A UNIFIED APPROACH TO REDUCED-ORDER MODELLING AND CONTROL OF LARGE-SCALE SYSTEMS WITH MULTIPLE DECISION MAKERS

VIKRAM R. SAKSENA
AT&T Bell Laboratories, Holmdel, NJ 07733, U.S.A.

AND

J. B. CRUZ, JR.
Coordinated Science Laboratory, University of Illinois, 1101 W. Springfield Ave., Urbana, IL 61801, U.S.A.

SUMMARY

This paper presents a unified approach to reduced-order modelling and control of large-scale systems with multiple decision makers. The 'core' of a large game problem is identified through the information structure and the class of structure preserving strategies. The induced decomposition results in a low-order game problem and two decentralized optimal control problems. An example of a two-area power system demonstrates the computational attractiveness of the proposed design methodology.

KEY WORDS Differential games Control of large-scale systems Reduced-order modelling Two-area power system Decentralized optimal control

INTRODUCTION

The problem of efficient management and control of large-scale systems has been extremely challenging to control engineers. There are essentially two main issues of concern. The modelling issue is complicated owing to the large dimension of the system. The crucial problem here is one of model simplification, i.e. how to obtain a simplified low-order model of the system which results in an acceptable control design. This problem is intimately related to notions of time-scales, weak-coupling and controllability—observability. The control design issue is complicated by the presence of multiple decision makers having possibly conflicting goals and possessing decentralized information. The crucial problem here is to obtain optimal multicontroller strategies under non-classical information patterns and various co-operative and non-co-operative solution concepts. The intricate relationship between the modelling and strategy design issues introduces additional complexities not encountered while considering each problem in isolation. This is because many aspects of the system structure are variant under the control actions. Many cases of ill-posed closed-loop designs based on reduced-order

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models have been reported. The complexities become more involved when there are multiple decision makers as opposed to a centralized decision maker. This is because each decision maker’s perception of the system structure and dynamics may be altered by the actions of the other decision makers. Hence, any approach towards the development of an efficient design methodology must treat the modelling and strategy design issues in a unified framework. Such a unified framework has been formulated and developed through the concept of multimodelling. The multimodelling concept allows the decision makers to use different simplified models of the same large-scale system, and provides procedures for obtaining solutions which are close to the optimum under certain assumptions on the system dynamics and information structure.

The crux of the problem in multimodelling is to identify the ‘core’ where there is a strong interaction among all the decision makers, and other low-order subproblems where the interactions are weak. This leads to the possibility of decentralized strategy design by the decision makers using several low-order models of the same system. Such a decomposition may be achieved through the time-scale and weak-coupling properties of the system, the controllability–observability structure of the model or through some other considerations.

In this paper we extend the observability-based decomposition approach for Nash problems to co-operative situations under the Pareto solution concept. The applicability of this decomposition approach requires a certain degree of co-operation among the decision makers. Therefore, from a practical point of view, it is important to formulate this approach for Pareto problems. We shall demonstrate the practical importance of this approach through an example of a two-area power system.

PROBLEM FORMULATION AND STRUCTURAL DECOMPOSITION

Consider a linear system controlled by two decision makers

\[ \dot{x} = Ax + B_1 u_1 + B_2 u_2; \quad x(0) = x_0 \]  
\[ y_i = C_i x; \quad i = 1, 2 \]  
\[ \text{dim } x = n, \text{ dim } u_i = m_i, \text{ dim } y_i = p_i \]

The variables \( y_i \) will be referred to as the ‘observation set’ of the decision makers. These are in fact the controlled variables ‘observed’ through the performance indices

\[ J_i (v_1, v_2) = \left\{ \frac{1}{2} \int_0^\infty (y_i^T y_i + u_i^T R_i u_i) \, dt \mid u_i(t) = v_i(\cdot) \right\}; \quad i = 1, 2 \]

where \( v_i(\cdot) \) is an admissible strategy of decision maker \( i \). We shall assume that the decision makers have access to the full state of the system.

The decision makers are to select optimal strategies \( \{ v_1^*, v_2^* \mid v_i^* \in \Gamma_i; i = 1, 2 \} \) such that the Pareto cost

\[ J = \sum_{i=1}^2 \alpha_i J_i; \quad 0 < \alpha_i < 1; \quad \alpha_1 + \alpha_2 = 1 \]

is minimized. \( \{ \Gamma_i; i = 1, 2 \} \) are the admissible strategy sets of the decision makers, to be specified later.

If \( \{ \Gamma_i; i = 1, 2 \} \) are the sets of all possible linear feedback strategies, then, assuming that the
triple

\[
\begin{bmatrix}
A_i & [B_1 B_2] & [C_1] \\
B_i & C_2 \\
C_1 & C_2
\end{bmatrix}
\]

is stabilizable–detectable, the Pareto-optimal solution is given by

\[
\nu_i = -\frac{1}{\alpha_i} R_i^{-1} B_i^T P x; \quad i = 1, 2
\]

(4)

where \( P \) is the positive-semidefinite solution of the Riccati equation

\[
A_i^T P + P A_i + \sum_{i=1}^{2} \left[ \alpha_i C_i^T C_i - \frac{1}{\alpha_i} P B_i R_i^{-1} B_i^T P \right] = 0
\]

(5)

Under the action of (4), the closed-loop system is asymptotically stable.

