Series Nash Solution of Two-Person, Nonzero-Sum, Linear-Quadratic Differential Games

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Abstract. It is well-known that the Nash equilibrium solution of a two-person, nonzero-sum, linear differential game with a quadratic cost function can be expressed in terms of the solution of coupled generalized Riccati-type matrix differential equations. For high-order games, the numerical determination of the solution of the nonlinear coupled equations may be difficult or even impossible when the application dictates the use of small-memory computers. In this paper, a series solution is suggested by means of a parameter imbedding method. Instead of solving a high-order matrix-Riccati equation, a lower-order matrix-Riccati equation corresponding to a zero-sum game is solved. In addition, lower-order linear equations have to be solved. These solutions to lower-order equations are the coefficients of the series solution for the nonzero-sum game. Cost functions corresponding to truncated solutions are compared with those for exact Nash equilibrium solutions.

1. Introduction

Consider a two-person, linear differential game described by

\[ \dot{x} = Ax + B_1u_1 + B_2u_2, \]  
\[ x(t_0) = x_0, \]

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where the $n$-vector $x$ is the state of the game, the $m_1$-vector $u_1$ is the strategy of Player 1, the $m_2$-vector $u_2$ is the strategy of Player 2, $A, B_1, B_2$ are $n \times n, n \times m_1, n \times m_2$ matrices whose elements are piecewise continuous in time $t$, and $t_0$ is a fixed instant of time. The cost function for Player $i$ is

$$J_i(u_1, u_2) = \frac{1}{2} \langle x, S_i x \rangle + \frac{1}{2} \int_{t_0}^t \left[ \langle x, Q_i x \rangle + \langle u_1, R_{i,1} u_1 \rangle + \langle u_2, R_{i,2} u_2 \rangle \right] dt,$$ (3)

$i = 1, 2$, where $j = 2$ when $i = 1$ and $j = 1$ when $i = 2$. The terminal time $t_f$ is fixed, and $Q_i, R_{i,1}, R_{i,2}$ are symmetric matrices whose elements are piecewise continuous in $t$. Furthermore, $R_{11} > 0, R_{22} > 0, R_{12} < 0, R_{21} < 0, S_{1f} \geq 0, S_{2f} \leq 0, Q_1 \geq 0$, and $Q_2 \leq 0$. All of the above matrices are assumed to be known to both players.

Whereas minmax strategies are natural choices for zero-sum games, the latitude for reasonable strategies for nonzero-sum games is much wider. For example, depending on the nature of the information available to each player and the possibility to cooperation or noncooperation, different attributes for the strategies may be desirable. Minmax, noninferior, and Nash equilibrium solutions have been investigated before (Refs. 1–2). Furthermore, open-loop and feedback Nash strategies generally lead to different values for the cost functions (Ref. 2).

For nonzero-sum linear games with quadratic cost functions, the minmax strategies and noninferior strategies are obtained by solving decoupled Riccati-type matrix equations. However, the Nash strategies are obtained from coupled Riccati-type matrix equations.

A strategy set $(u_1^*, u_2^*)$ is called a Nash equilibrium strategy set if

$$J_i(u_1, u_2) \geq J_i(u_1^*, u_2^*), \quad i = 1, 2.$$ (4)

$$J_1(u_1^*, u_2) \geq J_2(u_1^*, u_2^*).$$ (5)

If $J_1 + J_2 = 0$, the game is zero-sum. It has been shown (Ref. 1) that, if $u_1$ and $u_2$ are required to be feedback functions of $x$, the Nash equilibrium strategies are

$$u_i^* = -R_{i,2} B_i S_i x,$$ (6)

where $S_i$ satisfies the generalized matrix-Riccati equations

$$S_t = -A'S_t - S_t A - Q_i + S_t B_i R_{i,1} B_i'S_t + S_t B_i R_{i,2} B_i/S_i$$

$$+ S_t B_i R_{i,2} B_i'S_t - S_i B_i R_{i,2} R_{i,1} B_i'B_i/S_i, \quad S_i(t_f) = S_{if}.\quad (7)$$
i = 1, 2, where \( j = 2 \) when \( i = 1 \) and \( j = 1 \) when \( i = 2 \). Sufficient conditions which guarantee the existence of solutions \( S_1 \) and \( S_2 \) for \( t_0 \leq t \leq t_f \) are given by Rhodes (Ref. 3).

Equation (7) for \( i = 1, 2 \) represents coupled nonlinear matrix differential equations. \( S_1 \) and \( S_2 \) are \( n \times n \) matrices; but, since they are symmetric, there are \( n(n + 1) \) different variables. If \( n \) is large, the numerical determination of the solution of these \( n(n + 1) \) coupled nonlinear differential equations could be quite formidable. Furthermore, in applications where the solutions have to be obtained on-line using limited computational capability, the solution of these \( n(n + 1) \) nonlinear differential equations may not be feasible.

In this paper, a parameter imbedding method is employed to obtain series solutions for \( S_1 \) and \( S_2 \). Instead of \( n(n + 1) \) nonlinear equations, only \( \frac{1}{2}n(n + 1) \) nonlinear equations have to be solved. Higher-order terms in the series require the solution of \( \frac{1}{2}n(n + 1) \) linear equations for each term. However, the set of linear equations for each term has the same homogeneous part and only the forcing terms are different. The cost functions using the truncated strategies will be compared with the exact cost functions. As in all imbedding methods, a bonus of the calculation is that a wider class of problems is solved. The imbedding parameter introduced in the paper allows an examination of a two-person game which is zero-sum when the parameter is zero; and, when the parameter has a value of unity, the solution is an approximation for the original problem. However, since the solution is obtained as a power series in the parameter, a range of nonzero-sum games with varying degrees of asymmetry from the zero-sum condition \( J_1 = -J_2 \) is automatically studied. Thus, the sensitivity of the solution to a change in the asymmetry from the zero-sum condition is available.

