ELECTROMECHANICAL MOTION DEVICES

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McGraw-Hill Book Company

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CHAPTER 6

SYNCHRONOUS MACHINES

6.1 INTRODUCTION

Nearly all electric power is generated by synchronous machines driven either by hydroturbines or steam turbines or combustion engines. Just as the induction machine is the workhorse when it comes to converting energy from electric to mechanical, the synchronous machine is the principal means of converting energy from mechanical to electric. Although nearly all electric power is generated with three-phase synchronous machines, their electrical and electromechanical behavior can be predicted from the equations which describe the two-phase salient-pole synchronous machine. In particular, with only slight modifications, these equations can be used to predict the performance of large hydroturbine and steam turbine synchronous generators, synchronous motors, and reluctance motors used in low-power drive systems. It is for this reason that we will focus our attention on the two-phase machine since the work involved is less than with the three-phase device. However, those who wish to study the three-phase synchronous machine may do so since there is a section devoted to it near the end of the chapter.

The rotor of a synchronous machine is equipped with a field winding and one or more short-circuited windings which we will refer to as *damper windings*. In general, the rotor windings have different electrical characteristics. Moreover, the rotor of a salient-pole synchronous machine is magnetically asymmetrical. Owing to these rotor asymmetries, a change of variables offers no advantage in the case of the rotor variables. However, we will find beneficial to define a change of variables or transformation for the voltage currents, and flux linkages of the stator circuits. In effect, this transformatio replaces these stator variables with fictitious variables associated with circui fixed in the rotor.

In this chapter, the voltage and electromagnetic torque equations are fir established for the synchronous machine in machine variables. Referen frame theory is then used to establish the machine equations with the stat variables transformed to a reference frame fixed in the rotor (Park equations). The equations which describe the steady-state behavior are th derived from these equations. Attention is also given to the two-phase reluc ance motor which finds wide use in control system applications.

6.2 TWO-PHASE SYNCHRONOUS MACHINE

A two-pole two-phase salient-pole synchronous machine is shown in Fig. 6.2- The stator windings are identical, sinusoidally distributed windings, as d scribed in Chap. 4. The electrical characteristics of the rotor of a synchrono machine may be approximated by a field winding (*fd* winding) and sho circuited damper or amortisseur windings (*kq* and *kd* windings). Although t damper windings are shown with provisions to apply a voltage, they are, fact, short-circuited windings which represent the paths for induced rot currents. In particular, these short-circuited windings represent squirrel-ca type windings (short-circuited copper bars) forged below the surface of t rotor or current paths in the iron of solid-iron rotors. Laminated salient-p rotors with cage damper windings are used in machines with a large number poles while solid-iron round rotors with or without cage-type damper windi are used in high-speed (two- or four-pole) machines. In any event, electrical characteristics of the equivalent damper windings may be determin by test. We will assume that the damper windings are approximated by sinusoidally distributed windings displaced 90°. The *kd* winding has the sa magnetic axis as the *fd* winding, it has N_{kd} equivalent turns with resistance The magnetic axis of *kq* winding is 90° ahead of the magnetic axis of the *fd* *kd* windings. It has N_{kq} equivalent turns and r_{kq} resistance. It is importan mention that the rotor configuration shown in Fig. 6.2-1 for a two-ph machine is the same for any multiphase two-pole synchronous machin some cases, a more accurate representation of the electrical characteristi the rotor is achieved by assuming that two or more damper windings exi each axis (i.e., *kq*1, *kq*2, . . . and *kd*1, *kd*2, . . .). We will consider only th and *kd* windings; the modifications and extensions necessary to accommo any number of rotor windings are straightforward [1].

The quadrature axis (q axis) and direct axis (d axis) are introduced in 6.2-1. The q axis is the magnetic axis of the *kq* winding whereas the d a the magnetic axis for the *fd* and *kd* windings. The q and d axes are reser