When \( n \) is large, it is computationally difficult to solve (5). Therefore, when we wish to generate a set of Pareto-optimal solutions for different values of \( \alpha_i \) (as is usually the case), the above approach may not be feasible since it requires repeated solutions of the Riccati equation. The decomposition approach leads to a partial non-interaction among the decision makers by appropriately constraining their admissible strategy sets. As a result of this partial non-interaction, the decision makers are able to obtain a subset of their control gains through decentralized computations. The rest of the control gains are obtained by solving a set of coupled equations. All the equations are of lower dimension than (5).

The observation sets (1b) of the decision makers induce a certain observability decomposition on the state space. We can identify this decomposition explicitly either by performing chained aggregation sequentially with respect to each decision maker’s observation set or, equivalently, by performing a similarity transformation directly following a procedure dual to the one in Reference 12 where a controllability decomposition was achieved.

The transformed system is represented by

\[
\begin{align*}
\dot{x} &= A \bar{x} + B_1 u_1 + B_2 u_2; \quad x(0) = \bar{x}_0 \\
y_i &= C_i \bar{x}; \quad i = 1, 2
\end{align*}
\]

(6a)

(6b)

where

\[
A = \begin{bmatrix}
A_{11} & 0 & A_{13} \\
0 & A_{22} & A_{23} \\
0 & 0 & A_{33}
\end{bmatrix}
\]

\[
C_1 = [C_{11} \ 0 \ 0]
\]

\[
C_2 = [0 \ C_{22} \ C_{23}]
\]

\[
B_i = \begin{bmatrix}
B_{i1} \\
B_{i2} \\
B_{i3}
\end{bmatrix}; \quad i = 1, 2
\]

and

\[
[A_{ii}, C_{ii}], \left[ \begin{bmatrix}
A_{ii} & \bar{A}_{i3} \\
0 & \bar{A}_{33}
\end{bmatrix}, [C_{ii} C_{i3}] \right]; \quad i = 1, 2
\]

are observable pairs.
The eigenvalues of $\hat{A}_{ii}$ ($i = 1, 2$) represent the modes which are observable only to decision maker $i$, but not to decision maker $j$, whereas the eigenvalues of $\hat{A}_{33}$ represent the modes which are observable to both decision makers. We have neglected the modes which are unobservable to both the decision makers since, in a well-formulated problem, these modes are stable and do not contribute to the cost.

The input structure specified by the matrices $\hat{B}_1$ and $\hat{B}_2$ is not in a form suitable for our analysis. We make input space transformations and appropriately overlap the input structure with the observability decomposition. We assume that the pairs $\{ (\hat{A}_{ii}, \hat{B}_{ii}), (\hat{A}_{ij}, \hat{B}_{ij}); i = 1,2 \}$ are controllable so that there exist matrices $G_1$ and $G_2$ such that the input space transformation $\{ \hat{u}_i = G_i \hat{u}_i; i = 1,2 \}$ gives the new input matrices the form

$$\hat{B}_1 = \hat{B}_1 G_1 = \begin{bmatrix} \hat{B}_{11} & \hat{B}_{14} \\ 0 & \hat{B}_{12} \\ 0 & \hat{B}_{13} \end{bmatrix}, \quad \hat{B}_2 = \begin{bmatrix} 0 & \hat{B}_{21} \\ \hat{B}_{22} & \hat{B}_{24} \\ 0 & \hat{B}_{23} \end{bmatrix}$$

where the pairs $\{ (\hat{A}_{ii}, \hat{B}_{ii}); i = 1, 2 \}$ are controllable.

Before performing the input space transformation, we might need to perform another state space transformation, but this can be done without destroying the observability decomposition. This is to put the system in an appropriate basis such that $X = R_1 \oplus R_2 \oplus \varnothing$ where $X$ is the state space and $R_i$ is a controllability subspace of decision maker $i$. The input space transformations $G_i$ identify explicitly the control channels through which the individually observable modes are completely controllable.13

The system and the performance indices, after the observability decomposition and the input space transformation, take the form

$$\dot{x} = \hat{A}x + \hat{B}_1 \hat{u}_1 + \hat{B}_2 \hat{u}_2; \ x(0) = \bar{x}_0$$

$$y_i = \hat{C}_i x; \quad i = 1, 2$$

$$J_i = \frac{1}{2} \int_0^\infty (y_i^T y_i + \bar{u}_i^T \hat{R}_i \bar{u}_i) \, dt; \quad i = 1, 2$$

where $\hat{R}_i = G_i^T R_i G_i$. We shall assume that

$$\hat{R}_i = \begin{bmatrix} \hat{R}_{ii} & 0 \\ 0 & \hat{R}_{ij} \end{bmatrix} > 0; \quad i, j = 1, 2; \quad i \neq j$$

We are now in a position to define the admissible strategy sets for the decision makers. The admissible strategy sets that we are particularly interested in here will be referred to as 'structure preserving strategies'.

