_2)$. We call $\mathbf{m}$ the joint message. Note that since each of the two source messages, i.e. m_1 and m_2, consists of $R_1T/2$ information bits, the joint message $\mathbf{m}$, has a total of R_1T information bits in it. As before, we denote the destination signature corresponding to the joint message $\mathbf{m}$, by $\mathbf{s}(\mathbf{m})$, i.e.,

$$\mathbf{y} = \mathbf{s}(\mathbf{m}) + \mathbf{n}.$$

In order to decode the source messages and produce the ACK/NACK signals, the destination uses a *joint* bounded distance decoder. This decoder is identical to the one devised for the ARQ-DDF relay protocol (refer to Lemma 1), with the only modification that the joint message $\mathbf{m}$ takes the role of m everywhere, e.g., $\varphi^\ell(\cdot)$ is now defined as (compare to (15))

$$\varphi^\ell(\mathbf{y}^\ell) \triangleq \arg\min_{\mathbf{m}} |\mathbf{y}^\ell - \mathbf{s}^\ell(\mathbf{m})|^2, \quad \text{for} \quad L \geq \ell \geq 1.$$

In the proposed decoder, the destination provides a total of one bit of feedback, for the two sources, per transmission round. Therefore, there is no need for defining *source-specific* $\varphi(\cdot)$ and $\psi(\cdot)$ functions. With the encoder and decoder defined, one can now follow the same steps taken in the proof of Lemma 1 to show that (compare to (23))

$$P_E \dot{\leq} \rho^{-d_{\text{DDF-MAR}}(\frac{r_1}{L})}, \tag{58}$$

and that (compare to (34))

$$p(\ell) \dot{\leq} \rho^{-d_{\text{DDF-MAR}}(\frac{r_1}{\ell})}, \quad \text{for} \quad L > \ell > 0. \tag{59}$$

Now, (58) and (59), together with the fact that for $1 > r_e \geq 0$, $d_{\text{DDF-MAR}}(\frac{r_e}{L}) = 2 - \frac{r_e}{L}$ (refer to (4)), result in (compare to (36))

$$P_E \dot{\leq} \rho^{-(2-\frac{r_e}{L})}, \quad \text{for} \quad 1 > r_e \geq 0. \tag{60}$$

It is then straightforward to use Lemma 11 of [6] to show that there are codes in the ensemble $\{C(\rho)\}$, that achieve (60). This proves that the upper bound (57) is achievable and thus, completes the proof.

5.4 Proof of Theorem 4

A simple min-cut max-flow examination reveals that the optimal diversity gain for this channel is upper bounded by

$$d_{\text{CVMA}}(r_e, L) \leq \min\{d_{2\times 2}(r_e, L), d_{1\times 3}(\frac{r_e}{2}, L)\}, \tag{61}$$

where $d_{m\times n}(\cdot,\cdot)$ denotes the optimal diversity gain for an $m \times n$ ARQ MIMO channel. Now, (61) results in (6) and the proof of the converse part is complete.

To prove the achievability part, let $C(\rho) = \{C_1(\rho), C_2(\rho)\}$ denote the random codes used by the two sources. These are codes of length $2T$ symbols, rate $R_1/4$ BPCU, and generated by an i.i.d complex Gaussian random process of mean zero and variance E. Let us also denote the two messages to be sent by m_1 and m_2. Note that each message consists of $R_1T/2$ information bits, such that the joint message $\mathbf{m} \triangleq (m_1, m_2)$ consists of a total of R_1T bits. We denote the codewords corresponding to m_1 and m_2 by $\mathbf{x}_1(m_1)$ and $\mathbf{x}_2(m_2)$. As before, the destination signatures of m_1, m_2 and $\mathbf{m}$ are denoted by $\mathbf{S}(m_1)$, $\mathbf{S}(m_2)$ and $\mathbf{S}(\mathbf{m})$, respectively. Thus

$$\begin{aligned} \mathbf{Y} &= \mathbf{S}(\mathbf{m}) + \mathbf{N}, \quad \text{or} \\ \mathbf{Y} &= \mathbf{S}(m_1) + \mathbf{S}(m_2) + \mathbf{N}, \end{aligned}$$

where $\mathbf{Y} \in \mathbb{C}^{2\times 2T}$ and $\mathbf{N} \in \mathbb{C}^{2\times 2T}$ represent the destination received signal and additive noise, respectively. We denote the signal received through antenna $j \in \{1, 2\}$ by $\mathbf{y}_j$. Similarly, the contribution of message m_i $i \in \{1, 2\}$, to the signal received through antenna j, is denoted by $\mathbf{s}_j(m_i)$. As before, we use the superscript ℓ to denote the portion of the signal that corresponds to the first ℓ rounds of transmission.