The use of parameter imbedding for obtaining series solutions for almost-zero-sum games has been investigated before (Ref. 4). The method in Ref. 4, when applied to the linear-quadratic differential game considered in this paper, will yield the same zeroth-order term in the series as the imbedding method below. The remaining terms in the series are different for the two imbedding methods, but the two series will yield the same solution for unity value of the imbedding parameter. The method discussed below leads to simpler calculation of the series.

In optimal control, parameter imbedding has been used to achieve computation reduction for the design of large-scale systems (Ref. 5). The method is particularly useful for weakly-coupled systems, because a few terms in the series yield a performance index which is close to the optimal one (Ref. 5).
2. Series Solution by Parameter Imbedding

Consider the equations

\[ \dot{S}_i = -A'S_i - S_iA - Q_i + S_iB_iR_{ii}^{-1}B_i'S_i + S_iB_iR_{ij}^{-1}B_j'S_j + S_iB_iR_{ij}^{-1}B_j'S_i \]
\[ - S_iB_iR_{ii}^{-1}R_iR_{ji}^{-1}B_j'S_i, \quad S_i(e, t_j) = S_i^t, \]  

\[ i = 1, 2, \text{ where } j = 2 \text{ when } i = 1 \text{ and } j = 1 \text{ when } i = 2, \text{ where} \]

\[ \dot{S}_i^t = \frac{1}{4}(S_i^t - S_j^t) + \frac{1}{4}e(S_i^t + S_j^t), \]
\[ \dot{Q}_i = \frac{1}{4}(Q_i - Q_j) + \frac{1}{4}e(Q_i + Q_i), \]
\[ \dot{R}_{ii} = \frac{R_{ii}^{-1}}{4}[(R_{ii} - R_{ij}) + \frac{1}{4}e(R_{ii} + R_{ij})] R_{ii}^{-1}, \]
\[ \dot{R}_{ij} = \frac{R_{ij}^{-1}}{4}[(R_{ij} - R_{ii}) + \frac{1}{4}e(R_{ij} + R_{ii})] R_{ij}^{-1}R_{jj}, \]

and \( e \) is a scalar parameter. Clearly, (8) reduces to (7) when \( e = 1 \). Assume the existence and uniqueness of the solution of (8) for all \( t \) in \([t_0, t_f]\) and for all \( e \) in an interval \( I \) which includes \([0, 1]\). Since the right-hand side of (8) is a polynomial in \( S_1, S_2, e \), it follows that \( S_1 \) and \( S_2 \) are infinitely differentiable with respect to \( e \) for all \( t \) in \([t_0, t_f]\) and all \( e \) in \( I \) (Ref. 6). Hence, \( S_1 \) and \( S_2 \) are analytic with respect to \( e \) in \( I \).

Let the solutions be expanded about \( e = 0 \), as follows:

\[ S_1(e, t) = \sum_{i=0}^{\infty} \left( \left[ \frac{\partial S_1(e, t)}{\partial e^i} \right]_e=0 (e^i/!) \right), \]
\[ S_2(e, t) = \sum_{i=0}^{\infty} \left( \left[ \frac{\partial S_2(e, t)}{\partial e^i} \right]_e=0 (e^i/!) \right). \]

In this section, the equations that must be satisfied by the coefficients in (13)–(14) are presented. The convergence of these series in \( I \) is guaranteed by the analyticity of \( S_1 \) and \( S_2 \). When (13) and (14) are evaluated at \( e = 1 \), the solution for the original nonzero-sum game is obtained.

2.1. Calculation of \( S_1(0, t) \) and \( S_2(0, t) \). The zeroth-order terms are the solutions of (8) for \( i = 1, 2 \), with \( e \) set equal to zero, that is,

\[ \dot{S}_i = -A'S_i - S_iA - \frac{1}{4}(Q_i - Q_j) + \frac{1}{4}S_i(E_i - F_i) S_i + \frac{1}{4}S_i(E_j - F_j) S_j \]
\[ + \frac{1}{4}S_i(E_i - F_i) S_i + \frac{1}{4}S_i(E_j - F_j) S_j, \quad S_i(0, t_j) = \frac{1}{4}(S_i^t - S_j^t). \]
where

\[ E_i = R_{ii}^{-1} B_i', \]  \hspace{1cm} (16) 

\[ F_i = R_{ii}^{-1} R_{il} R_{il}^{-1} B_i', \]  \hspace{1cm} (17) 

\( j = 2 \) when \( i = 1 \) and \( j = 1 \) when \( i = 2 \). The dependence on \( t \) has been left out for convenience in writing. Although (15) for \( i = 1, 2 \) are coupled equations, \( S_1(0, t) \) and \( S_2(0, t) \) can be obtained by solving only the lower-order matrix equation given below. Sufficient conditions which guarantee the existence and uniqueness of the solution of the reduced equation are discussed in Section 3.

Consider the equation

\[ \dot{S} = -A' \dot{S} - \dot{SA} - \frac{1}{2}(Q_1 - Q_2) + \frac{1}{2}\dot{S}(E_1 - F_1 - E_2 + F_2)\dot{S}, \quad \dot{S}(t_r) = \frac{1}{2}(S_{1r} - S_{2r}), \]  \hspace{1cm} (18)

where \( E_i \) and \( F_i \) are defined in (16)-(17). It is easily verified that

\[ S_1(0, t) = \dot{S} \]  \hspace{1cm} (19)

and

\[ S_2(0, t) = -\dot{S} \]  \hspace{1cm} (20)

satisfy (15) for \( i = 1, 2 \). Thus, \( S_1(0, t) \) and \( S_2(0, t) \) can be obtained by solving (18); and, since \( \dot{S} \) is symmetric, this entails solving \( \frac{1}{2}n(n + 1) \) coupled nonlinear scalar equations.