**Definition**

A structure preserving strategy set is the set of all linear state feedback strategies which preserve the observability decomposition (6) of the closed-loop system. Specifically,

$$\Gamma_1 = \left\{ \nu_1 \mid \nu_1(x) = -F_1 \hat{x} = -\begin{bmatrix} F_{11} & 0 \\ 0 & 0 \end{bmatrix} \hat{x} \right\}$$

$$\Gamma_2 = \left\{ \nu_2 \mid \nu_2(x) = -F_2 \hat{x} = -\begin{bmatrix} 0 & F_{22} \\ 0 & 0 \end{bmatrix} \hat{x} \right\}$$

We shall now show that the Pareto-optimal design of structure-preserving strategies results in a partial non-interaction among the decision makers.
PARETO-OPTIMAL SOLUTION

To find the Pareto-optimal solution, we need to find a pair \( \nu_i^* \in \Gamma_i; \ i = 1,2 \) such that the cost (3) is minimized. Substituting \( \bar{u}_i = \nu_i(x) \) from (10) in (8) and (9), we obtain

\[
\dot{x} = \bar{A} \bar{x}; \quad \bar{x}(0) = \bar{x}_0 \tag {11a}
\]

\[y_i = \bar{C}_i \bar{x}; \quad i = 1,2 \tag {11b}
\]

\[
J_i = \frac{1}{2} \int_0^\infty (\bar{x}^T Q_i \bar{x}) \, dt; \quad i = 1,2 \tag {12}
\]

where

\[Q_i = \bar{C}_i^T \bar{C}_i + \bar{F}_i^T \bar{R}_i \bar{F}_i; \quad i = 1,2 \]

and the closed-loop system matrix is

\[
\bar{A} = A - \bar{B}_1 F_1 - \bar{B}_2 F_2 = \\
\begin{bmatrix}
\bar{A}_{11} - \bar{B}_{11} F_{11} & 0 & \bar{A}_{13} - \bar{B}_{11} F_{13} - \bar{B}_{14} F_{31} - \bar{B}_{21} F_{32} \\
0 & \bar{A}_{22} - \bar{B}_{22} F_{22} & \bar{A}_{23} - \bar{B}_{22} F_{23} - \bar{B}_{24} F_{32} - \bar{B}_{12} F_{31} \\
0 & 0 & \bar{A}_{33} - \bar{B}_{13} F_{31} - \bar{B}_{23} F_{32}
\end{bmatrix}
\]

\[
\tilde{A} = \\
\begin{bmatrix}
\bar{A}_{11} & 0 & \bar{A}_{13} \\
0 & \bar{A}_{22} & \bar{A}_{23} \\
0 & 0 & \bar{A}_{33}
\end{bmatrix}
\tag {13}
\]

The optimal solution \( \{ F_1^*, F_2^*, F_3^*; i = 1,2 \} \) will depend in general on the initial conditions \( \bar{x}_0 \). To remove this explicit dependence, we follow Reference 14 and assume that the initial conditions are random with

\[E[\bar{x}_0 \bar{x}_0^T] = N > 0 \tag {14}\]

and modify the cost functionals to be

\[J_i = \frac{1}{2} E\left(\int_0^\infty (\bar{x}^T Q_i \bar{x}) \, dt\right); \quad i = 1,2 \tag {15}\]

Introduce \( M \) and \( L \in \mathbb{R}^{n \times n} \) defined by

\[\frac{1}{2} \bar{x}_0^T M \bar{x}_0 = \sum_{i=1}^2 \frac{\alpha_i}{2} \int_0^\infty (\bar{x}^T Q_i \bar{x}) \, dt \tag {16}\]

\[L = \int_0^\infty E[\bar{x}(t)\bar{x}^T(t)] \, dt \tag {17}\]

For any given pair \( \{ F_1, F_2 \} \) such that \( \text{Re} \lambda(\tilde{A}) < 0, \ M \geq 0 \) and \( L > 0 \) satisfy the matrix Lyapunov equations

\[M \tilde{A} + \tilde{A}^T M + \sum_{i=1}^2 \alpha_i Q_i = 0 \tag {18}\]

\[\tilde{A} L + L \tilde{A}^T + N = 0 \tag {19}\]
Partition $M$, $L$ and $N$ appropriately

\[
M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{12} & M_{22} & M_{23} \\ M_{13}^T & M_{23} & M_{33} \end{bmatrix}
\]
\[
L = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{12}^T & L_{22} & L_{23} \\ L_{13}^T & L_{23} & L_{33} \end{bmatrix}
\]
\[
N = \begin{bmatrix} N_{11} & N_{12} & N_{13} \\ N_{12}^T & N_{22} & N_{23} \\ N_{13}^T & N_{23} & N_{33} \end{bmatrix}
\]

Applying the matrix minimum principle\(^\text{15}\), the optimal $F_{ii}^*$, $F_{i3}^*$, $F_{3i}^*$ for the Pareto solution can be shown to satisfy (for $i = 1, 2$)

\[
F_{ii}^* = \hat{R}_{ii}^{-1}\hat{B}_{ii}^TM_{ii}
\]  
(20a)

\[
F_{i3}^* = \hat{R}_{i3}^{-1}\hat{B}_{i3}^TM_{i3}
\]  
(20b)

\[
F_{3i}^* = \hat{R}_{3i}^{-1}\hat{B}_{3i}^T\left[\frac{1}{\alpha_i}M_{33} + M_{33}^TL_{i3}L_{i3}^{-1}\right] + \hat{R}_{i3}^{-1}\hat{B}_{i3}^T[M_{i3} + M_{ii}L_{i3}L_{i3}^{-1}]
\]  
(20c)

where

\[
M_{ii}\tilde{A}_{ii} + \tilde{A}_{ii}^TM_{ii} + \tilde{C}_{ii}^T\tilde{C}_{ii} - M_{ii}\tilde{B}_{ii}\hat{R}_{ii}^{-1}\hat{B}_{ii}^TM_{ii} = 0
\]  
(21a)

