Identification of PV Source Models

Dhruv Kapoor, Ali Keyhani and Abir Chatterjee
The Ohio State University
Columbus, Ohio 43210
Keyhani@ece.osu.edu

***Abstract*—**Smart Grids are one of the solutions proposed to handle today's increasing energy demands in a reliable, efficient, and clean fashion. Smart Grids often utilize a number of sources that are not a part of the main utility supply to either augment the supply, or to feed power back to the mains. This report attempts to develop a system for modeling of a solar cell array that would likely function as a part of an entire smart grid installation. The objective is to develop in thorough detail a method to find the parameters of the single-exponential model using only values provided in the data sheet of the panel. The report expounds step by step modeling procedure with clarity and in greater mathematical detail, and provides a thorough and intensive set of algorithms to successfully solve the equations developed herein.

I. INTRODUCTION

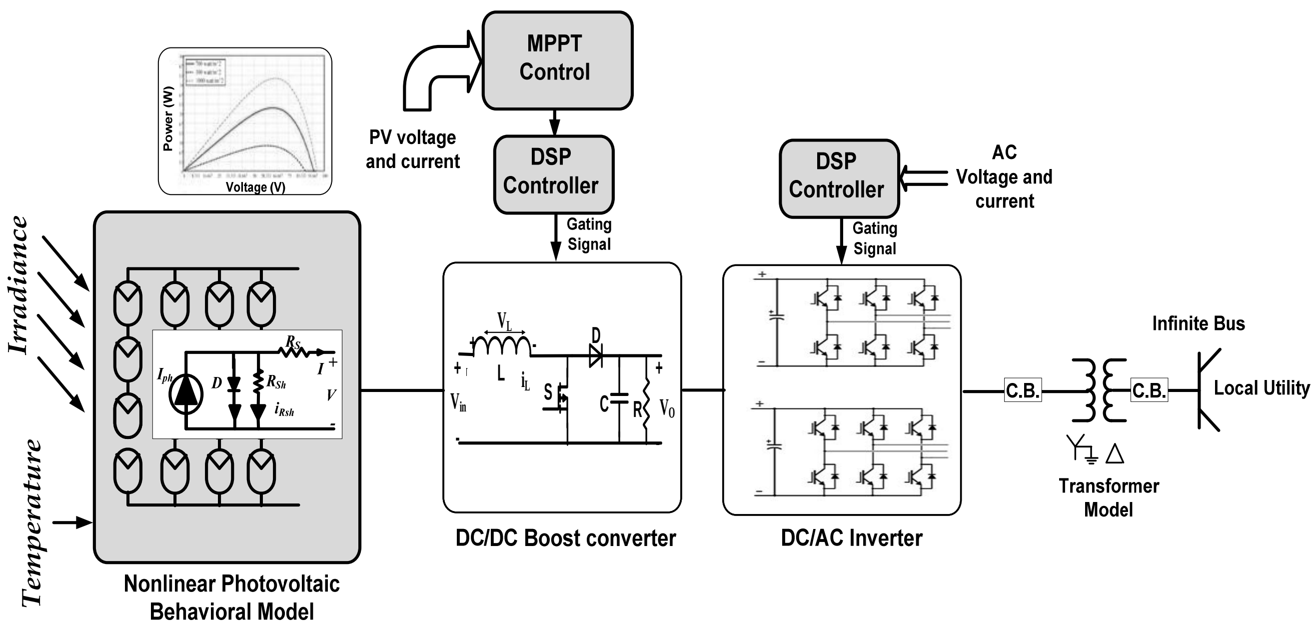

Fig. 1. The structure of a basic Smart Grid installation showing the PV panel connected to the grid

In recent years, numerous countries have turned toward solar power installations to meet their national energy requirements. As of 2008, 13.4 GW of PV capacity is installed in IEA PVPS countries, an annual growth rate of 71%, with Germany leading in terms of installed capacity, followed by Spain [26]. Importantly, this growth has fueled the increase in the share of grid-connected systems, with only 5.5% of the systems in IEA PVPS countries off the grid [26]. In light of these developments, it is pertinent that unreliable, inefficient, and aging utility grids be replaced by cleaner, more connected, and more robust systems. One of the solutions proposed is the Smart Grid. With an ever increasing number of localized, grid-connected, solar and wind installations, often at consumers' homes, the grids of tomorrow need to be able to integrate these systems efficiently to boost the installed capacity, while providing bi-directional metering to compensate and encourage local producers of power. Smart Grids have been described as [27] grids that are more connected, automated, and coordinated between suppliers, consumers and networks, whether at the local or long-distance transmission level.

With the increasing focus on Smart Grids, developing faster and more accurate models for each of the individual components of the Smart Grid system can lead to improvements in the way Smart Grids are developed and implemented. As can be discerned from Fig. 1, a systematic approach that deals with each of the individual components of the Smart Grid system is called for.

Given the importance of solar power within the Smart Grid system, it is imperative that better models be found that can more accurately track the behavior of various components of a PV system, among them, the PV cell itself, inverters, and storage batteries. This report focuses on the extraction of parameters for the single-diode model of a PV cell [28] using only data supplied in the manufacturers' data sheets for the formulation of the equations involved in the problem, while not compromising on the accuracy or efficacy of the model. The report then goes to analyze possible methods of solution of these implicit equations, and modes of determination of initial values to be used in their solution.

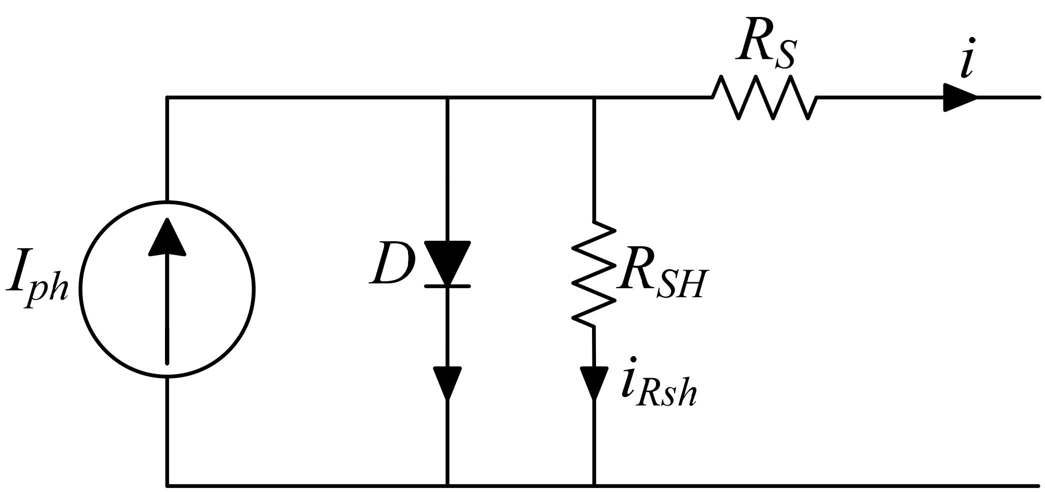

Fig. 2. Equivalent circuit of a photovoltaic cell using the single exponential model

Shown above is the single-diode model of the PV cell, with the general current-voltage characteristics based on the single exponential model given by:

$$I = I_{ph} - I_o\left(e^{\frac{V+IR_s}{n_s V_t}} - 1\right) - \frac{V + IR_s}{R_{sh}} \tag{1}$$

V_t, the junction thermal voltage, is given as:

$$V_t = \frac{AkT_{STC}}{q} \tag{2}$$

As shall be elucidated on later, it is conducive to express the equations in terms of V_t rather than A. The value of A can be determined easily if V_t is found, by simply rearranging the terms of Eq. (2) to give:

$$A = \frac{qV_t}{kT_{STC}} \tag{3}$$

A table with a brief descriptions of all the symbols used is provided below.

