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Formulation of Richardson’s Model of Arms Race from a Differential Game Viewpoint

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I. INTRODUCTION

In his book [1] on arms race, Lewis Fry Richardson proposed a dynamic model describing the phenomenon of armament buildup between two nations. The model is based on the following argument: If \( x_1(t) \) and \( x_2(t) \) are measures of the armament levels of two countries, then for each nation its armament level will tend

(i) to increase proportionately to the armament level of the other nation;
(ii) to decrease proportionately to the economic burden corresponding to its own armament; and
(iii) to increase guided by its grievances and hatred towards the other nation.

In mathematical terms the model is described by the set of linear differential equations

\[
\begin{align*}
\dot{x}_1 &= \sigma x_2 - \alpha x_1 + g \\
\dot{x}_2 &= \rho x_1 - \gamma x_2 + h
\end{align*}
\]

where \( \sigma \) and \( \rho \) are called "defence coefficients", \( \alpha \) and \( \gamma \) "fatigue coefficients" and \( g \) and \( h \) "grievance coefficients".

McGuire [2], and more recently Brito [3], considered a model in which the economic as well as the political aspects of arms race are considered in a single framework of resource allocation. The model proposed in [3] is dynamic in nature and considers two nations whose armament levels (or stocks) \( x_1(t) \) and \( x_2(t) \) satisfy the differential equations

\[
\begin{align*}
\dot{x}_1 &= -\beta_1 x_1 + Z_1 \\
\dot{x}_2 &= -\beta_2 x_2 + Z_2,
\end{align*}
\]

where \( Z_1(t) \) and \( Z_2(t) \) are the expenditures on weapons at time \( t \) by each nation and \( \beta_1 x_1 \) and \( \beta_2 x_2 \) \((\beta_1 \geq 0\) and \( \beta_2 \geq 0\)) are the resources necessary to maintain and operate the weapons \( x_1 \) and \( x_2 \) respectively. This model also assumes that each nation determines its weapons expenditures \( Z_i \) in such a way as to maximize a total objective function of the form

\[
J_i = \int_0^\infty e^{-\gamma t} U_i(Z_i, D_i(x_1, x_2)) dt, \quad i = 1, 2,
\]

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where $r$ is the discount rate, $D_i$ is the $i$th country's index of defence and $U_i$ is the instantaneous utility of the $i$th country. Each nation does not have complete freedom in the selection of its weapons expenditures since $Z_i$ must not exceed the net national product $Y_i$. This constraint is expressed by the equation:

$$Z_i + C_i = Y_i, \quad i = 1, 2,$$

...(6)

where $C_i$ is the consumption of the $i$th nation. These variables and several assumptions justifying the validity of the model are described in detail in [3] and will not be repeated here.

For the infinite horizon problem, equation (5), it is usually of interest to examine whether the problem has an equilibrium point and whether this equilibrium point is stable. In [3] this was done from a purely optimal control point of view where the equilibrium point was defined on the assumption that each nation maximizes its objective function against a given fixed value of armament level of the other nation. In other words, it is assumed that for a fixed $\bar{x}_j$ for nation $j$, nation $i$ computes its best weapons expenditure policy $Z_i$ by maximizing $J_i$ when $x_j$ is fixed at $\bar{x}_j$. In a similar manner, nation $j$ is assumed to compute its best $Z_j$ by maximizing $J_j$ when $x_i$ is fixed at $\bar{x}_i$. The equilibrium point is then defined as the solution of $F_1(\bar{x}_1, \bar{x}_2) = F_2(\bar{x}_1, \bar{x}_2) = 0$ where

$$\dot{x}_1 = F_1(x_1, x_2)$$
$$\dot{x}_2 = F_2(x_1, x_2)$$

are the equations of motion when the optimal policies are substituted in (3)-(4). Thus these are steady-state equilibrium points.

