ECE 842-Lecture Random Variables

Random Variables: A variable which takes on values at random; and may be thought of as a function of the outcomes of some random experiment.

Let x be a random variable, the outcome of a probalistic experiment. The manner of specifying the probability with which different values are taken by the random variable is the probability distribution function $P_r(x)$ and it is defined by

$$P_r(x) = P_r (X \leq x)$$

Or by the probability density function f(x)

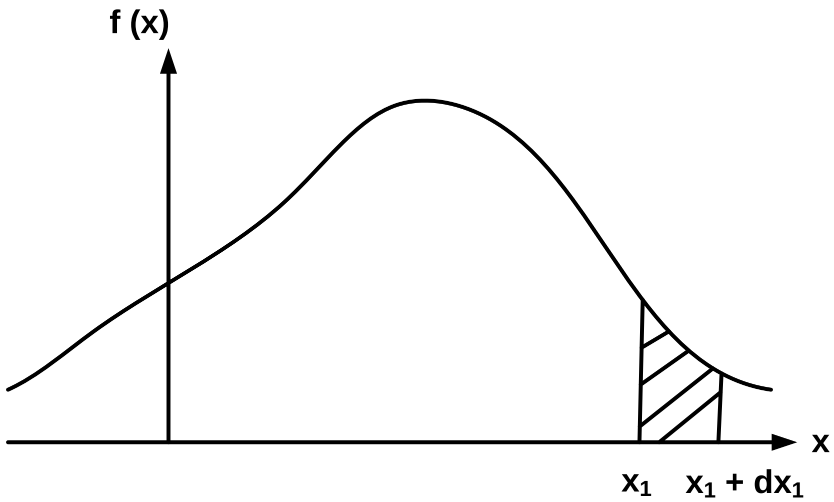
$$f(x) = \frac{dP_r(x)}{dx}$$

Let f(x) denotes probability density function of x.

Then

$f(x_1)\, dx_1$ = Probability that the value of x will be in the interval

$$x_1 \leq x \leq x_1 + d\,x_1$$



$f(x) \geq 0$ and $\int_{-\infty}^{+} f(x)dx = 1$

Types of Random Processes

Two types of random variables can be considered:

a) Colored: Values of random process correlated in time.
b) White: Values of random process uncorrelated in time.

Colored noise: By observing past values of a colored random sequence for process, one may predict its future behavior.

White noise: It is impossible to predict future behavior of white noise by observing past values,

Discrete white noise: Discrete white noise is a time sequence of independent random variables.

x(1), x(2), ……………….. x(k), ………………….. x(j) ………………

Mean E [x(k)] = $\bar{x}$ $$\hat{E}[x(k)] = \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} x(k)$$

Covariance Cov [x(k) , x(j)] =

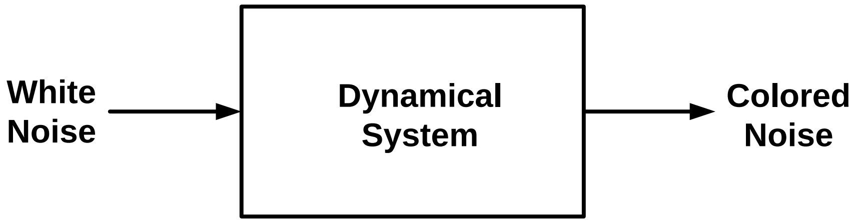
$$E\{ [x(k) - \bar{x}(k)] [x(j) - \bar{x}(j)] \}$$