Table 1. Summary of Notations

$\boldsymbol{I_{ph}}$	Photo-generated current at STC
$\boldsymbol{I_o}$	Dark saturation current at STC
$\boldsymbol{R_s}$	Panel series Resistance
$\boldsymbol{R_{sh}}$	Panel parallel (shunt) resistance
$\boldsymbol{n_s}$	Number of cells in the panel connected in series
$\boldsymbol{V_t}$	Junction thermal voltage
$\boldsymbol{A}$	Diode quality (ideality) factor
$\boldsymbol{k}$	Boltzmann's constant
$\boldsymbol{T_{STC}}$	Temperature at STC[1] in Kelvin
$\boldsymbol{q}$	Charge of the electron

The term '-1' in (1), being much smaller than the exponential term, is generally ignored.

II. OBTAINING THE EXPRESSIONS FOR SINGLE EXPONENTIAL PARAMETERS

Deriving the set of equations that will later be studied is relatively simple, and the method to derive them is adopted from [28]. The five parameters to be determined are I_{ph}, I_o, R_s, R_{sh}, and A. Since A can be expressed easily in terms of V_t, q, k, and T_{stc}, of which only the former is unknown, the approach will be to first obtain V_t, and then solve (3) for A.

Since the objective was to derive expressions for the parameters that only require datasheet values, three key points of the V-I characteristic will be employed: the points at which circuit is short-circuited and open-circuited, and the maximum power point. Table 2 summarizes the symbols provided in the datasheet pertaining to these points of interest.

Table 2. Datasheet Values Used

$\boldsymbol{I_{sc}}$	Short-circuit current at STC
$\boldsymbol{V_{oc}}$	Open-circuit voltage at STC

[1] STC denoted the standard conditions used to measure the nominal output power of photovoltaic cells. The cell junction temperature at STC is 25℃, the irradiance level is 1000W/m^2, and the reference air mass is 1.5 solar spectral irradiance distribution.

$\boldsymbol{V_{mpp}}$	Voltage at the Maximum Power Point (MPP) at STC
$\boldsymbol{I_{mpp}}$	Current at the MPP at STC

To begin with, equation (1) is written at these three points of interest, thus giving:

$$I_{sc} = I_{ph} - I_o e^{\frac{I_{sc}R_s}{n_s V_t}} - \frac{I_{sc}R_s}{R_{sh}} \tag{4}$$

$$I_{mpp} = I_{ph} - I_o e^{\frac{V_{mpp}+I_{mpp}R_s}{n_s V_t}} - \frac{V_{mpp} + I_{mpp}R_s}{R_{sh}} \tag{5}$$

$$I_{oc} = 0 = I_{ph} - I_o e^{\frac{V_{oc}}{n_s V_t}} - \frac{V_{oc}}{R_{sh}} \tag{6}$$

Since the MPP corresponds to the point where the power is maximum on the V-I characteristic, we have:

$$\left.\frac{dP}{dV}\right|_{\substack{V=V_{mpp}\\ I=I_{mpp}}} = 0 \tag{7}$$

Since there are five parameters that need to be determined, a fifth equation is still needed. The derivative of the current with the voltage at short-circuit is given as the negative of the reciprocal of R_{sh_o}, which, as shall be shown later, is the same as the parameter R_{sh}.

$$\left.\frac{dI}{dV}\right|_{I=I_{sc}} = -\frac{1}{R_{sh_o}} \tag{8}$$

Hence, five equations in five variables have been established. Since the remainder of the development will focus solely on finding an easy and efficient way to solve the equations, it will be conducive to reduce the clutter above and represent the development in a more lucid, yet mathematically rigorous fashion. In order to do establish this form, x will represent the vector of unknowns, and a will represent the vector of known values, and the transformation will take the form:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} I_{ph} \\ I_o \\ V_t \\ R_s \\ R_{sh} \\ R_{sh_o} \end{bmatrix} \tag{9}$$

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} I_{sc} \\ V_{oc} \\ V_{mpp} \\ I_{mpp} \\ n_s \end{bmatrix} \tag{10}$$

Note that although it is A that is the actual parameter that needs to be determined, V_t has been listed as the third unknown since the expressions will be in terms of V_t, which is related to A by a constant through Eq. (2).

The symbols for current, voltage, and power, while not representing any parameters in themselves, play an important role in reducing the problem mathematically. In keeping with the changes in other variables, these will now be represented by the vector y as follows:

$$y = \begin{bmatrix} I \\ V \\ P \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{11}$$

After applying the change to a more mathematically conducive form, the equation of the single exponential model, Eq. (1), changes to:

$$y_1 = x_1 - x_2 \left(e^{\frac{y_2 + y_1 x_4}{a_5 x_3}} - 1 \right) - \frac{y_2 + y_1 x_4}{x_5} \tag{12}$$

As mentioned earlier, the term '-1', in the bracket in the second term of Eq. (12), is insignificant compared to the exponential term, and can generally be ignored, giving:

$$y_1 = x_1 - x_2 e^{\frac{y_2 + y_1 x_4}{a_5 x_3}} - \frac{y_2 + y_1 x_4}{x_5} \tag{13}$$

The five basic equations, from (4) to (8), take the form:

$$a_1 = x_1 - x_2 e^{\frac{a_1 x_4}{a_5 x_3}} - \frac{a_1 x_4}{x_5} \tag{14}$$

$$a_4 = x_1 - x_2 e^{\frac{a_3 + a_4 x_4}{a_5 x_3}} - \frac{a_3 + a_4 x_4}{x_5} \tag{15}$$

$$0 = x_1 - x_2 e^{\frac{a_2}{a_5 x_3}} - \frac{a_2}{x_5} \tag{16}$$

$$\left. \frac{dy_3}{dy_2} \right|_{\substack{y_2 = a_3 \\ y_1 = a_4}} = 0 \tag{17}$$

$$\left. \frac{dy_1}{dy_2} \right|_{y_1 = a_1} = -\frac{1}{x_6} \tag{18}$$

Table 3 below sums up all the changes made.