\[
M_{i3}\tilde{A}_{i3} + M_{i3}\tilde{A}_{i3}^T + \tilde{A}_{i3}^TM_{i3} + \tilde{C}_{i3}^T\tilde{C}_{i3} + F_{i3}^T\hat{R}_{i3}F_{i3} = 0
\]  
(21b)

\[
M_{33}\tilde{A}_{33} + \tilde{A}_{33}^T + \sum_{i=1}^{2}\alpha_i(M_{33}^T\tilde{A}_{i3} + \tilde{A}_{i3}^TM_{33} + \tilde{C}_{i3}^T\tilde{C}_{i3} + F_{i3}^T\hat{R}_{i3}F_{i3})
\]
\[+ \alpha_1F_{31}^T\hat{R}_{12}F_{31} + \alpha_2F_{32}^T\hat{R}_{21}F_{32} = 0
\]  
(21c)

\[
\tilde{A}_{i3}^TL_{i3} + \tilde{A}_{i3}^TL_{i3}^T + L_{i3}\tilde{A}_{33}^T + N_{i3} = 0
\]  
(22a)

\[
\tilde{A}_{33}^TL_{33} + L_{33}\tilde{A}_{33}^T + N_{33} = 0
\]  
(22b)

{$\{\tilde{A}_{ii}^*, \tilde{A}_{i3}^*, \tilde{A}_{33}^*; i = 1, 2\}$ are the optimal system matrices. The controllability–observability of the triple $((\tilde{A}_{ii}, \tilde{B}_{ii}, \tilde{C}_{ii}); i = 1, 2)$ guarantees $\text{Re}\lambda(\tilde{A}_{ii}^*) < 0; i = 1, 2$. For the solution to be well-defined, we need only to verify that $\text{Re}\lambda(\tilde{A}_{33}^*) < 0$. It can be readily seen from (20a) and (21a) that $F_{ii}^*$ is the solution of an optimal state regulator problem with parameters $(\tilde{A}_{ii}^*, \tilde{B}_{ii}, \tilde{C}_{ii}, \hat{R}_{ii})$.

The following proposition highlights the multimodel nature of the Pareto solution.

**Proposition 1**

Given the linear system (8) controlled by two decision makers, their performance indices (9), and the Pareto cost (3), the design of structure preserving Pareto strategies leads to the low-order optimization problem

\[
\min J = \int_{z_{10}}^{z_{20}} \left\{ \frac{1}{2} \sum_{i=1}^{3} \alpha_i \int_0^\infty (y_i^Ty_i + \bar{u}_i^T\hat{R}_i\bar{u}_i) \, dt \right\}
\]

subject to

\[
\bar{u}_i = \nu_i(z_i) = \begin{bmatrix} F_{ii} & F_{i3} \\ 0 & F_{3i} \end{bmatrix} z_i
\]
where

\[
\begin{align*}
\dot{z}_i &= \begin{bmatrix} \tilde{A}_{i1} & \tilde{A}_{i3} - \tilde{B}_{i2} F_{3j} \\ 0 & \tilde{A}_{33} - \tilde{B}_{i2} F_{3j} \end{bmatrix} z_i + \begin{bmatrix} \tilde{B}_{i1} \\ 0 \end{bmatrix} \bar{u}_i; \quad z_i(0) = z_{i0} \\
y_i &= \begin{bmatrix} C_{i1} & C_{i3} \end{bmatrix} z_i \\
E[z_{i0} z_{j0}^T] &= \begin{bmatrix} N_{i1} & N_{i3} \\ N_{31} & N_{33} \end{bmatrix} \\
i, j &= 1, 2; \quad i \neq j
\end{align*}
\]

This optimization problem results in a partial non-interaction between the decision makers. Its solution is given by the set of equations (20)–(22).

The partial non-interaction is achieved because each decision maker can evaluate his control gain \( F^*_i \) independently in a decentralized manner by solving equations (20a) and (21a). The control gains \( (F^*_i, F^*_j; i = 1, 2) \) are then obtained by solving the set of equations (20b), (20c), (21b), (21c) and (22). The gains \( F^*_i \) are independent of the Pareto parameters \( \alpha_i \) and need to be computed only once, whereas the gains \( F^*_3 \) and \( F^*_i \) depend on \( \alpha_i \) and need to be computed repeatedly for each \( \alpha_i \). It should be pointed out that the equations for computing \( F^*_3 \) and \( F^*_i \) are not in any standard form. In general, we cannot guarantee the existence of a solution to these equations. But in many practical cases, such as the power system example of this paper, these equations are of a significantly lower dimension than the dimension of the overall system and can be solved by ‘brute force’ methods.

Notice that the reduced-order model of each decision maker is influenced by the control gains of the other decision maker. This ‘anticipative’ feature has been demonstrated elsewhere in reduced-order modeling of multiple decision maker problems.\(^6\),\(^16\)

We have succeeded in identifying the ‘core’ of a high-order game problem where the decision makers actually interact. The important point to note is that the core was \emph{induced} through the information structure and a judicious choice of the admissible strategies. This fact signifies the cohesive structure of the modelling and strategy design issues.

**DECOUPLING OF COMPLETELY OBSERVABLE SYSTEMS**

In situations when the system is completely observable through the observation set of each decision maker, the ‘core’ is the full problem itself. But in some cases, the observability decomposition can be achieved through the use of state feedback. The role of decoupling controls has been studied in detail.\(^10\) Here, we only briefly outline the procedure.

