On The Han-Kobayashi Region For The Interference Channel

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Abstract

In this correspondence, a simplified description of the Han-Kobayashi rate region for the general interference channel is derived. Using this result, we establish the equivalence between the Han-Kobayashi and Chong-Motani-Garg recently discovered region. Moreover, a tighter bound for the cardinality of the time-sharing auxiliary random variable emerges from our simplified description.

I. Background

The interference channel (IC) models the situation where M unrelated senders try to communicate their separate messages to M different receivers via a common channel as shown in Fig. 1. In this model, there is no cooperation between any of the senders or receivers, and hence, the transmission of from each sender to its corresponding receiver is viewed as interference by the other sender-receiver pairs. In this paper, we limit ourselves to the two-user IC. The study of the IC was first initiated by Shannon [1], and was further studied by Ahlswede [2]. In [3], Carleial determined an improved achievable rate region for the IC. Later, Han and Kobayashi established the best achievable rate region to date for the general IC [4]. Except for the Gaussian IC under strong interference [4]–[6], a class of discrete additive degraded IC [7], a class of deterministic IC [8] and the discrete memoryless IC with strong interference [9], the capacity of the general IC remains unknown to date.

A. Definitions and Notations

In our notation, a discrete random variable U is assumed to take values u in a finite set $\mathcal{U}$. We use $\|\mathcal{U}\|$ to denote the cardinality of $\mathcal{U}$, and $p(u)$ to denote the probability distribution function of U on $\mathcal{U}$. Vectors are denoted with boldface letters, e.g. $\mathbf{x}^n$, where the i'th element of a vector $\mathbf{x}^n$ is denoted by x_i.

Definition 1: Let $(X_1, X_2, ..., X_k)$ denote a finite collection of discrete random variables with some fixed joint distribution, $p(x_1, x_2, ..., x_k)$, $(x_1, x_2, ..., x_k) \in \mathcal{X}_1 \times \mathcal{X}_2 \times ... \times \mathcal{X}_k$. Let S denote an ordered subset of these random variables and consider n independent copies of S. The set $A_\epsilon^{(n)}$ of ϵ-typical n-sequences $(\mathbf{x}_1^n, \mathbf{x}_2^n, ..., \mathbf{x}_k^n)$ is defined as

$$\begin{aligned}&A_\epsilon^{(n)}(X_1, X_2, ..., X_k)\\&= \left\{(\mathbf{x}_1^n, \mathbf{x}_2^n, ..., \mathbf{x}_k^n) : \left|-\frac{1}{n}\log p(\mathbf{s}^n) - H(S)\right| < \epsilon, \forall S \subseteq \{X_1, X_2, ..., X_k\}\right\}.\end{aligned}$$

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Definition 2: A two-user discrete IC consists of two input alphabets $\mathcal{X}_1$ and $\mathcal{X}_2$, two output alphabets $\mathcal{Y}_1$ and $\mathcal{Y}_2$, and a probability transition function $p\left(.,.|x_1,x_2\right)$. The IC is said to be memoryless if $p\left(\mathbf{y}_1^n,\mathbf{y}_2^n|\mathbf{x}_1^n,\mathbf{x}_2^n\right) = \prod_{i=1}^{n} p\left(y_{1i},y_{2i}|x_{1i},x_{2i}\right)$. Since there is no cooperation between the receivers, the capacity region of the discrete memoryless IC depends only on the conditional marginal distributions.

$$p\left(y_1|x_1,x_2\right) = \sum_{y_2\in\mathcal{Y}_2} p\left(y_1y_2|x_1,x_2\right), \quad p\left(y_2|x_1,x_2\right) = \sum_{y_1\in\mathcal{Y}_1} p\left(y_1y_2|x_1,x_2\right).$$

Definition 3: A $\left(2^{nR_1},2^{nR_2},n\right)$ code for an interference channel with independent information consists of two sets of integers $V_1 = \left\{1,2,...,2^{nR_1}\right\}$ and $V_2 = \left\{1,2,...,2^{nR_2}\right\}$ called the message sets, two encoding functions

$$f_1 : V_1 \mapsto \mathcal{X}_1^n \quad \text{and} \quad f_2 : V_2 \mapsto \mathcal{X}_2^n$$

and two decoding functions

$$g_1 : \mathcal{Y}_1^n \mapsto V_1 \quad \text{and} \quad g_2 : \mathcal{Y}_2^n \mapsto V_2.$$

Definition 4: The average probability of error is defined as the probability that the decoded message is not equal to the transmitted message, i.e.,

$$P_e^{(n)} = \frac{1}{2^{n(R_1+R_2)}} \sum_{(v_1,v_2)\in V_1\times V_2} \Pr\left(g_1\left(Y_1^n\right)\neq v_1 \text{ or } g_2\left(Y_2^n\right)\neq v_2|\left(v_1,v_2\right) \text{ sent}\right)$$

where (V_1,V_2) are assumed to be uniformly distributed over $\left\{1,2,...,2^{nR_1}\right\}\times\left\{1,2,...,2^{nR_2}\right\}$.

Definition 5: A rate pair (R_1,R_2) is said to be achievable for the interference channel if there exists a sequence of $\left(2^{nR_1},2^{nR_2},n\right)$ codes with $P_e^{(n)}\to 0$.

Definition 6: The discrete-time additive white Gaussian IC, shown in Fig. 2, is described by

$$Y_1 = c_{11}X_1 + c_{21}X_2 + Z_1$$
$$Y_2 = c_{12}X_1 + c_{22}X_2 + Z_2$$

where the input and output signals are real, the coefficients c_{ij} are real constants, and the noise terms Z_1 and Z_2 are zero-mean Gaussian random variables. Also, the mean value of X_1^2 and X_2^2 cannot exceed P_1 and P_2 respectively, i.e.,

$$\mathbb{E}\left[X_1^2\right]\leq P_1 \quad \text{and} \quad \mathbb{E}\left[X_2^2\right]\leq P_2.$$

In [3], it was shown that any Gaussian IC can be reduced to a standard form by an appropriate transformation, where $c_{11}^2 = c_{22}^2 = 1$ and $\mathbb{E}\left[Z_1^2\right] = \mathbb{E}\left[Z_2^2\right] = 1$. The capacity of the Gaussian IC is not known, except for the case of no interference, where $c_{21}^2 = c_{12}^2 = 0$, for the case of strong interference, where $c_{21}^2\geq 1$ and $c_{12}^2\geq 1$ and for the one-sided Gaussian IC under strong interference, where $c_{12}^2 = 0$ and $c_{21}^2\geq 1$ or $c_{21}^2 = 0$ and $c_{12}^2\geq 1$.

