Image Processing:
4. Optical Flow

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Motion estimation
- Optical flow is used to compute the motion of the pixels of an image sequence. It provides a dense (point to point) pixel correspondence.
- Correspondence problem: determine where the pixels of an image at time $t$ are in the image at time $t+1$.
- Large number of applications.

Two important definitions
- Motion field: “the 2-D projection of a 3-D motion onto the image plane.”
- Optical flow: “the apparent motion of the brightness pattern in an image sequence.”

The method of Horn and Schunck
- This is the most fundamental optical flow algorithm.
- As you will see, it has several important flaws that makes its use inappropriate in a large number of applications.
- Most of the other algorithms proposed to date are based on the formulation advanced by Horn and Schunck.
If the brightness is assumed to be constant from frame to frame, then the motion associated to each pixel \((x, y)\) of an image \(I\) can be modeled as:

\[
I(x, y, z) = I(x + u\delta t, y + v\delta t, t + \delta t)
\]

This is known as the data conservation constraint.

The 1st-order Taylor expansion

\[
I_{x}u + I_{y}v + I_{t} = 0
\]

\[
E_{D} = \iint_{R} (I_{x}u + I_{y}v + I_{t})^2 \, dx \, dy
\]

The solution can be obtained by minimizing the functional:

\[
E_{D} + \lambda E_{S}
\]

Minimization

\[
\iint_{R} F(u, v, u_{x}, u_{y}, v_{x}, v_{y}) \, dx \, dy
\]

To minimize the above integral, we can use calculus of variations. The Euler equations are:

\[
F_{u} - \frac{\partial}{\partial x} F_{v} - \frac{\partial}{\partial y} F_{v} = 0
\]

\[
F_{v} - \frac{\partial}{\partial x} F_{u} - \frac{\partial}{\partial y} F_{u} = 0
\]

And we want to minimize the expression:

\[
F = (u_{x}^2 + u_{y}^2) + (v_{x}^2 + v_{y}^2) + \lambda (I_{x}u + I_{y}v + I_{t})^2
\]

The discrete case

- We can estimate the derivatives, \(I_{x}\) and \(I_{y}\), by using the following discrete approximation:

\[
I_{x} = \frac{1}{4} \left( I_{i+1,j,k} + I_{i-1,j+1,k} + I_{i+1,j+1,k} + I_{i+1,j+1,k+1} \right)
- \frac{1}{4} \left( I_{i,j,k} + I_{i+1,j+1,k} + I_{i,j,k+1} + I_{i,j,k+1} \right)
\]

Note that:

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
\]

Limitations

• The assumptions embedded in Horn & Schunck formulation are generally inappropriate.
• E.g. both constraints are violated at motion boundaries – known as motion discontinuities.
• The aperture problem: to gain accuracy $R$ needs to be large, however, the larger $R$ is the most probable that our assumptions become invalid.

Robust Statistics: Black & Anandan

• It is possible to regard motion discontinuities as outliers.
• We can discard outliers only if we can detect them.
• Outliers normally deviate largely from the mean motion.

Same constraints as in Horn & Schunck

$$E_n = \int \int \rho(I_u + I_v + I) dx dy$$

$$E_s = \int \int \rho(V(a,v)) dx dy$$

New estimator, e.g.:

$$\rho(x, \sigma) = \log \left( 1 + \frac{1}{2} \left( \frac{x}{\sigma} \right)^2 \right)$$
Examples

\[ \lambda = 1 \]

\begin{itemize}
  \item The methods seen above assume no changes in the illumination of a scene from frame to frame.
  \item This is not realistic; even if the illumination source(s) is (are) not moving.
  \item It is possible to modify the constraint used by the two preceding method.
\end{itemize}

Illumination Changes:
Negahdaripour

\begin{itemize}
  \item Illumination changes also violate the assumptions of \( E_0 \) and \( E_s \).
  \item B&A approach cannot handle large variations in lighting. Its formulation does not take this into account.
  \item We can easily incorporate this information in the form of a multiplier and an offset:
\end{itemize}

\[ I(x + u\delta, y + v\delta, t + \delta) = M(x, y, t)I(x, y, t) + C(x, y, t) \]
Gennert & Negahdaripour

\[ E_B = \iint_R (I_{xy} + I_{yx} - I_{x} - I_{y}) \, dx \, dy \]
\[ E_s = \iint_R \nabla(u,v) \, dx \, dy \]
\[ E_M = \iint_R \nabla m \, dx \, dy \]
\[ E_C = \iint_R \nabla c \, dx \, dy \]
\[ E = \lambda_1 E_B + \lambda_2 E_s + \lambda_3 E_M + \lambda_4 E_C \]

Closed solution

Negahdaripour has proposed the following closed-form solution:

\[ \sum_i \begin{bmatrix} I_{xx} & I_{xy} & -I_{x} & -I_{y} \\ I_{yx} & I_{yy} & -I_{y} & -I_{x} \\ -I_{x} & -I_{y} & I & 1 \\ -I_{y} & -I_{x} & 1 & I \end{bmatrix} \begin{bmatrix} \partial x \\ \partial y \\ \partial u \\ \partial v \end{bmatrix} = \begin{bmatrix} -I_{x}I_{y} \\ -I_{x}I_{y} \\ H_x \\ H_y \end{bmatrix} \]
Error estimation

$$
Err_{vel} = \sum \frac{||V_{estimate} - V_c||}{n_i} \\
Err_{ang} = \sum \arccos \left( \frac{V_{estimate} \cdot V_c}{||V_{estimate}|| ||V_c||} \right)
$$

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### Flower sequence

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<td>H&amp;S</td>
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### Yosemite sequence

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<td>41.999</td>
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Some Examples