Suppose that after appropriate state space, input space and output space transformations, the system can be put in the form\(^10,11\)

\[
\begin{align*}
\dot{x} &= \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} \\ \tilde{A}_{31} & \tilde{A}_{32} & \tilde{A}_{33} \end{bmatrix} x + \begin{bmatrix} B_{11} & B_{14} \\ 0 & B_{12} \\ 0 & B_{13} \end{bmatrix} u_1 + \begin{bmatrix} 0 & B_{21} \\ B_{22} & B_{24} \\ 0 & B_{23} \end{bmatrix} u_2 \\
y_1 &= \begin{bmatrix} C_{11} & 0 & C_{13} \end{bmatrix} x \\
y_2 &= \begin{bmatrix} 0 & C_{22} & C_{23} \end{bmatrix} x
\end{align*}
\]

with \( (\tilde{B}_{ii}, \tilde{B}_{ij}, i, j = 1, 2) \) being square and non-singular. Now if the decision makers use the
strategies,

\[ \dot{u}_1 = \begin{bmatrix} 0 & -B^{-1}_{11}(A_{12} - B_{21}B^{-1}_{23}A_{32}) & 0 \\ -B^{-1}_{13}A_{31} & 0 & 0 \end{bmatrix} \dot{x} + \dot{u}_1 \]  
\[ (24a) \]

\[ \dot{u}_2 = \begin{bmatrix} -B^{-1}_{22}(A_{21} - B_{12}B^{-1}_{13}A_{31}) & 0 & 0 \\ 0 & -B^{-1}_{23}A_{32} & 0 \end{bmatrix} \dot{x} + \dot{u}_2 \]  
\[ (24b) \]

then the resulting partially closed-loop system has the form

\[ \dot{x} + \begin{bmatrix} A_{11} - B_{14}B^{-1}_{13}A_{31} & 0 & \tilde{A}_{13} \\ 0 & A_{22} - B_{24}B^{-1}_{23}A_{32} & A_{23} \\ 0 & 0 & \tilde{A}_{33} \end{bmatrix} \dot{x} + \begin{bmatrix} B_{11} & B_{14} \\ 0 & B_{12} \\ 0 & B_{13} \end{bmatrix} \dot{u}_1 + \begin{bmatrix} 0 & B_{21} \\ 0 & B_{24} \end{bmatrix} \dot{u}_2 \]  
\[ (25) \]

It can be readily seen that the system (25), (23b) has the desired form of (8).

It is significant to note that making the dimension of \( B_{ii} \) as large as possible results in a 'maximally decoupled' system, i.e. a system in which the decentralized control problems are of the highest possible dimension. Consequently, the 'core' problem is of lowest possible dimension.10.11

The use of decoupling control induces a degree of suboptimality if the performance indices are chosen a priori. This is because the decoupling control is chosen from a purely algebraic point of view without any optimality considerations. Since the decision makers are acting in cooperation, they would agree to the use of the decoupling controls if the resulting computational advantages compensate for the performance loss.

**AN EXAMPLE**

We shall now consider the control design problem of a two-area power system with each area being under a different control authority. We shall first transform the system into our desired form (8) and then obtain Pareto strategies by solving equations (20)–(22).

A two-area power system with each area containing two thermal plants is constructed from Reference 17. The system is modelled by

\[ \dot{x} = Ax + Bu_1 + Bu_2 \]
\[ y_i = C_i x; \quad i = 1, 2 \]
\[ (26) \]

where \( x \in \mathbb{R}^{19}, u_1 \in \mathbb{R}^2, u_2 \in \mathbb{R}^2, y_1 \in \mathbb{R}^2, y_2 \in \mathbb{R}^2 \). The state, control and output variables are defined in the Appendix. The system matrices are given by

\[
A = \begin{bmatrix}
A^{(1)}_{11} & 0 & A^{(1)}_{13} & 0 & 0 & 0 & 0 \\
0 & A^{(1)}_{22} & A^{(1)}_{23} & 0 & 0 & 0 & 0 \\
A^{(1)}_{31} & A^{(1)}_{32} & -0.1124 & -0.083 & 0 & 0 & 0 \\
0 & 0 & 22.21 & 0 & -22.21 & 0 & 0 \\
0 & 0 & 0 & 0.083 & 0 & -0.1124 & 0 & 0 \\
0 & 0 & 0 & 0 & A^{(2)}_{13} & A^{(2)}_{11} & 0 \\
0 & 0 & 0 & 0 & 0 & A^{(2)}_{23} & A^{(2)}_{22} \\
\end{bmatrix}
\]

\[ A^{(1)}_{11} = A^{(2)}_{11} = A^{(1)}_{22} = A^{(2)}_{22} = \begin{bmatrix} -2 & 0 & 0 & 0 \\
4.75 & -5 & 0 & 0 \\
0 & 0.167 & -0.167 & 0 \\
0 & 0 & 2 & -2 \end{bmatrix} \]
\[
A_{13}^{(1)} = A_{13}^{(2)} = A_{23}^{(1)} = A_{23}^{(2)} = [-4 \ 0 \ 0]^T
\]
\[
A_{51}^{(1)} = A_{51}^{(2)} = A_{32}^{(1)} = A_{32}^{(2)} = [0 \ 0.01 \ 0.0093 \ 0.014]
\]
\[
B_1 = \begin{bmatrix}
B_{11}^{(1)} & 0 \\
0 & B_{22}^{(1)} \\
0_{11 \times 2} & 0_{11 \times 2}
\end{bmatrix}; \quad B_2 = \begin{bmatrix}
0_{11 \times 2} \\
B_{11}^{(2)} & 0 \\
0 & B_{22}^{(2)}
\end{bmatrix}
\]
\[
B_{11}^{(1)} = B_{11}^{(2)} = B_{22}^{(1)} = B_{22}^{(2)} = \begin{bmatrix}
4 \\
0 \\
0 \\
0
\end{bmatrix}
\]
\[
C_1 = [0_{2 \times 8} \ 1 \ 0 \ 0_{2 \times 9}]; \quad C_2 = [0_{2 \times 9} \ 1 \ 0 \ 0_2 \times 8]
\]