B. The Han-Kobayashi Region

In [4], Han and Kobayashi introduced 5 auxiliary random variables Q, U_1, W_1, U_2 and W_2, defined on arbitrary finite sets $\mathcal{Q}$, $\mathcal{U}_1$, $\mathcal{W}_1$, $\mathcal{U}_2$ and $\mathcal{W}_2$, respectively. In Han-Koyashi coding strategy, sender TX_1 splits the message V_1 into (V_{11},V_{12}), where $V_{11} = \left\{1,2,...,2^{nS_1}\right\}$ and $V_{12} = \left\{1,2,...,2^{nT_1}\right\}$. Similarly, sender TX_2 splits the message V_2 into (V_{21},V_{22}), where $V_{21} = \left\{1,2,...,2^{nT_2}\right\}$ and $V_{22} = \left\{1,2,...,2^{nS_2}\right\}$. This split aims at allowing each of the receivers to decode partial information from its non-intended sender. Hence, V_{12} represents the message intended for receiver RX_1 which can also be decoded by receiver RX_2, and similarly, V_{21} represents the message intended for receiver RX_2 which can also be decoded by receiver RX_1. Here, the auxiliary random variable U_1 serves to carry the message V_{12}, while the auxiliary random variable W_1 serves to carry the message V_{11}. The same applies to the auxiliary random variables U_2 and W_2. Hence, the encoding functions f_1 and f_2 are given by

$$f_1 : V_1 = (V_{11}, V_{12}) \mapsto \mathcal{X}_1^n \quad \text{and} \quad f_2 : V_2 = (V_{21}, V_{22}) \mapsto \mathcal{X}_2^n,$$

where the function f_1 consists of three functions f_{11}, f_{12} and f_{13} defined as follows

$$f_{11} : V_{11} \mapsto \mathcal{W}_1^n, \quad f_{12} : V_{12} \mapsto \mathcal{U}_1^n \quad \text{and} \quad f_{13} : \mathcal{W}_1^n \times \mathcal{U}_1^n \mapsto \mathcal{X}_1^n$$

Similarly, f_2 decomposes into the following three components

$$f_{21} : V_{21} \mapsto \mathcal{U}_2^n \quad f_{22} : V_{22} \mapsto \mathcal{W}_2^n, \quad \text{and} \quad f_{23} : \mathcal{W}_2^n \times \mathcal{U}_2^n \mapsto \mathcal{X}_2^n.$$

In a nutshell, this strategy is basically an application of Cover's superposition coding technique [10] and was first used by Carleial [3] in the context of the Gaussian IC. Carleial made use of a sequential decoder, otherwise known as the stripping decoder. In this approach, Receiver RX_1 decodes either U_1 or U_2 first before decoding W_1, whereas receiver RX_2 decodes either U_1 or U_2 first before decoding W_2. On the other hand, Han and Kobayashi the more powerful joint decoder where Receiver RX_1 decodes U_1, U_2 and W_1 simultaneously, and receiver RX_2 decodes U_1, U_2 and W_2 simultaneously. In addition, Han and Kobayashi introduced a time-sharing parameter Q instead of using the convex-hull operation. The time-sharing parameter Q includes, as a speical case, the TDM/FDM strategy introduced by Carleial [3] for the Gaussian IC. Next, we state the achievable rate region of Han and Kobayashi, $\mathcal{R}^o_{\mathrm{HK}}$, as described in [4][1]

Let $\mathcal{P}^*$ be the set of probability distributions $P^*(.)$ that factor as

$$\begin{aligned} &P^*(q, u_1, w_1, u_2, w_2, x_1, x_2) \\ =&p(q)\, p(u_1|q)\, p(w_1|q)\, p(u_2|q)\, p(w_2|q)\, p(x_1|u_1, w_1, q)\, p(x_2|u_2, w_2, q). \end{aligned}$$

Suppose we fix $P^*(.)$. Consider receiver RX_1 and the set of non-negative rate-tuples (S_1, T_1, S_2, T_2) denoted by $\mathcal{R}^{(o,1)}_{\mathrm{HK}}(P^*)$ that satisfy

$$S_1 \le I(W_1; Y_1|U_1U_2Q) \quad (1)$$
$$T_1 \le I(U_1; Y_1|W_1U_2Q) \quad (2)$$
$$T_2 \le I(U_2; Y_1|W_1U_1Q) \quad (3)$$
$$S_1 + T_1 \le I(U_1W_1; Y_1|U_2Q) \quad (4)$$
$$S_1 + T_2 \le I(W_1U_2; Y_1|U_1Q) \quad (5)$$
$$T_1 + T_2 \le I(U_1U_2; Y_1|W_1Q) \quad (6)$$
$$S_1 + T_1 + T_2 \le I(U_1W_1U_2; Y_1|Q). \quad (7)$$

Similarly, let $\mathcal{R}^{(o,2)}_{\mathrm{HK}}(P^*)$ be the set of non-negative rate-tuples (S_1, T_1, S_2, T_2) that satisfy (1)-(7) with the indexes 1 and 2 swapped everywhere. For a set $\mathcal{S}$ of 4-tuples (S_1, T_1, S_2, T_2), let $\prod(\mathcal{S})$ be the set of (R_1, R_2) such that $R_1 = S_1 + T_1$ and $R_2 = S_2 + T_2$ for some $(S_1, T_1, S_2, T_2) \in \mathcal{S}$. We have the following result.

Theorem 1 (Han-Kobayashi): The set

$$\mathcal{R}^o_{\mathrm{HK}} = \prod \left(\bigcup_{P^* \in \mathcal{P}^*} \mathcal{R}^{(o,1)}_{\mathrm{HK}}(P^*) \cap \mathcal{R}^{(o,2)}_{\mathrm{HK}}(P^*) \right) \quad (8)$$

is an achievable rate region for the discrete memoryless IC.

Proof: Refer to [4]. ■

II. The Main Result

Our main contribution is the following compact description of the Han-Kobayashi achievable rate region:

[1]We use superscript "o" and "c" to differentiate the original description of the Han-Kobayashi region from our compact description.

Theorem 2: For a fixed $P_1^* \in \mathcal{P}_1^*$, let $\mathcal{R}_{\text{HK}}^c(P_1^*)$ be the set of (R_1, R_2) satisfying

$$R_1 \leq I(X_1; Y_1|U_2Q) \tag{9}$$
$$R_2 \leq I(X_2; Y_2|U_1Q) \tag{10}$$
$$R_1 + R_2 \leq I(X_1U_2; Y_1|Q) + I(X_2; Y_2|U_1U_2Q) \tag{11}$$
$$R_1 + R_2 \leq I(X_1; Y_1|U_1U_2Q) + I(X_2U_1; Y_2|Q) \tag{12}$$
$$R_1 + R_2 \leq I(X_1U_2; Y_1|U_1Q) + I(X_2U_1; Y_2|U_2Q) \tag{13}$$
$$2R_1 + R_2 \leq I(X_1U_2; Y_1|Q) + I(X_1; Y_1|U_1U_2Q) + I(X_2U_1; Y_2|U_2Q) \tag{14}$$
$$R_1 + 2R_2 \leq I(X_2; Y_2|U_1U_2Q) + I(X_2U_1; Y_2|Q) + I(X_1U_2; Y_1|U_1Q). \tag{15}$$

Then we have

$$\mathcal{R}_{\text{HK}}^c = \bigcup_{P_1^* \in \mathcal{P}_1^*} \mathcal{R}_{\text{HK}}^c(P_1^*). \tag{16}$$

is an achievable rate region for the interference channel. Furthermore, $\mathcal{R}_{\text{HK}}^c = \mathcal{R}_{\text{HK}}^o$ and the region remains invariant if we impose the following constraints on the cardinalities of the auxiliary sets:

$$\|\mathcal{U}_1\| \leq \|\mathcal{X}_1\| + 6, \quad \|\mathcal{U}_2\| \leq \|\mathcal{X}_2\| + 6 \quad \text{and} \quad \|\mathcal{Q}\| \leq 8. \tag{17}$$