Next, we describe the decoder. Since the performance analysis for the optimal decoder seems intractable, in the sequel, we describe a *suboptimal* bounded distance decoder and analyze its performance. Obviously, this analysis provides a *lower bound* on the diversity gain achieved through the protocol. To describe the decoder, let us label the source and the receiving antenna that are connected through the channel with the highest signal to interference (due to the other source) and noise ratio by s (s stands for superior), while labeling the remaining source and receiving antenna by i (i stands for inferior). This means that,

$$\frac{|g_{ss}|^2\rho}{|g_{is}|^2\rho+\sigma^2} \geq \max\{\frac{|g_{si}|^2\rho}{|g_{ii}|^2\rho+\sigma^2}, \frac{|g_{is}|^2\rho}{|g_{ss}|^2\rho+\sigma^2}, \frac{|g_{ii}|^2\rho}{|g_{si}|^2\rho+\sigma^2}\}, \tag{62}$$

where, e.g., g_{si} denotes the gain of the channel connecting source s to receive antenna i. Now, the decoder $\{\varphi, \psi\}$, uses the two sets of functions, $\varphi = \{\varphi_j^1, \varphi_s^1, \varphi_j^2, \varphi_i^2\}$ and $\psi = \{\psi_j^1, \psi_s^1\}$ to decode the messages, and produce the ACK/NACK feedback bits, as follows:

1. At the end of the first round, the decoder uses φ_j^1 to *jointly* decode the two messages, i.e.

$$\varphi_j^1(\mathbf{Y}^1) \triangleq \arg\min_{\mathbf{m}} ||\mathbf{Y}^1 - \mathbf{S}^1(\mathbf{m})||^2. \tag{63}$$

   We denote the event that $\varphi_j^1(\mathbf{Y}^1)$ is different from the actual joint message sent, by E_j^1.

2. To decide whether it has correctly decoded the two messages or not, the decoder uses ψ_j^1, where ψ_j^1 outputs a one, if $\mathbf{m}$ is the *unique* joint message satisfying

$$||\mathbf{Y}^1 - \mathbf{S}^1(\mathbf{m})||^2 \leq 2T(1+\delta)\sigma^2. \tag{64}$$

   In (64), δ is some positive value. In any other case, ψ_j^1 outputs a zero. Now, if $\psi_j^1(\mathbf{Y}^1) = 1$, then the decoder sends back ACK signals to *both* of the users, declaring $\varphi_j^1(\mathbf{Y}^1)$ as the decoded joint message. This causes the two sources to start transmission of their next messages. Otherwise, it proceeds to the next step as described below. We denote the event $\psi_j^1(\mathbf{Y}^1) = 1$, by A_j^1.

3. At the end of the first round and in the event of failure in jointly decoding the two messages, i.e., $\overline{A_j^1}$, the decoder uses φ_s^1 to decode the superior message, treating the inferior source's contribution as interference. In doing so, the decoder only utilizes the signal it has received through its s antenna, i.e.

$$\varphi_s^1(\mathbf{y}_s^1) \triangleq \arg\min_{m_s} |\mathbf{y}_s^1 - \mathbf{s}_s^1(m_s)|^2. \tag{65}$$

   We denote the event that $\varphi_s^1(\mathbf{y}_s^1)$ is different from the actual superior message sent, by E_s^1.

4. To decide whether it has correctly decoded the superior message or not, the decoder uses ψ_s^1, where ψ_s^1 outputs a one, if m_s is the *unique* superior message satisfying

$$|\mathbf{y}_s^1 - \mathbf{s}_s^1(m_s)|^2 \leq T(1+\delta)(|g_{is}|^2\rho + \sigma^2). \tag{66}$$

   In (66), δ is some positive value. In any other case, ψ_s^1 outputs a zero. Now, if $\psi_s^1(\mathbf{y}_s^1) = 0$, the decoder requests for a second round of transmission by sending back NACK signals to *both* of the sources. However, if $\psi_s^1(\mathbf{y}_s^1) = 1$, then the decoder sends back an ACK signal to the superior source, declaring $\varphi_s^1(\mathbf{y}_s^1)$ as the decoded superior message, while requesting a second round of transmission for the inferior message by sending back a NACK signal to the inferior source. We denote the event $\psi_s^1(\mathbf{y}_s^1) = 1$, by A_s^1.

5. At the end of the second round and conditioned on successful decoding of the superior message in the first round, i.e., $A_s^1 \cap \overline{A_j^1}$, the decoder declares $\varphi_i^2(\mathbf{Y}^2, \varphi_s^1(\mathbf{y}_s^1))$ as the decoded inferior message where

$$\varphi_i^2(\mathbf{Y}^2, \varphi_s^1(\mathbf{y}_s^1)) \triangleq \arg\min_{m_i} ||\mathbf{Y}^2 - \mathbf{S}^2(\varphi_s^1(\mathbf{y}_s^1)) - \mathbf{S}^2(m_i)||^2. \quad (67)$$

In (67), $\varphi_s^1(\mathbf{y}_s^1)$ is the decoded superior message as given by (65). We denote the event that $\varphi_i^2(\mathbf{Y}^2, \varphi_s^1(\mathbf{y}_s^1))$ is different from the actual inferior message sent, by E_i^2.