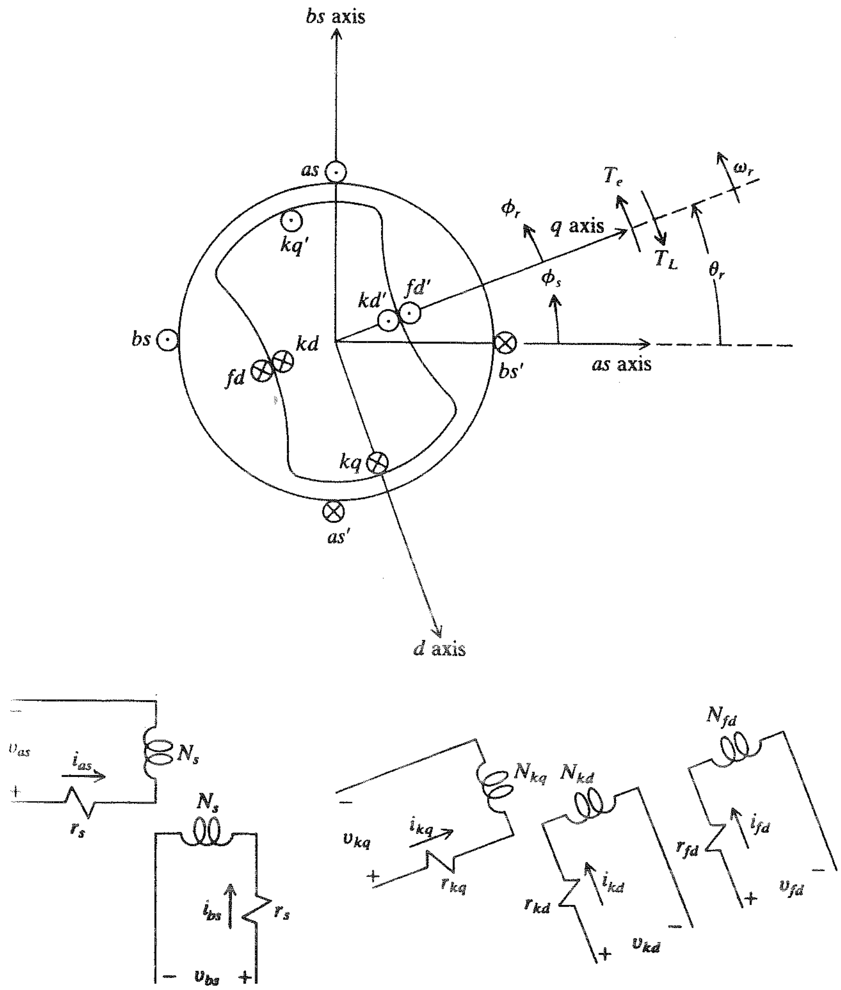


FIGURE 6.2-1
Two-pole two-phase salient-pole synchronous machine.

denote the rotor magnetic axes of a synchronous machine since, over the years, they have been associated with the physical structure of the rotor quite independent of any transformation. The angular displacement about the stator is denoted ϕ_s and it is referenced to the *as* axis. The angular displacement about the rotor is ϕ_r, which is referenced to the q axis. The electrical angular velocity of the rotor is ω_r and θ_r is the electrical angular displacement of the rotor measured from the *as* axis to the q axis. Thus, a given point on the rotor surface at the angular position ϕ_r may be related to an adjacent point on the inside stator surface with angular position ϕ_s as

$$\boxed{\phi_s = \phi_r + \theta_r} \tag{6.2-1}$$

The electromechanical torque T_e and the load torque T_L are also shown in Fig. 6.2-1. We are aware from Chap. 2 that T_e is assumed positive in the positive direction of θ_r. The load torque is positive in the opposite direction, opposing rotation.

The stator of a synchronous machine is symmetrical; however, the rotor is asymmetrical from two standpoints. The rotor windings are not identical since, in general, they do not have the same number of turns and the same value of resistance. Also, owing to the nonuniform air gap of the salient-pole synchronous machine, the magnetic characteristics of the q and d axes are not the same.