Proof: We first reduce the set of inequalities in Theorem 1 to those in Lemma 1 using Fourier-Motzkin elimination. It is then straightforward to see that $\mathcal{R}_{\text{HK}}^o \subseteq \mathcal{R}_{\text{HK}}^c$. In order to prove that $\mathcal{R}_{\text{HK}}^c \subseteq \mathcal{R}_{\text{HK}}^o$, we make use of Lemma 2. The assertion about the cardinalities of $\|\mathcal{U}_1\|$, $\|\mathcal{U}_2\|$ and $\|\mathcal{Q}\|$ follows directly from the application of Caratheodory's theorem to the expressions (9)-(15). ■

Before proceeding to Lemma 1, we need to derive a few simple results about $\mathcal{R}_{\text{HK}}^o$. It can be readily shown that the rate region $\mathcal{R}_{\text{HK}}^o$ suffers no reduction by assuming deterministic encoding functions rather than probabilistic functions [4]. Hence, for any $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$, $p(x_1|u_1, w_1, q)$ and $p(x_2|u_2, w_2, q)$ either equals to 0 or 1. Since $(Q, U_1, W_1) \to X_1 \to (Y_1, Y_2)$ and $(Q, U_2, W_2) \to X_2 \to (Y_1, Y_2)$ form Markov chains, we can write the following

$$\begin{aligned}
&I(W_1; Y_1|U_1U_2Q) \\
&= H(Y_1|U_1U_2Q) - H(Y_1|U_1W_1U_2Q) \\
&= H(Y_1|U_1U_2Q) - H(Y_1|X_1U_1W_1U_2Q) \\
&= H(Y_1|U_1U_2Q) - H(Y_1|X_1U_2Q) \\
&= I(X_1; Y_1|U_1U_2Q).
\end{aligned} \tag{18}$$

Following along the same lines, we can write the following equalities

$$I(W_1U_2; Y_1|U_1Q) = I(X_1U_2; Y_1|U_1Q) \tag{19}$$
$$I(W_2; Y_2|U_1U_2Q) = I(X_2; Y_2|U_1U_2Q) \tag{20}$$
$$I(W_2U_1; Y_2|U_2Q) = I(X_2U_1; Y_2|U_2Q). \tag{21}$$

On applying the above equalities together with Fourier-Motzkin elimination, we obtain the following result:

Lemma 1: Let $\mathcal{P}_1^*$ be the set of probability distributions $P_1^*(.)$ that factor as

$$P_1^*(q, u_1, u_2, x_1, x_2) = p(q)\, p(x_1u_1|q)\, p(x_2u_2|q). \tag{22}$$

For a fixed $P_1^* \in \mathcal{P}_1^*$, let $\mathcal{R}_{\text{HK}}^o(P_1^*)$ be the set of all rate pairs (R_1, R_2) satisfying

$$R_1 \leq I(X_1; Y_1|U_2Q) \tag{23}$$
$$R_1 \leq I(U_1; Y_2|X_2Q) + I(X_1; Y_1|U_1U_2Q) \tag{24}$$
$$R_2 \leq I(X_2; Y_2|U_1Q) \tag{25}$$
$$R_2 \leq I(U_2; Y_1|X_1Q) + I(X_2; Y_2|U_1U_2Q) \tag{26}$$
$$R_1 + R_2 \leq I(X_1U_2; Y_1|Q) + I(X_2; Y_2|U_1U_2Q) \tag{27}$$

$$R_1 + R_2 \leq I(X_1; Y_1|U_1U_2Q) + I(X_2U_1; Y_2|Q) \quad (28)$$
$$R_1 + R_2 \leq I(X_1U_2; Y_1|U_1Q) + I(X_2U_1; Y_2|U_2Q) \quad (29)$$
$$2R_1 + R_2 \leq I(X_1U_2; Y_1|Q) + I(X_1; Y_1|U_1U_2Q) + I(X_2U_1; Y_2|U_2Q) \quad (30)$$
$$R_1 + 2R_2 \leq I(X_2; Y_2|U_1U_2Q) + I(X_2U_1; Y_2|Q) + I(X_1U_2; Y_1|U_1Q). \quad (31)$$

Finally, we have

$$\mathcal{R}^o_{\mathrm{HK}} = \bigcup_{P_1^* \in \mathcal{P}_1^*} \mathcal{R}^{'}_{\mathrm{HK}}(P_1^*). \quad (32)$$

Furthermore, the region $\mathcal{R}^o_{\mathrm{HK}}$ remains invariant if we impose the following constraints on the cardinalities of the auxiliary sets:

$$\|\mathcal{U}_1\| \leq \|\mathcal{X}_1\| + 6, \quad \|\mathcal{U}_2\| \leq \|\mathcal{X}_2\| + 6 \quad \text{and} \quad \|\mathcal{Q}\| \leq 10. \quad (33)$$

Proof: Refer to Appendix I for a detailed proof. ■

The equivalence between $\mathcal{R}^c_{\mathrm{HK}}$ and $\mathcal{R}^o_{\mathrm{HK}}$ emerges from the following lemma.

Lemma 2: For a fixed $P_1^* \in \mathcal{P}_1^*$, $\mathcal{R}^c_{\mathrm{HK}}(P_1^*) \subseteq \mathcal{R}^o_{\mathrm{HK}}(P_1^*) \cup \mathcal{R}^o_{\mathrm{HK}}(P_1^{**}) \cup \mathcal{R}^{'}_{\mathrm{HK}}(P_1^{***})$ where

$$P_1^{**} = \sum_{u_1 \in \mathcal{U}_1} P_1^* \quad \text{and} \quad P_1^{***} = \sum_{u_2 \in \mathcal{U}_2} P_1^*.$$

Proof: Suppose (R_1, R_2) is in $\mathcal{R}^c_{\mathrm{HK}}(P_1^*)$ but not in the $\mathcal{R}^o_{\mathrm{HK}}(P_1^*)$. Without loss of generality, we assume

$$I(U_1; Y_2|X_2Q) + I(X_1; Y_1|U_1U_2Q) < R_1 < I(X_1; Y_1|U_2Q). \quad (34)$$

By substituting $U_1 = \phi$ in Lemma 1, we see that $\mathcal{R}^o_{\mathrm{HK}}(P_1^{**})$ consists of all rate pairs (R_1, R_2) such that

$$\begin{aligned}
R_1 &\leq I(X_1; Y_1|U_2Q), \\
R_2 &\leq I(X_2; Y_2|Q), \\
R_2 &\leq I(U_2; Y_1|X_1Q) + I(X_2; Y_2|U_2Q), \\
R_1 + R_2 &\leq I(X_1U_2; Y_1|Q) + I(X_2; Y_2|U_2Q).
\end{aligned}$$

