Design of Minimum Sensitivity Systems

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Abstract—A method for the design of linear time-invariant multivariable minimum sensitivity systems is presented. The method utilizes a quadratic form in the parameter-induced output errors as a performance index to be minimized, with the constraint that a prescribed nominal transfer function matrix be obtained. An essential ingredient in the procedure is the use of a comparison sensitivity matrix. Two advantages that follow from the use of the sensitivity matrix are:

1) Physical realization constraints on the compensators may be included in the design.
2) The computational aspects of the problem are relatively simple and may be carried out routinely using any parameter optimization algorithm.

A nontrivial multivariable example illustrates the procedure. The design method may be viewed as the second part in a two-part procedure, where the first part is the determination of a desired nominal transfer function. A two-degree-of-freedom feedback structure is used to realize an optimum or desired nominal closed-loop transfer matrix, as well as a minimum in a sensitivity index.

I. Introduction

An important reason for the use of feedback structures in control system design is the possibility of reducing undesirable parameter variation effects. In this paper, a design method yielding a low-sensitivity linear time-invariant system is presented. The method is an extension of an approach suggested by Mazer[1] and further developed by Gonzales,[2] and utilizes a sensitivity matrix and corresponding scalar sensitivity performance index involving parameter-induced errors introduced by Cruz and Perkins.[3] In this approach, the design problem is...
divided into two steps:

1) The nominal system transfer function matrix is specified. For example, the transfer function may result from the optimization of an index.

2) A two-degree-of-freedom feedback structure realizing this nominal transfer-function matrix for nominal parameter values is chosen to optimize a scalar sensitivity index for a specified system input. Physical realization constraints on the controller are included in this optimization.

Many important design problems can be attacked meaningfully using this two-step approach.

The introduction of the sensitivity matrix converts the problem to a parameter optimization problem. A minimax solution to this problem is proposed in this paper. The use of the sensitivity matrix results in an important separation of the computational problem. The maximization of the index with respect to the plant parameters is performed only once and is outside the loop performing the iterative minimization on the controller parameters. This separation vastly simplifies the design.

A nontrivial example illustrates the procedure in detail.

II. Problem Statement

Consider the linear time-invariant multivariable control system shown in Fig. 1. The plant is characterized by the transfer function matrix \( P(s, \alpha) \) which is rational in \( s \), where \( s \) is the complex Laplace transform variable, and \( \alpha \) is a plant parameter whose components are unknown, but time-invariant. The compensating networks, represented by the transfer function matrices \( G(s) \) and \( H(s) \), are parameter-invariant and rational in \( s \). It is supposed that the system transfer function matrix \( T \) is specified to be \( T_n \) when the plant parameters are at their nominal values \( \alpha = \alpha_0 \):

\[
T_n = T(s, \alpha_0) = \left[ I + P(s, \alpha_0)G(s)H(s) \right]^{-1}P(s, \alpha_0)G(s)
\]  

(1)

The matrix \( T_n \) could be specified to obtain some desired time response, or to optimize some performance index, for example.

In any physical realization of this system, \( \alpha \) will differ from \( \alpha_0 \), and, thus, the output will differ from the desired output. The output error induced by the parameter variation is

\[
E(s, \alpha) = C(s, \alpha_0) - C(s, \alpha),
\]  

(2)

where

\[
\alpha = \alpha_0 + \Delta \alpha.
\]  

(3)

A measure of the effects of this parameter variation is the scalar sensitivity index,[1]

\[
J = \int_0^\infty e^t(t, \alpha)Qe(t, \alpha)dt,
\]  

(4)

where \( e(t, \alpha) \) is the inverse Laplace transform of \( E(s, \alpha) \), \( Q \) is a positive definite weighting matrix, and the prime denotes the matrix transpose. Using Parseval's theorem, (4) becomes

\[
J = \frac{1}{2\pi j} \int_{-i\infty}^{+i\infty} E'(-s, \alpha)QE(s, \alpha)ds.
\]  

(5)

In (5) the parameter-induced error is evaluated for a specific system input \( R \). Notice that \( J \) is a functional of \( G \) and \( H \). \( G \) and \( H \) are to be selected to minimize (5) subject to certain restrictions. These restrictions depend on the details of the specified system being designed. Typically, the following constraints may be imposed in a realistic design.