Equation (18) is the matrix-Riccati equation corresponding to the zero-sum differential game (1) with cost functions

\[ J_a = \frac{1}{4}(J_1 - J_2), \]  \hspace{1cm} (21) 

\[ J_s = \frac{1}{2}(J_2 - J_1), \]  \hspace{1cm} (22)

when \( J_1 \) and \( J_2 \) are given in (3). This is identical to the zeroth-order term that would be obtained if the imbedding procedure in Ref. 4 is applied.

2.2. Calculation of First-Order Coefficients. Differentiating (8) with respect to \( \epsilon \), setting \( \epsilon = 0 \), and substituting \( S_2(0, t) = -S_1(0, t) \), we have

\[ \partial S_1/\partial \epsilon = -[A' - \frac{1}{2}S_1(E_1 - F_1 - E_2 + F_2)](\partial S_1/\partial \epsilon) \]

\[ -[\partial S_1/\partial \epsilon][A - \frac{1}{2}(E_1 - F_1 - E_2 + F_2)S_1] \]

\[ -\frac{1}{4}(Q_1 + Q_2) + \frac{1}{2}S_1(E_1 + F_1 - 3E_2 - 3F_2)S_1, \]

\[ [\partial S_1/\partial \epsilon]_{t_r} = \frac{1}{2}(S_{1r} + S_{2r}). \]  \hspace{1cm} (23)
$S_i(0, t)$ can be obtained from the solution of (18), and then substituted in (23). Thus, $\partial S_i/\partial \epsilon$ can be obtained by solving the linear equation (23). Note that $\partial S_i/\partial \epsilon$ is symmetric, so that the solution of (23) involves $\frac{1}{2}n(n+1)$ linear scalar equations. Notice that, for $i = 1, 2$, the homogeneous portion of (23) remains the same. Only the forcing terms change.

The coefficients $\partial S_i/\partial \epsilon$ and $\partial S_i/\partial \epsilon$ described above are different from those that would be obtained if the imbedding procedure in Ref. 4 were to be applied. The equations resulting from (23) for $i = 1, 2$ are much simpler, because $\epsilon$ enters linearly in the coefficients of the matrix equations in (8), whereas in the corresponding matrix equations using Starr's imbedding method, $\epsilon$ would enter nonlinearly.

2.3. Calculation of kth-Order Coefficients. By repeated differentiation of (8) and setting $\epsilon = 0$, we have

$$\partial^k \tilde{S}_i/\partial \epsilon^k = -[A' - \frac{1}{2}(E_i - F_i - E_0 + F_0)][(\partial^p S_i/\partial \epsilon^p)
- (\partial^p S_i/\partial \epsilon^p) \times [A - \frac{1}{2}(E_i - F_i - E_0 + F_0) S_i]
+ \frac{1}{2} \sum_{p=1}^{k-1} [k! / p!(k-p)] [(\partial^p S_i/\partial \epsilon^p)(E_i - F_i)(\partial^k-r S_i/\partial \epsilon^k-r)]
+ (\partial^p S_i/\partial \epsilon^p)(E_i - F_i)(\partial^k-r S_i/\partial \epsilon^k-r)
+ (\partial^p S_i/\partial \epsilon^p)(E_i - F_i)(\partial^k-r S_i/\partial \epsilon^k-r) + (\partial^p S_i/\partial \epsilon^p)(E_i - F_i)(\partial^k-r S_i/\partial \epsilon^k-r)
+ \frac{1}{2} \sum_{p=0}^{k-1} [k! / p!(k-p)] [(\partial^p S_i/\partial \epsilon^p)(E_i + F_i)(\partial^k-p-1 S_i/\partial \epsilon^k-p-1)]
+ (\partial^p S_i/\partial \epsilon^p)(E_i + F_i)(\partial^k-p-1 S_i/\partial \epsilon^k-p-1)
+ (\partial^p S_i/\partial \epsilon^p)(E_i + F_i)(\partial^k-p-1 S_i/\partial \epsilon^k-p-1) -
(\partial^p S_i/\partial \epsilon^p)(E_i + F_i)(\partial^k-p-1 S_i/\partial \epsilon^k-p-1)],
\ [\partial^p S_i/\partial \epsilon^p],_i = 0, \quad (24)$$

for $k \geq 2, i = 1, 2$.

Equation (24), for $i = 1, 2, j = 2$ if $i = 1$ and $j = 1$ if $i = 2$, provides an algorithm for solving for the $k$th-order partial derivatives of $S_i$ and $S_2$, based on prior calculations of the partial derivatives of $S_i$ and $S_2$ up to order $k - 1$. Notice that (24), for $i = 1, 2$, are not coupled, so far as the calculation of the $k$th-order partial derivatives is concerned. Furthermore, the homogeneous part of (24) does not change with $i$ nor $k$. In fact, the homo-
geneous part is the same for all $k \geq 1$. Only the forcing terms change for the
calculation of the various partial derivatives.