$$= \gamma_{ij}$$

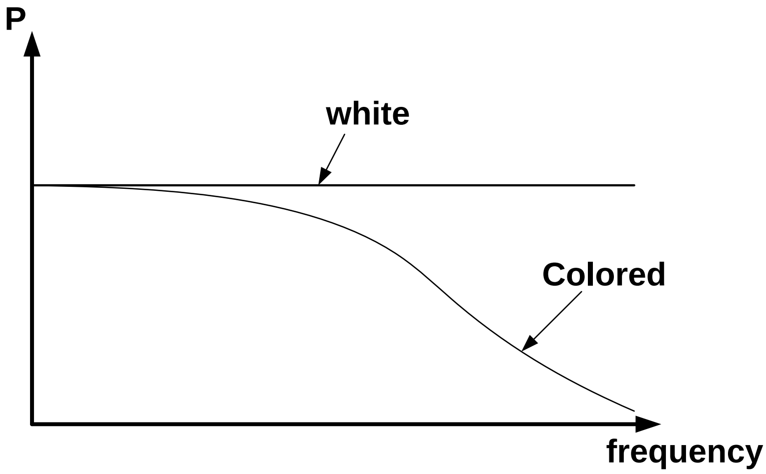
$\gamma_{ij} = \gamma$ if i = j

$\gamma_{ij} = 0$ if i ≠ j

Continuous white noise: It does not exist in nature. It is useful in mathematical modeling.



Continuous white noise has constant power at all frequencies.



If x(t) is a continuous time white noise

$$E[x(t)] = \bar{x}(t) \quad \text{well defined}$$

$$E\{[x(t) - \bar{x}(t)][x(\tau) - \bar{x}(\tau)]\}$$

$$= \gamma(t)\,\delta(t-\tau)$$

$$\gamma(t) = \gamma\,\delta(0) \quad \text{if} \quad t = \tau$$

$$\gamma(t) = 0 \quad \text{if} \quad t \neq \tau$$

Therefore, continuous white noise is uncorrelated in time and each instant of time has infinite variance.

Definition: The expected value of a random variable x is defined as

$$E[x] = \int_{-\infty}^{\infty} f(x)dx$$

The variance is defined as

$$\text{Var}[x] = E[(x - \bar{x})^2] = \int_{-\infty}^{\infty} (x - \bar{x})^2 f(x)dx$$

Also

$$\text{Cov}[x; x] = \text{Var}(x) = \gamma$$

$$\gamma = \frac{[(x(1)-\bar{x})^2 + (x(2)-\bar{x})^2 + \ldots\ldots\ldots\ldots\ldots}{N}$$

Example. A random variable x has the density function

$$f(x) = \frac{c}{x^2+1}, \text{ where } \quad -\infty < x < \infty$$

a) Find the value of the constant c.

b) Find the probability x^2 lies between 1/3 and 1.

a) We must have $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{-\infty}^{\infty} \frac{cdx}{x^2+1} = c\tan^{-1} x\Big|_{-\infty}^{\infty} = c[\frac{\pi}{2} - (-\frac{\pi}{2})] = 1$$

So that c = 1/π

b) If 1/3 ≤ x ≤ 1, then either $\frac{\sqrt{3}}{2} \le x \le 1$

Or $-1 \le x \le -\frac{\sqrt{3}}{2}$. (Note $\sqrt{\frac{1}{3}} = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$)

Thus the required probability is

$$\frac{1}{\pi}\int_{-1}^{-\sqrt{3}/3} \frac{dx}{x^2+1} + \frac{1}{\pi}\int_{\sqrt{3}/3}^{1} \frac{dx}{x^2+1} = \frac{2}{\pi}\int_{\sqrt{3}/3}^{1} \frac{dx}{x^2+1} = \frac{2}{\pi}[\tan^{-1}(1) - \tan^{-1}(\frac{\sqrt{3}}{3})] = \frac{2}{\pi}(\frac{\pi}{4} - \frac{\pi}{6}) = \frac{1}{6}$$

Joint densities. Assume x and y are random variables with joint probability density function.