6. Finally, in case of failure in decoding the superior message at the end of the first round, i.e., $\overline{A_s^1} \cap \overline{A_j^1}$, the decoder declares $\varphi_j^2(\mathbf{Y}^2)$ as the decoded joint message where

$$\varphi_j^2(\mathbf{Y}^2) \triangleq \arg\min_{\mathbf{m}} ||\mathbf{Y}^2 - \mathbf{S}^2(\mathbf{m})||^2. \quad (68)$$

We denote the event that $\varphi_j^2(\mathbf{Y}^2)$ is different from the actual joint message sent, by E_j^2.

Having described the decoder, we next characterize its average error probability P_E. Toward this end, we first notice that

$$P_E \leq P_{E|\overline{E_r}} + P_{E_r},$$

where E_r and $\overline{E_r}$ denote the event that the relaying source makes an error in decoding the inferior message, and its complement, respectively. Since the superior source only starts relaying after the mutual information between its received signal and inferior source's transmitted signal exceeds $R_1T/2$, we have (e.g., refer to Theorem 10.1.1 in [8])

$$P_{E_r} \leq \epsilon, \quad \text{for any} \quad \epsilon > 0.$$

This means that

$$P_E \dot{\leq} P_{E|\overline{E_r}}.$$

For the sake of notational simplicity, in the sequel, we denote $P_{E|\overline{E_r}}$ by P_E. To characterize P_E, we write

$$P_E = P_{E_j^1, A_j^1} + P_{E_s^1, A_s^1, \overline{A_j^1}} + P_{E_i^2, \overline{E_s^1}, A_s^1, \overline{A_j^1}} + P_{E_j^2, \overline{A_s^1}, \overline{A_j^1}}. \quad (69)$$

To understand (69), note that the first two terms correspond to making an *undetected* error in decoding one or both of the messages at the end of the first round, while the last two terms correspond to making a decoding error after requesting for two rounds of transmission. Next, we upper bound (69) by

$$P_E \leq P_{E_j^1, A_j^1} + P_{E_s^1, A_s^1} + P_{E_i^2, \overline{E_s^1}} + P_{E_j^2, \overline{A_s^1}}. \quad (70)$$

We start evaluating (70) by characterizing $P_{E_j^1, A_j^1}$. By examining the definitions for events E_j^1 and A_j^1 (refer to (63) and (64)), and through an argument similar to the one given for (19), we get

$$P_{E_j^1, A_j^1 | g, h} \leq \Pr\{||\mathbf{N}^1||^2 > 2T(1+\delta)\sigma^2\},$$

which for large enough T gives (compare to (21))

$$P_{E_j^1, A_j^1|g,h} \leq \epsilon \quad \text{for any} \quad \epsilon > 0, \quad \text{or}$$
$$P_{E_j^1, A_j^1} \leq \epsilon \quad \text{for any} \quad \epsilon > 0. \tag{71}$$

Next, we characterize $P_{E_s^1, A_s^1}$. Toward this end, we first fix a channel realization. Then, through examining the definitions for events E_s^1 and A_s^1 (refer to (65) and (66)), and by pursuing the same steps which led to (19), we get

$$P_{E_s^1, A_s^1|g,h} \leq \Pr\{|\mathbf{s}_s^1(m_i) + \mathbf{n}_s^1|^2 > T(1+\delta)(|g_{is}|^2\rho + \sigma^2)\},$$

where m_i denotes the actual inferior message sent and $\mathbf{s}_s^1(m_i)$ represents its signature, at the end of the first round, at the superior antenna. Realizing that, conditioned on a certain channel realization, $|\mathbf{s}_s^1(m_i) + \mathbf{n}_s^1|^2$ has a central Chi-squared distribution with $2T$ degrees of freedom, we conclude that for large enough T, we have (compare to (21))

$$P_{E_s^1, A_s^1|g,h} \leq \epsilon \quad \text{for any} \quad \epsilon > 0, \quad \text{or}$$
$$P_{E_s^1, A_s^1} \leq \epsilon \quad \text{for any} \quad \epsilon > 0. \tag{72}$$

In other words, (71) and (72) mean that, through using long enough codes, one can make the probability of making undetected errors arbitrarily small. Note that for doing so, the bounded distance decoder does not employ any kind of cyclic redundancy check (CRC) techniques. Now, using (70), (71) and (72) we conclude

$$P_E \dot{\leq} P_{E_i^2, \overline{E_s^1}} + P_{E_j^2, \overline{A_s^1}}. \tag{73}$$

To characterize $P_{E_i^2, \overline{E_s^1}}$, we first fix a channel realization and then write

$$P_{E_i^2, \overline{E_s^1}|g,h} = P_{\overline{E_s^1}|g,h} P_{E_i^2|\overline{E_s^1},g,h},$$
$$\leq P_{E_i^2|\overline{E_s^1},g,h}. \tag{74}$$