However, from (9), we obtain

$$R_1 \leq I(X_1; Y_1|U_2Q),$$

and from (34) and (12), we obtain

$$R_2 < I(X_2; Y_2|Q),$$

and from (34) and (13), we obtain

$$\begin{aligned}
R_2 &< I(U_2; Y_1|U_1Q) + I(X_2; Y_2|U_2Q) \\
&\leq I(U_2; Y_1|X_1Q) + I(X_2; Y_2|U_2Q),
\end{aligned}$$

and from (34) and (14), we obtain

$$R_1 + R_2 \leq I(X_1U_2; Y_1|Q) + I(X_2; Y_2|U_2Q).$$

We see that (R_1, R_2) satisfying the above constraints are in $\mathcal{R}^o_{\mathrm{HK}}(P_1^{**})$. The proof for

$$R_2 > I(U_2; Y_1|X_1Q) + I(X_2; Y_2|U_1U_2Q)$$

follows exactly along the same lines. It readily follows that $\mathcal{R}^c_{\mathrm{HK}}(P_1^*) \subseteq \mathcal{R}^o_{\mathrm{HK}}(P_1^*) \cup \mathcal{R}^o_{\mathrm{HK}}(P_1^{**}) \cup \mathcal{R}^o_{\mathrm{HK}}(P_1^{***})$. ■

Finally, since $\mathcal{R}^c_{\mathrm{HK}}(P_1^*) \subseteq \mathcal{R}^o_{\mathrm{HK}}(P_1^*) \cup \mathcal{R}^o_{\mathrm{HK}}(P_1^{**}) \cup \mathcal{R}^o_{\mathrm{HK}}(P_1^{***})$, it immediately follows that $\mathcal{R}^c_{\mathrm{HK}} \subseteq \mathcal{R}^o_{\mathrm{HK}}$ and since $\mathcal{R}^o_{\mathrm{HK}} \subseteq \mathcal{R}^c_{\mathrm{HK}}$, we obtain our result $\mathcal{R}^c_{\mathrm{HK}} = \mathcal{R}^o_{\mathrm{HK}}$.

III. Discussion

In this section, we make a few remarks about our results.

Remark 1: Han and Kobayashi make use of the polymatroidal structure underlying the collection of bounds that specify the region $\mathcal{R}_{\mathrm{HK}}{}^{o}$, (1)-(7) and their counterparts at receiver RX_2, to convert them to a set of bounds on R_1, R_2, R_1+R_2, $2R_1+R_2$ and R_1+2R_2 [4, Theorem 4.1]. Even though Theorem 2 is just a different description of the Han Kobayashi rate region, it gives the simplest description of the best rate region to date. In addition, Theorem 2 also gives us a tighter bound for the cardinality of the auxiliary sets $\mathcal{Q}$, $\mathcal{U}_1$ and $\mathcal{U}_2$. Another interesting observation is that even though the coding technique requires the use of the auxiliary random variables W_1 and W_2, the rate region $\mathcal{R}_{\mathrm{HK}}$ does not depend on these auxiliary random variables. Hence, cardinality bounds on $\mathcal{W}_1$ and $\mathcal{W}_2$ are unnecessary.

Remark 2: Han and Kobayashi conjectured that the rate region formulated using the time-sharing parameter Q strictly enlarges the rate region formulated using the convex-hull operation, as suggested by a numerical example for the Gaussian IC. However, we note that the convex-hull formulation is in fact equivalent to the formulation using the time-sharing parameter Q for the discrete memoryless IC. Consider a rate point $\mathbf{R}$ satisfying the inequalities (9)-(15). We can rewrite the right hand side of the first inequality as follows

$$\begin{aligned} I\left(X_1;Y_1|U_2Q\right) &= \sum_{q\in\mathcal{Q}} p\left(q\right) I\left(X_1;Y_1|U_2Q=q\right) \\ &= \sum_{q\in\mathcal{Q}} p\left(q\right) I\left(X_1;Y_1|U_2\right)_{p_{1q}(u_1x_1)p_{2q}(u_2x_2)} \end{aligned} \tag{35}$$

We can similarly expand the other inequalities in the same way.

For simplicity, consider a rate pair as a vector and denote a pair of vectors satisfying inequalities (9)-(15), with $Q=\phi$ (ϕ is a constant) and for a specific input distribution $p_{1q}\left(u_1x_1\right)p_{2q}\left(u_2x_2\right)$, as $\mathbf{R}_q$. Then, by Theorem 2, with $Q=\phi$, $\mathbf{R}_q=\left(R_{1q},R_{2q}\right)$ is achievable. Since $\mathbf{R}$ satisfies (9)-(15) and we can expand all of the right hand sides as in (35), there exists a set of pairs $\mathbf{R}_q$ satisfying Theorem 2, with $Q=\phi$, such that

$$\mathbf{R} = \sum_{q\in\mathcal{Q}} p\left(q\right)\mathbf{R}_q. \tag{36}$$

Since a convex combination of achievable rates is achievable, so is $\mathbf{R}$ by using the convex-hull operation in Theorem 2, with $Q=\phi$, instead of the time-sharing formulation. Hence, we readily see that the convex-hull formulation is equivalent to the formulation using the time-sharing parameter Q for the discrete memoryless IC.

Nonetheless, we continue to formulate the rate region using the time-sharing parameter Q. Our motivation is the fact that this formulation can be readily extended to the Gaussian IC, with obvious modifications, where it **may** provide an enlarged region as compared to the convex-hull formulation. The reason is the two additional power constraints imposed on $\mathbb{E}\left[X_1^2\right]$ and $\mathbb{E}\left[X_2^2\right]$ in the Gaussian IC. Due to these two additional power constraints, when we use TDM/FDM for the Gaussian IC, where sender TX_1 and sender TX_2 use a fraction β and $1-\beta$ of the time/bandwidth with powers $\frac{P_1}{\beta}$ and $\frac{P_2}{1-\beta}$, respectively, certain rate points may be achievable which are possibly unattainable by the convex-hull formulation.

Remark 3: We observe that the Chong-Motani-Garg region, i.e., $\mathcal{R}_{\mathrm{CMG}}$, reported in [11], is equivalent to the Han-Kobayashi region. This equivalence sheds light on the two main insights behind our compact description of the Han-Kobayashi region. We first observe that for receiver RX_1, no decoding error is committed if the message $V_1=\left(V_{11},V_{12}\right)$ is decoded correctly but the message V_{21} is decoded wrongly. The same applies to receiver RX_2. This implies that constraint (3, and its counterpart for receiver RX_2, are unnecessary to drive the overall probability of error to ϵ. Moreover, the coding scheme considered in [11] use only 3 auxiliary random variables Q, U_1 and U_2 defined on arbitrary finite sets $\mathcal{Q}$, $\mathcal{U}_1$ and $\mathcal{U}_2$. The auxiliary random variables U_1 and U_2 now serve as cloud centers that can be distinguished by both receivers. For transmitter 1, instead of generating two independent codebooks with codewords

$\mathbf{U}_1^n(j)$ and $\mathbf{W}_1^n(k)$, for each codeword $\mathbf{U}_1^n(j)$, we generate a codebook with codewords $\mathbf{X}_1^n(j,k)$, where $j \in \{1,2,...,2^{nT_1}\}$ and $k \in \{1,2,...,2^{nS_1}\}$. This construction renders the constraints (2) and (6), and their counterparts for receiver RX_2, unnecessary. Combining these two observations yields the following result

Lemma 3: Let $\mathcal{P}_1^*$ be the set of probability distributions $P_1^*(.)$ that factor as

$$P_1^*(q,u_1,u_2,x_1,x_2) = p(q)\,p(u_1x_1|q)\,p(u_2x_2|q)\,, \tag{37}$$

and $\mathcal{R}_{\mathrm{CMG}}^{(1)}(P_1^*)$ be be the set of non-negative rate-tuples (S_1,T_1,S_2,T_2) that satisfy

$$S_1 \le I(X_1;Y_1|U_1U_2Q) \tag{38}$$
$$S_1 + T_2 \le I(U_2X_1;Y_1|U_1Q) \tag{39}$$
$$S_1 + T_1 \le I(X_1;Y_1|U_2Q) \tag{40}$$
$$S_1 + T_1 + T_2 \le I(U_2X_1;Y_1|Q)\,. \tag{41}$$

Similarly, let $\mathcal{R}_{\mathrm{CMG}}^{(2)}(P_1^*)$ be the set of non-negative rate-tuples (S_1,T_1,S_2,T_2) that satisfy (38)-(41) with the indexes 1 and 2 swapped. Then, the set given by

$$\mathcal{R}_{\mathrm{CMG}} = \bigcup_{P_1^* \in \mathcal{P}_1^*} \mathcal{R}_{\mathrm{CMG}}(P_1^*) \tag{42}$$

is an achievable rate region for the discrete memoryless IC.