With balanced steady-state stator currents, an air gap mmf (mmf_s) is established which rotates about the air gap of a two-pole machine at ω_e, the angular velocity of the stator currents (4.4-11). Now, the damper windings are short-circuited and, for the machine to operate as a synchronous machine, a dc voltage is applied to the *fd* winding (field winding) by a brush and slip ring arrangement. The resulting field current i_{fd} establishes an air gap mmf (mmf_r) which is fixed with respect to the rotor. The air gap mmf (poles) established by the field winding must rotate at the same angular velocity as the rotating air gap mmf (poles) established by the stator windings in order to produce a nonzero average electromagnetic torque during steady-state operation. Therefore, the rotor must rotate in synchronism with the air gap mmf established by the stator windings ($\omega_r = \omega_e$); hence, the name synchronous machine. The main torque production mechanism is this interaction of the air gap mmf established by the stator currents (mmf_s) and the air gap mmf due to the direct current flowing in the field winding (mmf_r). However, electromagnetic torque (reluctance torque) is also developed at synchronous speed due to the nonuniform air gap (salient-pole rotor). The so-called salient-pole construction is common for slower speed machines (large number of poles) such as hydroturbine generators. In this type of rotor construction, the field winding is wound upon the rotor surface, as shown in Fig. 6.2-1, and the air gap is nonuniform to make room for the placement of the field winding. Therefore the q-axis magnetic path has a higher reluctance than the d-axis magnetic path. Now, in Chap. 2 we learned that torque is produced in a reluctance machine to align the minimum-reluctance path of the rotor with the mmf produced by the stator. Let us apply this principle to the salient-pole synchronous machine. There is an electromagnetic torque developed to align the minimum-reluctance path (d axis) with the resultant air gap mmf ($\text{mmf}_s + \text{mmf}_r$). We will find that in the case of salient-pole synchronous machines, the reluctance torque is a small part of the total torque developed. However, two facts warrant mentioning; first in high-speed synchronous machines (two-, four-, six-pole machines) the field windings are generally embedded in rotor slots and the air gap is, for the most part, uniform (round rotor). It is apparent that, in the case of round-rotor synchronous machine, the reluctance torque is not present

Second, if the field winding (*fd* winding) is removed from the salient-pole synchronous machine shown in Fig. 6.2-1, it would be a two-phase reluctance machine, which is used widely in low-power drive systems.

We have yet to discuss the damper windings. It was found early on that a synchronous machine with only a field winding on the rotor and without provisions for induced currents to circulate in the rotor iron would tend to oscillate about synchronous speed in a slowly damped manner following any slight disturbance. Adding damper windings (short-circuited rotor windings) provided the desired damping. To explain this damping action, let us compare the torque developed by the damper windings to that developed by an induction motor. The damper windings are short-circuited as are the rotor windings of an induction motor and, as we discussed in Chap. 4, currents are induced in these rotor (damper) windings whenever the speed of the rotor differs from the angular velocity of the rotating air gap mmf established by the stator currents (mmf_s). (It is assumed that you have not studied Chap. 5, but that you have read the material in Sec. 4.5 on induction machines.) Since the damper windings are not symmetrical and since the air gap is not uniform, the steady-state torque due to the interaction of the currents induced in the damper windings and mmf_s will pulsate; however, an average torque will occur. Now, the main torque of a synchronous machine is developed at synchronous speed because of the interaction of mmf_s and mmf_r. At synchronous speed, current is not induced in the damper windings and, hence, "induction motor" torque is not developed. However, if for any reason the speed of the rotor should vary around synchronous speed because of a disturbance, currents will be induced in the damper windings and the torque developed due to induction motor action, although small, will damp oscillations of the rotor speed. That is, a slight slowing down (speeding up) of the rotor will produce an induction motor torque to accelerate (decelerate) the rotor back to synchronous speed.

Although it is generally necessary to start large synchronous machines by auxiliary means, smaller-horsepower synchronous machines and reluctance motors develop sufficient induction motor torque because of the damper windings to accelerate the machine to near synchronous speed. During this starting period, the field winding of the synchronous machine is also short-circuited, hence, it too provides some induction motor torque. The synchronous machine will accelerate to near synchronous speed, whereupon it will operate as an induction machine developing the average torque necessary to satisfy the no-load losses. The field winding is then open-circuited and a dc voltage is applied to its terminals by means of a brush and slip ring combination. The machine then "pulls in" to synchronism with the rotating air gap mmf established by the stator currents and, thus, operates as a synchronous machine.

The damper windings of reluctance motors are often designed so that the device will develop sufficient induction motor torque to accelerate the rotor, sometimes under load, from stall to near synchronous speed. If the load is not

