Lecture # 5 Model Order Identification

ECE 842

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Consider a zero mean {Y(.)} process which can modeled by ARMA (n, m)

$Y(k) = \phi_1 Y(k-1) + \ldots\ldots\ldots \phi_n Y(k-n) + \phi_{n+1} W(k-1) + \ldots\ldots\ldots\ldots + \phi_{n+m} W(k-m) + W(k)$

Multiply the above by Y(k+j) and take E{ }

$Y(k+j) = \phi_1 Y(k+j-1) + \ldots\ldots\ldots \phi_n Y(k+j-n) + \phi_{n+1} W(k+j-1) + \ldots\ldots\ldots\ldots + \phi_{n+m} W(k+j-m) + W(k+j)$

$E [Y(k) Y(k+j)] = \phi_1 E [Y(k) Y(k+j-1)] + \ldots\ldots\ldots + \phi_n E [Y(k) Y(k+j-n)] + \phi_{n+1} E [Y(k) W(k+j-1)] + \ldots\ldots\ldots\ldots + \phi_{n+m} E [Y(k) Y(k+j-m)] + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\gamma (j) = \phi_1 \gamma (j-1) + \ldots\ldots\ldots \phi_n \gamma (j-n) + \phi_{n+1} \gamma_{yw} (j-1) + \ldots\ldots\ldots\ldots + \phi_{n+m} \gamma_{yw} (j-m) + \gamma_{yw} (j)$

Note that Y(k-j) depends only on the disturbances up to the time k-j, therefore for $j \geq m+1$

$$\gamma_{yw}(j-1)\big|_{j=m+1} = \gamma_{yw}(m) = 0$$

$$or \quad j \geq m+1$$

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$$\gamma_{yw}(j-m)\big|_{j=m+1} = \gamma_{yw}(1) = 0$$

Also $$\rho(j) = \frac{\gamma(j)}{\gamma_0}$$

$\rho (j) = \phi_1 \rho (j-1) + \phi_1 \rho (j-2) + \ldots\ldots\ldots + \phi_n \rho (j-n)$ for $j \geq m+1$

$\rho (j) = \phi_1 \rho (j-1) + \phi_1 \rho (j-2) + \ldots\ldots\ldots + \phi_n \rho (j-n)$
$*+ \phi_{n+1} \rho_{yw} (j-1) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + \phi_{n+m} \rho_{yw} (j-m)$ for $j \leq m$

For pure ARMA (0, m)

$$\phi_1 = \phi_2 = \ldots\ldots\ldots\ldots.. = \phi_n = 0$$

* will reduce to

$$\rho\,(j) = \phi_{n+1}\,\rho_{yw}\,(j-1) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + \phi_{n+m}\,\rho\,(j-m) \qquad \text{for } j \le m$$

$$\rho\,(j) = 0 \qquad \text{for } j > m$$

If m = m* (say 2), then **

$$\rho\,(j) = 0 \qquad \text{for } j \le m^*$$

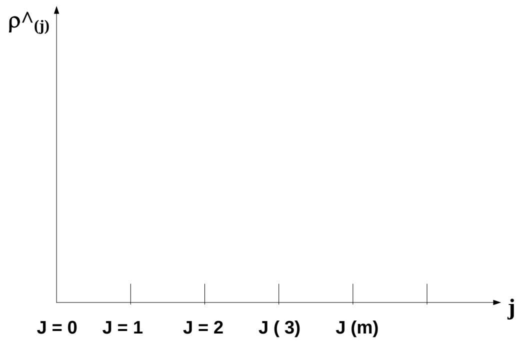
$$\rho\,(j) \ne 0 \qquad \text{for } j > m^*$$

Which means that autocorrelation functions of an MA(m*) process are zero or "Cut off" after the m*th lag.