Table 3. Table of Transformations

$\mathbf{I_{ph}}$	$\boldsymbol{x_1}$	Photo-generated current at STC
$\mathbf{I_o}$	x_2	Dark saturation current at STC
$\mathbf{V_t}$	x_3	Junction thermal voltage

$\mathbf{R_s}$	x_4	Panel series Resistance
$\mathbf{R_{sh}}$	x_5	Panel parallel (shunt) resistance
$\mathbf{R_{sh_o}}$	x_6	Panel parallel (shunt) resistance
$\mathbf{I_{sc}}$	a_1	Short-circuit current at STC
$\mathbf{V_{oc}}$	a_2	Open-circuit voltage at STC
$\mathbf{V_{mpp}}$	a_3	Voltage at the Maximum Power Point (MPP) at STC
$\mathbf{I_{mpp}}$	a_4	Current at the MPP at STC
$\mathbf{n_s}$	a_5	Number of cells in the panel connected in series
I	y_1	Current
V	y_2	Voltage
P	y_3	Power

III. FURTHER DEVELOPMENT AND REDUCTION INTO SIMPLER FORM

The objective of this section, as indicated earlier, will be to solve the derivatives in (17) and (18) to obtain a set of equations that will be relatively simple to solve for the required parameters. To begin with, Eq. (16) can be rearranged as:

$$x_1 = x_2 e^{\frac{a_2}{a_5 x_3}} + \frac{a_2}{x_5} \tag{19}$$

x_1 can now be inserted from Eq. (19) into (13) and (14) respectively.

$$y_1 = x_2\left(e^{\frac{a_2}{a_5 x_3}} - e^{\frac{y_2 + y_1 x_4}{a_5 x_3}}\right) + \frac{a_2 - y_2 - y_1 x_4}{x_5} \tag{20}$$

$$a_1 = x_2\left(e^{\frac{a_2}{a_5 x_3}} - e^{\frac{a_1 x_4}{a_5 x_3}}\right) + \frac{a_2 - a_1 x_4}{x_5} \tag{21}$$

To simplify the solution, the second term in the parenthesis in Eq. (21), being insignificant compared to the first, will be dropped, to result in:

$$a_1 = x_2 e^{\frac{a_2}{a_5 x_3}} + \frac{a_2 - a_1 x_4}{x_5} \tag{22}$$

Rearranging the terms of (22) to obtain an equation in terms of x_2,

$$x_2 = \left(a_1 - \frac{a_2 - a_1 x_4}{x_5}\right) e^{-\frac{a_2}{a_5 x_3}} \tag{23}$$

Inserting x_1 from (19) and x_2 from (23) into Eq. (15),

$$a_4 = a_1 - \frac{a_3 + a_4 x_4 - a_1 x_4}{x_5} - \left(a_1 - \frac{a_2 - a_1 x_4}{x_5}\right) e^{\frac{a_3 + a_4 x_4 - a_2}{a_5 x_3}} \tag{24}$$

In order to resolve Eq. (17), the derivative can be expressed as:

$$\left.\frac{dy_3}{dy_2}\right|_{\substack{y_2 = a_3 \\ y_1 = a_4}} = \frac{d(y_1 y_2)}{dy_2} \tag{25}$$

At this juncture, it would be helpful to return for a moment to what the mathematical abstractions in the vector *y* represent physically, Re-quoting Eq. (11).

Clearly, since there are no storage elements (the capacitance of the diode junction can be safely ignored), the power is related simply to current and voltage as:

$$P = IV \tag{26}$$

Applying the terminology in (11) to Eq. (26),

$$y_3 = y_1 y_2 \tag{27}$$

Making use of (27), Eq. (25) can be simplified as:

$$\left.\frac{dy_3}{dy_2}\right|_{\substack{y_2=a_3\\ y_1=a_4}} = \frac{d(y_1 y_2)}{dy_2} = y_1 + \frac{dy_1}{dy_2} y_2 \tag{28}$$

In order to evaluate the second term on the right hand side of Eq. (28), the derivative of Eq. (13)with respect to y_2 is needed. Eq. (13) can be written as:

$$y_1 = f(y_1, y_2) \tag{29}$$

Taking the differential of both sides results in:

$$dy_1 = dy_1 \frac{\partial f(y_1, y_2)}{\partial y_1} + dy_2 \frac{\partial f(y_1, y_2)}{\partial y_2} \tag{30}$$

And thus rearranging the terms in Eq. (30), we obtain the derivative of y_1 with respect to y_2:

$$\frac{dy_1}{dy_2} = \frac{\frac{\partial}{\partial y_2} f(y_1, y_2)}{1 - \frac{\partial}{\partial y_1} f(y_1, y_2)} \tag{31}$$

Using the result obtained in (31), Eq. (28) transforms to:

$$\frac{dy_3}{dy_2} = a_4 + \frac{a_3 \frac{\partial}{\partial y_2} f(y_1, y_2)}{1 - \frac{\partial}{\partial y_1} f(y_1, y_2)} \tag{32}$$

Using the function $f(y_1, y_2)$ from the right hand side of Eq. (24) to solve the equation above, we obtain:

$$\begin{aligned} &\left.\frac{dy_3}{dy_2}\right|_{y_1=a_4} = 0 \\ &= a_4 + a_3 \frac{-\frac{(a_1 x_5 - a_2 + a_1 x_4) e^{\frac{a_3 + a_4 x_4 - a_2}{a_5 x_3}}}{a_5 x_3 x_5} - \frac{1}{x_5}}{1 + \frac{(a_1 x_5 - a_2 + a_1 x_4) e^{\frac{a_3 + a_4 x_4 - a_2}{a_5 x_3}}}{a_5 x_3 x_5} x_4 + \frac{x_4}{x_5}} \end{aligned} \tag{33}$$

Thus far, four of the equations that constitute the five basic equations of the single exponential model, Eq. (14) to (18), have been dealt with. In order to proceed with the last and final equation, Eq. (18), it must first be shown that x_6 approximately equals x_5. To begin with, Eq. (13) is differentiated with respect to y_1 [29] to give:

$$-\frac{dy_2}{dy_1}\left(\frac{x_2}{a_5 x_3} e^{\frac{y_2 + y_1 x_4}{a_5 x_3}} + \frac{1}{x_5}\right) = 1 + \frac{x_2 x_4}{a_5 x_3} e^{\frac{y_2 + y_1 x_4}{a_5 x_3}} + \frac{x_4}{x_5} \tag{34}$$

At short circuit conditions, referring to Table 3, and from Eq. (18), we have:

$$y_2 = 0$$
$$y_1 = a_1$$
$$\left.\frac{dy_1}{dy_2}\right|_{\substack{y_1=a_1\\ y_2=0}} = -\frac{1}{x_6} \tag{35}$$

Thus, at short circuit, Eq. (34) becomes:

$$(x_6 - x_4)\left(\frac{x_2}{a_5x_3}e^{\frac{a_1x_4}{a_5x_3}} + \frac{1}{x_5}\right) - 1 = 0 \tag{36}$$

In order to proceed further, the approximations below, which hold true for most cells under AM1 illumination, will need to be made [5].

$$x_5, x_6 \gg x_4$$
$$\frac{x_2}{a_5x_3}e^{\frac{a_1x_4}{a_5x_3}} \ll \frac{1}{x_5} \tag{37}$$