Now, using (67), it is straightforward to verify that (refer to [1])

$$P_{PE_i^2|\overline{E_s^1},g,h} \leq \left(1 + \frac{1}{2}\rho(|g_{is}|^2 + |g_{ii}|^2)\right)^{-(T+T')} \times$$
$$\left(1 + \frac{1}{2}\rho(|g_{ss}|^2 + |g_{si}|^2 + |g_{is}|^2 + |g_{ii}|^2) + \frac{1}{4}\rho^2 \det(GG^H)\right)^{-(T-T')}, \tag{75}$$

where

$$G \triangleq \begin{bmatrix} g_{ss} & g_{is} \\ g_{si} & g_{ii} \end{bmatrix}.$$

In (75), T' is the number of symbol intervals, in the second round, that the superior source needs to listen to the inferior one, before decoding its message, i.e.

$$T' \triangleq \min\{T, \lceil \frac{TR_1}{2\log_2(1 + |h|^2 c\rho)} \rceil\}. \tag{76}$$

Since GG^H is a positive semi-definite matrix, we have

$$P_{PE_i^2|\overline{E_s^1},g,h} \leq \left(1 + \frac{1}{2}\rho(|g_{is}|^2 + |g_{ii}|^2)\right)^{-(T+T')} \times$$
$$\left(1 + \frac{1}{2}\rho(|g_{ss}|^2 + |g_{si}|^2 + |g_{is}|^2 + |g_{ii}|^2)\right)^{-(T-T')}. \tag{77}$$

Note that in deriving (77) from (75), we have ignored the term $\frac{1}{4}\rho^2 \det(GG^H)$. As a consequence, the derived upper bound may be loose. However, this is indeed necessary, for the sake of analysis tractability. Now, let us define v_{kl}, where $k, l \in \{s, i\}$, as the exponential order of $1/|g_{kl}|^2$, and u, as the exponential order of $1/|h|^2$. Then, using (77) and realizing that there are a total of $\rho^{\frac{Tr_1}{2}}$ codewords in the inferior source's code-book, we derive

$$P_{E_i^2|\overline{E_s^1},v,u} \dot{\leq} \rho^{-T[(1-\min\{v_{is},v_{ii}\})^+(1+f)+(1-\min\{v_{ss},v_{si},v_{is},v_{ii}\})^+(1-f)-\frac{r_1}{2}]}, \tag{78}$$

where $f \triangleq T'/T$. Using (76), we have

$$f = \min\{1, \frac{r_1}{2(1-u)^+}\}. \tag{79}$$

An argument similar to that given for (43), reveals that if we define the outage event O_i^{2+} as

$$O_i^{2+} \triangleq \{(v_{ss}, \cdots, u) \in \mathbb{R}^{5+} | (1 - \min\{v_{is}, v_{ii}\})^+(1 + f)+ \\ (1 - \min\{v_{ss}, v_{si}, v_{is}, v_{ii}\})^+(1 - f) \leq \frac{r_1}{2}\}, \tag{80}$$

then

$$P_{E_i^2|\overline{E_s^1}} \dot{\leq} P_{O_i^2}, \tag{81}$$

where (recall (48))

$$P_{O_i^2} \dot{=} \rho^{-d_i(r_1)}, \quad \text{with} \quad d_i(r_1) \triangleq \inf_{O_i^{2+}} \{v_{ss} + v_{si} + v_{is} + v_{ii} + u\}. \tag{82}$$

In writing (82), we have used the fact that

$$v_{ss} + v_{si} + v_{is} + v_{ii} = v_{11} + v_{12} + v_{21} + v_{22}.$$

Obviously, $(v_{ss}, v_{si}, v_{is}, v_{ii})$ should also satisfy

$$1 - v_{ss} - (1 - v_{is})^+ \geq \max\{1 - v_{si} - (1 - v_{ii})^+, 1 - v_{is} - (1 - v_{ss})^+, \\ 1 - v_{ii} - (1 - v_{si})^+\}, \tag{83}$$

which is the counterpart of (62), stated in terms of the channel exponential orders. Now, to characterize $d_i(r_1)$, we first derive $\inf_{O_i^{2+}}\{v_{ss} + v_{si} + v_{is} + v_{ii}\}$, as a function of f, i.e.

$$\inf_{O_i^{2+}} \{v_{ss} + v_{si} + v_{is} + v_{ii}\}(f) = \lambda_i(f),$$

where

$$\lambda_i(f) \triangleq \begin{cases} 4 - \frac{r_1}{1-f}, & 1 - \frac{r_1}{2} > f \geq 0 \\ \frac{4-r_1}{1+f}, & 1 \geq f \geq 1 - \frac{r_1}{2} \end{cases}. \tag{84}$$

On the other hand, from (79) we get

$$\inf\{u\}(f) = \lambda(f), \quad \text{where} \quad \lambda(f) \triangleq 1 - \frac{r_1}{2f}, \quad f \geq \frac{r_1}{2}. \tag{85}$$

Then, from (82), we get

$$d_i(r_1) = \inf_{1 \geq f \geq \frac{r_1}{2}} \lambda_i(f) + \lambda(f).$$

The right hand side of this expression can be derived using (84) and (85), i.e.