Therefore for determining the order of MA model

1) Calculate $\hat{\rho}(j), \quad j = 1,2,.........m, m+1,................$

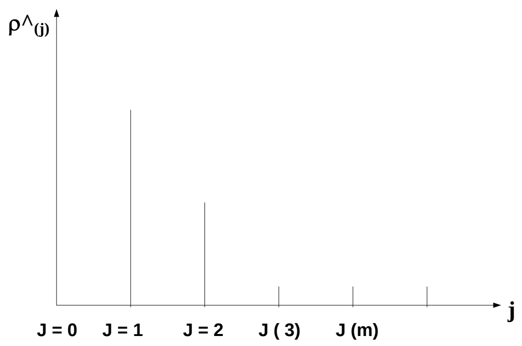
2) Plot



3) If the process is pure MA (i.e. can be modeled by a pure MA model), then for a m = m*

$$\hat{\rho}(m^*+1) \;\; is \;\; \text{"}Cut \;\; off\text{"}$$

Ex. Given {Y(.)} and $\hat{\rho}(j) \qquad j = 1,2,.........., m, m+1$



Determine MA model which should be used as candidate for this process.

Sol. $m^* = 2$; MA(2)

Suppose we consider a pure autoregressive process (i.e. ARMA (n, 0)), then

$$\phi_{n+1} = \phi_{n+2} = \ldots\ldots + \phi_{n+m} = 0$$

$$\rho(j) = \phi_1 \rho(j-1) + \phi_2 \rho(j-2) + \ldots\ldots + \phi_n \rho(j-n)$$

If n = 1, then

$$\rho(j) = \phi_1 \rho(j-1) \qquad \text{when } (n = 1)$$

Note that when n = 1, then

$$Y(k) = \phi_1 Y(k-1)$$

$$Y(k-1) = \phi_1 Y(k-2)$$

$$Y(k-2) = \phi_1 Y(k-3)$$

.

$$Y(k-n) = \phi_1 Y(k-n-1)$$

Similarly

$$\rho(j-1) = \phi_1 \rho(j-2)$$

$$\rho(j-2) = \phi_1 \rho(j-3)$$

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Which results in

$$\rho(j) = \phi_1^2 \rho(j-2) = \phi_1^3 \rho(j-3) = \phi_1^j$$

But

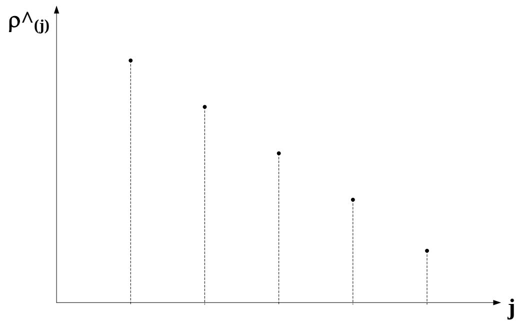
$$-1 \le \rho(j) \le 1$$

Therefore

$$-1 \le \phi_1 \le 1$$

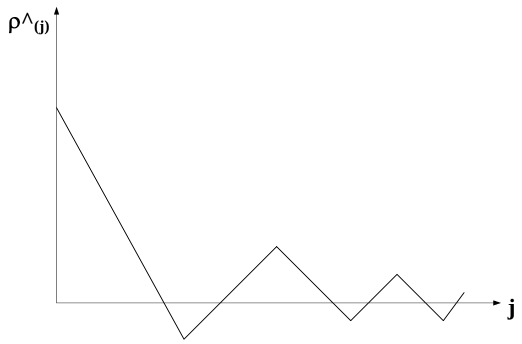
$\rho(j)$ is said to "tail off" as j increases for ϕ_1 positive

$$\rho(j) = \phi_1^j$$



Autocorrelation functions decays exponentially

For ϕ_1 negative



Autocorrelation functions decays exponentially and oscillates

The tailing – off pattern holds true for any pure AR(n) process.