Using the approximations in (37) in Eq. (36),

$$x_6 = x_5 \tag{38}$$

With the help of this approximation, we can eliminate the parameter x_6, which is not found on solar cell datasheets. Making use of Eq. (38), Eq. (18) can be rewritten in terms of x_5.

$$\left.\frac{dy_1}{dy_2}\right|_{y_1=a_1} = -\frac{1}{x_5} \tag{39}$$

Finally, to obtain the last equation, the right hand side of Eq. (31) is evaluated at $y_1 = a_1$, and $\frac{dy_1}{dy_2}$ is substituted from Eq. (39) into the result of the previous operation.

$$-\frac{1}{x_5} = \frac{-\frac{(a_1x_5 - a_2 + a_1x_4)e^{\frac{a_1x_4-a_2}{a_5x_3}}}{a_5x_3x_5} - \frac{1}{x_5}}{1 + \frac{(a_1x_5 - a_2 + a_1x_4)e^{\frac{a_1x_4-a_2}{a_5x_3}}}{a_5x_3x_5}x_4 + \frac{x_4}{x_5}} \tag{40}$$

IV. METHODS OF SOLUTION OF OBTAINED EQUATIONS

Fortunately, the development above has resulted in three equations which are completely independent of two variables. In other words, three of the equations obtained are in terms of only three variables, thus enabling the time required to solve the expressions to be cut down drastically.

It follows that the approach should be to find the three variables, x_3, x_4, and x_5, that comprise the equations (24), (33), and (40) first, and then use these to find the other two variables using equations (19) and (23). However, there is yet another useful modification that can be taken advantage of to make finding the solution simpler. Eq. (24) can be rearranged to result in an explicit equation in x_3 as a function of x_4, x_5, and *a*:

$$x_3 = \frac{a_4x_4 + a_3 - a_2}{a_5 log\left(\frac{-a_4x_4 + a_1x_4 - a_4x_5 + a_1x_5 - a_3}{a_1x_4 + a_1x_5 - a_2}\right)} \tag{41}$$

It will be now helpful to introduce a vector, $f(x) = 0$, that comprises the equation above, and equations (33) and (40), rearranged in the form of $f_i(x) = 0$.

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix}$$

$$= \begin{bmatrix} -x_3 + \dfrac{a_4 x_4 + a_3 - a_2}{a_5 log\left(\dfrac{-a_4 x_4 + a_1 x_4 - a_4 x_5 + a_1 x_5 - a_3}{a_1 x_4 + a_1 x_5 - a_2}\right)} \\ \\ a_4 + a_3 \dfrac{-\dfrac{(a_1 x_5 - a_2 + a_1 x_4) e^{\frac{a_3 + a_4 x_4 - a_2}{a_5 x_3}}}{a_5 x_3 x_5} - \dfrac{1}{x_5}}{1 + \dfrac{(a_1 x_5 - a_2 + a_1 x_4) e^{\frac{a_3 + a_4 x_4 - a_2}{a_5 x_3}}}{a_5 x_3 x_5} x_4 + \dfrac{x_4}{x_5}} \\ \\ \dfrac{1}{x_5} + \dfrac{-\dfrac{(a_1 x_5 - a_2 + a_1 x_4) e^{\frac{a_1 x_4 - a_2}{a_5 x_3}}}{a_5 x_3 x_5} - \dfrac{1}{x_5}}{1 + \dfrac{(a_1 x_5 - a_2 + a_1 x_4) e^{\frac{a_1 x_4 - a_2}{a_5 x_3}}}{a_5 x_3 x_5} x_4 + \dfrac{x_4}{x_5}} \end{bmatrix} = 0 \qquad (42)$$

Expressing x_3 in an explicit fashion, as in (41), has the advantage that an initial value for x_3 need not be chosen separately, since its initial value can be found from the initial values of x_4 and x_5, and the known quantities that comprise the vector a. Another possible advantage is that the expression for x_3, Eq. (41), can be substituted into $f_2(x)$ and $f_3(x)$ to yield two equations in two variables, albeit long, thus offering another opportunity to speed up the solution of the implicit equations.

Numerous methods can be adopted to solve the system described above, each with its own merits and demerits. Given below is an overview of some of the methods surveyed.

1. **The Gauss Seidel Method:** The Gauss Seidel method is a fixed point iteration technique generally applied to linear systems, but can also be adapted for non-linear systems. Since the equations need to be of the form $x = F(x)$ for the Gauss Seidel method to be applicable, $f_2(\theta)$ and $f_3(\theta)$ will need to be rearranged. Iterations will take the form:

$$x^{(k+1)} = F(x^{(k)}) \qquad (43)$$

   The iteration can be halted if a suitable norm, $\left\| x^{(k+1)} - x^{(k)} \right\|$, is less than a predetermined tolerance, or if a certain maximum number of iterations is exceeded, or even if the solution appears to diverge by, say, entering the complex domain (which is possible since some of the equations will involve logarithms). Of these, only the former will constitute a successful solution; in case a solution is not obtained, a different set of initial values will need to be chosen.

   The three equations for which the Gauss Seidel method will be applied are derived by rearranging the equations in (42) as described above, resulting in:

$$x_3 = \frac{a_4x_4 + a_3 - a_2}{n_s log\left(\frac{-a_4x_4 + a_1x_4 - a_4x_5 + a_1x_5 - a_3}{a_1x_4 + a_1x_5 - a_2}\right)}$$

$$x_4 = \frac{-a_3 + a_2 + a_5x_3 log\left(\frac{a_5(a_4x_4 + a_4x_5 - a_3)x_3}{-a_4a_1x_4^2 - a_4a_1x_5x_4 + a_1x_4a_3 + a_1x_5a_3 + a_4a_2x_4 - a_3a_2}\right)}{a_4} \quad (44)$$

$$x_5 = \frac{a_5x_3x_4 + a_5x_3x_5 + x_4e^{\frac{a_1x_4-a_2}{a_5x_3}}(a_1x_4 + a_1x_5 - a_2)}{a_5x_3 + e^{\frac{a_1x_4-a_2}{a_5x_3}}(a_1x_4 + a_1x_5 - a_2)}$$

As can be observed above, initial values need only be chosen for x_4 and x_5, since x_3 is an explicit function of these two and a.

Thus, we have the equations for the Gauss Seidel method in hand. Using the brief description above, an algorithm to solve the equations using Gauss Seidel is detailed below.

I. The variables a_1 through a_5 are assigned the known values given in the datasheet, according to Table of Transformations given above. The Boltzmann constant, k, the temperature at standard test conditions, T_{STC}, and the charge on an electron, q, are assigned standard values, taking into according to the test conditions and the system of units in use.

II. During the second step, the flag '*convergentSolution*' is said to FALSE, When this is flagged one, the main loop controlling the Gauss Seidel process terminates with successful convergence. Next, the *tolerance* variable is set, which, as explained in the description above, causes the Gauss Seidel loop to exit with the *convergentSolution* flag set to TRUE when the norm $\|x^{(k+1)} - x^{(k)}\|$ is less than the specified value of *tolerance*.