In this paper, a different approach for solving this problem is taken. Naturally because of the presence of two performance indices $J_1$ and $J_2$ each depending \(^1\) on $Z_1$ and $Z_2$, the above model (3)-(6) is best formulated and solved within the framework of differential game theory rather than optimal control theory. The equilibrium solution considered in this paper is different in nature from the one considered in [3]. It is more realistic because it takes into consideration the element of competition that usually exists in a problem of this nature. We shall assume that the nations are seeking weapons expenditure policies satisfying the well-known Nash equilibrium solution concept of game theory [4]. The Nash solution is an equilibrium point in the sense that a unilateral attempt on the part of one nation to deviate will not increase its objective function. In order to stress the difference with [3], this paper will concentrate on a finite horizon problem where the results of [3] are not applicable. The infinite horizon case then follows immediately by taking the limit as the terminal time approaches infinity. The equilibrium solution obtained will still be different from that of [3]. In this paper we distinguish between two types of control structures: closed-loop (or feedback) controls of the form $Z(t, x_1(t), x_2(t))$ and open-loop controls of the form $Z_i(t)$. It is well known [5] that the Nash optimal trajectories for these two forms are generally different. This distinction is very important for this class of problems since a meaningful solution will have to be of the closed-loop form. The necessary conditions for optimality for the closed-loop solution are presented and it is shown that if the objective functions (5) are quadratic, then the resulting optimal closed-loop model will be of the Richardson type. The "defence", "fatigue" and "hatred" coefficients are then easily determined from the optimality conditions (which will be a set of differential equations in this case) and naturally are a function of the various weighting factors in the objective functions. Some results with regards to the stability of the resulting Richardson model for the infinite horizon problem are also presented.

\(^1\) $J_1$ depends implicitly on $Z_2$ through $x_2$ and $J_2$ depends implicitly on $Z_1$ through $x_1$. 
2. FORMULATION OF THE PROBLEM AS A DIFFERENTIAL GAME

Roughly speaking, a differential game is a system whose states are described by a set of differential equations and where there is more than one controller each trying to maximize a different objective (or utility) function. In our case the states of the system are the weapons stocks $x_1$ and $x_2$ and the controls are the weapons expenditures $Z_1$ and $Z_2$. The differential equations relating the controls $i$ to the states are given in (3)-(4) and we shall start by considering utility functions defined over a finite horizon, of the form

$$J_i(Z_1, Z_2) = \int_0^{t_f} e^{-\alpha t} U_i(Z_0, D_i(x_1, x_2)) dt, \quad i = 1, 2,$$

where $t_f$ is finite and fixed. The case of infinite horizon can then be treated by taking the limit as $t_f \to \infty$ and by ensuring that (7) converges to a finite value.

The first question that arises, when attempting to solve the above problem is: how is optimality defined? Since each objective function depends on both $Z_1$ and $Z_2$ it is no longer sufficient to say that each nation wants to maximize its objective function. A more precise definition of optimality is needed. Several such definitions, sometimes referred to as rationales, have been studied in the differential games literature. Among these we mention the Pareto, Nash, Stackelberg and Minimax solutions [5-9]. These definitions, in fact, reflect the various ways the nations may react to each other's armament builds in order to maximize their objective functions. For instance, if cooperation is present then a Pareto solution, which is arrived at by negotiation and which is better for the other nations than any other solution in its neighborhood, may be desirable. However, guarantees must exist to make sure that this solution is enforced on both parties. On the other hand, if cooperation is not present, and if both nations are at the same level of strength, then the Nash solution, which is secure against any attempt by any one nation to unilaterally alter its strategy, may be preferable. If, however, one nation is stronger than the other and is capable of assuming the position of a leader by imposing on the other a solution which is more favourable to itself, then the Stackelberg solution may be more desirable. Finally, if each nation is pessimistic to the extent of assuming that the other nation is trying to oppose it rather than maximize its own objective function, then a minimax solution would be more natural. Among all these solution rationales, only the Nash solution is considered in this paper. We therefore assume that the two nations do not cooperate and that they are both at the same level of strength.

Once the rationale has been selected, there are two forms according to which each nation can obtain its control. On the one hand, $Z_i$ can be chosen in open-loop form, as a function of time only (i.e. $Z_i(t)$), which means that each nation, at time $t = 0$, precalculates the time history of its armament expenditure and commits itself to this function for all future times. On the other hand $Z_i$ can be chosen in closed-loop form as a function of time and of the states of the system (i.e. $Z_i(t, x_1(t), x_2(t))$); which means that each nation, at time $t = 0$, precalculates how its armaments expenditures should depend on the weapons stocks of both nations for all future times. In this case the actual amounts of armament expenditures $Z_i$ at time $t$ will be determined only after the values of the weapons stocks $x_1(t)$ and $x_2(t)$ at time $t$ are known.