1) \( G \) and \( H \) must represent parameter-independent (fixed) stable compensating systems.

2) It may be desirable to avoid differentiation in either \( G \) or \( H \).

3) The final value of the parameter-induced error must be zero for (4) to be a meaningful measure of sensitivity.

4) The number of poles and zeros of \( G \) and \( H \) may be further restricted for simplicity of design, desired asymptotic frequency behavior, avoidance of infinite gain, etc.

Such constraints might be incorporated by a priori specification of the orders of the numerator and denominator polynomials in the elements of \( G \) and \( H \). The problem is then reduced to the optimum selection of the coefficients in these polynomials. The main difficulty, however, is in the loss of freedom involved in so detailed a specification of \( G \) and \( H \). How is it known that a smaller value of (5) cannot be obtained using some other choice of \( G \) and \( H \) that are equally acceptable from the realizability viewpoint? Moreover, this approach presents certain computational difficulties, as it involves the solution of simultaneous nonlinear algebraic equations with complicated side constraints due to the requirement that \( G \) and \( H \) yield a specified \( T_n \). These difficulties are reduced by the introduction of the sensitivity matrix.
III. Design Procedure

Consider an open-loop system, Fig. 2, having the same nominal plant input and the same nominal plant output as the closed-loop system of Fig. 1. Such systems are called nominally equivalent. It has been shown that the parameter variation errors are related by

\[ E(s, \alpha) = S E_o(s, \alpha), \]

where

\[ E_o(s, \alpha) = C_o(s, \alpha) - C_0(s, \alpha) \]

is the open-loop parameter-induced error, \( E(s, \alpha) \) as defined in (2), and where the sensitivity matrix \( S \) is given by

\[ S = [I + P(s, \alpha)G(s)H(s)]^{-1}. \]

In the following, only differentially small parameter variations \( \Delta \alpha = d\alpha \) will be considered. For this case, \( E_0 \) and \( E \) are differentially small. Thus, to first order, (6) becomes

\[ E(s, \alpha) = [I + P(s, \alpha_0)G(s)H(s)]^{-1} E_o(s, \alpha). \]

Therefore, the sensitivity matrix depends only on the nominal plant:

\[ J = \frac{1}{2\pi j} \int_{-\infty}^{\infty} E_o^*(-s, \alpha)S'(-s)Q(S)E_o(s, \alpha)ds. \]

The sensitivity index \( J \) now may be regarded as a functional of the sensitivity matrix \( S \), to be minimized by choice of \( S \). Recall that the input \( R \) is fixed. Once \( S \) is obtained, \( G \) and \( H \) can be found from (7) and (1):

\[ PG = S^{-1}T \]

and

\[ TH = I - S. \]

In the event (10) and (11) have nonunique solutions for \( G \) and \( H \), the designer has additional freedom in the choice of compensation networks. Equations (10) and (11) also allow physical realizability conditions on \( G \) and \( H \) to be incorporated easily into \( S \), as will be illustrated by the example in Section IV.

However, \( S \) is not completely at the designer's disposal. An examination of (8) and (1) reveals that the denominator polynomial of all entries in the matrix \( S \) is the same as the denominator polynomial of all entries in the specified system transfer function matrix \( T_n \), namely, \( \det [I + PGH] \). Thus, only the numerator polynomials in the matrix \( S \) are free. The minimum sensitivity problem, then, has been expressed as one of parameter optimization, the parameters being the coefficients in the numerator polynomials in \( S \), and the performance index optimized being the scalar sensitivity index (9).

