Lecture #4 Autocorrelation

ECE842

Ali Keyhani

Please note the following

γ (j) = E [Y(k) Y(k+j)]

γ (-j) = E [Y(k) Y(k-j)]

$$\gamma(j) = \frac{1}{N}\sum_{k=1}^{N-j} Y(k) \quad Y(k+j)$$

Ex. 1 Let N = 5, j = 1

0, 0, 0, 0, 0, Y(1), Y(2), Y(3), Y(4), Y(5), 0, 0, 0

For j = 1 $\hat{\gamma}_1 = \hat{\gamma}(1) = \frac{1}{5}\sum_{k=1}^{4}[Y(k) \quad Y(k+1)]$

$$\hat{\gamma}_1 = \hat{\gamma}(1) = \frac{1}{5}[Y(1) \quad Y(2) + Y(2) \quad Y(3) + Y(3) \quad Y(4) + Y(4) \quad Y(5)]$$

Ex. 2 N = 5, j = -1

$$\hat{\gamma}_{-1} = \hat{\gamma}(-1) = \frac{1}{5}\sum_{k=1}^{5-(-1)}[Y(k) \quad Y(k-1)]$$

$$\hat{\gamma}(-1) = \frac{1}{5}[Y(1) \quad Y(0) + Y(2) \quad Y(1) + Y(3) \quad Y(2) + Y(4) \quad Y(3) + Y(5) \quad Y(4) + Y(6) \quad Y(5)]$$

Therefore

$$\hat{\gamma}(1) = \hat{\gamma}(-1) = \frac{1}{5}[Y(1) \quad Y(2) + Y(2) \quad Y(3) + Y(3) \quad Y(4) + Y(4) \quad Y(5)]$$

$$\hat{\rho}(j) = \hat{\rho}(-j) = \frac{\hat{\gamma}(j)}{\hat{\gamma}_0}$$

Auto-covariance matrix

Let $\{Y(k)\}_{k=1}^{N}$ zero mean process

The auto-covariance matrix C(N) is

Let $\gamma_0 = \gamma(0)$, $\gamma_1 = \gamma(1)$ ……….. and so on

$$
C(N) = \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 \cdots\cdots\cdots \gamma_{N-1} \\ \gamma_1 & \gamma_0 & \gamma_1 \cdots\cdots\cdots \gamma_{N-2} \\ \gamma_2 & \gamma_1 & \gamma_0 \cdots\cdots\cdots \gamma_{N-3} \\ . & & \\ . & & \\ \gamma_{N-1} & \gamma_{N-2} & \gamma_{N-3} \cdots\cdots\cdots \gamma_0 \end{bmatrix}
$$

The autocorrelation matrix P(N):

$$
P(N) = \frac{1}{\gamma_0}\ C(N) = \begin{bmatrix} 1 & \rho_1 & \rho_2 \cdots\cdots\cdots \rho_{N-1} \\ \rho_1 & 1 & \rho_1 \cdots\cdots\cdots \rho_{N-2} \\ \rho_2 & \rho_1 & 1 \cdots\cdots\cdots \rho_{N-3} \\ . & & \\ . & & \\ \rho_{N-1} & \rho_{N-2} & \rho_{N-3} \cdots\cdots\cdots 1 \end{bmatrix}
$$

For a stationary process, the autocorrelation matrix is positive definite. i.e.

For N = 2 $\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{vmatrix} \succ 0 \quad \rightarrow 1 - {\rho_1}^2 \succ 0$

For N = 3 $\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{vmatrix} \succ 0 \quad \rightarrow \begin{vmatrix} 1 & \rho_2 \\ \rho_2 & 1 \end{vmatrix} \succ 0$

$$
\begin{vmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{vmatrix} \succ 0
$$