It is well known that, as a result of Bellman's principle of optimality [10], the open-loop and closed-loop controls coincide in optimal control problems. However, as was shown in [6], this result does not generally carry through to differential game problems. The reason for this difference is clearly related to the structure of information available

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1 For the rest of this paper, the terms controls and weapons expenditures will be used interchangeably. Similarly for the terms states and weapons stocks.

2 The Nash solution is also known as the Cournot or Equilibrium solution.
to each player [6-8]. This point is of central importance here because the nature of the problem necessitates that the optimal weapons expenditure be generated in closed-loop form. It is very natural to expect a nation to determine its weapons expenditures based on its rival's armament level and its own weapons stock at any given point in time. Very seldom does a nation commit itself to a predetermined armament expenditure policy (as in the open-loop case). For this reason, only the closed-loop Nash solution will be considered in this paper. Its mathematical definition is:

**Definition.** If there exist controls \( Z_i^*(t, x_i(t), x_2(t)) \) and \( Z_2^*(t, x_1(t), x_2(t)) \) such that

\[
J_1[Z_i^*(t, x_i(t), x_2(t)), Z_2^*(t, x_1(t), x_2(t))] \geq J_1[Z_i(t, x_i(t), x_2(t)), Z_2(t, x_1(t), x_2(t))] \quad \ldots (8)
\]

\[
J_2[Z_i^*(t, x_i(t), x_2(t)), Z_2^*(t, x_1(t), x_2(t))] \geq J_2[Z_i(t, x_i(t), x_2(t)), Z_2(t, x_1(t), x_2(t))] \quad \ldots (9)
\]

then the pair \( \{Z_i^*(t, x_i(t), x_2(t)), Z_2^*(t, x_1(t), x_2(t))\} \) is called a closed-loop Nash solution of the problem.

Thus by adopting a Nash strategy, each nation safeguards itself against any attempts by the other nation to deviate from its Nash policy (i.e. to cheat) in order to improve its total performance. In fact, if one nation holds fast to its Nash policy then the other nation cannot find a better policy than its Nash policy. It is also implicit in this equilibrium concept that each nation is assumed to know that the other nation is also using a Nash strategy.

3. NECESSARY CONDITIONS FOR THE EXISTENCE OF A NASH SOLUTION

Before we derive the necessary conditions for the existence of a closed-loop Nash solution, several simplifying assumptions concerning the control functions \( Z_i(t) \) will be made. First, we will avoid the equality constraints of (6) by assuming that there is an upper bound on the weapons expenditures for each nation, which is its net national product. That is, we shall assume that

\[
Z_i(t) \leq Y_i(t), \quad i = 1, 2. \quad \ldots (10)
\]

Note that the above inequality allows \( Z_i(t) \) to take negative values. If this happens, for example for nation \( i \), then it can be interpreted as a situation where this nation is transforming its weapons for peaceful usage thus increasing its consumption beyond the net national product. Second, in order to avoid unnecessary technical details, it will be assumed that the utility functions \( U_i \) are strictly concave in \( Z_i \) and that the "optimal" solution is always in the interior \(^1\) of the admissible set (10). In order to determine the necessary conditions, assume that nation \( j \) \(^2\) has determined its \( Z_j^*(t, x_1(t), x_2(t)) \), then nation \( i \) will be faced with an ordinary maximization problem (8)-(9) which can be solved in terms of the Hamiltonian:

\[
H_i = e^{-\alpha t} U_i[Z_i, D_i(x_1, x_2)] + p_{11}(-\beta_1 x_1 + Z_1) + p_{12}(-\beta_2 x_2 + Z_2). \quad \ldots (11)
\]