If the plant parameter variations are represented by \( d\alpha \in \alpha \), and if the \( S \) matrix numerator coefficients are represented by the vector \( \beta \), with \( \beta \in \beta \), then the sensitivity index \( J \) may be regarded as a function \( J(\beta, d\alpha) \) of the plant parameter deviations and the sensitivity numerator coefficients. Consequently, the optimization of (9) proposed here may be indicated by

\[ J^0 = \min_{\beta \in \beta} \max_{d\alpha \in \alpha} J(\beta, d\alpha). \]

For a single-input single-output system case, the sensitivity matrix (8) becomes the familiar Bode transfer function sensitivity. If the plant contains only one parameter \( \alpha \), the problem becomes that considered by Mazer. This case simplifies considerably because the scalar sensitivity index (9) is homogeneous in \( (d\alpha)^2 \). Thus, the optimum parameters \( \beta \) are independent of \( (d\alpha) \), and the minimum of \( J \) with respect to \( \beta \) may be found without first maximizing with respect to \( d\alpha \). The more complicated situation of several variable plant parameters, but still single-input single-output, has been studied by Gonzales.

The procedure for the multivariable, multiparameter case is illustrated in detail in Section IV. The key to the simplicity of the procedure is (6), which expresses the closed-loop error as the output of a system whose transfer function matrix is \( S \) and whose input is the open-loop error of the structure of Fig. 2. The matrix \( S \) is independent of \( d\alpha \), while the input \( E_0 \) is independent of \( \beta \). Sensitivity models are employed to generate \( E_0 \). The numerator coefficients in \( S \) are then adjusted iteratively until (9) is minimized, i.e., maximized over \( d\alpha \) and minimized over the numerator coefficients of \( S \). The procedure is relatively simple to implement on a digital computer, as the example indicates.

IV. A Multivariable Example

In this section, an example of control system design using the procedure of Section III is considered. The problem is to design a noninteracting control system for a rotating dc to ac converter. The field voltages for the two machines are considered to be the two control inputs, and the speed and the rms generator voltage are outputs. The inputs will be unit step functions. This system with fixed parameters has been considered by Peschon. A detailed derivation of the plant equations for variable parameters is given in the Appendix. Small variations about a static operating point are considered, and thus the plant may be described by linear differential equations:

\[ \dot{y}_1 + [5v_0^2 - v_0'^2]y_1 + [2v_0']y_2 = -6 \left[ \frac{5v_0^2 - v_0'^2}{\sqrt{5v_0^2 + v_0'^2}} \right] u_1 \]

Fig. 2. Multivariable open-loop control system. \( C_o(s, \alpha) = T_o(s, \alpha) \)

\( R(s) \quad G_o(s) \quad F(s, \alpha) \quad C_o(s, \alpha) \)

of the plant parameter variations and the sensitivity numerator coefficients. Consequently, the optimization of (9) proposed here may be indicated by

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and
\[ y_2 = [v_0']y_1 + 2y_3 - [2v_0']y_1 = \left[ \frac{12v_0}{5v_0^3 + v_0'^3} \right] u_2, \tag{14} \]

where
\( y_1 \) = incremental speed output
\( y_2 \) = incremental rms voltage output \( \tag{15} \)

and
\( u_4 \) = incremental dc machine field voltage
\( u_3 \) = incremental ac machine field voltage. \( \tag{16} \)

The plant parameters subject to variation are the static operating values of the two normalized field voltages \( v_0 \) and \( v_0' \), as indicated. Note that operating points for nonlinear systems can be regarded as parameters in linearized models for these systems. The nominal values of these normalized parameters are both unity, and the nominal plant transfer function becomes
\[
P_s(s) = \begin{bmatrix} -4 & -4 \\ s + 6 & (s + 2)(s + 6) \\ 4 & 2(s + 4) \\ s + 6 & (s + 2)(s + 6) \end{bmatrix}. \tag{17} \]

Acceptable closed-loop response is obtained using a nominal system transfer function matrix (see Peschon, \( \text{p. 101} \)):
\[
T_s(s) = \begin{bmatrix} 8 \\ s^2 + 4s + 8 & 0 \\ 0 & 2 \\ s^2 + 2s + 2 \end{bmatrix}. \tag{18} \]

The nominal system is to be decoupled, with each channel exhibiting second-order response having a damping ratio of 0.707.