Approximate Nash feedback strategies are obtained by truncating the
series in (13)–(14), setting $\epsilon = 1$, and substituting these approximate values of
$S_1$ and $S_2$ in (16). See summary in Fig. 1. The zeroth-order terms of $S_1$ and $S_2$
are the exact solutions when the game is zero-sum, that is, $Q_1 = -Q_2$,
$E_1 = F_1$, $E_2 = -F_2$, and $S_{1f} = -S_{2f}$. If the game is not zero-sum, but if
the norms of $E_1 + F_1$, $E_2 + F_2$, $Q_1 + Q_2$, and $S_{1f} + S_{2f}$ are much smaller
than the corresponding norms of $E_1 - F_1$, $E_2 - F_2$, $Q_1 - Q_2$, and
$S_{1f} - S_{2f}$, the game is called an almost-zero-sum game. One would intuitively
expect that, for almost-zero-sum games, an approximation for \( S_1 \) and \( S_2 \) using only a small number of terms would yield cost functions close to the exact Nash equilibrium cost functions. The degree of approximation of the Nash equilibrium cost functions is discussed in Section 4.

3. Dependence of Nash Cost Functions on Imbedding Parameter

In Section 2, parameter imbedding was introduced in the generalized matrix-Riccati equations in such a way that, when \( \epsilon = 1 \), the imbedded generalized matrix-Riccati equation (8) reduces to the original generalized matrix-Riccati equation (7). Hence, the series in (13)–(14) when \( \epsilon = 1 \) are the Nash solutions for the cost functions in (3). One might ask if the series in (13)–(14) for any \( \epsilon \neq 1 \) could correspond to Nash solutions for some other cost functions. It is readily verified that indeed (13)–(14) are Nash solutions for the differential game (1) and cost functions

\[
J_i = \frac{1}{2}x_i^T S_{ii} x_i + \frac{1}{2} \int_{t_0}^{t_f} (x_i^T \mathbf{Q} x_i + u_i^T \mathbf{R}_i u_i + u_i^T \mathbf{R}_{ij} u_j) dt,
\]

where \( S_{ii} \), \( \mathbf{Q}_i \), \( \mathbf{R}_{ii} \), \( \mathbf{R}_{ij} \) are defined in (9), (10), (11), (12); \( i = 1, 2 \), \( j = 2 \) when \( i = 1 \) and \( j = 1 \) when \( i = 2 \). One way of verifying that (13)–(14) are the Nash solutions for (25) for the differential game (1) is to apply the result in Ref. 1 for obtaining the generalized matrix-Riccati equation. Furthermore, the cost functions \( f_1 \) and \( f_2 \) reduce to \( f_1 \) and \( f_2 \) of (3) when \( \epsilon = 1 \) and satisfy the zero-sum condition \( f_1 + f_2 = 0 \) when \( \epsilon = 0 \).

Although \( \epsilon \) enters linearly in \( S_{ii} \), \( \mathbf{Q}_i \), \( \mathbf{Q}_l \), \( \mathbf{Q}_q \), the dependences on \( \epsilon \) of \( \mathbf{R}_{ii} \), \( \mathbf{R}_{li} \), \( \mathbf{R}_{lj} \), \( \mathbf{R}_{lj} \) are much more complicated. However, the matrices in the imbedded generalized matrix-Riccati equation are linear in \( \epsilon \). This linearity simplifies not only the calculation of the power series for \( S_1 \) and \( S_2 \) but also the calculation of \( f_1 \) and \( f_2 \). If the imbedding method in Ref. 4 were to be applied, the imbedded weighting matrices in \( f_1 \) and \( f_2 \) would be linear in \( \epsilon \), but the matrices in the imbedded generalized matrix-Riccati equations would have complex dependences on \( \epsilon \).

The Nash strategies for Players 1 and 2 for the cost function in (25) are

\[
u_i = -\mathbf{R}^T_{ii} \mathbf{B}_i S_i(\epsilon, t) x_i,
\]

where \( S_i(\epsilon, t) \) and \( S_i(\epsilon, t) \) are solutions of (8) for \( i = 1, 2 \). The solutions cover a class of games ranging from a zero-sum game when \( \epsilon = 0 \) to the original nonzero-sum game when \( \epsilon = 1 \).

In Section 2, it has been assumed that there exists a unique solution for the imbedded generalized matrix-Riccati equations for all \( t \) in \([t_0, t_f]\) and for
all $\epsilon$ in an interval that includes $[0, 1]$. This ensures the existence and uniqueness of a Nash solution for the imbedded nonzero-sum game with cost functions in (25). Sufficient conditions which guarantee existence of the generalized matrix-Riccati equations for the nonzero-sum game were derived by Rhodes (Ref. 3). In terms of the notation of this paper, these conditions are $S_{1f} + S_{2f} \geq 0$, $\bar{Q}_1 + \bar{Q}_2 \geq 0$, $\bar{R}_{11} + \bar{R}_{21} \geq 0$, $\bar{R}_{22} + \bar{R}_{12} \geq 0$, for all $t$ in $[t_0, t_f]$ and all $\epsilon$ in $I$, and the existence and uniqueness of the solutions of the matrix-Riccati equations for two zero-sum games, one with cost $J_1$ and another with cost $-J_2$. Furthermore, a sufficient condition for the existence and uniqueness of the solution of the matrix-Riccati equation for a zero-sum game with a cost function $J_1$ is that the relative controllability matrix

$$\int_t^{t_f} \phi(t_r, s)[\bar{B}_1, \bar{B}_1^* + \bar{B}_2, \bar{B}_2^*] \phi'(t_r, s) \, ds \quad (27)$$

is positive semidefinite for all $t$ in $[t_0, t_f]$ and for all $\epsilon$ in $I$ (Ref. 3). Similarly, for a zero-sum game with cost function $-J_2$, a sufficient condition is that the relative controllability matrix

$$-\int_t^{t_f} \phi(t_r, s)[\bar{B}_1, \bar{B}_1^* + \bar{B}_2, \bar{B}_2^*] \phi'(t_r, s) \, ds \quad (28)$$