It is well known \(^3\) that the maximizing \( Z_i^*(t, x_1(t), x_2(t)) \) must satisfy the conditions

\[
\dot{p}_{11} = -\frac{\partial H_i}{\partial x_1} = -e^{-\alpha t} \frac{\partial U_i}{\partial D_i} \frac{\partial D_i}{\partial x_1} + \beta_1 p_{11} - \frac{\partial Z_j}{\partial x_1} p_{1j}, \quad p_{11}(t_f) = 0 \quad \ldots (12)
\]

\[
\dot{p}_{12} = -\frac{\partial H_i}{\partial x_2} = -e^{-\alpha t} \frac{\partial U_i}{\partial D_i} \frac{\partial D_i}{\partial x_2} + \beta_2 p_{12} - \frac{\partial Z_j}{\partial x_2} p_{1j}, \quad p_{12}(t_f) = 0 \quad \ldots (13)
\]

\(^1\) If this assumption is violated and the optimal solution happens to be on the boundary of (10), then the Kuhn-Tucker techniques can be used. However, since this is not a central point in this paper, and for the sake of simplicity, it will be avoided.

\(^2\) Here, we assume that if \( j = 1 \) then \( i = 2 \) and if \( j = 2 \) then \( i = 1 \).
and
\[ 0 = \frac{\partial H_i}{\partial Z_i} = e^{-\eta} \frac{\partial U_i}{\partial Z_i} + p_i. \]  
...(14)

Reciprocally, similar conditions can be derived for \( Z_j^i(t, x_1(t), x_2(t)) \) when the control of nation \( i \) is \( Z_j^i(t, x_1(t), x_2(t)) \). In conclusion, the closed-loop Nash solution
\[ \{Z_1^1(t, x_1(t), x_2(t)), Z_2^1(t, x_1(t), x_2(t))\} \]
must therefore satisfy conditions (12)-(14) simultaneously with \( i = 1, 2 \). By defining new adjoint variables \( q_{ij} = \mu_i e^{\lambda t} \), (12)-(14) can be written in the following more compact form:
\[ q_{1j} = - \frac{\partial U_1}{\partial D_1} \frac{\partial D_1}{\partial x_i} + (\beta_j + r)q_{1j} - \frac{\partial Z_2}{\partial x_j} q_{12}, \quad q_{1j}(t_f) = 0, \quad j = 1, 2 \]  
...(15)

\[ q_{2j} = - \frac{\partial U_2}{\partial D_2} \frac{\partial D_2}{\partial x_i} + (\beta_j + r)q_{2j} - \frac{\partial Z_1}{\partial x_j} q_{21}, \quad q_{2j}(t_f) = 0, \quad j = 1, 2 \]  
...(16)

\[ q_{ii} = - \frac{\partial U_i}{\partial Z_i}, \quad i = 1, 2. \]  
...(17)

Note that the terms \( \partial Z_2/\partial x_j \) and \( \partial Z_1/\partial x_j \) account for the closed-loop nature of the controls. Equations (15)-(17) form a set of partial differential equations which is also a two-point boundary value problem and is generally very difficult to solve. Nevertheless, the solution of these equations \( Z_1^i(t, x_1, x_2) \) and \( Z_2^i(t, x_1, x_2) \) and equations (3)-(4) produces a system of the form:
\[ \dot{x}_1 = F_1(x_1, x_2, t) \]  
...(18)

\[ \dot{x}_2 = F_2(x_1, x_2, t). \]  
...(19)

These generally non-linear differential equations, even though they may have an interpretation similar to Richardson's model, are difficult to analyse. It is therefore of interest if the following question can be answered. Is there a class of objective functions for which the Nash optimal system (18)-(19) takes the form of Richardson's model? This question is of significant practical importance because it provides another approach to the problem of determining the coefficients when the armament variations of a two-nation system is modelled by Richardson's equations. These coefficients are generally not known for future times; however, they can be the result of an optimization (game) problem which is taking place between the two nations.

We shall answer the previous question only partially, by claiming that if the objective functions are quadratic functions of the states and the controls then (18)-(19) will reduce to a set of linear differential equations as in Richardson's model (1)-(2), except that the coefficients are time varying. Furthermore, if \( t_f \to \infty \), then these coefficients will reduce to constants that can be obtained from the solution of a set of nonlinear algebraic equations. This will be done in the following section.