After two steps of chained aggregation and one input-space transformation, we obtain the following representation:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} = \begin{bmatrix}
F_{11}^{(i)} & 0 & F_{13} \\
0 & F_{21}^{(i)} & F_{23} \\
F_{31} & F_{32} & F_{33}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} + \begin{bmatrix}
G_{11}^{(i)} & G_{12}^{(i)} \\
0 & 0 \\
0 & G_{31}^{(i)}
\end{bmatrix}
\begin{bmatrix}
\bar{u}_1^{(i)} \\
\bar{u}_2^{(i)}
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
0 & G_{21}^{(i)} & G_{22}^{(i)} \\
0 & G_{31}^{(i)} & G_{32}^{(i)}
\end{bmatrix}
\begin{bmatrix}
\bar{u}_1^{(i)} \\
\bar{u}_2^{(i)}
\end{bmatrix}
\]

\[
\dot{x}_1 \in \mathbb{R}^6; \quad \dot{x}_2 \in \mathbb{R}^6; \quad \dot{x}_3 \in \mathbb{R}^7
\]

where

\[
F_{11}^{(i)} = \begin{bmatrix}
-5 & 4.75 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 2 & 0 & 0 \\
0 & 0 & 0 & -0.167 & 0.167 & 0 \\
0 & 0 & 0 & 0 & -5 & 4.75 \\
0 & 0 & 0 & 0 & 0 & -2
\end{bmatrix}; \quad i = 1, 2
\]

\[
F_{33} = \begin{bmatrix}
-0.1124 & -0.083 & 0 & 1 & 0 & 0 & 0 \\
22.21 & 0 & -22.21 & 0 & 0 & 0 & 0 \\
0 & 0.083 & -0.1124 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -2 & 1 & 0 & 0 \\
-0.38 & 0 & 0 & 0 & -0.167 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 1 \\
0 & 0 & 0 & -0.38 & 0 & 0 & 0 & -0.167
\end{bmatrix}
\]

\[
F_{31} = \begin{bmatrix}
0_{4 \times 6} \\
F_{31}^{(1)} \\
0_{2 \times 6}
\end{bmatrix}; \quad F_{32} = \begin{bmatrix}
0_{6 \times 6} \\
F_{32}^{(1)} \\
0_{2 \times 6}
\end{bmatrix}
\]

\[
F_{2}^{(i)} = [0.136 \ -0.222 \ 0 \ 0 \ 0.136 \ -0.222]; \quad i = 1, 2
\]

\[
F_{13} = [0_{6 \times 3} \ | \ D \ | \ 0_{6 \times 3}]; \quad F_{23} = [0_{6 \times 5} \ | \ D \ | \ 0_{6 \times 1}]
\]

\[
D = [0 \ -4 \ 0 \ 0 \ 0 \ -4]^T
\]
\[ G_{11}^{(i)} = [0 -4 0 0 0 4]^T; \quad i = 1, 2 \]
\[ G_{12}^{(i)} = [0 4 0 0 0 0]^T; \quad i = 1, 2 \]
\[ G_{31} = [0 0 0 0 0 0] \]
\[ G_{32} = [0 0 0 0 0 0 0 0 0 0 0 0]^T \]

Now we need to apply a decoupling control to cancel out the terms \( F_{31} \) and \( F_{32} \) in (27). The decoupling control is chosen to be

\[ \ddot{u}_2^{(i)} = [-0.716 \quad 1.168 \quad 0 \quad 0 \quad -0.716 \quad 1.168] \ddot{x}_i + \ddot{u}_2^{(i)}; \quad i = 1, 2 \quad (28) \]

Substituting (28) in (27), we obtain

\[ \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = \begin{bmatrix} \dot{F}_{11}^{(1)} & 0 & \dot{F}_{13} \\ 0 & \dot{F}_{22}^{(2)} & \ddot{x}_2 \\ 0 & 0 & \dot{F}_{33}^{(3)} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} G_{11}^{(1)} G_{12}^{(1)} \\ 0 & 0 & G_{31} \\ 0 & 0 & G_{32} \end{bmatrix} \begin{bmatrix} \ddot{u}_1^{(1)} \\ \ddot{u}_2^{(1)} \\ \ddot{u}_3^{(3)} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \ddot{u}_1^{(2)} \\ \ddot{u}_2^{(2)} \end{bmatrix} \quad (29) \]

where

\[ \dot{F}_{11}^{(1)} = \dot{F}_{11}^{(2)} = \begin{bmatrix} -5 & 4.75 & 0 & 0 & 0 & 0 \\ -2.864 & 2.612 & 0 & 0 & -2.864 & 4.612 \\ 0 & 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & -0.167 & 0.167 & 0 \\ 0 & 0 & 0 & 0 & -5 & 4.75 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix} \]

Now the system is precisely in a form suitable for our design techniques. The frequency deviations in the two areas and the tie-line power flow comprise part of the core variables \( \ddot{x}_i \). The variables \( \ddot{x}_1 \) and \( \ddot{x}_2 \) are the residual variables associated with each area.