is positive semidefinite for all $t$ in $[t_0, t_f]$ and for all $\epsilon$ in $I$. The matrix $\phi(t, \tau)$ is the transition matrix for the system in (1). From the results in Ref. 3, a sufficient condition for the existence and uniqueness of the solution for the resulting zero-sum game when $\epsilon$ is set to zero in (25) is that

$$\int_t^{t_f} \phi(t_r, s)[\bar{B}_1, \bar{R}_{11} - \bar{R}_{21} \bar{R}_{11}^{-1} \bar{B}_1^* - \bar{B}_2, \bar{R}_{22} - \bar{R}_{12} \bar{R}_{11}^{-1} \bar{R}_{22} \bar{R}_{22}^{-1} \bar{B}_2^*] \phi'(t_r, s) \, ds \quad (29)$$

is positive semidefinite for all $t$ in $[t_0, t_f]$. But if (27)–(28) are satisfied for all $\epsilon$ in $I$ (including $\epsilon = 0$), it follows that (29) is also satisfied. Hence, (27)–(28) guarantee the existence and uniqueness of the solution of (18), and hence of $S_0(0, t)$ and $S_2(0, t)$. The higher-order terms in the series of (13)–(14) are solutions of (23)–(24), which are linear differential equations with continuous coefficients and continuous forcing functions. Hence, there exist unique solutions to these equations.

4. Degree of Approximation of Cost Functions

In this section, the effects of truncation of the strategies of the players on the cost functions are examined. The power series of $J_1$ and $J_2$ in $\epsilon$ define the
functions for all $\epsilon$; and, when these are evaluated at $\epsilon = 1$, the values are the Nash cost functions for the original nonzero-sum game. The cost functions for truncated strategies will be investigated by comparing the coefficients of their MacLaurin series with those for Nash cost functions.

**Case (a).** Both players use the Nash strategies of (26) for $i = 1, 2$. Denoting the Nash cost function for Players 1 by $J_{1a}$ and that for Player 2 by $J_{2a}$, the cost functions are (Ref. 1)

$$J_{ia} = \frac{1}{2}x^T S_i x, \quad i = 1, 2,$$

(30)

where $S_i$ is given by (8). The next three cases are compared with this exact Nash equilibrium situation.

**Case (b).** Player 1 uses the strategy

$$u_i = -\bar{R}_{ii}^{-1}B_i M_i x,$$

(31)

where

$$M_i = \sum_{i=0}^{m-1} (e^i,i)[\partial^i S_j(0, t)/\partial \epsilon^i],$$

(32)

but Player 2 uses the exact Nash strategy in (26). Denote the cost function for Player 1 by $J_{1b}$ and that for Player 2 by $J_{2b}$.

Since the cost functions for linear-quadratic differential games with linear feedback strategies are always quadratic in $x$, $J_{1b}$ and $J_{2b}$ are of the form

$$J_{ib} = \frac{1}{2}x^T P_i x,$$

(33)

where $P_i$ satisfies (Ref. 1)

$$\dot{P}_i = -(A - \bar{E}_i M_i - \bar{E}_i S_i)P_i - P_i(A - \bar{E}_i M_i - \bar{E}_i S_i) - \bar{Q}_i - G_i \bar{E}_i G_i - G_i \bar{F}_i G_i,$$

$$P_i(\epsilon, t_0) = \bar{S}_{ii},$$

(34)

where

$$\bar{E}_i = \frac{1}{2}(E_i - F_i) + \frac{1}{2} \epsilon(E_i + F_i), \quad F_i = -\frac{1}{2}(E_i - F_i) + \frac{1}{2} \epsilon(E_i - F_i),$$

(35)

$$G_1 = M_1, \quad G_2 = S_1,$$

(36)

and, as before, $j = 2$ when $i = 1$ and $j = 1$ when $i = 2$. $E_i$ and $F_i$ are defined in (16)–(17), and $\bar{S}_{ij}$ is defined in (9).
It can be shown that
\[
[\partial^i J_{1b}/\partial e^i]_{e=0} = [\partial^i J_{1a}/\partial e^i]_{e=0}, \quad i = 0, 1, \ldots, 2m - 1, \tag{37}
\]
\[
[\partial^i J_{2b}/\partial e^i]_{e=0} = [\partial^i J_{2a}/\partial e^i]_{e=0}, \quad i = 0, 1, \ldots, m. \tag{38}
\]

The approximation property in (37) is the same as for optimal control (Refs. 7–8). Thus, if the first \(m\) terms of the MacLaurin series for \(u_1\) are equal to the first \(m\) terms of the Nash solution (26) with \(i = 1\) for Player 1, and if Player 2 uses the Nash strategy in (26) with \(i = 2\), then the first \(2m\) terms of the MacLaurin series of \(J_{1b}\) will be equal to the first \(2m\) terms of the MacLaurin series of the exact Nash cost function \(J_{1a}\). Player 2 will not achieve the exact payoff function \(J_{2a}\), but (38) will be satisfied. Relation (37) is proved using the same arguments as in Ref. 8. That is, \(P_i - S_i = R_i\) is formed. If the differential equations for \(P_i\) and \(S_i\) are used, it is straightforward to show that \(\partial^i T_i/\partial e^j = 0\) at \(e = 0\), \(k = 0, \ldots, 2m - 1\) when \(i = 1\) and \(k = 0, \ldots, m\) when \(i = 2\).