III. During the critical third step, the initial values are chosen for variables x_4 and x_5 using the guidelines in the next section. This step may have to be revisited if the program is to be modified after an unsuccessful run, since the initial values can cause the algorithm to diverge, or converge to the wrong point.

IV. Finally, the Gauss Seidel algorithm is implemented, in steps V to XI.

V. A loop is initiated that terminates either when a certain threshold number of iterations have been completed, or when the *convergentSolution* flag is set to TRUE, indicating a possible successful solution. Upon termination, the program skips to step XII.

VI. Within the loop described above, the value of x_3 is calculated first, using the explicit equation for x_3 in (44) above.

VII. Making use of the newly calculated values of x_3, the interim values for x_4 and x_5 are found out using Eq. (44).

VIII. If this is the first iteration of the algorithm, the program skips to step X.

IX. The norm of the difference, $\left\| x_i - x_{i_{previous}} \right\|$ is found out for $i = 3$, 4, and 5. If each of these norms is less than the tolerance, the flag *convergentSolution* is set to TRUE, and the loop will terminate.

X. $x_{i_{previous}}$ is assigned the value of x_i, for each $i = 3$, 4, and 5. Thus, the values of x_i calculated in the current iteration are preserved, so as to be compared with the new values of x_i, to check for convergence.

XI. The algorithm returns to step V, where the loop will either continue, or exit, depending upon the conditions imposed.

XII. Although the loop has terminated, it is possible that a convergent solution has not been found, and instead the maximum number of iterations has been reached. Thus, the flag *convergentSolution* is checked. If this flag is FALSE, a message stating that no convergent solution could be found is displayed, and the program is terminated.

XIII. Since this step is reached only if a solution is found, the values of x_3, x_4, and x_5 are assumed to be accurate to the tolerance specified. Using these three, x_2 is determined by the explicit equation in (23), and x_1 is determined subsequently with Eq. (19).

XIV. Finally, the value of A, the diode quality factor, is determined using Eq. (3).

XV. Thus, the values of the five parameters, x_1, x_2, A, x_4, and x_5, have been determined.

As is evident from the algorithm above, the maximum number of total iterations should be set high enough to gauge if the iterations are resulting in a convergent solution. The tolerance value will depend, of course, on the accuracy of the data sheet values being used, and thus it is unlikely that a tolerance of less than 10^{-6}, or whereabouts, will be needed. The algorithm is most likely to fail if the initial values are too far off for the algorithm to converge. Exactly how far off they should be for the algorithm to fail is hard to say, but the next section attempts to highlight methods that have been found to be more likely to result in a convergent solution.

2. **Gauss Seidel with SOR/SUR:** While the Gauss Seidel algorithm has been seen to work well for most cases, it leaves a lot to be desired in terms of speed of convergence, and convergence when a good set of initial values cannot be found. A variant of the method, detailed here, is found to address these issues without the hassle of modifying the Gauss Seidel algorithm too much.

V. SELECTING INITIAL VALUES

Choosing initial values that aren’t good enough approximations of the desired single exponential parameters might result in the algorithm diverging, or converging at the wrong point. Different algorithms have different tolerances; some may converge even if the initial values aren’t very close to final result, while others require extremely precise approximations to the true values. The methods of Successive Under Relaxation and Successive Over Relaxation, described above, are unique in that they include a parameter that can be adjusted depending on how good an approximation is available. Described below are initialization techniques that have been found empirically to work well with a number of cases tested.

1. **Random Initialization:** Interestingly, initializing x_4 and x_5 as 0 works well with most cases, for example, Cases 1 through 3 described in Section VI below. Convergence is not as fast as it could be if the initial values are closer to the final results, but with modern computing, the difference in time taken is hardly noticeable. However, although a minority, there might, nevertheless, be a few cases for which this simple scheme of initialization breaks down. It is still helpful to have initialization with 0 as the 'default' scheme of initialization, with other methods used only if this one fails to converge.
2. **Initialization at Certain Select Points:** Since the values of x_4 and x_5 are generally known to be confined to certain broad sets of values, a trial-and-error process can be used to find the value that results in convergence. In general, x_4 is much smaller than one, and is only close to, or larger than 1 when the number of cells in series, represented by a_5, is quite substantial, for example, larger than 50. x_5 is usually much larger than x_4, and is generally above 100, and above 1000 for modules with a large number of cells, say, larger than 50. While this process is rather tedious, it only has to be applied to the rare cases where convergence might not occur as expected with Random Initialization.

VI. CASE STUDIES

While the development of the equations for deriving the parameters of the single diode model, and the description of the algorithm for solving the equations obtained have been mathematically rigorous, it is nevertheless important to test the procedure developed on real life cases. A study of three solar panels was done using the methods described in Section IV, with data for all three panels obtained from the datasheets provided by the manufacturers of the respective panels. As was stated earlier, no data apart from that provided in the datasheet was used.

All the cases studied, Cases 1 through 3, converged with 0 as the initial value for x_4 and x_5. Table 4 details the cases studied and the results obtained. And also lists the number of iterations the Gauss Seidel method needed to converge.

Table 4. Results of Case Studies

	Case 1 BP-MSX 120	Case 2 PV-MF165EB3	Case 3 GaAsGe Single Junction Solar Cells
a_1	3.87	7.36	0.0305
a_2	42.1	30.4	1.025
a_3	33.7	24.2	0.9
a_4	3.56	6.83	0.0286

a_5	72	50	1
x_1	3.87	7.36	0.0305
x_2	0.322e-6	0.104e-6	0.120e-12
A	1.398	1.310	1.523
x_4	0.473	0.251	5.228e-6
x_5	1.367e+3	1.168e+3	1.307e+3
Number of Iterations with Gauss Seidel	7367	23476	30396

Note that all these results are for standard illumination and temperature, that is, at STC. The number of iterations is very strongly dependent on how close the initial values are to the final result, and therefore faster convergence can be obtained if better initial values are chosen.

An important step in verifying the results obtained is to plot the I-V characteristics of each of the cases to see if they match with the expected I-V characteristics of a solar module. However, potting the curve in itself poses a challenge because of the implicit nature of Eq. (12). In order to speed up the computation, it is desirable to avoid solving the implicit equation at each of the points on the graph. Therefore, an alternative method has to be found to solve the equation in (12) successively for a large number of points without sacrificing on the accuracy of the plot. To begin with, Eq. (12) is rearranged as:

$$y_1 = \frac{\left(x_1 - x_2\left(e^{\frac{y_2+y_1x_4}{a_5V_t}} - 1\right) - \frac{y_2}{x_5}\right)}{\left(1 + \frac{x_4}{x_5}\right)} \qquad (45)$$

While the exponential term indicates that changes in y_1, or y_2, will result in a relatively large swing in the value of the expression, a very small change in y_1, or y_2, will not change the result of the expression by a lot. Building on this idea, it is natural to conclude that the previous value of y_1 can be used for the exponential term, with care being taken to ensure that the size of each individual 'step' is very small. A brief algorithm is provided below.