$f(x_1, y_1)\, dx_1 dy_1$ = Probability that $x_1 \le x \le x_1 + d\,x_1$

and $y_1 \le y \le y_1 + d\,y_1$

and

$$E[x] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)dxdy$$

$$E[y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)dxdy$$

$$\text{Variance}[x] = \Sigma_x = E[(x - \bar{x})^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\bar{x})^2 f(x,y)dxdy$$

$$\text{Variance}[y] = \Sigma_y = E[(y - \bar{y})^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y-\bar{y})^2 f(x,y)dxdy$$

Covariance and Correlation

Covariance of two random variables x and y are defined as

$$\Sigma_{xy} = \text{Cov}[\,x\,;\,y\,] = E[\,(x - \bar{x})\,(y - \bar{y})\,]$$

$$\text{Cov}[\,x\,;\,y\,] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\bar{x})(y-\bar{y})\,f(x,y)\,dxdy$$

Correlation of two random variables x and y is defined as

$$\rho(x,y) = \frac{\sum_{xy}}{\sqrt{\sum_{x}\sum_{y}}}$$

$$-1 \le \rho(x, y) \le 1$$

Random Vectors. Consider n – dimensional random vector.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}$$

where $x_1, x_2, \ldots\ldots\ x_n$ are scalar random variables, with joint density function

$$f(x) = f(x_1, x_2, \ldots\ldots\ x_n)$$

$$E[x] = \bar{x} = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$E[x] = \begin{bmatrix} E\{x_1\} \\ E\{x_2\} \\ . \\ . \\ E\{x_n\} \end{bmatrix}$$

$$E[x_i] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_i\, f(x_1, \ldots\ldots\ldots x_n)\,dx_1 \ldots\ldots\ldots\ldots dx_n$$

Covariance Matrix. Consider a random vector X.

$$X \in R^n \quad \text{then}$$

Σ_x = Cov [x ; x] = E [(x - $\bar{x}$) (x - $\bar{x}$)^T]

$$\sum_x = \begin{bmatrix} Cov[x_1;x_1] & Cov[x_1;x_2]\ldots\ldots Cov[x_1;x_n] \\ Cov[x_1;x_2] & Cov[x_2;x_2]\ldots\ldots Cov[x_2;x_n] \\ . & \\ . & \\ Cov[x_1;x_n] & Cov[x_2;x_n]\ldots\ldots Cov[x_n;x_n] \end{bmatrix}$$

Where

$$\text{Cov} (x_1 ; x_1) = \text{Var} [x_1] = \int_{-\infty}^{\infty} (x_1 - \bar{x}_1)^2 f(x)dx$$

$$\text{Cov} (x_1 ; x_2) = E [(x_1 - \bar{x}_1) (x_2 - \bar{x}_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_1 - \bar{x}_1)(x_2 - \bar{x}_2) f(x,y)dxdy$$

Covariance Calculation Given

$$Z = A x + B y$$

Let

Σ_z = Cov [z ; z] = Cov (zz^T) = Cov [(Ax + By) (Ax + By)^T]

= Cov (Axx^T A^T) + Cov (Byy^T B^T) = Cov (A) Cov (xx^T) Cov (A^T) + Cov (B) Cov (yy^T) Cov (B^T)

= A Cov (xx^T) A^T + B Cov (yy^T) B^T

Σ_x = Cov [x ; x]

Σ_y = Cov [y ; y]

Then

$\Sigma_z = A \Sigma_x A^T + B \Sigma_y B^T$

Gaussian Random Variables

Let x be a scalar Gaussian random variable

$$E [x] = \bar{x}, \text{Var} [x] = \Sigma$$

Gaussian density function

$$f(x) = \frac{1}{2\pi\sqrt{\sum}} \exp\{-\frac{1}{2\sum}(x-\bar{x})^2\}$$

Gaussian Random Vectors. Let x be n – dimensional Gaussian random vector

$$\bar{x} = E[x],\ \Sigma = Cov[x ; x]$$

Then the Gaussian Density function is

$$f(x) = (2\pi)^{-n/2} (\det \Sigma)^{-1/2} . \exp\{\{-\frac{1}{2}(x-\bar{x})^T \sum^{-1}(x-\bar{x})\}$$

Note that if x and y are independent Gaussian random vectors, then the random vector

$$Z = A X + B y$$

is also Gaussian