The Partial Autocorrelation

Initially, we do not know which order of autoregressive process to fit to an observed sequence. Assume the process can be modeled by a pure AR(k).

$$\rho(j) = \rho`_{k1}\,\rho(j-1) + \ldots\ldots\ldots\ldots.. + \rho`_{k(k-1)}\,\rho(j-k+1) + \rho`_{kk}\,\rho(j-k); \qquad j = 1, 2, \ldots\ldots k$$

Example

Let k = 2

j = 1, 2

$$\rho(1) = \rho`_{21}\,\rho(0) + \rho`_{21}\,\rho(1-2) \rightarrow \rho(1) = \rho`_{21}\,\rho(0) + \rho`_{21}\,\rho(1)$$

$$\rho(2) = \rho`_{21}\,\rho(1) + \rho`_{22}\,\rho(0) \rightarrow \rho(2) = \rho`_{21}\,\rho(1) + \rho`_{22}\,\rho(1)$$

$$\begin{bmatrix} \rho(1) \\ \rho(2) \end{bmatrix} = \begin{bmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{bmatrix} \begin{bmatrix} \rho`_{21} \\ \rho`_{22} \end{bmatrix}$$

For k =3

j = 1, 2, 3

$$\begin{bmatrix} \rho(1) \\ \rho(2) \\ \rho(3) \end{bmatrix} = \begin{bmatrix} 1 & \rho(1) & \rho(2) \\ \rho(1) & 1 & \rho(1) \\ \rho(2) & \rho(1) & 1 \end{bmatrix} \begin{bmatrix} \rho`_{31} \\ \rho`_{32} \\ \rho`_{33} \end{bmatrix}$$

In general

$$\begin{bmatrix} 1 & \rho(1) & \rho(2)\ldots\ldots\ldots\ldots & \rho(k-1) \\ \rho_1 & 1 & \rho(1)\ldots\ldots\ldots\ldots & \rho(k-2) \\ \cdot & & & \\ \cdot & & & \\ \rho(k-1) & \rho(k-2)\ldots\ldots\ldots\ldots\ldots 1 & & \end{bmatrix} \begin{bmatrix} \rho`_{k1} \\ \rho`_{k2} \\ \vdots \\ \rho`_{kk} \end{bmatrix} = \begin{bmatrix} \rho(1) \\ \rho(2) \\ \cdot \\ \cdot \\ \cdot \\ \rho(k) \end{bmatrix}$$

Or

$$[\ P_{(k)}\ ]\ [\ \rho`_k\ ] = [\ \rho_{(k)}\ ]$$

Solving * for k = 1, 2, 3 successively, we obtain

For k = 1 $[\ 1\ ]\ [\ \rho`_{11}\ ] = [\ \rho_{(1)}\ ]$

$$\rho`_{11} = \rho_{(1)}$$

For k = 2

$$\begin{bmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{bmatrix} \begin{bmatrix} \rho`_{21} \\ \rho`_{22} \end{bmatrix} = \begin{bmatrix} \rho(1) \\ \rho(2) \end{bmatrix}$$

$$\rho`_{22} = \frac{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & \rho(2) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{vmatrix}} = \frac{\rho(2) - \rho(1)^2}{1 - \rho(1)^2}$$

For k = 3

$$\rho`_{33} = \frac{\begin{vmatrix} 1 & \rho(1) & \rho(1) \\ \rho(1) & 1 & \rho(2) \\ \rho(2) & \rho(1) & \rho(3) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & \rho(2) \\ \rho(1) & 1 & \rho(1) \\ \rho(2) & \rho(1) & 1 \end{vmatrix}}$$

The quantity $\rho`_{k(k)}$; regarded as a function of the lag k, is called the partial correlation function.

Comments

- For an AR(r*), the partial autocorrelation function $\rho`_{k(k)}$ will be nonzero for k less than or equal (n*) and zero for k greater than n*.
- The partial autocorrelation function of a n*the order AR process has a cut off after lag P.

Estimation of the partial autocorrelation function

$$\hat{\rho}(j) = \hat{\rho}`_{k1}\hat{\rho}(j-1) + \hat{\rho}`_{k2}\hat{\rho}(j-2) + \dots\dots\dots\dots\dots\dots + \hat{\rho}`_{k(k-1)}\hat{\rho}(j-k+1) + \hat{\rho}`_{kk}\hat{\rho}(j-k)$$