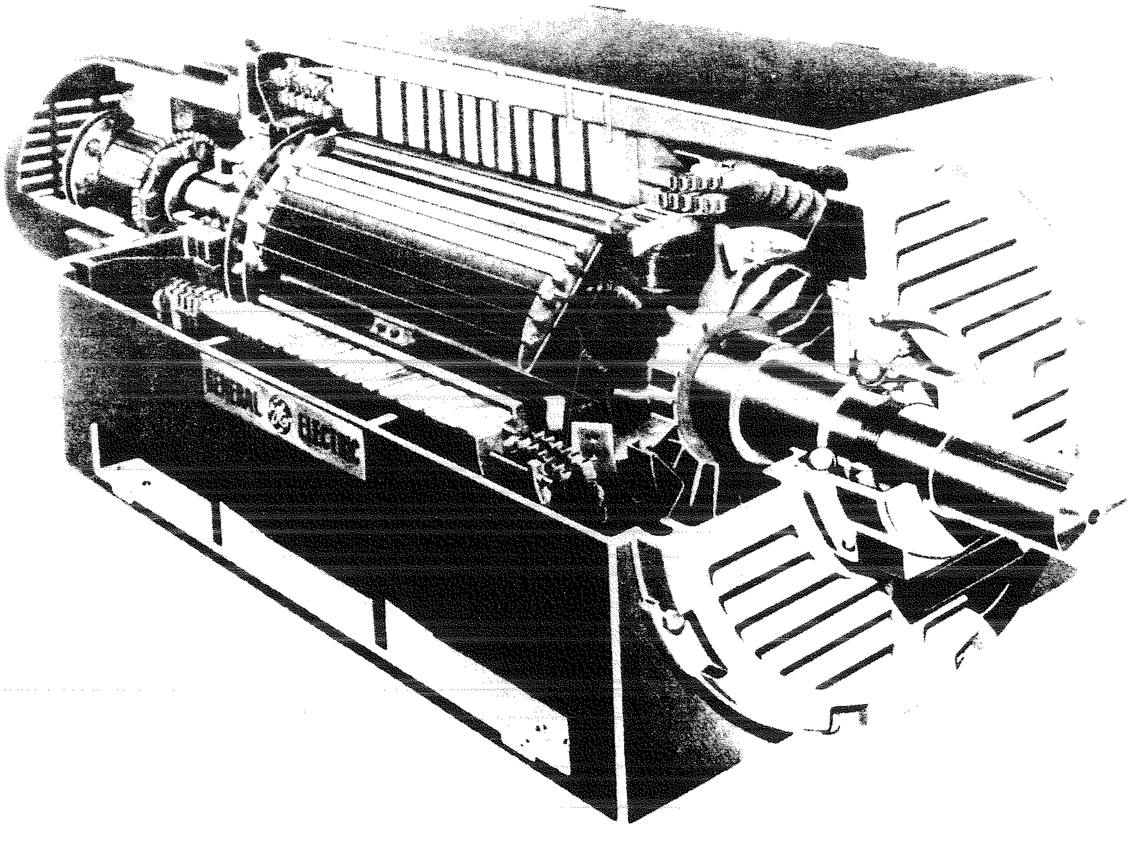


FIGURE 6.2-2 [illegible] synchronous machine. (*Courtesy of General Electric.*)

too large, the reluctance torque will then pull the rotor in step with the rotating air gap mmf established by the stator currents, and the device will operate as a reluctance machine.

Torque is torque by whatever means it is developed and, perhaps, we should not emphasize the separation of torque into three types since the system is nonlinear and superposition cannot be applied. Nevertheless, this separation is helpful and, as we proceed, we will be able to identify what we have called the induction motor torque, the reluctance torque, and the torque due to the interaction of mmf_s and mmf_r, all of which can occur in the machine shown in Fig. 6.2-1.

A four-pole three-phase salient-pole synchronous machine is shown in Fig. 6.2-2. Note the dc machine connected to the shaft for purposes of supplying voltage to the field winding of the synchronous machine. Note also, the squirrel-cage damper windings embedded in the pole faces. Figure 6.2-3 shows the stator and rotor of a miniature two-pole three-phase alternator with an alnico permanent-magnet rotor. This device produces 12 W at 4200 r/min to supply aircraft instruments. It mounts on an aircraft engine auxiliary drive pad where temperatures can be as high as 350°F.

SP6.2-1. Express mmf_r for the two-pole two-phase synchronous machine shown in Fig. 6.2-1. [$\text{mmf}_r = -(N_{fd}/2)i_{fd} \sin(\phi_s - \theta_r)$]
SP6.2-2. A dc voltage is applied to the *fd* winding of the machine shown in Fig. 6.2-1.