Proof: Refer to Appendix II. ■

We can see that $\mathcal{R}_{\mathrm{CMG}} = \mathcal{R}_{\mathrm{HK}^o}$ through the following simple argument. First, since we can choose a fixed P_1^* such that

$$P_1^*(q,u_1,u_2,x_1,x_2) = \sum_{w_1\in\mathcal{W}_1, w_2\in\mathcal{W}_2} P^*(q,u_1,u_2,w_1,w_2,x_1,x_2) \tag{43}$$

we readily see that $\mathcal{R}_{\mathrm{HK}}^o(P^*) \subseteq \mathcal{R}_{\mathrm{CMG}}(P_1^*)$, and hence $\mathcal{R}_{\mathrm{HK}}^o \subseteq \mathcal{R}_{\mathrm{CMG}}$. Second, the bounds (38)-(41) and their counterparts at receiver RX_2 can be again simplified using Fourier-Motzkin elimination [12, Theorem 3] to yield the reduced set of bounds in Theorem 2 which establishes $\mathcal{R}_{\mathrm{CMG}} = \mathcal{R}_{\mathrm{HK}}^c$.

Remark 4: It is interesting to note the only differences between $\mathcal{R}_{\mathrm{CMG}}^o$ and $\mathcal{R}_{\mathrm{HK}}^o$ lie in the bounds for R_1 and R_2. This observation allows for answering the question posed by Kramer in [12] on the existence of $P_1^* \in \mathcal{P}_1^*$ such that $\mathcal{R}_{\mathrm{CMG}}(P_1^*) \subsetneq \mathcal{R}_{\mathrm{HK}}^o(P_1^*)$ for certain ICs. For the Gaussian IC, when we set $|\mathcal{Q}| = 1$, we can easily determine parameters where $\mathcal{R}_{\mathrm{CMG}}(P_1^*) \subsetneq \mathcal{R}_{\mathrm{HK}}^o(P_1^*)$. We assume the following customary restriction on the input signals where U_1, U_2, X_1 and X_2 are Gaussian random variables

$$\frac{\mathbb{E}\left[U_1^2\right]}{\mathbb{E}\left[X_1^2\right]} = \alpha, \qquad \frac{\mathbb{E}\left[U_2^2\right]}{\mathbb{E}\left[X_2^2\right]} = \beta \tag{44}$$

such that $\alpha \in [0,1]$, $\beta \in [0,1]$, $\mathbb{E}\left[X_1^2\right] = P_1$ and $\mathbb{E}\left[X_2^2\right] = P_2$. From Fig. 3, we see that when $P_1 = P_2 = 6$, $c_{12}^2 = c_{21}^2 = 0.4$ and $\alpha = \beta = 0.5$, $\mathcal{R}_{\mathrm{HK}}^{'}(P_1^*) \subsetneq \mathcal{R}_{\mathrm{CMG}}(P_1^*)$.

IV. Conclusion

Our main contribution is a simplified description of the celebrated Han-Kobyashi inner bound on the capacity region of the interference channel. This description sheds more light on the role of the auxiliary random variables and the corresponding coding/decoding strategy.

APPENDIX I
PROOF OF LEMMA 1

When we apply the Fourier-Motzkin elimination on the inequalities (1)-(7) and their counterparts at receiver RX_2, we obtain the following 21 bounds on R_1, R_2, R_1+R_2, $2R_1+R_2$ and R_1+2R_2

$$R_1 \le I(X_1;Y_1|U_2Q) \tag{45}$$
$$R_1 \le I(U_1;Y_2|X_2Q) + I(X_1;Y_1|U_1U_2Q) \tag{46}$$
$$R_1 \le I(U_1;Y_1|W_1U_2Q) + I(X_1;Y_1|U_1U_2Q) \tag{47}$$
$$R_2 \le I(X_2;Y_2|U_1Q) \tag{48}$$
$$R_2 \le I(U_2;Y_1|X_1Q) + I(X_2;Y_2|U_1U_2Q) \tag{49}$$
$$R_2 \le I(U_2;Y_2|W_2U_1Q) + I(X_2;Y_2|U_1U_2Q) \tag{50}$$
$$R_1+R_2 \le I(X_1;Y_1|U_1U_2Q) + I(U_1U_2;Y_1|W_1Q) + I(X_2;Y_2|U_1U_2Q) \tag{51}$$
$$R_1+R_2 \le I(X_1;Y_1|U_1U_2Q) + I(U_1U_2;Y_2|W_2Q) + I(X_2;Y_2|U_1U_2Q) \tag{52}$$
$$R_1+R_2 \le I(X_1;Y_1|U_1U_2Q) + I(X_2U_1;Y_2|Q) \tag{53}$$
$$R_1+R_2 \le I(U_1;Y_1|W_1U_2Q) + I(X_1U_2;Y_1|U_1Q) + I(X_2;Y_2|U_1U_2Q) \tag{54}$$
$$R_1+R_2 \le I(X_1U_2;Y_1|U_1Q) + I(X_2;Y_2|U_1U_2Q) + I(U_1;Y_2|X_2Q) \tag{55}$$
$$R_1+R_2 \le I(X_1U_2;Y_1|U_1Q) + I(X_2U_1;Y_2|U_2Q) \tag{56}$$
$$R_1+R_2 \le I(X_1U_2;Y_1|Q) + I(X_2;Y_2|U_1U_2Q) \tag{57}$$
$$R_1+R_2 \le I(X_1;Y_1|U_1U_2Q) + I(U_2;Y_1|X_1Q) + I(U_1X_2;Y_2|U_2Q) \tag{58}$$
$$R_1+R_2 \le I(X_1;Y_1|U_1U_2Q) + I(U_2;Y_2|U_1W_2Q) + I(U_1X_2;Y_2|U_2Q) \tag{59}$$
$$2R_1+R_2 \le 2I(X_1;Y_1|U_1U_2Q) + I(U_1U_2;Y_1|W_1Q) + I(U_1X_2;Y_2|U_2Q) \tag{60}$$
$$2R_1+R_2 \le 2I(X_1;Y_1|U_1U_2Q) + I(U_1U_2;Y_2|W_2Q) + I(U_1X_2;Y_2|U_2Q) \tag{61}$$
$$2R_1+R_2 \le I(X_1U_2;Y_1|Q) + I(X_1;Y_1|U_1U_2Q) + I(X_2U_1;Y_2|U_2Q) \tag{62}$$
$$R_1+2R_2 \le 2I(X_2;Y_2|U_1U_2Q) + I(U_1U_2;Y_2|W_2Q) + I(U_2X_1;Y_1|U_1Q) \tag{63}$$
$$R_1+2R_2 \le 2I(X_2;Y_2|U_1U_2Q) + I(U_1U_2;Y_1|W_1Q) + I(U_2X_1;Y_1|U_1Q) \tag{64}$$
$$R_1+2R_2 \le I(U_1X_2;Y_2|Q) + I(X_2;Y_2|U_1U_2Q) + I(U_2X_1;Y_1|U_1Q)\,. \tag{65}$$