4. SOLUTION WITH QUADRATIC OBJECTIVE FUNCTIONS

In this section, quadratic objective functions of the following form will be considered:
\[ J_1 = \frac{1}{2} \int_0^{t_f} e^{-\eta}[-R_1(W_1 - Z_1)^2 - Q_1(x_1 - a_1x_2 - v_1)^2] dt \]  
...(20)

\[ J_2 = \frac{1}{2} \int_0^{t_f} e^{-\eta}[-R_2(W_2 - Z_2)^2 - Q_2(x_2 - a_2x_1 - v_2)^2] dt, \]  
...(21)
where \( R_1 \) and \( R_2 \) are strictly positive real numbers; \( Q_1, Q_2, a_1 \) and \( a_2 \) are non-negative real numbers and \( W_1 \) and \( W_2 \) are such that \( W_1 < Y_1 \) and \( W_2 < Y_2 \). Quadratic objective functions have been widely used in the application of control theory to economic systems [13]. The interpretation of these objective functions is that each nation wants to narrow the gap between its armament level and a linear function of its opponent's armament level while at the same time minimizing its armament expenditures. There are two points, however, that need to be examined in order to correctly interpret these objective functions. First, the choice of \( W_i \) must be such that for fixed \( x_1 \) and \( x_2 \), \( U_i \) decreases if \( Z_i \) increases in the region of interest. This situation is clearly illustrated in Figure 1; and as a result it seems logical for each nation to select a negative value for \( W_i \) and expect that \( Z_i^* \) will always be to the right of \( W_i \). If that is the case, then this objective function will be meaning-
ful in the region of interest \((W_i < Z_j < Y_j)\), if not then a different value of \(W_i\) must be chosen. Second, the indices of defence \(D_i = x_i - a_i x_j - v_i\) are plotted on the \((x_1, x_2)\) plane in Figure 2. The quantity \(v_i\) is the desired armament level of nation \(i\) when the armament level of nation \(j\) is zero. These lines give the most desirable armament level for each nation for any given armament level of the other nation; and hence, the interpretation of the defence part of the objective functions is that at each instant of time, nation \(i\) is trying to attract point \((x_1, x_2)\) towards line \(D_i\) while nation \(2\) is trying to attract it towards line \(D_2\). In other words, the line \(x_i = a_i x_j + v_i\) is the desired target level of nation \(i\) given \(x_j\). This is illustrated in Figure 2.

Applying the necessary conditions (15)-(17), we get:

\[
\dot{q}_{11} = Q_1(x_1 - a_1 x_2 - v_1) + (\beta_1 + r) q_{11} - \frac{\partial Z_1}{\partial x_1} q_{12}, \quad q_{11}(t_f) = 0 \quad \text{(22)}
\]

\[
\dot{q}_{12} = -a_1 Q_1(x_1 - a_1 x_2 - v_1) + (\beta_2 + r) q_{12} - \frac{\partial Z_2}{\partial x_2} q_{12}, \quad q_{12}(t_f) = 0 \quad \text{(23)}
\]

\[
\dot{q}_{21} = -a_2 Q_2(x_2 - a_2 x_1 - v_2) + (\beta_1 + r) q_{21} - \frac{\partial Z_1}{\partial x_1} q_{21}, \quad q_{21}(t_f) = 0 \quad \text{(24)}
\]

\[
\dot{q}_{22} = Q_2(x_2 - a_2 x_1 - v_2) + (\beta_2 + r) q_{22} - \frac{\partial Z_2}{\partial x_2} q_{22}, \quad q_{22}(t_f) = 0 \quad \text{(25)}
\]

\[
q_{11} = R_1(Z_1 - W_i) \quad \text{and} \quad q_{22} = R_2(Z_2 - W_j). \quad \text{(26)}
\]

The above system of differential equations can be easily solved for

\[Z^*_i(t, x_1(t), x_2(t)), \quad i = 1, 2,\]

by introducing the transformations:

\[
q_{11} = -K_{11} x_1 - K_{12} x_2 - F_1, \quad q_{21} = -E_{11} x_1 - E_{12} x_2 - G_1, \quad \text{(27)}
\]

\[
q_{12} = -K_{21} x_1 - K_{22} x_2 - F_2, \quad q_{22} = -E_{21} x_1 - E_{22} x_2 - G_2. \quad \text{(28)}
\]