The nineteenth-order game in its original form (26) may be computationally intractable. But in the form (29), and allowing only structure-preserving strategies, we need only to solve two sixth-order optimal control problems and one seventh-order problem where decision makers interact.

For the Pareto-optimal design, the cost functionals are chosen to be

\[ J_i = \frac{1}{2} \int_0^\infty (\ddot{x}_i^T Q_i \ddot{x}_i + \ddot{x}_i^T Q_i \ddot{x}_i + \ddot{u}_i^T R_i \ddot{u}_i) \; dt; \quad i = 1, 2 \]

with

\[ Q_{11} = \text{diag} (10, 10, 10, 10, 1, 1) \]
\[ Q_{22} = \text{diag} (12, 15, 10, 5, 5, 5) \]
\[ Q_{13} = \text{diag} (10, 7, 0, 0, 0, 0) \]
\[ Q_{23} = \text{diag} (0, 5, 10, 0, 0, 0) \]
\[ R_1 = \text{diag} (10, 25) \]
\[ R_2 = \text{diag} (5, 20); \quad \text{Cov} (\ddot{x}_0) = N = I \]

**Case 1: Pareto cost** \( J = 0.4 J_1 + 0.6 J_2 \)

The optimal gains \( F_{ii}^{*} \) are first obtained from optimal control problems (20a) and (21a).

\[ F_{11}^{*} = [-0.167 -0.722 \quad 0.0132 \quad 0.571 \quad 0.043 \quad 0.13] \]
\[ F_{22}^{*} = [-0.402 -1.082 \quad 0.017 \quad 0.528 \quad 0.042 \quad 0.714] \]
Then the optimal gains $F_{13}^*$ and $F_{31}^*$ are obtained from the coupled equations (20b), (20c), (21b), (21c) and (22).

$$
F_{13}^* = \begin{bmatrix}
-4.65 & -16.28 & 1.37 & 12.44 & -5.53 & 0 & 0 \\
-6.62 & -21.35 & 3.04 & 0 & 0 & -10.92 & 4.15 \\
16.5 & 0.589 & -16.44 & -0.653 & -0.068 & 0 & 0 \\
-17.08 & 0.568 & 16.33 & 0 & 0 & -8.21 & -0.121
\end{bmatrix}
$$

The closed-loop eigenvalues become $-0.2 \pm j0.51$, $-0.24 \pm j0.48$, $-0.39 \pm j0.05$, $-0.52 \pm j0.07$, $-1.03 \pm j1.5$, $-1.99$, $-2$, $-2.09$, $-2.21$, $-2.21$, $-5.18 \pm j1.92$ and $-7.09 \pm j1.96$. The optimal feedback strategies are

$$
\nu_1^* = \begin{bmatrix}
-0.722 & 0.167 & 0.263 & -0.591 & -0.13 & -0.043 & -0.571 & -0.013 \\
-1.168 & 0.716 & 0.03 & -0.03 & -1.168 & -0.716 & 0 & 0
\end{bmatrix}
$$

$$
\nu_2^* = \begin{bmatrix}
0.314 & -1.014 & -0.144 & -1.082 & 0.402 & -0.197 & 0.519 \\
0.811 & -0.027 & 0.776 & -1.168 & 0.716 & -0.057 & 0.037
\end{bmatrix}
$$

**Case 2: Pareto cost $J = 0.1 J_1 + 0.9 J_2$**

The gains $F_{11}^*$ do not change. The gains $F_{13}^*$ and $F_{31}^*$ are

$$
F_{13}^* = \begin{bmatrix}
-6.72 & -18.33 & 4.47 & 10.92 & -6.71 & 0 & 0 \\
-5.91 & -19.38 & 3.72 & 0 & 0 & -13.08 & 3.17 \\
21.26 & 4.715 & -20.38 & -5.05 & -2.313 & 0 & 0 \\
-15.15 & 0.481 & 14.77 & 0 & 0 & -0.514 & -0.131
\end{bmatrix}
$$

The closed-loop eigenvalues become $-0.13 \pm j0.56$, $-0.172$, $-0.39 \pm j0.05$, $-0.52 \pm j0.07$, $-0.61$, $-1.03 \pm j1.15$, $-1.99$, $-2$, $-2.21$, $-2.21$, $-3.12$, $-5.18 \pm j1.92$ and $-7.09 \pm j1.96$. The optimal feedback strategies are

$$
\nu_1^* = \begin{bmatrix}
-0.722 & 0.167 & 0.412 & -0.716 & -0.13 & -0.043 & -0.571 & -0.013 \\
-1.168 & 0.716 & 0.068 & -0.11 & -1.168 & -0.716 & 0 & 0
\end{bmatrix}
$$

$$
\nu_2^* = \begin{bmatrix}
0.198 & 0.812 & -0.133 & -1.082 & 0.402 & -0.156 & 0.363 \\
0.626 & -0.012 & 0.494 & -1.168 & 0.716 & -0.032 & 0.024
\end{bmatrix}
$$
Case 3: Pareto cost $J = 0.9 J_1 + 0.1 J_2$