**Case (c).** Player 1 uses the strategy in (31)–(32) and Player 2 uses the strategy
\[
u_3 = -\tilde{R}_{23}^{-1} B_3' M_3 x, \tag{39}
\]
\[
M_2 = \sum_{i=0}^{k-1} (e^i/i!) [\partial^i S_2(0, t)/\partial e^i]. \tag{40}
\]

Denoting the cost functions for Players 1 and 2 by \(J_{1c}\) and \(J_{2c}\), we see that
\[
J_{1c} = \frac{1}{2} x' L_1 x, \quad i = 1, 2, \tag{41}
\]
where \(L_i\) satisfies
\[
L_i = -(A - E_1 M_1 - E_2 M_2)' L_i - L_i (A - E_3 M_1 - E_2 M_2) - Q_i - M_1 E_1 M_1 - M_2 E_2 M_1,
\]
\[
L_i(e, t) = S_{ij}. \tag{42}
\]

It can be shown that
\[
[\partial^i J_{1c}/\partial e^i]_{e=0} = [\partial^i J_{1a}/\partial e^i]_{e=0}, \quad i = 0, 1, \ldots, \min \left\{ \frac{2m - 1}{k} \right\}, \tag{43}
\]
and
\[
[\partial^i J_{2c}/\partial e^i]_{e=0} = [\partial^i J_{2a}/\partial e^i]_{e=0}, \quad i = 0, 1, \ldots, \min \left\{ \frac{2k - 1}{m} \right\}. \tag{44}
\]
The derivations of (43)–(44) are analogous to those of (37)–(38). Thus, if both players use first-order corrections on nominally zero-sum strategies, the resulting cost functions match the exact Nash cost functions to second order.

**Case (d).** Player 1 uses the truncated strategy in (31)–(32) and Player 2 uses the optimal strategy which minimizes \( J_2 \) knowing that Player 1 is using the strategy in (31)–(32).

The optimal strategy for Player 2 is

\[
  u_2 = -R_{22}^{-1} B_1 K_1 x,
\]

where

\[
  \dot{K}_2 = -K_2 (A - E_t M_1) - (A - E_t M_1) K_2 + K_2 E_t K_2 - Q_2 - M_1 F_2 M_1,
\]

\[
  K_2(e, t_3) = \bar{S}_{tr}.
\]

The cost functions are

\[
  J_{1d} = \frac{1}{2} x' K_i x, \quad i = 1, 2,
\]

where \( K_i \) satisfies

\[
  \dot{K}_i = -K_i (A - E_t M_1 - E_t K_2) - (A - E_t M_1 - E_t K_2)' K_i - M_1 E_t M_1 - Q_1 - K_2 F_2 K_2,
\]

\[
  K_i(e, t_3) = \bar{S}_{tr}.
\]

The cost functions \( J_{1d} \) and \( J_{2d} \) have the property that

\[
  [\partial^i J_{1d}/\partial e^j]_{e=0} = [\partial^i J_{2d}/\partial e^j]_{e=0}, \quad i = 0, 1, \ldots, m,
\]

\[
  [\partial^i J_{2d}/\partial e^j]_{e=0} = [\partial^i J_{2d}/\partial e^j]_{e=0}, \quad i = 1, 2, \ldots, m.
\]

Cases (b), (c), (d) are for Player 1 using a truncated strategy and Player 2 using either Nash, truncated, or optimal strategy knowing that Player 1 uses a truncated strategy. By symmetry, the roles of Players 1 and 2 may be interchanged to investigate two other possible strategies for Player 1 when Player 2 uses truncated strategy. The comparisons in the above cases are with respect to the exact Nash strategy of case (a). Cost functions for cases (b), (c), (d) may be compared among themselves instead of the one for case (a) by deriving differential equations for \( P_t - L_t, P_t - K_t \), and \( L_t - K_t \), \( i = 1, 2 \), and investigating to what order their partial derivatives with respect to \( e \) are zero identically.
The results are
\[
[\partial^i J_{1b}/\partial c^i]_{c=0} = [\partial^i J_{1d}/\partial c^i]_{c=0}, \quad i = 0, 1, \ldots, k, \tag{51}
\]
\[
[\partial^i J_{2b}/\partial c^i]_{c=0} = [\partial^i J_{2d}/\partial c^i]_{c=0}, \quad i = 0, 1, \ldots, k, \tag{52}
\]
\[
[\partial^i J_{1b}/\partial c^i]_{c=0} = [\partial^i J_{1d}/\partial c^i]_{c=0}, \quad i = 0, 1, \ldots, m, \tag{53}
\]
\[
[\partial^i J_{2b}/\partial c^i]_{c=0} = [\partial^i J_{2d}/\partial c^i]_{c=0}, \quad i = 0, 1, \ldots, 2m - 1, \tag{54}
\]
\[
[\partial^i J_{1e}/\partial c^i]_{c=0} = [\partial^i J_{1d}/\partial c^i]_{c=0}, \quad i = 0, 1, \ldots, \min\left\{\frac{m}{k}\right\}, \tag{55}
\]
\[
[\partial^i J_{2e}/\partial c^i]_{c=0} = [\partial^i J_{2d}/\partial c^i]_{c=0}, \quad i = 0, 1, \ldots, \min\left\{\frac{m}{k}\right\}. \tag{56}
\]

For a given initial state, the functions may be plotted against the scalar parameter \(\epsilon\). \(J_{1b}(\epsilon)\) and \(J_{2b}(\epsilon)\) are the exact Nash cost functions for a class of almost-zero-sum games, including a zero-sum game when \(\epsilon = 0\) and the original nonzero-sum game when \(\epsilon = 1\). Equations (37), (38), (43), (44), (49), (50) indicate the closeness of the curves for \(J_{1b}, J_{1e}, J_{1d}\) to \(J_{1a}\) and the closeness of the curves for \(J_{2b}, J_{2e}, J_{2d}\) to \(J_{2a}\). Equations (51)–(56) show that the different truncated strategies compared among themselves yield cost functions which are close to each other. For example, in (51)–(52), it is seen that the truncation order for Player 1 has no effect on the comparison of the cost functions between Cases (b) and (c). In fact, when \(k \rightarrow \infty\), Case (c) becomes Case (b), so that \(J_{1c}\) becomes identical to \(J_{1b}\) and \(J_{1e}\) becomes identical to \(J_{2b}\). Similarly, from (53)–(54), it is seen that, when \(m \rightarrow \infty\) Cases (b), (d), (a) become the same. For a comparison of Cases (c) and (d), (55)–(56) show that the cost function differences depend on \(m\) and \(k\) because the strategy of Player 2 in Case (d) depends on the truncation \(m\) of Player 1 and the strategies are both truncated for Case (c) and these depend on both \(m\) and \(k\).