Then (26) gives

\[Z^*_1 = -\frac{K_{11}}{R_1} x_1 - \frac{K_{12}}{R_1} x_2 - \left(\frac{F_1}{R_1} - W_i\right) \quad \text{(29)}
\]

and

\[Z^*_2 = -\frac{E_{21}}{R_2} x_1 - \frac{E_{22}}{R_2} x_2 - \left(\frac{G_2}{R_2} - W_2\right) \quad \text{(30)}
\]

and when (27)-(30) are substituted in (22)-(25), a set of differential equations, with boundary conditions at \(t_f\), that the unknowns \(K_{ij}\), \(E_{ij}\), \(F_i\) and \(G_i\) must satisfy can be easily derived.\(^1\)

Thus with the closed-loop controls (29)-(30), the dynamic behaviour of the armament levels (3)-(4) will then follow the differential equations

\[
\dot{x}_1 = -\left(\beta_1 + \frac{K_{11}}{R_1}\right) x_1 - \frac{K_{12}}{R_1} x_2 - \left(\frac{F_1}{R_1} - W_1\right) \quad \text{(31)}
\]

\[
\dot{x}_2 = -\left(\beta_2 + \frac{E_{22}}{R_2}\right) x_1 - \frac{E_{22}}{R_2} x_2 - \left(\frac{G_2}{R_2} - W_2\right). \quad \text{(32)}
\]

Comparing these equations with (1)-(2), the Richardson's coefficients are now easily identifiable. However, it should be noted at this stage that the fact that the coefficients

\(^1\) For the sake of brevity, these equations are omitted from the paper.
in (31)-(32) arc time varying is not a restriction, but rather a result of the constraint that $t_f$ was chosen to be finite.

We now briefly discuss the infinite horizon case ($t_f \rightarrow \infty$). A possible requirement in this case is to ensure that the objective functions in (7) with $t_f \rightarrow \infty$ are finite.\footnote{If the objective functions are allowed to tend to infinity, then the overtaking principle to order alternative paths [13] can be used and conditions (33) will no longer be necessary.} It is well known that this can be achieved with the same necessary conditions (15)-(17) except [11] that the boundary conditions should be changed to:

$$
limit_{t \rightarrow \infty} e^{-rt}q_{i1}(t)x_i(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow 0} e^{-rt}q_{i2}(t)x_i(t) = 0 \quad \ldots (33)$$

for $j = 1, 2$. In the linear quadratic case, this is reflected in the parameters $K_{ii}$, $E_{ii}$, $F_i$ and $G_i$ in (29)-(30) becoming constants, that can be obtained by solving a set of non-linear algebraic equations (see Appendix). In this case, for (33) to be satisfied, it is sufficient to ensure that the equilibrium point $(x_{1e}, x_{2e})$ of (31)-(32) is asymptotically stable. The following proposition gives some conditions for this to be achieved.

**Proposition 1.** If the parameters $K_{ii}$, $E_{ii}$, $F_i$ and $G_i$ in (29)-(30) satisfy either

(a) $K_{11} > 0$ and $K_{11}K_{22} - K_{12}^2 > 0$

or

(b) $E_{11} > 0$ and $E_{11}E_{22} - E_{12}^2 > 0$

then the equilibrium point $(x_{1e}, x_{2e})$ of the infinite horizon Richardson's model (31)-(32) will be asymptotically stable.

**Proof.** The proof is given in the Appendix.

Thus the above proposition suggests that the closed-loop Nash controls are obtained from a solution of the non-linear algebraic equations (see Appendix) satisfying either condition (a) or (b). The existence of such a solution, however, is not yet guaranteed, and it remains as a problem for future research. We also note it is obvious, from the way it is constructed, that this infinite horizon equilibrium point is different from that of [2].

An interesting property of the infinite horizon solution is that the resulting Richardson’s model has constant, rather than time varying, coefficients. In a real system, however, it is not possible to expect that this model will describe the armament levels of the two nations for a long period of time. This is so because it is not possible to assume that the armament expenditure policies will be kept the same, long after an equilibrium situation has been reached (unless, perhaps, if this equilibrium situation is total disarmament, i.e. the origin). This new situation can still be accommodated in this model. For instance, if, after an equilibrium level of armament $(x_{1e}, x_{2e})$ has been reached, the two nations desire to change their policies (for example to further reduce their arms stocks, in a disarmament situation), then a new infinite horizon problem can be solved based on different weighting factors, in the quadratic objective functions, that are more suitable for the new situation. The new expenditure policies will then produce a new Richardson’s model which will better describe the armament (or disarmament) race for the period of time until the new equilibrium point is reached.