I. a_1 through a_5 are initialized based on the values provided in the datasheet, and x_1, x_2, A, x_4, and x_5 are initialized using the results obtained from the solution of the equations described in (42). k, T_{stc}, and q are constants and are initialized with their respective values. V_t is calculated using Eq. (2).

II. The variable *numberOfSteps* is initialized. The higher its value, the more accurate will be the resulting plot.

III. y_1 and y_2 are declared as vecors of length *numberOfSteps*. Since the first 'point' on the curve is the short circuit point, $y_1(1)$, the first element of the vector y_1, is initialized as a_1, and $y_2(1)$ is initialized as 0.

IV. The variable $Dely_2$ is initialized as a_2 divided by *numberOfSteps*.

V. A loop is initialized that runs from $i = 2$ to $i = numberOfSteps$. During each iteration, the value of $y_2(i)$ is calculated as the sum of the previous value of y_2, $y_2(i-1)$, and $Dely_2$. Eq. (45) is used to calculate each value of $y_1(i)$, with $y_1(i-1)$ being used on the right side of Eq. (45) to calculate each $y_1(i)$. As is evident here, *numberOfSteps* needs to be a large enough number to negate the inaccuracy introduced by this approximation.

VI. Finally, after the loop ends, the vectors y_1 and y_2 are used to plot the I-V characteristics of the solar module. Since y_2 represents the voltage V, the plot generally uses it as the x-axis.

Obtaining the plots is thus a simple matter of implementing the algorithm above for whatever solar module the plot is desired. The plots of the I-V characteristics of the three modules described in Cases 1 through 3 are given below.