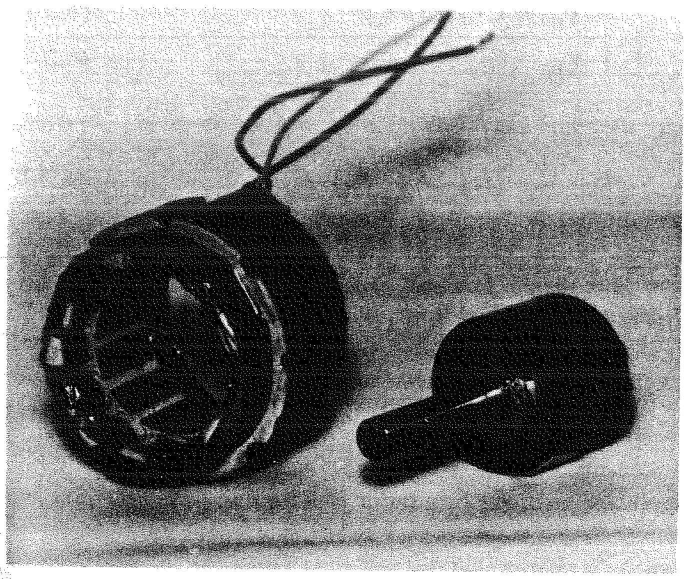


FIGURE 6.2-3
Stator and rotor of a two-pole 12-W 4200-r/min permanent-magnet synchronous machine. (*Courtesy of Vickers ElectroMech.*)

The damper windings are short-circuited and the machine is driven at ω_r, counterclockwise. Assume that the stator currents are balanced 60-Hz currents with $\tilde{I}_{as} = -j\tilde{I}_{bs}$. Determine the frequency of the currents flowing in the damper windings. $[\omega_r + \omega_e]$

6.3 VOLTAGE EQUATIONS AND WINDING INDUCTANCES

The voltage equations for the two-pole two-phase salient-pole synchronous machine shown in Fig. 6.2-1 may be expressed as

$$v_{as} = r_s i_{as} + \frac{d\lambda_{as}}{dt} \tag{6.3-1}$$

$$v_{bs} = r_s i_{bs} + \frac{d\lambda_{bs}}{dt} \tag{6.3-2}$$

$$v_{kq} = r_{kq} i_{kq} + \frac{d\lambda_{kq}}{dt} \tag{6.3-3}$$

$$v_{fd} = r_{fd} i_{fd} + \frac{d\lambda_{fd}}{dt} \tag{6.3-4}$$

$$v_{kd} = r_{kd} i_{kd} + \frac{d\lambda_{kd}}{dt} \tag{6.3-5}$$

The above equations may be written in matrix form as

$$\mathbf{v}_{abs} = \mathbf{r}_s \mathbf{i}_{abs} + p\boldsymbol{\lambda}_{abs} \tag{6.3-6}$$

$$\mathbf{v}_{qdr} = \mathbf{r}_r \mathbf{i}_{qdr} + p\boldsymbol{\lambda}_{qdr} \tag{6.3-7}$$

where

$$(\mathbf{f}_{abs})^T = [f_{as} \quad f_{bs}] \tag{6.3-8}$$

$$(\mathbf{f}_{qdr})^T = [f_{kq} \quad f_{fd} \quad f_{kd}] \tag{6.3-9}$$

In the above equations, the s and r subscripts denote variables associated with the stator and rotor windings, respectively, and p is the operator d/dt. Also

$$\mathbf{r}_s = \begin{bmatrix} r_s & 0 \\ 0 & r_s \end{bmatrix} \tag{6.3-10}$$

$$\mathbf{r}_r = \begin{bmatrix} r_{kq} & 0 & 0 \\ 0 & r_{fd} & 0 \\ 0 & 0 & r_{kd} \end{bmatrix} \tag{6.3-11}$$

A review of matrix algebra is given in Appendix D. The flux linkage equation may be expressed as

$$\lambda_{as} = L_{asas} i_{as} + L_{asbs} i_{bs} + L_{askq} i_{kq} + L_{asfd} i_{fd} + L_{askd} i_{kd} \tag{6.3-12}$$

$$\lambda_{bs} = L_{bsas} i_{as} + L_{bsbs} i_{bs} + L_{bskq} i_{kq} + L_{bsfd} i_{fd} + L_{bskd} i_{kd} \tag{6.3-13}$$

$$\lambda_{kq} = L_{kqas} i_{as} + L_{kqbs} i_{bs} + L_{kqkq} i_{kq} + L_{kqfd} i_{fd} + L_{kqkd} i_{kd} \tag{6.3-14}$$

$$\lambda_{fd} = L_{fdas}i_{as} + L_{fdbs}i_{bs} + L_{fdkq}i_{kq} + L_{fdfd}i_{fd} + L_{fdkd}i_{kd} \qquad (6.3\text{-}15)$$

$$\lambda_{kd} = L_{kdas}i_{as} + L_{kdbs}i_{bs} + L_{kdkq}i_{kq} + L_{kdfd}i_{fd} + L_{kdkd}i_{kd} \qquad (6.3\text{-}16)$$