The first step in the design is to form the sensitivity matrix \( S \) and to apply realization constraints for \( G \) and \( H \) via (10) and (11). Noting that the poles of the entries of \( S \) are the same as the poles of the entries of the nominal system transfer function matrix,
\[
S = \frac{1}{D(s)} \begin{bmatrix} \beta_1s^4 + \beta_2s^3 + \beta_3s^2 + \beta_4s + \beta_5 \\
\beta_6s^4 + \beta_7s^3 + \beta_8s^2 + \beta_9s + \beta_{10} \\
\beta_{11}s^4 + \beta_{12}s^3 + \beta_{13}s^2 + \beta_{14}s + \beta_{15} \\
\beta_{16}s^4 + \beta_{17}s^3 + \beta_{18}s^2 + \beta_{19}s + \beta_{20} \end{bmatrix} \tag{19} \]

where
\[
D(s) = (s^2 + 4s + 8)(s^2 + 2s + 2). \]

For the existence of (4),
\[
\lim_{s \to \infty} e(t, v_0, v_0') = 0 \tag{20} \]

for all \( v_0, v_0' \). Thus, it is required that
\[
\lim_{s \to 0} sE(s, v_0, v_0') = 0. \tag{21} \]

From (6), it can be shown that
\[
E = SdPU_0, \tag{22} \]

where \( U_0 \) is the open-loop plant input,
\[
U_0 = G_0R. \tag{23} \]

For this problem,
\[
R = \begin{bmatrix} V(s) \\ V'(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{s} \\ \frac{1}{s} \end{bmatrix}. \tag{24} \]

So, if (21) is to be satisfied for all \( dP \),
\[
\lim_{s \to \infty} S(s) = 0. \tag{25} \]

Thus
\[
\beta_5 = \beta_{10} = \beta_{15} = \beta_{20} = 0. \tag{26} \]

From (11), it can be seen that a reasonable bandwidth realization constraint is
\[
\lim_{s \to \infty} S(s) = I, \tag{27} \]

as this will ensure that the entries of \( H \) will increase with frequency no faster than the entries of \( T \) decrease. Applying (27) to (19) yields
\[
\beta_1 = \beta_{10} = 1 \\
\beta_5 = \beta_{11} = 0. \tag{28} \]

The forms of \( G \) and \( H \) may be checked using (10) and (11). The form of \( G \) is satisfactory, since each entry contains more poles than zeros. If differentiation is permitted, the structure of \( H \) resulting from (11) will be satisfactory. If differentiation is not acceptable, further constraints among the \( \beta 's \) may be imposed to eliminate the undesirable terms. For this example, it will be assumed that differentiation in \( H \) is acceptable.

The final realization constraints to be imposed concern the stability of the \( G \) and \( H \) compensators. The poles of the entries of \( H \) do not depend on the \( \beta 's \), and thus the stability of \( H \) is not affected by the choice of \( \beta 's \). The poles of the entries of \( G \) are affected by the \( \beta 's \), however, and a check for stability must be incorporated in the design procedure.

Thus, the conditions of (26) and (28) result in the sensitivity matrix
\[
S(s) = \begin{bmatrix} s \\ (s^2 + 4s + 8)(s^2 + 2s + 2) \end{bmatrix} \begin{bmatrix} s^2 + 2s + 2 \\
\beta_1s^4 + \beta_2s^3 + \beta_3s^2 + \beta_4s + \beta_5 \\
\beta_6s^4 + \beta_7s^3 + \beta_8s^2 + \beta_9s + \beta_{10} \\
\beta_{11}s^4 + \beta_{12}s^3 + \beta_{13}s^2 + \beta_{14}s + \beta_{15} \\
\beta_{16}s^4 + \beta_{17}s^3 + \beta_{18}s^2 + \beta_{19}s + \beta_{20} \end{bmatrix}. \tag{29} \]
Hence, there are 12 $\beta$'s available for adjustment.