Though the number of bounds seems daunting, when we examine the inequalities carefully, we see that all the inequalities, except for those specified in Lemma 1, are redundant. For example, the bound for R_1 given by (47) is redundant due to (45).

$$\begin{aligned}
&I(U_1;Y_1|W_1U_2Q) + I(X_1;Y_1|U_1U_2Q)\\
&= H(U_1|W_1U_2Q) - H(U_1|W_1U_2Y_1Q) + I(X_1;Y_1|U_1U_2Q)\\
&= H(U_1|U_2Q) - H(U_1|W_1U_2Y_1Q) + I(X_1;Y_1|U_1U_2Q)\\
&\ge H(U_1|U_2Q) - H(U_1|U_2Y_1Q) + I(X_1;Y_1|U_1U_2Q)\\
&= I(U_1;Y_1|U_2Q) + I(X_1;Y_1|U_1U_2Q)\\
&= I(X_1;Y_1|U_2Q)\,.
\end{aligned} \tag{66}$$

Similarly, the bound for R_2 given by (50) is redundant due to (48). We can also show that the bounds for R_1+R_2 are redundant except for (53), (56) and (57). For example, the bound for R_1+R_2 given by (51) is redundant due to (57).

$$\begin{aligned}
&I(X_1;Y_1|U_1U_2Q) + I(U_1U_2;Y_1|W_1Q) + I(X_2;Y_2|U_1U_2Q)\\
&= I(X_1;Y_1|U_1U_2Q) + H(U_1U_2|Q) - H(U_1U_2|Y_1W_1Q) + I(X_2;Y_2|U_1U_2Q)\\
&\ge I(X_1;Y_1|U_1U_2Q) + H(U_1U_2|Q) - H(U_1U_2|Y_1Q) + I(X_2;Y_2|U_1U_2Q)\\
&= I(X_1;Y_1|U_1U_2Q) + I(U_1U_2;Y_1|Q) + I(X_2;Y_2|U_1U_2Q)\\
&= I(U_2X_1;Y_1|Q) + I(X_2;Y_2|U_1U_2Q)\,.
\end{aligned} \tag{67}$$

Similarly, the bound for R_1+R_2 given by (52) is redundant due to (53), (54) is redundant due to (57),

(55) is redundant due to (56), (58) is redundant due to (56) and (59) is redundant due to (53). Following along the same lines, the bound for $2R_1 + R_2$ due to (60) is redundant due to (46) and (53), (61) is redundant due to (62) while the bound for $R_1 + 2R_2$ due to (63) is redundant due to (49) and (57), (64) is redundant due to (65).

APPENDIX II
PROOF OF LEMMA 3

Codebook Generation: Generate a codeword Q^n of length n, generating each element i.i.d according to $\prod_{i=1}^{n} p(q)$. For the codeword $\mathbf{Q}^n$, generate $2^{nR_{12}}$ independent codewords $\mathbf{U}_1^n(j)$, $j \in \{1, 2, ..., 2^{nR_{12}}\}$, generating each element i.i.d according to $\prod_{i=1}^{n} p(u_{1i}|q_i)$. For the codeword $\mathbf{Q}^n$, and each of the codeword $\mathbf{U}_1^n(j)$, generate $2^{nR_{11}}$ i.i.d codewords $\mathbf{X}_1^n(j,k)$, $k \in \{1, 2, ..., 2^{nR_{11}}\}$, generating each element i.i.d according to $\prod_{i=1}^{n} p(x_{1i}|q_i, u_{1i}(j))$. For the codeword $\mathbf{Q}^n$, generate $2^{nR_{21}}$ independent codewords $\mathbf{U}_2^n(l)$, $l \in \{1, 2, ..., 2^{nR_{21}}\}$, generating each element i.i.d according to $\prod_{i=1}^{n} p(u_{2i}|q_i)$. For the codeword $\mathbf{Q}^n$, and each of the codeword $\mathbf{U}_2^n(l)$, generate $2^{nR_{22}}$ i.i.d codewords $\mathbf{X}_2^n(l,m)$, $m \in \{1, 2, ..., 2^{nR_{22}}\}$, generating each element i.i.d according to $\prod_{i=1}^{n} p(x_{2i}|q_i, u_{2i}(l))$.

Encoding: For encoder 1, to send the codeword pair (j,k), send the corresponding codeword $\mathbf{X}_1^n(j,k)$. For encoder 2, to send the codeword pair (l,m), send the corresponding codeword $\mathbf{X}_2^n(l,m)$.

Decoding: Receiver 1 determines the unique $\left(\hat{j},\hat{k}\right)$ and a $\hat{l}$ such that

$$\left(\mathbf{U}_1^n\left(\hat{j}\right), \mathbf{X}_1^n\left(\hat{j},\hat{k}\right), \mathbf{U}_2^n\left(\hat{l}\right), \mathbf{Y}_1^n\right) \in A_\epsilon^{(n)}\left(U_1, X_1, U_2, Y_1\right). \tag{68}$$

Receiver 2 determines the unique $\left(\hat{\hat{l}},\hat{m}\right)$ and a $\hat{\hat{j}}$ such that

$$\left(\mathbf{U}_2^n\left(\hat{\hat{l}}\right), \mathbf{X}_2^n\left(\hat{\hat{l}},\hat{m}\right), \mathbf{U}_1^n\left(\hat{\hat{j}}\right), \mathbf{Y}_2^n\right) \in A_\epsilon^{(n)}\left(U_2, X_2, U_1, Y_2\right). \tag{69}$$

Analysis of the Probability of Error: We consider only the decoding error of probability for receiver RX_1. The same analysis applies for receiver RX_2. By the symmetry of the random code construction, the conditional probability of error does not depend on which pair of indices is sent. Thus the conditional probability of error is the same as the unconditional probability of error. So, without loss of generality, we assume that $(j,k) = (1,1)$ and $(l,m) = (1,1)$ was sent.