$$d_i(r_1) = \begin{cases} 3 - r_1, & 1 > r_1 \geq 0 \\ \frac{2(4-r_1)}{2+r_1}, & 2 \geq r_1 \geq 1 \end{cases}. \tag{86}$$

This, together with (82), (81) and (74), completes the characterization of $P_{E_i^2,\overline{E_s^1}}$, i.e.

$$P_{E_i^2,\overline{E_s^1}} \dot{\leq} \rho^{-d_i(r_1)}. \tag{87}$$

To characterize the second term of (73), i.e., $P_{E_j^2,\overline{A_s^1}}$, we first split the event E_j^2 (refer to (68)) into

$$E_j^2 = E_{jsi}^2 \cup E_{ji}^2 \cup E_{js}^2,$$

where E_{jsi}^2, E_{ji}^2 and E_{js}^2 represent the events that, at the end of the second round, the joint decoder makes errors in decoding both of the messages, only the inferior message and only the superior message, respectively. Since these three events are mutually exclusive, we have

$$P_{E_j^2,\overline{A_s^1}} = P_{E_{jsi}^2,\overline{A_s^1}} + P_{E_{ji}^2,\overline{A_s^1}} + P_{E_{js}^2,\overline{A_s^1}}. \tag{88}$$

Characterization of the first term is very straightforward,

$$\begin{aligned} P_{E_{jsi}^2,\overline{A_s^1}} &\leq P_{E_{jsi}^2}, \\ &\doteq \rho^{-d_{2\times 2}(\frac{r_1}{2})}. \end{aligned} \tag{89}$$

Characterization of $P_{E_{ji}^2,\overline{A_s^1}}$, though, is a little bit more involved. In particular, notice that

$$P_{E_{ji}^2,\overline{A_s^1}} \leq \min\{P_{E_{ji}^2}, P_{\overline{A_s^1}}\}. \tag{90}$$

Now, conditioned on a certain channel realization, $P_{PE_{ji}^2|g,h}$ can be upper bounded as (refer to [1])

$$P_{PE_{ji}^2|g,h} \leq \left(1 + \frac{1}{2}\rho(|g_{is}|^2 + |g_{ii}|^2)\right)^{-2T}.$$

Realizing that there are a total of $\rho^{\frac{Tr_1}{2}}$ pairs of codewords in each source's code book, we get

$$P_{E_{ji}^2|v,u} \dot{\leq} \rho^{-2T[(1-\min\{v_{is},v_{ii}\})^+ - \frac{r_1}{4}]}. \tag{91}$$

Now, examining (91) reveals that if we define O_{ji}^{2+} as

$$O_{ji}^{2+} \triangleq \{(v_{ss}, \cdots, u) \in \mathbb{R}^{5+} | (1 - \min\{v_{is}, v_{ii}\})^+ \leq \frac{r_1}{4}\}, \tag{92}$$

then for all channel realizations $(v_{ss}, \cdots, u) \in \mathbb{R}^{5+}$ *not* included in O_{ji}^{2+}, $P_{E_{ji}^2|v,u}$ can be made arbitrary small, provided that T is large enough, i.e.

$$P_{E_{ji}^2|v,u} \leq \epsilon, \quad \text{for any} \quad \epsilon > 0 \quad \text{and} \quad (v_{ss}, \cdots, u) \in \overline{O_{ji}^2} \cap \mathbb{R}^{5+}. \tag{93}$$

Next, we turn our attention to $P_{\overline{A_s^1}}$. To characterize this probability, we first fix a channel realization. Then, by comparing the definition of A_s^1 (refer to (66)) to that of A^ℓ (refer to (16)), and pursuing the exact same arguments which led (26) to (34), we conclude

$$P_{\overline{A_s^1}|g,h} \dot{\leq} P_{E_s^1|g,h}. \tag{94}$$

But $P_{PE_s^1|g,h}$ is given by (notice how the inferior source's contribution is treated as interference),

$$P_{PE_s^1|g,h} \leq \left(1 + \frac{1}{2}\rho(\frac{|g_{ss}|^2}{1+\rho|g_{is}|^2})\right)^{-T},$$

which, in terms of channel exponentials, translates into

$$P_{E_s^1|v,u} \dot{\leq} \rho^{-T\left[\left(1-v_{ss}-(1-v_{is})^+\right)^+ - \frac{r_1}{2}\right]}. \tag{95}$$

Now, from (94) and (95), we conclude

$$P_{\overline{A_s^1}|v,u} \dot{\leq} \rho^{-T\left[\left(1-v_{ss}-(1-v_{is})^+\right)^+ - \frac{r_1}{2}\right]}. \tag{96}$$

This means that if

$$O_s^{1+} \triangleq \{(v_{ss}, \cdots, u) \in \mathbb{R}^{5+} | \left(1 - v_{ss} - (1-v_{is})^+\right)^+ \leq \frac{r_1}{2}\}, \tag{97}$$

then for all channel realizations $(v_{ss}, \cdots, u) \in \mathbb{R}^{5+}$, *not* included in O_s^{1+}, $P_{\overline{A_s^1}|v,u}$ can be made arbitrary small, provided that T is large enough, i.e.