For this example, the plant parameter variations are given by the tolerance ranges

\begin{align*}
0.8 & \leq v_0 \leq 1.2 \\
0.8 & \leq v'_0 \leq 1.2. \quad (30)
\end{align*}

The actual minimax calculation is carried out by computer. The calculation is simplified because the index $J(\beta, da)$ of (12) is convex in $da$. Furthermore, because of (30), the plant parameter space is a convex polyhedron in Euclidean space (a rectangle in this two-parameter example). If this polyhedron is denoted by $\Omega$ and the set of its vertices by $\Omega_0$, then it has been shown (Gonzales[5] and Salmon[6]) that if $\beta$ satisfies

\begin{equation}
\max_{da \in \Omega_0} J(\beta, da) = \min_{\beta \in \Theta} \max_{da \in \Omega_0} J(\beta, da), \quad (31)
\end{equation}

then $\beta$ is the minimax solution, that is,

\begin{equation}
J^*(\beta) = \min_{\beta \in \Theta} \max_{da \in \Omega_0} J(\beta, da). \quad (32)
\end{equation}

For this example, only two vertices, rather than four, need be considered because of the symmetry of the tolerance ranges (30), coupled with the quadratic nature of $J$ with respect to $da$.

The basic procedure, then, is to compute the maximum with respect to $da$ of $J(\beta, da)$ for the vertices $da \in \Omega_0$ and then minimize with respect to $\beta$. The computation of $J$ and the resulting minimization could be done for the closed-loop system of Fig. 1. The computation is considerably simplified, however, if the open-loop error is calculated first, and then related to closed-loop error, and hence $J$, by the sensitivity matrix $S$. The open-loop error, to first order, is

\begin{equation}
e_0 = dc_0 = \begin{bmatrix} dy_1 \\ dy_2 \end{bmatrix}, \quad (33)
\end{equation}

where $y_1$ is the speed and $y_2$ is the rms generator voltage. However,

\begin{align*}
dy_1 &= \frac{\partial y_1}{\partial v_0} dv_0 + \frac{\partial y_1}{\partial v'_0} dv'_0 \\
dy_2 &= \frac{\partial y_2}{\partial v_0} dv_0 + \frac{\partial y_2}{\partial v'_0} dv'_0. \quad (34)
\end{align*}

The partial derivatives appearing in (34) are open-loop sensitivity and can be readily computed from sensitivity models.[6] The iterative computational procedure is shown as a flow chart in Fig. 3. The values of the quantities at the two vertices are denoted by superscripts $A$ and $B$.

For this example, the minimax solution was obtained using a computer procedure devised by Salmon.[6] The minimization scheme implemented was proposed by Rosenbrock.\[11\]

The main virtues of using the sensitivity matrix $S$ in this procedure are twofold. First, to compute $e_0, e_0$ need be computed only once for each $da \in \Omega_0$, since this calculation is done outside the iterative loop for $\beta$. Only $S$ is varied as $\beta$ changes. The second advantage is that the minimization with respect to $G$ and $H$ is reduced to a parameter minimization with respect to $\beta$.

For the rotating converter example, the optimum values of the $\beta$'s of (29) are:

\begin{align*}
\beta_2 &= 2.588 & \beta_{12} &= -0.2915 \\
\beta_3 &= 2.855 & \beta_{13} &= -0.7754 \\
\beta_4 &= 0.2888 & \beta_{14} &= -0.7332 \\
\beta_5 &= 0.5194 & \beta_{15} &= 2.490 \\
\beta_6 &= 1.297 & \beta_{16} &= 2.643 \\
\beta_7 &= 1.391 & \beta_{17} &= 0.6319. \quad (35)
\end{align*}

The corresponding compensators are

\begin{align*}
G &= \frac{1}{s\Delta(s)} \begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{bmatrix} \quad (37)
\end{align*}

\begin{align*}
H &= \begin{bmatrix} 3.41s^5 + 15.1s^3 + 37.7s^2 & + 16 \\
8(s^2 + 2s + 2) & -s(0.519s^2 + 1.297s + 1.39) \end{bmatrix} \\
& \quad \begin{bmatrix} 8(s^2 + 2s + 2) \\
3.51s^3 + 15.4s^2 + 23.4s + 16 \end{bmatrix} \\
& \quad \begin{bmatrix} 2(s^2 + 4s + 8) \\
2(s^2 + 4s + 8) \end{bmatrix} \quad (36)
\end{align*}
where

\[
g_{11}(s) = -8(s^2 + 2s + 2)(0.25s^4 + 3.245s^3 \\
+ 5.782s^2 + 4.305s - 0.127)
\]