We have an error if the correct codewords, $\{\mathbf{U}_1^n(1), \mathbf{X}_1^n(1,1), \mathbf{U}_2^n(1)\}$ are not jointly typical with the received sequence. If incorrect codewords $\left\{\mathbf{U}_1^n\left(\hat{j}\right), \mathbf{X}_1^n\left(\hat{j},\hat{k}\right), \mathbf{U}_2^n\left(\hat{l}\right)\right\}$ are jointly typical with the received codeword, i.e., $\hat{j} \neq 1$ or $\hat{k} \neq 1$, an error is also declared. However, no error is declared if $\left\{\mathbf{U}_1^n(1), \mathbf{X}_1^n(1,1), \mathbf{U}_2\left(\hat{l} \neq 1\right)\right\}$ are jointly typical with the received sequence. Define the following event

$$E_{jkl} = \left\{\left(\mathbf{U}_1^n(j), \mathbf{X}_1^n(j,k), \mathbf{U}_2^n(l), \mathbf{Y}_1^n\right) \in A_\epsilon^{(n)}\right\}. \tag{70}$$

Then by the union of events bound,

$$\begin{aligned}
P_e^{(n)} &= P\left(E_{111}^c \bigcup \cup_{(j,k)\neq(1,1)} E_{jkl}\right) \\
&\leq P\left(E_{111}^c\right) + \sum_{j\neq1,k=1,l=1} P\left(E_{j11}\right) + \sum_{j\neq1,k=1,l\neq1} P\left(E_{j1l}\right) + \sum_{j=1,k\neq1,l=1} P\left(E_{1k1}\right) \\
&\quad + \sum_{j=1,k\neq1,l\neq1} P\left(E_{1kl}\right) + \sum_{j\neq1,k\neq1,l=1} P\left(E_{jk1}\right) + \sum_{j\neq1,k\neq1,l\neq1} P\left(E_{jkl}\right) \\
&\leq P\left(E_{111}^c\right) + 2^{nR_{12}} 2^{-n(I(X_1;Y_1|U_2Q)-4\epsilon)} + 2^{n(R_{12}+R_{21})} 2^{-n(I(U_2X_1;Y_1|Q)-4\epsilon)} \\
&\quad + 2^{nR_{11}} 2^{-n(I(X_1;Y_1|U_1U_2Q)-4\epsilon)} + 2^{n(R_{11}+R_{21})} 2^{-n(I(U_2X_1;Y_1|U_1Q)-4\epsilon)} \\
&\quad + 2^{n(R_{11}+R_{12})} 2^{-n(I(X_1;Y_1|U_2Q)-4\epsilon)} + 2^{n(R_{11}+R_{12}+R_{21})} 2^{-n(I(U_2X_1;Y_1|Q)-4\epsilon)}.
\end{aligned} \tag{71}$$

Since $\epsilon > 0$ is arbitrary, the conditions of Theorem 3 imply that each term tends to 0 as $n \to \infty$. Refer to Appendix III for a detailed analysis of the error probabilities. The above bound shows that the average probability of error, averaged over all choices of codebooks in the random code construction, is arbitrarily small. Hence there exists at least one code $\mathcal{C}^*$ with arbitrarily small probability of error.

APPENDIX III
COMPUTATION OF THE ERROR PROBABILITY

We only consider the error probability of receiver RX_1. For $j \neq 1$, we have

$$\begin{aligned}
P(E_{j11}) &= P\left(\left(\mathbf{Q}^n, \mathbf{U}_1^n(j), \mathbf{X}_1^n(j,1), \mathbf{U}_2^n(1), \mathbf{Y}_1^n\right) \in A_\epsilon^{(n)}\right) \\
&= \sum_{\left(\mathbf{q}^n,\mathbf{u}_1^n,\mathbf{x}_1^n,\mathbf{u}_2^n,\mathbf{y}_1^n\right)\in A_\epsilon^{(n)}} p\left(\mathbf{u}_1^n, \mathbf{x}_1^n|\mathbf{q}^n\right) p\left(\mathbf{u}_2^n, \mathbf{y}_1^n|\mathbf{q}^n\right) p\left(\mathbf{q}^n\right) \\
&\leq \left|A_\epsilon^{(n)}\right| 2^{-n(H(U_1X_1|Q)-\epsilon)} 2^{-n(H(U_2Y_1|Q)-\epsilon)} 2^{-n(H(Q)-\epsilon)} \\
&\leq 2^{-n(H(U_1X_1|Q)+H(U_2Y_1|Q)+H(Q)-H(QU_1X_1U_2Y_1)-4\epsilon)} \\
&= 2^{-n(I(X_1;Y_1|QU_2)-4\epsilon)}.
\end{aligned}$$

For $j \neq 1$, $k \neq 1$ we have

$$\begin{aligned}
P(E_{jk1}) &= P\left(\left(\mathbf{Q}^n, \mathbf{U}_1^n(j), \mathbf{X}_1^n(j,k), \mathbf{U}_2^n(1), \mathbf{Y}_1^n\right) \in A_\epsilon^{(n)}\right) \\
&= \sum_{\left(\mathbf{q}^n,\mathbf{u}_1^n,\mathbf{x}_1^n,\mathbf{u}_2^n,\mathbf{y}_1^n\right)\in A_\epsilon^{(n)}} p\left(\mathbf{u}_1^n, \mathbf{x}_1^n|\mathbf{q}^n\right) p\left(\mathbf{u}_2^n, \mathbf{y}_1^n|\mathbf{q}^n\right) p\left(\mathbf{q}^n\right) \\
&\leq \left|A_\epsilon^{(n)}\right| 2^{-n(H(U_1X_1|Q)-\epsilon)} 2^{-n(H(U_2Y_1|Q)-\epsilon)} 2^{-n(H(Q)-\epsilon)} \\
&\leq 2^{-n(H(U_1X_1|Q)+H(U_2Y_1|Q)+H(Q)-H(QU_1X_1U_2Y_1)-4\epsilon)} \\
&= 2^{-n(I(X_1;Y_1|QU_2)-4\epsilon)}.
\end{aligned}$$

For $k \neq 1$ we have

$$\begin{aligned}
P(E_{1k1}) &= P\left(\left(\mathbf{Q}^n, \mathbf{U}_1^n(1), \mathbf{X}_1^n(1,k), \mathbf{U}_2^n(1), \mathbf{Y}_1^n\right) \in A_\epsilon^{(n)}\right) \\
&= \sum_{\left(\mathbf{q}^n,\mathbf{u}_1^n,\mathbf{x}_1^n,\mathbf{u}_2^n,\mathbf{y}_1^n\right)\in A_\epsilon^{(n)}} p\left(\mathbf{x}_1^n|\mathbf{q}^n\mathbf{u}_1^n\right) p\left(\mathbf{u}_2^n\mathbf{y}_1^n|\mathbf{q}^n\mathbf{u}_1^n\right) p\left(\mathbf{q}^n\mathbf{u}_1^n\right) \\
&\leq \left|A_\epsilon^{(n)}\right| 2^{-n(H(X_1|QU_1)-\epsilon)} 2^{-n(H(U_2Y_1|QU_1)-\epsilon)} 2^{-n(H(QU_1)-\epsilon)} \\
&\leq 2^{-n(H(X_1|QU_1)+H(U_2Y_1|QU_1)+H(QU_1)-H(QU_1X_1U_2Y_1)-4\epsilon)} \\
&= 2^{-n(I(X_1;Y_1|QU_1U_2)-4\epsilon)}.
\end{aligned}$$