From the above four cases, it is seen that, if Player 1 uses a truncated strategy, and if Player 2 uses a Nash strategy, the cost function of Player 2 is not as close to the exact Nash value as that of Player 1, in the sense of (37) and (38). Player 2 will achieve the same degree of approximation to \(m\)th order in the sense of (44) by truncating his strategy to \(k\)th order, where \(k\) is more than half as large as \(m\). If Player 2 uses the optimal strategy which minimizes \(J_2\) knowing that Player 1 uses a truncated strategy, his cost function would not be too different from what he would obtain had he used a truncated Nash strategy, in the sense of (44) and (50).

In solving for the cost functions in MacLaurin series form, various partial derivatives with respect to \(c\) of \(S_t, P_t, L_t, K_t\) are needed. However, the homogeneous parts of the differential equations for these partial derivatives evaluated
at \( \epsilon = 0 \) are all the same. The forcing terms are easily formed because of the simple manner in which \( \epsilon \) appears in the weighting matrices of the equations.

5. Examples

Example 5.1. A velocity-controlled pursuit–evasion game as in Ref. 1 is considered here to illustrate the procedure. Specifically,

\[
t = v_p - v_e,
\]

(57)

Fig. 2. \( S_1(t) \) and \( S_2(t) \) for various degrees of truncation for Example 1.
\[ J_p = \frac{1}{2} \sigma_p r_f r_f + \frac{1}{2} \int_0^t (v_p' v_p / c_p + v_e' v_e / c_e) \, dt, \quad \text{(58)} \]

\[ J_e = -\frac{1}{2} \sigma_e r_f r_f + \frac{1}{2} \int_0^t (v_e' v_e / c_e + v_p' v_p / c_p) \, dt. \quad \text{(59)} \]

Consider the case \( \sigma_p^2 = \sigma_e^2 = c_p c_e = 1, \ a = c_p / c_e = 0, \ b = c_e / c_p = 0, \ \alpha^2 = c_p / c_e = 4, \) and \( t_f = 0.5. \) The zeroth-order term is found from (18) to be

\[ \dot{S} = 0.75 \dot{S}_e^3, \quad S(0.5) = 1. \quad \text{(60)} \]

Thus,

\[ S_1(0, t) = \dot{S}(t), \quad S_2(0, t) = -\dot{S}(t). \quad \text{(61)} \]

The first-order terms are found from (23) to be

\[ \dot{S}_1 / \dot{S}_e = 1.5 S_1(0, t)(\dot{S}_1 / \dot{S}_e) + 0.25[S_1(0, t)]^3, \quad [\dot{S}_1 / \dot{S}_e]_{t=0.5} = 0, \quad \text{(62)} \]

\[ \dot{S}_2 / \dot{S}_e = 1.5 S_1(0, t)(\dot{S}_2 / \dot{S}_e) - 2.75[S_1(0, t)]^3, \quad [\dot{S}_2 / \dot{S}_e]_{t=0.5} = 0. \quad \text{(63)} \]

The second-order terms are found from (24) to be

\[ \ddot{S}_1 / \dot{S}_e^2 = 1.5 S_1(0, t)(\ddot{S}_1 / \dot{S}_e^2) + [2(\ddot{S}_1 / \dot{S}_e) + (\ddot{S}_1 / \dot{S}_e)(\dot{S}_2 / \dot{S}_e)] + 0.5(\dot{S}_2 / \dot{S}_e)^3 \]

\[ + [4S_1(\ddot{S}_1 / \dot{S}_e) + 4S_1(\ddot{S}_2 / \dot{S}_e) + 4S_1(\dot{S}_1 / \dot{S}_e) - S_2(\ddot{S}_2 / \dot{S}_e)], \quad (\ddot{S}_1 / \dot{S}_e^2)_{t=0.5} = 0. \quad \text{(64)} \]

\[ \ddot{S}_2 / \dot{S}_e^2 = 1.5 S_1(0, t)(\ddot{S}_2 / \dot{S}_e^2) + [0.5(\ddot{S}_2 / \dot{S}_e)^2 + 4(\ddot{S}_1 / \dot{S}_e)(\ddot{S}_2 / \dot{S}_e) + 2(\ddot{S}_1 / \dot{S}_e)^3] \]

\[ + [S_2(\ddot{S}_2 / \dot{S}_e) + 4S_1(\ddot{S}_2 / \dot{S}_e) + 4S_1(\ddot{S}_1 / \dot{S}_e) - 4S_1(\ddot{S}_1 / \dot{S}_e)], \quad (\ddot{S}_2 / \dot{S}_e^2)_{t=0.5} = 0. \quad \text{(65)} \]