6. CONCLUSIONS

In this paper, the problem of armament buildup between two nations, each seeking an armament expenditure policy to maximize its objective function, has been formulated as a differential game problem, and the Nash strategy has been suggested as a possible solution concept for the problem. It was also shown that if the objective functions of both nations are quadratic, then the closed-loop Nash policies reduce the system to the same structure as the well known Richardson’s model of arms race. This formulation is therefore suggested
as a possible procedure for determining the coefficients in Richardson's model based on a choice of the parameters in the quadratic objective functions. Some results concerning the stability of the model for an infinite horizon problem have also been pointed out.

APPENDIX

For the infinite horizon case, a possible solution candidate for equations (22)-(26) with boundary conditions (33) is the pair of armament policies given by (29)-(30) with the parameters $K_{ij}$, $E_{ij}$, $F_i$ and $G_i$ constants. Upon substituting (27)-(30) in (22)-(25), it can be easily shown that these parameters must satisfy $K_{12} = K_{21}$ and $E_{12} = E_{21}$ and that they can be obtained by solving the following set of algebraic equations:

\[
(2\beta_1 + r)K_{11} + \frac{K_{11}^2}{R_1} + 2 \frac{K_{12}E_{12}}{R_2} - Q_1 = 0 \quad \cdots (A.1)
\]

\[
(\beta_1 + \beta_2 + r)K_{12} + \frac{E_{12}K_{22}}{R_2} - \frac{K_{11}K_{12}}{R_1} + a_1Q_1 = 0 \quad \cdots (A.2)
\]

\[
(2\beta_2 + r)K_{22} + \frac{K_{22}^2}{R_1} + 2 \frac{E_{22}K_{22}}{R_2} - a_2^2Q_1 = 0 \quad \cdots (A.3)
\]

\[
(2\beta_1 + r)E_{11} + \frac{E_{11}^2}{R_2} + 2 \frac{E_{12}K_{11}}{R_1} - a_2^2Q_2 = 0 \quad \cdots (A.4)
\]

\[
(\beta_1 + \beta_2 + r)E_{12} + \frac{E_{11}E_{11}}{R_1} + \frac{E_{12}E_{22}}{R_2} + \frac{E_{12}K_{12}}{R_1} + a_2Q_2 = 0 \quad \cdots (A.5)
\]

\[
(2\beta_2 + r)E_{22} + \frac{E_{22}^2}{R_1} + 2 \frac{E_{12}K_{12}}{R_1} - Q_2 = 0 \quad \cdots (A.6)
\]

\[
\left(\beta_1 + r + \frac{K_{11}}{R_1}\right)F_1 + \frac{E_{12}}{R_2}F_2 + \frac{K_{12}}{R_2}G_2 - K_{11}W_1 - K_{12}W_2 - Q_1v_1 = 0 \quad \cdots (A.7)
\]

\[
\left(\beta_2 + r + \frac{E_{22}}{R_2}\right)F_2 + \frac{K_{12}}{R_1}F_1 + \frac{K_{22}}{R_2}G_2 - K_{12}W_1 - K_{22}W_2 + a_1Q_1v_1 = 0 \quad \cdots (A.8)
\]

\[
\left(\beta_1 + r + \frac{K_{11}}{R_1}\right)G_1 + \frac{E_{11}}{R_1}F_1 + \frac{E_{12}}{R_2}G_2 - E_{11}W_1 - E_{12}W_2 + a_2Q_2v_2 = 0 \quad \cdots (A.9)
\]

\[
\left(\beta_2 + r + \frac{E_{22}}{R_2}\right)G_2 + \frac{E_{12}}{R_1}F_1 + \frac{K_{12}}{R_1}G_1 - E_{12}W_1 - E_{22}W_2 - Q_2v_2 = 0 \quad \cdots (A.10)
\]