For $j \neq 1$, $l \neq 1$ we have

$$\begin{aligned}
P(E_{j1l}) &= P\left(\left(\mathbf{Q}^n, \mathbf{U}_1^n(j), \mathbf{X}_1^n(j,1), \mathbf{U}_2^n(l), \mathbf{Y}_1^n\right) \in A_\epsilon^{(n)}\right) \\
&= \sum_{\left(\mathbf{q}^n,\mathbf{u}_1^n,\mathbf{x}_1^n,\mathbf{u}_2^n,\mathbf{y}_1^n\right)\in A_\epsilon^{(n)}} p\left(\mathbf{u}_1^n\mathbf{x}_1^n\mathbf{u}_2^n|\mathbf{q}^n\right) p\left(\mathbf{y}_1^n|\mathbf{q}^n\right) p\left(\mathbf{q}^n\right) \\
&\leq \left|A_\epsilon^{(n)}\right| 2^{-n(H(U_1X_1U_2|Q)-\epsilon)} 2^{-n(H(Y_1|Q)-\epsilon)} 2^{-n(H(Q)-\epsilon)} \\
&\leq 2^{-n(H(U_1X_1U_2|Q)+H(Y_1|Q)+H(Q)-H(QU_1X_1U_2Y_1)-4\epsilon)} \\
&\leq 2^{-n(I(X_1U_2;Y_1|Q)-4\epsilon)}.
\end{aligned}$$

For $j \neq 1$, $k \neq 1$, $l \neq 1$ we have

$$\begin{aligned}
P(E_{jkl}) &= P\left(\left(\mathbf{Q}^n, \mathbf{U}_1^n(j), \mathbf{X}_1^n(j,k), \mathbf{U}_2^n(l), \mathbf{Y}_1^n\right) \in A_\epsilon^{(n)}\right) \\
&= \sum_{\left(\mathbf{q}^n,\mathbf{u}_1^n,\mathbf{x}_1^n,\mathbf{u}_2^n,\mathbf{y}_1^n\right)\in A_\epsilon^{(n)}} p\left(\mathbf{u}_1^n\mathbf{x}_1^n\mathbf{u}_2^n|\mathbf{q}^n\right) p\left(\mathbf{y}_1^n|\mathbf{q}^n\right) p\left(\mathbf{q}^n\right) \\
&\leq \left|A_\epsilon^{(n)}\right| 2^{-n(H(U_1X_1U_2|Q)-\epsilon)} 2^{-n(H(Y_1|Q)-\epsilon)} 2^{-n(H(Q)-\epsilon)} \\
&\leq 2^{-n(H(U_1X_1U_2|Q)+H(Y_1|Q)+H(Q)-H(QU_1X_1U_2Y_1)-4\epsilon)}
\end{aligned}$$

$$= 2^{-n(I(X_1 U_2;Y_1|Q)-4\epsilon)}.$$

For $k \neq 1$, $l \neq 1$ we have

$$\begin{aligned}
&P\left(E_{1kl}\right) = P\left(\left(\mathbf{Q}^n, \mathbf{U}_1^n\left(1\right), \mathbf{X}_1^n\left(1,k\right), \mathbf{U}_2^n\left(l\right), \mathbf{Y}_1^n\right) \in A_\epsilon^{(n)}\right) \\
&= \sum_{\left(\mathbf{q}^n,\mathbf{u}_1^n,\mathbf{x}_1^n,\mathbf{u}_2^n,\mathbf{y}_1^n\right)\in A_\epsilon^{(n)}} p\left(\mathbf{x}_1^n \mathbf{u}_2^n|\mathbf{q}^n \mathbf{u}_1^n\right) p\left(\mathbf{y}_1^n|\mathbf{q}^n \mathbf{u}_1^n\right) p\left(\mathbf{q}^n \mathbf{u}_1^n\right) \\
&\leq \left|A_\epsilon^{(n)}\right| 2^{-n(H(X_1 U_2|QU_1)-\epsilon)} 2^{-n(H(Y_1|QU_1)-\epsilon)} 2^{-n(H(QU_1)-\epsilon)} \\
&= 2^{-n(H(X_1 U_2|QU_1)+H(Y_1|QU_1)+H(QU_1)-H(QU_1 X_1 U_2 Y_1)-4\epsilon)} \\
&\leq 2^{-n(I(X_1 U_2;Y_1|QU_1)-4\epsilon)}.
\end{aligned}$$

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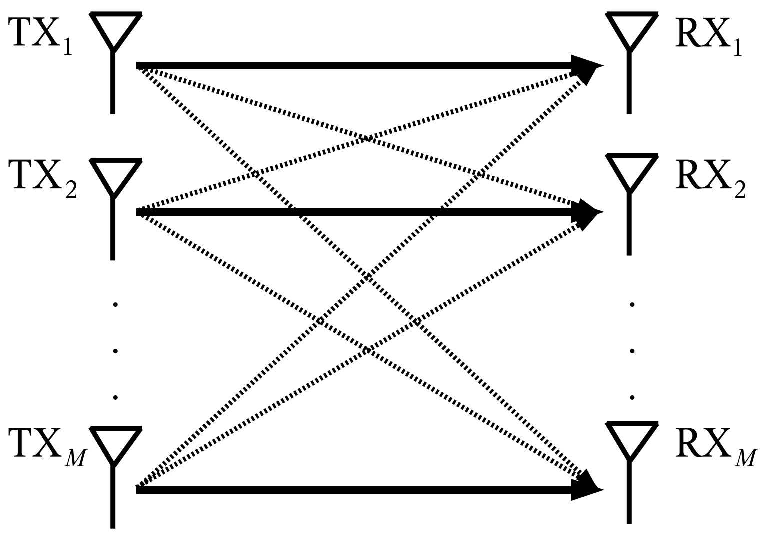

Fig. 1. A M-user interference channel

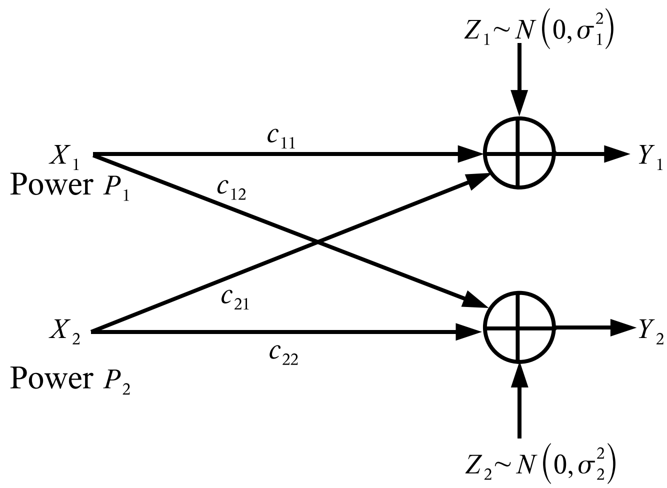

Fig. 2. The Gaussian interference channel

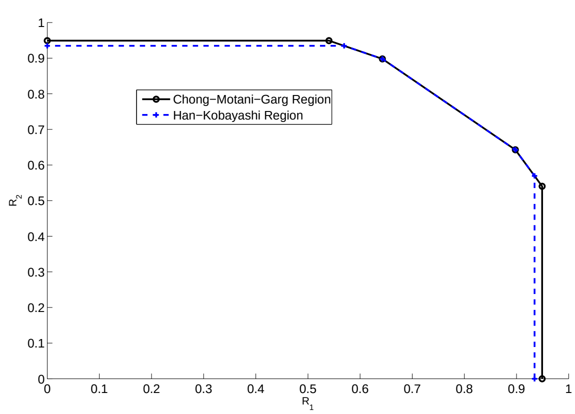

Fig. 3. An example where $\mathcal{R}_{\rm HK}\left(P_1^*\right) \subsetneq \mathcal{R}_{\rm CMG}\left(P_1^*\right)$