\[
g_{12}(s) = -2(s^2 + 8)(1.29s^3 + 4.50s^2 \\
+ 6.69s + 3.22)
\]

\[
g_{21}(s) = -8(s + 2)(s^2 + 2s + 2)(s^3 + 3.01s^2 \\
+ 3.94s + 2.02)
\]

\[
g_{22}(s) = 2(s^2 + 8)(s^3 + 2.64s^2 + 2.08s \\
- 0.444)(s + 2)
\]

\[
\Delta(s) = s^6 + 5.08s^5 + 12.1s^4 + 15.6s^3 + 11.7s^2 \\
+ 4.60s + 1.20.
\]

(38)

Response curves for step inputs are shown in Fig. 4. Responses are shown for several combinations of parameter variations, and for nominal parameter values.

For comparison purposes, responses are shown in Fig. 5 for a two-degree-of-freedom design suggested by Peschon[4] for achieving the same nominal transfer function matrix. Peschon uses the configuration of Fig. 1 with

\[
G(s) = \begin{bmatrix}
4 \\
-4
\end{bmatrix}.
\]

(39)

\[
H(s) = \begin{bmatrix}
1 \\
-s/4
\end{bmatrix}.
\]

(40)

The responses of Fig. 4 clearly exhibit less departure from nominal than do the responses of Fig. 5.

V. CONCLUSIONS

In this paper, a design procedure for a minimum sensitivity feedback system was proposed. The sensitivity index used is a quadratic form in the output error induced by plant parameters which are subject to variation. By using a comparison sensitivity matrix, the computation of the closed-loop error for a specified system input can be carried out by first computing the open-loop error, which depends on the plant parameter variations alone. The minimization with respect to the controllers reduces to a parameter optimization problem involving the comparison sensitivity matrix alone. The separation of the plant parameter variations from the control parameter perturbations greatly simplifies the computation of the minimum sensitivity controllers.

The method was applied to a multivariable dc to ac converter example. The system has two plant parameters subject to variation. Based on the design procedure, 12 control parameters were chosen to achieve minimum sensitivity.

Fig. 4. (a) Minimum sensitivity system output \(y_1(t)\) for unit step input \(u_1\) and unit step input \(u_2\) of dc to ac converter example for several plant parameters. The four plots correspond to \((v_0 = 1.0, \ t_0' = 1.0, \ t_0'' = 0.8, \ t_0''' = 0.8, \ t_0'''' = 1.2), \ (v_0 = 0.8, \ t_0' = 0.8, \ t_0'' = 1.2, \ t_0''' = 1.2), \ (v_0 = 0.8, \ t_0'' = 1.2), \ (t_0'' = 0.8, \ t_0'''' = 1.2). \) One division of the horizontal scale corresponds to one normalized time unit. Ten divisions of the vertical scale correspond to one unit output. (b) Minimum sensitivity system output \(y_2(t)\) for inputs, parameters, and scales as in (a).
In this Appendix, the linearized equations for the rotating converter are obtained. The converter consists of a dc and an ac machine connected in tandem, as shown in Fig. 6. The following nomenclature is introduced.

**Dc Machine**

\[
\begin{align*}
V &= \text{armature voltage} \\
I &= \text{armature current} \\
v &= \text{field voltage} \\
i &= \text{field current} \\
\omega &= \text{angular shaft velocity} \\
R &= \text{armature resistance} \\
r &= \text{field resistance}
\end{align*}
\]

Motor torque: \( T = k_3 I \), neglecting saturation in the dc machine.

Back emf: \( e_a = k_3 i \omega \), neglecting saturation.

Armature and field inductances are negligible.

**Ac Machine**

\[
\begin{align*}
V' &= \text{rms generator voltage} \\
I' &= \text{rms load current} \\
v' &= \text{field voltage} \\
i' &= \text{field current} \\
R_1 &= \text{load resistance} \\
r' &= \text{field resistance} \\
l' &= \text{field inductance}
\end{align*}
\]

Back emf constant = \( k_2 \) (neglect saturation).