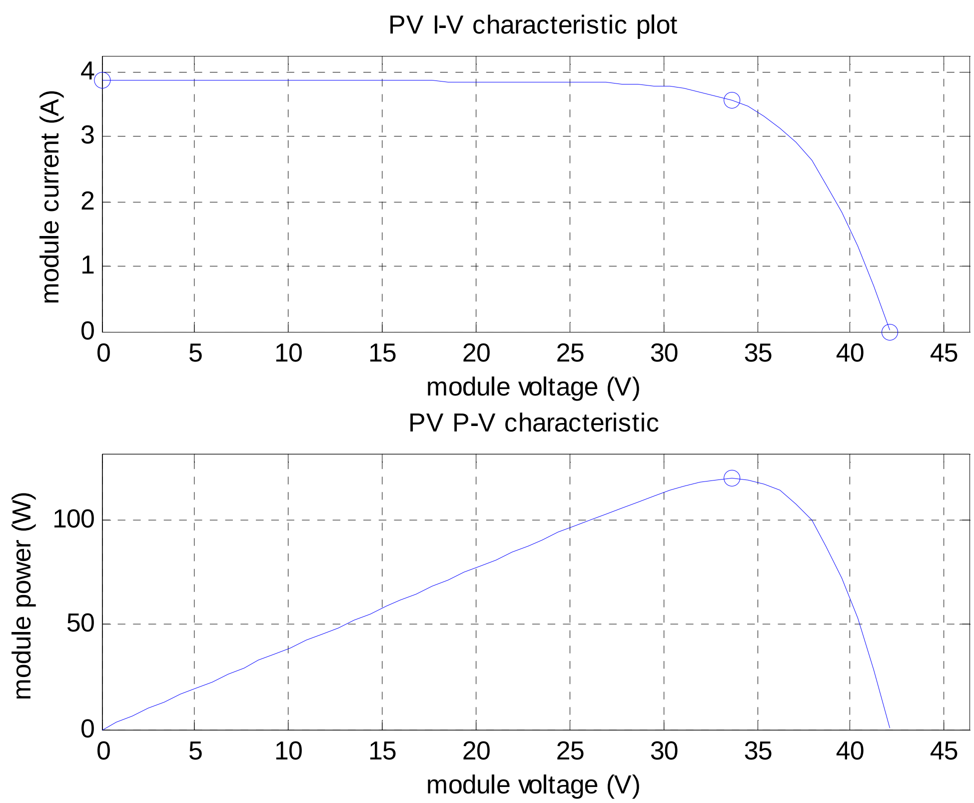

Fig. 3 The plot of the I-V characteristics of Case 1

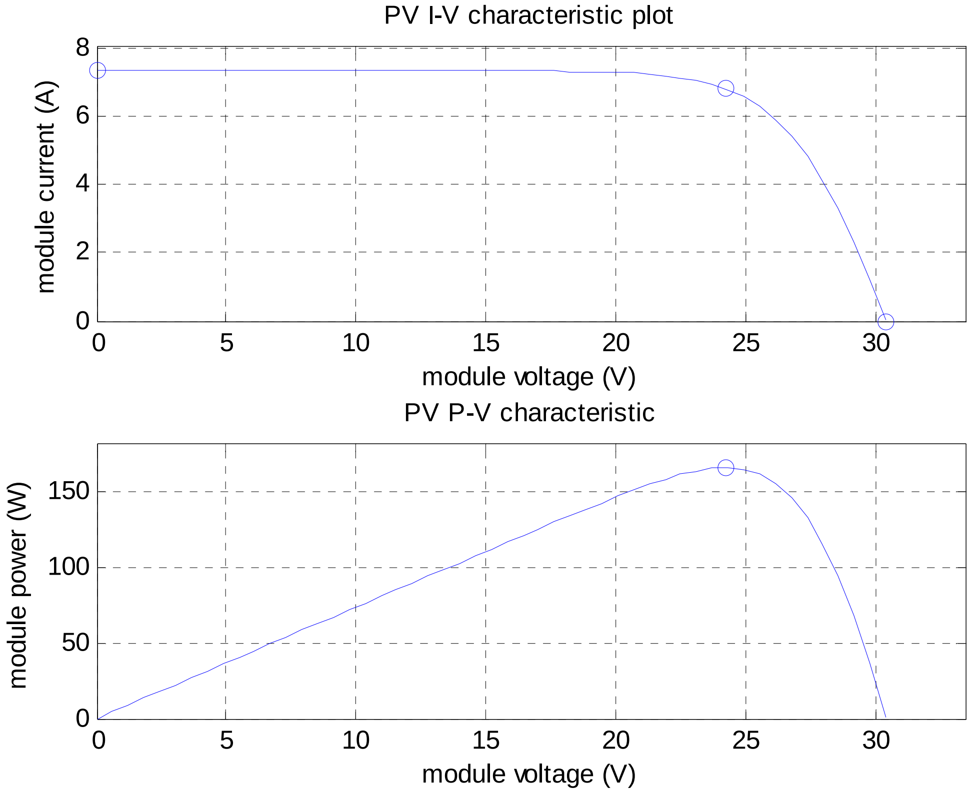

Fig. 4 The plot of the I-V characteristics of Case 2

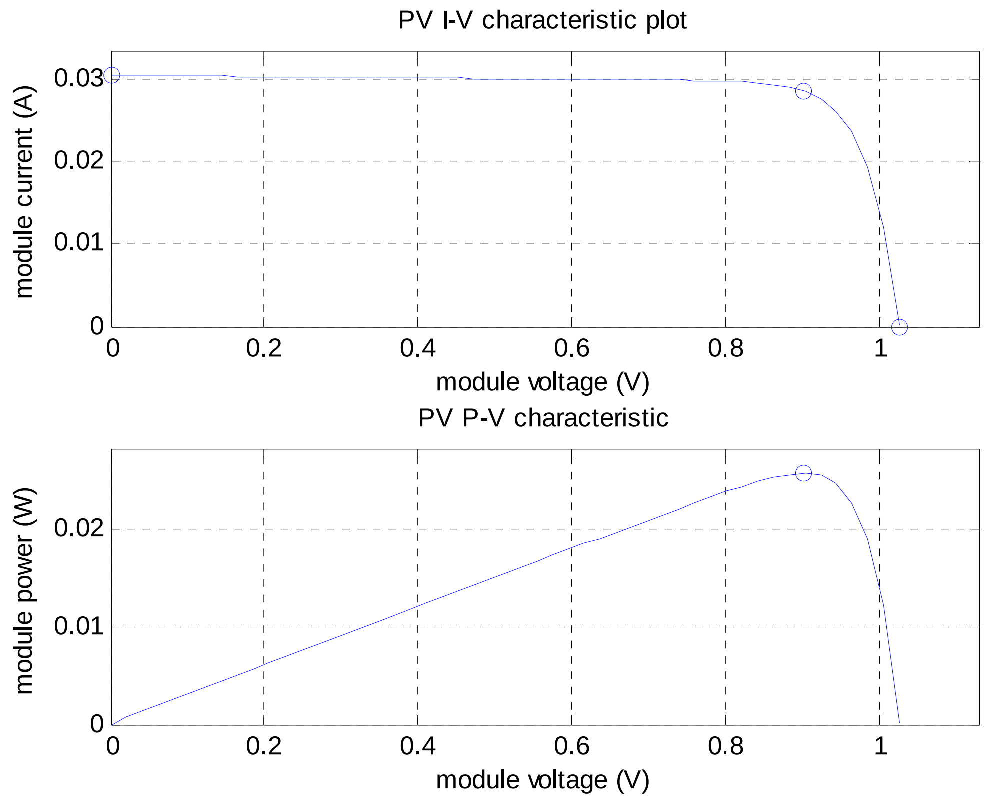

Fig. 5 The plot of the I-V characteristics of Case 3

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APPENDIX

1. **The MATLAB Program for the Gauss Seidel Algorithm described in Section IV above.**

```
a1 = 3.87;
```

```
a2 = 42.1;
a3 = 33.7;
a4 = 3.56;
a5 = 72;

k = 1.3806503e-23;
Tstc = 298;
q = 1.60217646e-19;

convergentSolution = 0;
i = 1;
tolerance = 1e-6;

x4 = 0;
x5 = 0;

while((i < 1000000) && (convergentSolution == 0))
    x3 = (a3+a4*x4-a2)/log(-(a4*x5-a1*x5+a3+a4*x4-a1*x4)/(a1*x5-a2+a1*x4))/a5;
    x4 = (-a3+a2+log(a5*x3*(a4*x5+a4*x4-a3)/(-a4*a1*x5*x4+a4*a2*x4-a4*a1*x4^2+a3*x5*a1-a2*a3+a3*a1*x4))*a5*x3)/a4;
    x5 = (a5*x3*x5+exp((a1*x4-a2)/a5/x3)*a1*x5*x4-exp((a1*x4-a2)/a5/x3)*a2*x4+exp((a1*x4-a2)/a5/x3)*a1*x4^2+x4*a5*x3)/(exp((a1*x4-a2)/a5/x3)*a1*x5-exp((a1*x4-a2)/a5/x3)*a2+exp((a1*x4-a2)/a5/x3)*a1*x4+a5*x3);

  if i ~= 1
      if (norm(x5 - x5Previous) < tolerance) && (norm(x4 - x4Previous) < tolerance) && (norm(x3 - x3Previous) < tolerance)
          convergentSolution = 1;
      end
  end
  x3Previous = x3;
  x4Previous = x4;
  x5Previous = x5;
  i = i + 1;
end

if convergentSolution == 0
  disp('Convergent solution not found')
```

```
else
    x2 = (a1 - (a2 - a1*x4)/x5) * exp(-1*a2/(a5*x3));
    x1 = x2 * exp(a2/(a5*x3)) + a2/x5;
    A = x3 * q / (k * Tstc);
    x = [x1; x2; A; x4; x5]
end
```

2. **The MATLAB Program for the Gauss Seidel Algorithm with SOR or SUR described in Section IV above.**

```
a1 = 3.87;
a2 = 42.1;
a3 = 33.7;
a4 = 3.56;
a5 = 72;

k = 1.3806503e-23;
Tstc = 298;
q = 1.60217646e-19;

convergentSolution = 0;
i = 1;
tolerance = 1e-6;
w = .6;

x4 = 0;
x5 = 0;
x3 = (a3+a4*x4-a2)/log(-(a4*x5-a1*x5+a3+a4*x4-a1*x4)/(a1*x5-
    a2+a1*x4))/a5;


while((i < 1000000) && (convergentSolution == 0))
    x3 = (1-w)*x3 + w*((a3+a4*x4-a2)/log(-(a4*x5-a1*x5+a3+a4*x4-
        a1*x4)/(a1*x5-a2+a1*x4))/a5);
    x4 = (1-w)*x4 + w*((-a3+a2+log(a5*x3*(a4*x5+a4*x4-a3)/(-
        a4*a1*x5*x4+a4*a2*x4-a4*a1*x4^2+a3*x5*a1-
        a2*a3+a3*a1*x4))*a5*x3)/a4);
    x5 = (1-w)*x5 + w*((a5*x3*x5+exp((a1*x4-a2)/a5/x3)*a1*x5*x4-exp((a1*x4-
        a2)/a5/x3)*a2*x4+exp((a1*x4-
        a2)/a5/x3)*a1*x4^2+x4*a5*x3)/(exp((a1*x4-a2)/a5/x3)*a1*x5-
        exp((a1*x4-a2)/a5/x3)*a2+exp((a1*x4-a2)/a5/x3)*a1*x4+a5*x3));
```

```
   if i ~= 1
          if (norm(x5 - x5Previous) < tolerance) && (norm(x4 - x4Previous) <
              tolerance) && (norm(x3 - x3Previous) < tolerance)
                   convergentSolution = 1;
      end
   end
   x3Previous = x3;
   x4Previous = x4;
   x5Previous = x5;
   i = i + 1;
end

if convergentSolution == 0
   disp('Convergent solution not found')
else
   x2 = (a1 - (a2 - a1*x4)/x5) * exp(-1*a2/(a5*x3));
   x1 = x2 * exp(a2/(a5*x3)) + a2/x5;
   A = x3 * q / (k * Tstc);
   x = [x1; x2; A; x4; x5]
end
```