Similarly, third-order and higher-order terms can be found by solving (24) for that order. Approximate strategies are then found from truncated series of (13)–(14) with \( \varepsilon = 1. \) See Fig. 2. The costs to pursuer and evader when fourth-order truncations are used are found to be \( 0.591 \times \frac{1}{2} r^2(0) \) and \( -0.241 \times \frac{1}{2} r^2(0), \) while the exact Nash strategies yield \( J_1 = 0.593 \times \frac{1}{2} r^2(0) \) and \( J_2 = -0.241 \times \frac{1}{2} r^2(0). \)

\[ ^4 \text{In the two examples of this section, } R_{12} = R_{11} = 0, \text{ so that the sufficient conditions for the existence of solutions for } S_1 \text{ and } S_2 \text{ in Ref. 3 are not satisfied. However, in these examples, it can be shown that the solutions for } S_1 \text{ and } S_2 \text{ exist.} \]
Example 5.2. An acceleration-controlled pursuit–evasion game, closely related to that given in Ref. 1, is considered. Specifically,

\[ \dot{r} = v, \quad r(0) = 1, \]  
\[ \dot{v} = a_p - a_e, \quad v(0) = 0, \]  
\[ J_p = r^2_r t + \frac{1}{2} \int_0^{0.3} (r^2 r + a_p^2 a_e) \, dt, \]  (68)

\[ J_e = -0.6r^2_r t + \frac{1}{2} \int_0^{0.3} (-1.2r^2 r + 0.4a_e^2 a_e) \, dt. \]  (69)

The presence of quadratic terms of the states in the integral has the following meaning: evader (pursuer) wants to maximize (minimize) the distance not only at the final instant but also during the course of pursuit. In general, this problem may not be transformed into one of velocity-controlled pursuit–evasion game. Thus, to obtain exact Nash strategies, it will be necessary to solve coupled Riccati-type matrix equations. Using zeroth-order truncations, one obtains \( J_1 = 1.615, J_2 = -0.884 \). Using first-order truncations, one obtains \( J_1 = 1.490, J_2 = -0.856 \). When second-order truncations are used, \( J_1 = 1.476, J_2 = -0.860 \). The exact Nash strategies yield \( J_1 = 1.478, J_2 = -0.862 \). Tables 1 and 2 show comparisons of \( S \).

Table 1. Comparison of \( S \) for Example 5.2 \( (t = 0.25) \).

<table>
<thead>
<tr>
<th>( t = 0.25 )</th>
<th>( S_1(11) )</th>
<th>( S_1(12) )</th>
<th>( S_1(22) )</th>
<th>( S_2(11) )</th>
<th>( S_2(12) )</th>
<th>( S_2(22) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact Nash strategy</td>
<td>2.30</td>
<td>0.543</td>
<td>0.133</td>
<td>-1.49</td>
<td>-0.336</td>
<td>-0.081</td>
</tr>
<tr>
<td>Zeroth-order truncation</td>
<td>1.89</td>
<td>0.437</td>
<td>0.106</td>
<td>-1.89</td>
<td>-0.437</td>
<td>-0.106</td>
</tr>
<tr>
<td>First-order truncation</td>
<td>2.32</td>
<td>0.548</td>
<td>0.134</td>
<td>-1.50</td>
<td>-0.338</td>
<td>-0.081</td>
</tr>
<tr>
<td>Second-order truncation</td>
<td>2.30</td>
<td>0.543</td>
<td>0.133</td>
<td>-1.49</td>
<td>-0.335</td>
<td>-0.081</td>
</tr>
</tbody>
</table>

Table 2. Comparison of \( S \) for Example 5.2 \( (t = 0) \).

<table>
<thead>
<tr>
<th>( t = 0 )</th>
<th>( S_1(11) )</th>
<th>( S_1(12) )</th>
<th>( S_1(22) )</th>
<th>( S_2(11) )</th>
<th>( S_2(12) )</th>
<th>( S_2(22) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact Nash strategy</td>
<td>2.96</td>
<td>1.34</td>
<td>0.647</td>
<td>-1.72</td>
<td>-0.712</td>
<td>-0.331</td>
</tr>
<tr>
<td>Zeroth-order truncation</td>
<td>2.26</td>
<td>0.990</td>
<td>0.471</td>
<td>-2.26</td>
<td>-0.990</td>
<td>-0.471</td>
</tr>
<tr>
<td>First-order truncation</td>
<td>3.15</td>
<td>1.44</td>
<td>0.694</td>
<td>-1.82</td>
<td>-0.760</td>
<td>-0.355</td>
</tr>
<tr>
<td>Second-order truncation</td>
<td>3.00</td>
<td>1.37</td>
<td>0.658</td>
<td>-1.70</td>
<td>-0.700</td>
<td>-0.325</td>
</tr>
</tbody>
</table>
6. Concluding Remarks

By a simple linear parameter imbedding of the generalized matrix-Riccati equations, strategies for two-person, linear-quadratic, nonzero-sum differential games are obtained in series form. For almost-zero-sum games, approximate Nash cost functions may be obtained with much less computation using lower-order equations, in contrast to the exact solution, which involves solving higher-order equations. The method requires solving a matrix-Riccati equation for a zero-sum game with half as many variables as the original problem and a set of linear equations with the same homogeneous part. The linear equations are also of lower dimensionality. Finally, the effect of truncating the series on the cost functions is discussed.

The numerical examples show that low-order truncations yield reasonably accurate results. In general, the method would be most useful when the cost functions correspond to almost-zero-sum games and when the dimensionality is high. Although realistic games are usually not linear with quadratic cost functions, one possible method of obtaining an approximation to Nash strategies is to expand the cost function and differential equations in Taylor series and consider strategies which are also Taylor's series. The first-order approximations in the control strategies are based on a linearized game with quadratic cost functions. This approach is analogous to that of Al'brekht and Lukes as applied to control problems (Refs. 9–10).

References