$$P_{\overline{A_s^1}|v,u} \leq \epsilon, \quad \text{for all} \quad \epsilon > 0 \quad \text{and} \quad (v_{ss}, \cdots, u) \in \overline{O_s^1} \cap \mathbb{R}^{5+}. \tag{98}$$

Now, using (90), (93) and (98) we conclude

$$P_{E_{ji}^2, \overline{A_s^1}|u,v} \leq \epsilon \quad \text{for all} \quad \epsilon > 0 \quad \text{and} \quad (v_{ss}, \cdots, u) \in \overline{O_{s,ji}^{1,2}} \cap \mathbb{R}^{5+}, \tag{99}$$

where

$$O_{s,ji}^{1,2+} \triangleq O_s^{1+} \cap O_{ji}^{2+}, \tag{100}$$

or

$$\begin{aligned} O_{s,ji}^{1,2+} = \{(v_{ss}, \cdots, u) \in \mathbb{R}^{5+} | (1 - \min\{v_{is}, v_{ii}\})^+ \leq \frac{r_1}{4}, \\ \left(1 - v_{ss} - (1-v_{is})^+\right)^+ \leq \frac{r_1}{2}\}. \end{aligned} \tag{101}$$

This means that

$$P_{E_{ji}^2, \overline{A_s^1}} \dot{\leq} P_{O_{s,ji}^{1,2}}, \tag{102}$$

where

$$P_{O^{1,2}_{s,ji}} \dot{=} \rho^{-d_{s,ji}(r_1)}, \quad \text{with} \quad d_{s,ji}(r_1) \triangleq \inf_{O^{12+}_{s,ji}} \{v_{ss} + \cdots + u\}. \tag{103}$$

Now, using (101), it is straightforward to show that

$$d_{s,ji}(r_1) = \begin{cases} 4 - 2r_1, & \frac{4}{3} > r_1 \geq 0 \\ 2 - \frac{r_1}{2}, & 2 \geq r_1 \geq \frac{4}{3} \end{cases}. \tag{104}$$

This, together with (103) and (102), completes the characterization of $P_{E^2_{ji}, \overline{A^1_s}}$, i.e.

$$P_{E^2_{ji}, \overline{A^1_s}} \dot{\leq} \rho^{-d_{s,ji}(r_1)}. \tag{105}$$

Characterization of $P_{E^2_{js}, \overline{A^1_s}}$, proceeds in a similar way. In particular, pursuing the exact same steps leading to (91), reveals that

$$P_{E^2_{js}|v,u} \dot{\leq} \rho^{-2T[(1-\min\{v_{ss}, v_{si}\})^+ - \frac{r_1}{4}]}.$$

This means that defining O^{2+}_{js} as

$$O^{2+}_{js} \triangleq \{(v_{ss}, \cdots, u) \in \mathbb{R}^{5+} | (1 - \min\{v_{ss}, v_{si}\})^+ \leq \frac{r_1}{4}\},$$

results in

$$P_{E^2_{js}|v,u} \leq \epsilon, \quad \text{for any} \quad \epsilon > 0 \quad \text{and} \quad (v_{ss}, \cdots, u) \in \overline{O^2_{js}} \cap \mathbb{R}^{5+}.$$

This, however, together with (98), gives

$$P_{E^2_{js}, \overline{A^1_s}|u,v} \leq \epsilon \quad \text{for all} \quad \epsilon > 0 \quad \text{and} \quad (v_{ss}, \cdots, u) \in \overline{O^{1,2}_{s,js}} \cap \mathbb{R}^{5+},$$

where

$$O^{1,2+}_{s,js} \triangleq O^{1+}_s \cap O^{2+}_{js}.$$

or

$$\begin{aligned} O^{1,2+}_{s,js} = \{(v_{ss}, \cdots, u) \in \mathbb{R}^{5+} | (1 - \min\{v_{ss}, v_{si}\})^+ \leq \frac{r_1}{4}, \\ \left(1 - v_{ss} - (1 - v_{is})^+\right)^+ \leq \frac{r_1}{2}\}, \end{aligned}$$

or

$$O^{1,2+}_{s,js} = \{(v_{ss}, \cdots, u) \in \mathbb{R}^{5+} | (1 - \min\{v_{ss}, v_{si}\})^+ \leq \frac{r_1}{4}\}. \tag{106}$$