Torque constant = \( k_3 \) (neglect saturation).

Armature inductance and resistance are negligible.

\( J \) = moment of inertia of both rotors.

Rotor mechanical damping and shaft springiness are negligible.

Using the above assumptions and nomenclature,

\[
\begin{align*}
V &= IR + k_3 i \omega \quad (41) \\
v &= ri \\
v'\frac{di'}{dt} + r'i' &= v' \quad (43) \\
V' &= k_3 i' \omega = l'R_1 \\
k_3 i'I &= J\omega + k_3 l'I'. \quad (45)
\end{align*}
\]

The variables \( v \) and \( v' \) are designated as inputs, and \( \omega \) and \( V' \) as outputs. \( V \) is to be fixed, a constant. Other variables are to be eliminated. Thus, (45) becomes

\[
k_1 \frac{v}{R} \left[ V - k_3 \frac{i \omega}{r} \right] = J\omega + k_2 \left( \frac{V'}{k_3} \right) \left( \frac{V'}{R_1} \right). \quad (46)
\]

Similarly, (43) becomes

\[
v' = r' \left[ \frac{V'}{k_3} \right] + l' \frac{d}{dt} \left( \frac{V'}{k_3} \right). \quad (47)
\]
resulting operating point can be found. Suppose, for example, that the desired nominal operating point is given by
\[ v_0 = 1.0 \quad \omega_0 = 1.0 \]
\[ v_0' = 1.0 \quad \omega_0' = 1.0 \]
\[ i_0 = 1.0 \quad I_0' = 1.0. \]

From (41) through (45),
\[ V_0 = R + k_1 \]
\[ 1 = r \]
\[ r' = 1 \]
\[ 1 = k_2 = R_i \]
\[ k_1 = k_2. \]

Hence,
\[ V_0 = R + 1 \]
\[ r = 1 \]
\[ r' = 1 \]
\[ R_i = 1 \]
\[ k_1 = k_2 = 1 \]

will yield the desired nominal operating point. If \( R=0.2 \), for example, then \( V_0=1.2 \). Two further machine parameters, which do not affect the static operating point, may be specified independently. Suppose
\[ J = 1, \quad J' = 0.5. \]

With the machine parameters chosen as in (56) and (57), (52) and (53) become, for any operating point,
\[ \dot{y}_1 + \left[ \frac{k_1 v_0^2 - V_0'^2}{R_0 \omega_0^2} \right] y_1 + \left[ \frac{2V_0'}{R_0 \omega_0} \right] y_2 = \left[ \frac{k_1 M_0}{R_0} - \frac{2k_1 v_0^2 \omega_0}{R_0 \omega_0^2} \right] u_1 \]
and
\[ \left[ \frac{1}{k_2 \omega_0} \right] \dot{y}_2 - \left[ \frac{V_0'}{k_2 \omega_0} \right] \dot{y}_1 + \left[ \frac{1}{\omega_0} \right] y_2 = \left[ \frac{V_0'}{\omega_0^3} \right] y_1 = u_2. \]

For the example treated in Section IV, \( v_0 \) and \( v_0' \) are selected as plant parameters. Hence, \( V_0' \) and \( \omega_0' \) must be expressed in terms of \( v_0 \) and \( v_0' \). This is done using (49) and (50) with machine parameters as in (56) and (57). Equations (58) and (59) then become
\[ \dot{y}_1 + \left[ 5v_0^2 - v_0'^2 \right] y_1 + \left[ 2v_0' \right] y_2 = -6 \left[ \frac{5v_0^2 - v_0'^2}{5v_0^2 + v_0'^2} \right] u_1 \]
\[ \dot{y}_2 + \left[ 5v_0^2 - v_0'^2 \right] y_1 + \left[ 2v_0' \right] y_2 = -6 \left[ \frac{5v_0^2 - v_0'^2}{5v_0^2 + v_0'^2} \right] u_2. \]
\[ y_2 - [v'_2]y_1 + 2y_2 - [2v'_2]y_1 = \left[ \frac{12v_0}{5v_0^2 + v_0^2} \right] u_2. \]  

These equations are used as the plant model in (13) and (14).

**References**


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