$F_{ii}^*$ remains the same. $F_{13}^*$ and $F_{3i}^*$ are

\[
F_{13}^* = \begin{bmatrix}
-2.82 & -13.19 & 1.11 & 13.73 & -3.88 & 0 & 0
\end{bmatrix}
\]

\[
F_{23}^* = \begin{bmatrix}
-7.75 & -25.62 & 6.51 & 0 & 0 & -11.21 & 5.34
\end{bmatrix}
\]

\[
F_{31}^* = \begin{bmatrix}
13.91 & 0.366 & -14.48 & -0.489 & -0.041 & 0 & 0
\end{bmatrix}
\]

\[
F_{32}^* = \begin{bmatrix}
-18.19 & 0.614 & 17.56 & 0 & 0 & -1.112 & -0.291
\end{bmatrix}
\]

The closed-loop eigenvalues become $-0.1 \pm j0.66$, $-0.158$, $-0.39 \pm j0.05$, $-0.52 \pm j0.07$, $-0.542$, $-0.98 \pm j1.52$, $-1.99$, $-2.16$, $-2.21$, $-2.21$, $-3.56$, $-5.18 \pm j1.92$ and

![Figure 1. Time responses for the case $J = 0.4 J_1 + 0.6 J_2$](image-url)
\(-7.09 \pm j1.96\). The optimal feedback strategies are

\[
\nu_1^* = \begin{bmatrix}
-0.722 & 0.167 & 0.115 & -0.482 & -0.13 & -0.043 & -0.571 & -0.013 \\
-1.168 & 0.716 & 0.019 & -0.023 & -1.168 & -0.716 & 0 & 0 \\
& & & & & & -0.106 & 0.518 & -0.06 \\
& & & & & & -0.613 & -0.014 & 0.593 \\
\end{bmatrix}
\]

\[
\nu_2^* = \begin{bmatrix}
0.523 & 1.131 & -0.393 & -1.082 & 0.402 & -0.403 & 0.626 \\
0.928 & -0.047 & 1.022 & -1.168 & 0.716 & -0.109 & 0.042 \\
& & & & & & -0.714 & -0.042 & -0.528 & -0.017 \\
& & & & & & -1.168 & -0.716 & 0 & 0 \\
\end{bmatrix}
\]

Figure 2. Time responses for the case \(J = 0.1 J_1 + 0.9 J_2\)
Figure 3. Time responses for the case $J = 0.9 \, J_1 + 0.1 \, J_2$

Notice that the strategy of each decision maker requires a knowledge of the states of his own area and only the frequency deviation of the other area; this feature is desirable from an implementation point of view. The time responses of the tie line power flow and frequency deviations of the two areas are plotted in Figures 1–3. It can be seen that the response of the frequency deviation corresponding to the area weighted lightly in the Pareto cost is more oscillatory; this is what one would expect. The response of the tie line power flow does not change significantly in the three cases.

We also computed the Pareto optimal solution given by (4) and (5). The computation times in this case were observed to be four to five times more than the times required to compute the structure preserving strategies.
CONCLUSIONS

In this paper we have formulated a unified approach to reduced-order modelling and control of large-scale systems with multiple decision makers. The key roles of the information structure and admissible strategies have been analysed. We have shown that the information structure together with the class of structure preserving strategies decomposes the overall game problem into a low-order game problem and two low-order decentralized optimal control problems. The reduced-order model of each decision maker incorporates explicitly some of the feedback gains of the other decision maker. This fact demonstrates the 'anticipative' nature of reduced-order models in multiple decision maker problems. The design methodology is applied to the control problem of a two-area power system. This example demonstrates the computational advantages of the design methodology.

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APPENDIX: MODEL VARIABLES OF THE POWER EXAMPLE

\[ x_{1}, x_{12} = \text{valve position displacement in first thermal unit of areas 1 and 2.} \]
\[ x_{2}, x_{13} = \text{power output displacement of HP turbine in first thermal unit of areas 1 and 2.} \]
\[ x_{3}, x_{14} = \text{power output displacement of IP turbine in first thermal unit of areas 1 and 2.} \]
\[ x_{4}, x_{15} = \text{power output displacement of LP turbine in first thermal unit of areas 1 and 2.} \]
\[ x_{5}, x_{16} = \text{valve position displacement in second thermal unit of areas 1 and 2.} \]
\[ x_{6}, x_{17} = \text{power output displacement of HP turbine in second thermal unit of areas 1 and 2.} \]
\[ x_{7}, x_{18} = \text{power output displacement of IP turbine in second thermal unit of areas 1 and 2.} \]
\[ x_{8}, x_{19} = \text{power output displacement of LP turbine in second thermal unit of areas 1 and 2.} \]
\[ x_{9}, x_{11} = \text{frequency deviation of areas 1 and 2.} \]
\[ x_{10} = \text{tie-line power flow connecting areas 1 and 2.} \]
\[ u_{1}^{(1)}, u_{1}^{(2)} = \text{set-point adjustment of first thermal unit in areas 1 and 2.} \]
\[ u_{2}^{(1)}, u_{2}^{(2)} = \text{set-point adjustment of second thermal unit in areas 1 and 2.} \]
\[ y_{1}^{(1)}, y_{2}^{(2)} = \text{frequency deviation of areas 1 and 2.} \]
\[ y_{2}^{(1)}, y_{1}^{(2)} = \text{tie-line power flow of areas 1 and 2.} \]

REFERENCES