Thus

$$P_{E^2_{js}, \overline{A^1_s}} \dot{\leq} \rho^{-d_{s,js}(r_1)}, \tag{107}$$

where

$$d_{s,js}(r_1) \triangleq \inf_{O^{1,2+}_{s,js}} \{v_{ss} + \cdots + u\}.$$

Now using (106), together with (83), it is a simple matter to show that

$$d_{s,js}(r_1) = 4 - r_1. \tag{108}$$

This completes the characterization of $P_{E^2_{js},\overline{A^1_s}}$. Next, we use (88), (89), (105) and (107) we conclude

$$P_{E^2_j,\overline{A^1_s}} \dot{\leq} \rho^{-d_{s,j}(r_1)}, \tag{109}$$

where

$$d_{s,j}(r_1) = \min\{d_{2\times 2}(r_1), d_{s,ji}(r_1), d_{s,js}(r_1)\}.$$

Using (104) and (108), however, we get

$$d_{s,j}(r_1) = d_{s,ji}(r_1). \tag{110}$$

Finally, (73), together with (87) and (109), gives

$$d_{\text{DDF-CVMA}}(r_1, 2) \geq \min\{d_i(r_1), d_{s,j}(r_1)\},$$

where $d_{\text{DDF-CVMA}}(r_1, 2)$ denotes the diversity gain achieved by the protocol. Now, using (86), (110) and (104) we get

$$d_{\text{DDF-CVMA}}(r_1, 2) \geq \begin{cases} 3 - r_1, & 1 > r_1 \geq 0 \\ 4 - 2r_1, & \frac{4}{3} > r_1 \geq 1 \\ 2 - \frac{r_1}{2}, & 2 > r_1 \geq \frac{4}{3} \end{cases} \tag{111}$$

Next, we prove that for $2 > r_1 \geq 0$, $r_{\text{e}} = r_1$. To do this, we only need to characterize $p(1)$ (recall (11)). Toward this end, we observe that

$$\begin{aligned} p(1) =& P_{\overline{A^1_j}}, \\ \doteq& P_{E^1_j}. \end{aligned} \tag{112}$$

$P_{E^1_j}$, however, is the joint error probability, for a multiple access channel with two single-antenna users and one double-antenna destination, which is known to be (refer to [7])

$$P_{E^1_j} \doteq \rho^{-(2-r_1)}. \tag{113}$$

Now, (113), together with (112), proves that $p(1)$ decays polynomially with ρ over the range $2 > r_1 \geq 0$. Thus

$$r_e = r_1 \text{ for } 2 > r_1 \geq 0. \tag{114}$$

From (114) and (111), we conclude

$$d_{\text{DDF-CVMA}}(r_e, 2) \geq \begin{cases} 3 - r_e, & 1 > r_e \geq 0 \\ 4 - 2r_e, & \frac{4}{3} > r_e \geq 1 \\ 2 - \frac{r_e}{2}, & 2 > r_e \geq \frac{4}{3} \end{cases} \tag{115}$$

Now, application of Lemma 11 of [6] shows that there are codes in the ensemble that achieve (112) and (115), *simultaneously*. This proves the achievability of (7).

Next, we prove the asymptotic optimality result. To do this, however, we first need to generalize the protocol. To extend the protocol to the case of L ARQ rounds, the random codes $C(\rho) = \{C_1(\rho), C_2(\rho)\}$ used by the two sources should be modified such that each one is of length LT symbols and rate $R_1/2L$ BPCU. Notice that these choices leave the total number of information bits per joint message equal to R_1T bits. The decoder used by the destination is also a direct extension of the one described in the proof of the achievability part. In particular, at the end of the ℓth round $(L \geq \ell \geq 1)$ and in the event that none of the two messages has yet been successfully decoded, the decoder uses φ_j^ℓ and ψ_j^ℓ to jointly decode the two messages (φ_j^ℓ and ψ_j^ℓ are defined in a way similar to (63) and (64)). If successful, the decoder sends back ACK signals to both of the users (i.e., event A_j^ℓ), otherwise it tries to decode the superior message, using φ_s^ℓ and ψ_s^ℓ (which are defined in a way similar to (65) and (66)). In the case that the decoder succeeds in decoding the superior message (i.e., event A_s^ℓ), an ACK signal is sent to the corresponding user. The description given so far pertains to the scenario where none of the messages is successfully decoded by the end of the $(\ell-1)$th round. If the superior message is already decoded, the decoder uses φ_i^ℓ $(L \geq \ell > 1)$ to decode the inferior message, i.e.

$$\varphi_i^\ell(\mathbf{Y}^\ell, m_s) \triangleq \arg\min_{m_i} ||\mathbf{Y}^\ell - \mathbf{S}^\ell(m_s) - \mathbf{S}^\ell(m_i)||^2, \tag{116}$$

where m_s denotes the decoded superior message. To decide whether it has decoded the inferior message error-free or not, the decoder uses ψ_i^ℓ, where ψ_i^ℓ outputs a one, if m_i is the *unique* message satisfying

$$||\mathbf{Y}^\ell - \mathbf{S}^\ell(m_s) - \mathbf{S}^\ell(m_i)||^2 \leq 2\ell T(1+\delta)\sigma^2.$$

In any other case, ψ_i^ℓ outputs a zero. Now, if $\psi_i^\ell(\mathbf{Y}^\ell, m_s) = 1$, then the decoder sends back an ACK signal to the inferior user (i.e., event A_i^ℓ), declaring $\varphi_i^\ell(\mathbf{Y}^\ell, m_s)$ as the decoded inferior message.