Since the above equations may have more than one solution, the purpose of Proposition 1 is to give sufficient conditions in order for a solution of (A.1)-(A.10) to lead to an asymptotically stable closed-loop system (and hence satisfy (33)). We shall prove Proposition 1 by constructing an appropriate Lyapunov function [14] for the system. Assume conditions (a) are satisfied, and consider the following function

\[ V(t, x_1, x_2) = e^{-\eta}(K_{11}x_1^2 + 2K_{12}x_1x_2 + K_{22}x_2^2). \]

It is clear that under conditions (a) this function is positive definite. Now

\[
\frac{dV}{dt} = -re^{-\eta}(K_{11}x_1^2 + 2K_{12}x_1x_2 + K_{22}x_2^2) + e^{-\eta}(2K_{11}x_1\dot{x}_1 + 2K_{12}x_1\dot{x}_2 + 2K_{12}x_1x_2 + 2K_{22}x_2\dot{x}_2),
\]
which, when evaluated along the trajectory (31)-(32) and after making use of (A.1)-(A.10) and omitting unnecessary algebraic manipulations, reduces to
\[ \frac{dV}{dt} = -\left( Q_1 + \frac{K_{11}^2}{R_1} \right) x_1^2 e^{-n} - \left( a_1^2 Q_1 + \frac{K_{12}^2}{R_1} \right) x_2^2 e^{-n} + 2a_1 Q_1 x_1 x_2 e^{-n}. \]

In order to show that \( dV/dt \) is negative definite, we need only show that the following two conditions are satisfied:

1. \( Q_1 + \frac{K_{11}^2}{R_1} > 0 \)

and

2. \( \left( Q_1 + \frac{K_{11}^2}{R_1} \right) (a_1^2 Q_1 + \frac{K_{12}^2}{R_1}) - a_1^2 Q_1^2 = \frac{K_{11}^2 K_{12}^2}{R_1^2} + Q_1 \frac{K_{12}^2}{R_1} + a_1^2 Q_1 \frac{K_{11}^2}{R_1} > 0. \)

Condition (1) follows from (a) and the assumption that \( Q_1 \geq 0 \). It is obvious that the expression in (2) is non-negative and therefore we need only show that it is not equal to zero. This can be shown by contradiction. Suppose it is equal to zero, then this can happen only if

(i) \( K_{11} = 0 \) and \( Q_1 = 0 \)

or

(ii) \( K_{12} = 0 \) and \( a_1 = 0. \)

Case (i) implies that (A.1) reduces to
\[ K_{11}^2 + R_1(2\beta_1 + r)K_{11} = 0, \]
which has \( K_{11} = 0 \) or \( K_{11} = -R_1(2\beta_1 + r) \) as solutions. Since \( \beta_1 \geq 0 \), both solutions contradict condition (a).

Case (ii) implies that (A.3) and (A.6) reduce to
\[ \left( 2\beta_2 + r + \frac{4E_{22}}{R_2} \right) K_{22} = 0 \]
and
\[ E_{22}^2 + R_2(2\beta_2 + r)E_{22} - R_2 Q_2 = 0. \]

When the solutions of the second equation for \( E_{22} \) are substituted in the first, we obtain
\[ (\pm \sqrt{(2\beta_2 + r)^2 + Q_2})K_{22} = 0. \]

Since \( Q_2 \) is assumed \( \geq 0 \), the quantity inside the square root sign is never zero and this equation gives \( K_{22} = 0 \), which also contradicts condition (a). Therefore, neither cases (i) or (ii) can hold and hence the expression in (2) is strictly positive. The negative definiteness of \( dV/dt \) is therefore established. Hence \( V \) is a Lyapunov function and the system is asymptotically stable. In a similar manner case (b) can be proved with the Lyapunov function
\[ V = e^{-n}(E_{11}x_1^2 + 2E_{12}x_1x_2 + E_{22}x_2^2). \]

Thus \( x_1 \rightarrow x_{1e}, x_2 \rightarrow x_{2e} \), the boundary conditions (33) are satisfied and the convergence of the integrals (20)-(21) with \( t_f \rightarrow \infty \) is therefore guaranteed.

REFERENCES