3. **The MATLAB program for plotting the I-V characteristics of a set of input variables and parameters using the method described in Section VI above.**

```
a1 = 3.87;
a2 = 42.1;
a3 = 33.7;
a4 = 3.56;
a5 = 72;

k = 1.3806503e-23;
Tstc = 298;
q = 1.60217646e-19;

x1 = 3.871339576378674;
x2 = 3.227050157847574e-007;
A = 1.397597184504120;
x4 = 0.472779988828421;
x5 = 1.365848626397404e+003;
```

```
Vt = A*k*Tstc/q;

numberOfSteps = 1000;
y1 = zeros(numberOfSteps, 1);
y2 = zeros(numberOfSteps, 1);
y1(1) = a1;
y2(1) = 0;

DelY2 = a2 / numberOfSteps;

for i = 2:numberOfSteps
   y2(i) = y2(i-1)+DelY2;
      y1(i) = (x1 - x2*(exp((y2(i)+y1(i-1)*x4)/(a5*Vt))- 1)-y2(i)/x5)/(1+x4/x5);
end

plot(y2, y1);
```

4. **Observations about the open-circuit case.**

At open-circuit conditions, as shall be shown below, the shunt resistance, comprising R_{sh} and the diode resistance r_D, is approximately r_D by virtue of the forward biased nature of operation.

$$r_D \parallel R_{sh} \cong r_D \tag{46}$$

Although the diode resistance r_D is variable over the entire I-V characteristic, the forward biased nature at open-circuit ensures that r_D would be restricted to a certain range. Thus,

$$r_D = \frac{V_D}{I_D} \cong \frac{0.7}{I_D} \tag{47}$$

where $V_D$is the voltage across the diode, and I_D the current through it. Referring to Fig. 2,

$$I_D = I_{ph} - \frac{V_D}{R_{sh}} = I_o\left(e^{\frac{V_D}{n_c V_t}} - 1\right) \tag{48}$$

Since the diode is expected to be in the forwards bias mode, V_D can be assumed to be approximately 0.7, hence leading to:

$$I_D = I_{ph} - \frac{0.7}{R_{sh}} \tag{49}$$

I_{ph} is generally observed to be greater than 10^{-2}, and R_{sh} greater than 10^2. Thus, a safe lower limit for I_D, making use of these observations and Eq. (49), would be 10^{-2}. Substituting this into Eq. (47),

$$r_D \leq \frac{.7}{.01} = 70 \tag{50}$$

As an even broader approximation,

$$r_D \leq 100 \tag{51}$$

Thus, from Eq. (46),

$$r_D \parallel R_{sh} \leq 100 \tag{52}$$

Therefore, at open-circuit, the shunt resistance R_{sh} has almost no bearing whatsoever on the open-circuit voltage V_{oc}. The series resistance R_s has no role to play either, since no current flows through it because of the open-circuit conditions. Thus, an excellent approximation to Fig. 2 at open-circuit conditions would be Fig. 6 below.

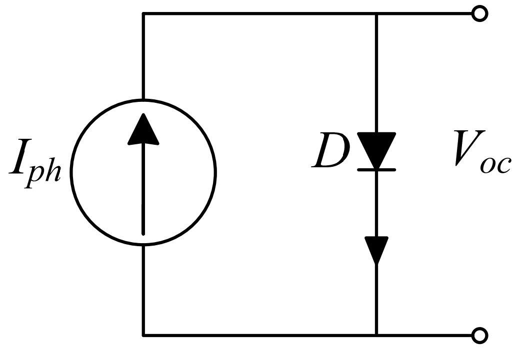

Fig. 6 An approximation to the equivalent circuit at open-circuit conditions

From Fig. 6,

$$I_{ph} = I_o\left(e^{\frac{V_{oc}}{n_c V_t}} - 1\right) \tag{53}$$

Rearranging Eq. (53) to obtain an explicit expression in V_{oc},

$$V_{oc} = n_s V_t ln\left(\frac{I_{ph}}{I_o} + 1\right) \tag{54}$$

5. **Observations about the short-circuit case.**

The short-circuit conditions, as described below, shall lead to an interestingly concise result for the value of I_{sc}. To begin with, the shunt resistance R_{sh}, which, at short-circuit conditions, will appear in parallel across R_s, can be largely ignored, since it is usually more than 10^3 times the series resistance. Thus, the circuit in Fig. 2 will appear as shown in Fig. 7 below.

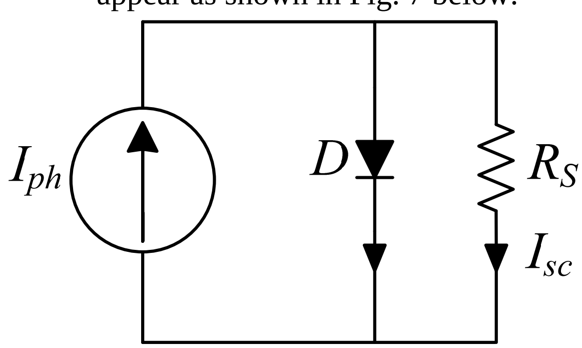

Fig. 7 The equivalent circuit at short circuit conditions with R_{sh} ignored

The current through the diode, I_D, will be related to V_D as:

$$I_D = I_{ph} - \frac{V_D}{R_s} = I_o \left(e^{\frac{V_D}{n_c V_t}} - 1 \right) \tag{55}$$

Since $V_D = I_{sc} R_s$, we have:

$$I_D = I_{ph} - I_{sc} = I_o \left(e^{\frac{I_{sc} R_s}{n_c V_t}} - 1 \right) \tag{56}$$

In a practical application of a PV system, series combination of modules is made to form a string and parallel combination of strings to form an array. From the study of a large number of systems, the following were observed: I_{ph} and I_o for the string will be the same as that of the module or cell. For an array, however, these currents will be increased as many times as there are strings in the array. The series and parallel resistances will assume a value given by the usual series and parallel combination of resistances. While it is difficult to gauge the effect of these factors in Eq. (56) above, it is easier to use the approximate form of the equation of the single-diode model, where the '-1' term in the above equation is ignored.

It will be helpful at this moment to return to the more mathematically conducive notation adopted earlier. Substituting the expression for x_2 from Eq. (23) into Eq. (19),

$$x_1 = a_1 \left(1 + \frac{x_4}{x_5} \right) \tag{57}$$

Since x_4, the series resistance, is generally much smaller than x_5, an excellent approximation for x_1 is obtained:

$$x_1 \cong a_1 \tag{58}$$

Therefore, in effect, almost no current flows through the diode D in Fig. 7, and therefore the short-circuit current is nearly equal to the photo-current I_{ph}. Table 4, which lists the results of the case studies, confirms this approximation.

While it may be tempting to conclude that the approximation in (58) and the equation in (57) will imply that only one initial value, either x_4 or x_5 needs to be chosen, this, unfortunately, is not so. This will become clearer when (57) is rearranged as either of the two equations that follow.

$$x_4 = \frac{x_5}{a_1} (x_1 - a_1) \tag{59}$$

$$x_5 = \frac{a_1 x_4}{x_1 - a_1} \tag{60}$$

As can be seen, approximating x_1 as a_1 will not help in reducing the number of initial values that need to be chosen.