Having described the decoder, we next characterize its error probability, P_E, as L grows to infinity. Note that P_E can be written as

$$P_E = \Sigma_{\ell=1}^L P_{E^\ell}, \tag{117}$$

where E^ℓ denotes the event of making an error at the end of the ℓth round. It is important to realize that for $\ell < L$, E^ℓ corresponds to an *undetected* error, i.e., one for which an ACK signal is sent back. Now, an argument similar to those given for (21), (71) and (72) reveals that for large enough T,

$$P_{E^\ell} \leq \epsilon, \quad \text{for} \quad L > \ell \geq 1.$$

This together with (117) gives

$$P_E \dot{\leq} P_{E^L}. \tag{118}$$

But

$$P_{E^L} \leq \Sigma_{\ell=1}^{L-1} P_{E_i^L, \overline{E_s^\ell}, A_s^\ell, \overline{A_j^\ell}} + P_{E_j^L, \overline{A_s^{L-1}}, \overline{A_j^{L-1}}}, \tag{119}$$

where $P_{E_i^L, \overline{E_s^\ell}, A_s^\ell, \overline{A_j^\ell}}$ upper-bounds the probability that the decoder makes an error after L rounds of transmission, conditioned on successful decoding of the superior message at the end of the ℓth round. $P_{E_j^L,\overline{A_s^{L-1}},\overline{A_j^{L-1}}}$, on the other hand, upper-bounds the decoder error probability after L rounds, given that it does not decode any of the messages by the $(L-1)$th round. To characterize $P_{E_i^L,\overline{E_s^\ell},A_s^\ell,\overline{A_j^\ell}}$, notice that

$$P_{E_i^L,\overline{E_s^\ell},A_s^\ell,\overline{A_j^\ell}} \leq \min\{P_{E_i^L,\overline{E_s^\ell},A_s^\ell}, P_{\overline{A_j^\ell}}\}, \quad \text{for} \quad L > \ell \geq 1. \tag{120}$$

Since we are only interested in the case where L grows to infinity, we can assume that L is even. Now for $\frac{L}{2} \geq \ell \geq 1$, we have

$$P_{E_i^L,\overline{E_s^\ell},A_s^\ell} \leq P_{E_i^L,\overline{E_s^{L/2}},A_s^{L/2}}. \tag{121}$$

This is because in the event $E_i^L\overline{E_s^\ell}A_s^\ell$, ℓ corresponds to the number of rounds during which the superior user is acting as a DDF relay for the inferior one. Likewise, for $\ell \geq \frac{L}{2}+1$

$$P_{\overline{A_j^\ell}} \leq P_{\overline{A_j^{L/2}}}. \tag{122}$$

This is because in the event $\overline{A_j^\ell}$, ℓ corresponds to the number of rounds during which the two users are simultaneously transmitting their messages. Clearly, increasing ℓ reduces $P_{\overline{A_j^\ell}}$ (notice that for exactly the same reason, $P_{E_j^L} \leq P_{E_j^{L/2}}$). Following the same steps leading to (34) reveals that

$$P_{\overline{A_j^{L/2}}} \dot{\leq} P_{E_j^{L/2}}, \tag{123}$$

which simply states that the probability of sending back a NACK signal is of the same exponential order as the probability of making a joint error. Now from equations (119) to (123), we conclude

$$P_{E^L} \dot{\leq} \frac{L}{2}(P_{E_i^L,\overline{E_s^{L/2}},A_s^{L/2}} + P_{\overline{E_j^{L/2}}}), \quad \text{for } L \text{ even}. \tag{124}$$

Comparing (124) with (73), we realize that P_{E^L} is of same exponential order as the error probability of the ARQ-DDF CVMA protocol, with two rounds of transmission and a destination first-round rate of $\frac{2R_1}{L}$ BPCU, i.e.

$$P_{E^L} \dot{\leq} \rho^{-d_{\text{DDF-CVMA}}(\frac{2r_1}{L},2)}, \quad \text{for} \quad 2 > r_1 \geq 0,$$

where $d_{\text{DDF-CVMA}}(r_1, 2)$ is given by (111). Now, letting L to grow to infinity, together with (118) and (111), gives

$$\lim_{L\to\infty} P_E \dot{\leq} \rho^{-3}, \quad \text{for} \quad 2 > r_1 \geq 0. \tag{125}$$

To prove that $r_e = r_1$, we notice that (recall (11))

$$\eta \leq \frac{R_1}{1+p(1)},$$

where

$$\begin{aligned} p(1) =& P_{\overline{A_j^1}}, \\ \doteq& \rho^{-(2-r_1)}. \end{aligned} \tag{126}$$

The last step of (126) follows from the same argument given for (113). This shows that $r_e = r_1, 2 > r_1 \geq 0$, which together with (125) proves (8). Note that the proof for existence of codes that achieve P_E and η simultaneously results from application of Lemma 11 of [6].

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