To establish stationary for a process, we need to calculate a lot of

Recursive Computation Variance (σ^2)

$$\sigma^2(k+1) = E[\, Y(k+1) - \bar{Y}(k+1)\,]^2$$

$$= E[\, Y^2(k+1) + \bar{Y}^2(k+1) - 2\bar{Y}(k+1)\, Y(k+1)]$$

$$= E[\, Y^2(k+1)] + \bar{Y}^2(k+1) - 2\bar{Y}(k+1)\, E[Y(k+1)]]$$

$$= E[\, Y^2(k+1)] + \bar{Y}^2(k+1) - 2\bar{Y}(k+1)\, \bar{Y}(k+1)$$

$$= E[\, Y^2(k+1)] - \bar{Y}^2(k+1)$$

$$\sigma^2(k+1) = \frac{Y^2(1) + Y^2(2) + \dots\dots\dots\dots\dots\dots Y^2(k) + Y^2(k+1)}{k+1} - \bar{Y}^2(k+1)$$

$$= \frac{k}{k+1}[\frac{Y^2(1) + \dots\dots\dots\dots\dots Y^2(k)}{k}] + \frac{Y^2(k+1)}{k+1} - \bar{Y}^2(k+1)$$

But

$$\sigma^2(k) = E[\, Y(k) - \bar{Y}(k)\,]^2 = E[\, Y(k)]^2 - \bar{Y}^2(k)$$

Therefore

$$E[\, Y(k)]^2 = \sigma^2(k) + \bar{Y}^2(k) = \frac{Y^2(1) + \dots\dots\dots\dots\dots Y^2(k)}{k}$$

Sub. For $E[\, Y(k)]^2$

$$\sigma^2(k+1) = \frac{k}{k+1}\sigma^2(k) + \frac{Y^2(k+1)}{k+1} + \frac{k}{k+1}\bar{Y}^2(k) - \bar{Y}^2(k+1) \qquad (1)$$

Let us compute $\frac{k}{k+1}\bar{Y}^2(k)$?

First

$$\bar{Y}(k+1) = \bar{Y}(k) + \frac{Y(k+1) - \bar{Y}(k)}{k+1}$$

$$[\bar{Y}(k+1)]\,(k+1) = \bar{Y}(k)\,(k+1) + Y(k+1) - \bar{Y}(k)$$

$$= (\bar{Y}(k))\,(k) + \bar{Y}(k) + Y(k+1) - \bar{Y}(k)$$

$$k\,[\bar{Y}(k)] = (k+1)\,[\bar{Y}(k+1)] - Y(k+1)$$

$$\bar{Y}(k) = \frac{k+1}{k}\,[\bar{Y}(k+1)] + \frac{Y(k+1)}{k}$$

Cal. $\bar{Y}^2(k)$

$$\bar{Y}^2(k) = \left(\frac{k+1}{k}\right)^2 [\bar{Y}^2(k+1)] + \left(\frac{Y^2(k+1)}{k}\right) - \frac{Y(k+1)}{k}\bar{Y}(k+1)Y(k+1)$$

Multiply the above by $\frac{k}{k+1}$

$$\frac{k}{k+1}\bar{Y}^2(k) = \frac{k+1}{k}\bar{Y}^2(k+1) + \frac{1}{k(k+1)}Y^2(k+1) - \frac{2}{k}\bar{Y}(k+1)Y(k+1) \qquad (2)$$

Sub. (2) in (1) and simplify

$$-1 + \frac{k+1}{k} = \frac{-k+k+1}{k} = \frac{1}{k}$$

$$\sigma^2(k+1) = \frac{k}{k+1}\sigma^2(k) + \frac{Y^2(k+1)}{k+1} + \frac{k+1}{k}\bar{Y}^2(k+1) + \frac{1}{k(k+1)}Y^2(k+1) - \frac{2}{k}\bar{Y}(k+1)Y(k+1) - \bar{Y}^2(k+1)$$

$$\sigma^2(k+1) = \frac{k}{k+1}\sigma^2(k) + \bar{Y}^2(k+1)\left[-1 + \frac{k+1}{k}\right] + Y^2(k+1)\left[\frac{1}{k+1} + \frac{1}{k(k+1)}\right] - \frac{2}{k}\bar{Y}(k+1)Y(k+1)$$

$$\frac{1}{k+1} + \frac{1}{k(k+1)} = \frac{k+1}{k(k+1)} = \frac{1}{k}$$

$$\sigma^2(k+1) = \frac{k}{k+1}\sigma^2(k) + \frac{1}{k}\bar{Y}^2(k+1) + \frac{1}{k}Y^2(k+1) - \frac{2}{k}\bar{Y}(k+1)Y(k+1)$$

$$= \frac{k}{k+1}\sigma^2(k) + \frac{1}{k}[Y(k+1) - \bar{Y}(k+1)]^2$$