In the case of a salient-pole device (nonuniform air gap), the self-inductances of the stator windings and the mutual inductances between stator windings are functions of θ_r. Although this is a review of the material in Chap. 1, let us consider L_{asas}. With $\theta_r = 0$, we see from Fig. 6.2-1 that the magnetizing inductance of L_{asas} is less than it would be when $\theta_r = \frac{1}{2}\pi$. Let the magnetizing inductance of the *as* winding be denoted L_{mq} when $\theta_r = 0$ since the q axis (high-reluctance path) is aligned with the magnetic axis of the *as* winding. Thus,

$$L_{asas} = L_{ls} + L_{mq} \qquad \theta_r = 0 \qquad (6.3\text{-}17)$$

where L_{ls} is the leakage inductance of the stator windings and

$$L_{mq} = \frac{N_s^2}{\mathcal{R}_{mq}} \qquad (6.3\text{-}18)$$

where $\mathcal{R}_{mq}$ is an equivalent reluctance of the magnetic path in the q axis. We called this $\mathcal{R}_m(0)$ in (1.7-24). Now, at $\theta_r = \frac{1}{2}\pi$ the d axis (low-reluctance path) is aligned with the magnetic axis of the *as* winding. Hence, denoting this magnetizing inductance as L_{md}, we can write

$$L_{asas} = L_{ls} + L_{md} \qquad \theta_r = \tfrac{1}{2}\pi \qquad (6.3\text{-}19)$$

where

$$L_{md} = \frac{N_s^2}{\mathcal{R}_{md}} \qquad (6.3\text{-}20)$$

where $\mathcal{R}_{md}$ is an equivalent reluctance of the magnetic path in the d axis. This is $\mathcal{R}_m(\frac{1}{2}\pi)$ in (1.7-25).

Since $\mathcal{R}_{mq} > \mathcal{R}_{md}$, $L_{mq} < L_{md}$, and we see that a minimum L_{asas} occurs at $\theta_r = 0$ and also again at $\theta_r = \pi$. Therefore, (6.3-17) is valid for $\theta_r = 0$ and π. Similarly, maximum L_{asas} occurs at $\theta_r = \frac{1}{2}\pi$ and again at $\theta_r = \frac{3}{2}\pi$; hence (6.3-19) applies for $\theta_r = \frac{1}{2}\pi$ and $\frac{3}{2}\pi$. The magnetizing inductance varies about an average value (which must be positive) and if we assume this variation to be sinusoidal, it would vary as a function of $2\theta_r$ (Fig. 1.7-3). Let L_A be the average value and L_B the amplitude of the sinusoidal variation about this average value. In this case,

$$L_{mq} = L_A - L_B \qquad (6.3\text{-}21)$$

$$L_{md} = L_A + L_B \qquad (6.3\text{-}22)$$

Substituting (6.3-18) and (6.3-20) for L_{mq} and L_{md}, respectively, into (6.3-21) and (6.3-22) and solving for L_A and L_B yields

$$L_A = \frac{N_s^2}{2}\left(\frac{1}{\mathcal{R}_{mq}} + \frac{1}{\mathcal{R}_{md}}\right) \qquad (6.3\text{-}23)$$

$$L_B = \frac{N_s^2}{2}\left(\frac{1}{\mathcal{R}_{mq}} - \frac{1}{\mathcal{R}_{md}}\right) \tag{6.3-24}$$

Assuming a sinusoidal variation, we can write (Fig. 1.7-3)

$$L_{asas} = L_{ls} + L_A - L_B \cos 2\theta_r \tag{6.3-25}$$

If the air gap were uniform as is the case in a round-rotor synchronous machine, $\mathcal{R}_{mq} = \mathcal{R}_{md}$ and, hence, from (6.3-24), $L_B = 0$.

By a similar procedure, it follows that, for the salient-pole device,

$$L_{bsbs} = L_{ls} + L_A + L_B \cos 2\theta_r \tag{6.3-26}$$

Note when $\theta_r = 0$, L_{asas} is a minimum according to (6.3-25) and, according to (6.3-26), L_{bsbs} is a maximum. This, of course, corresponds to that which is portrayed in Fig. 6.2-1.

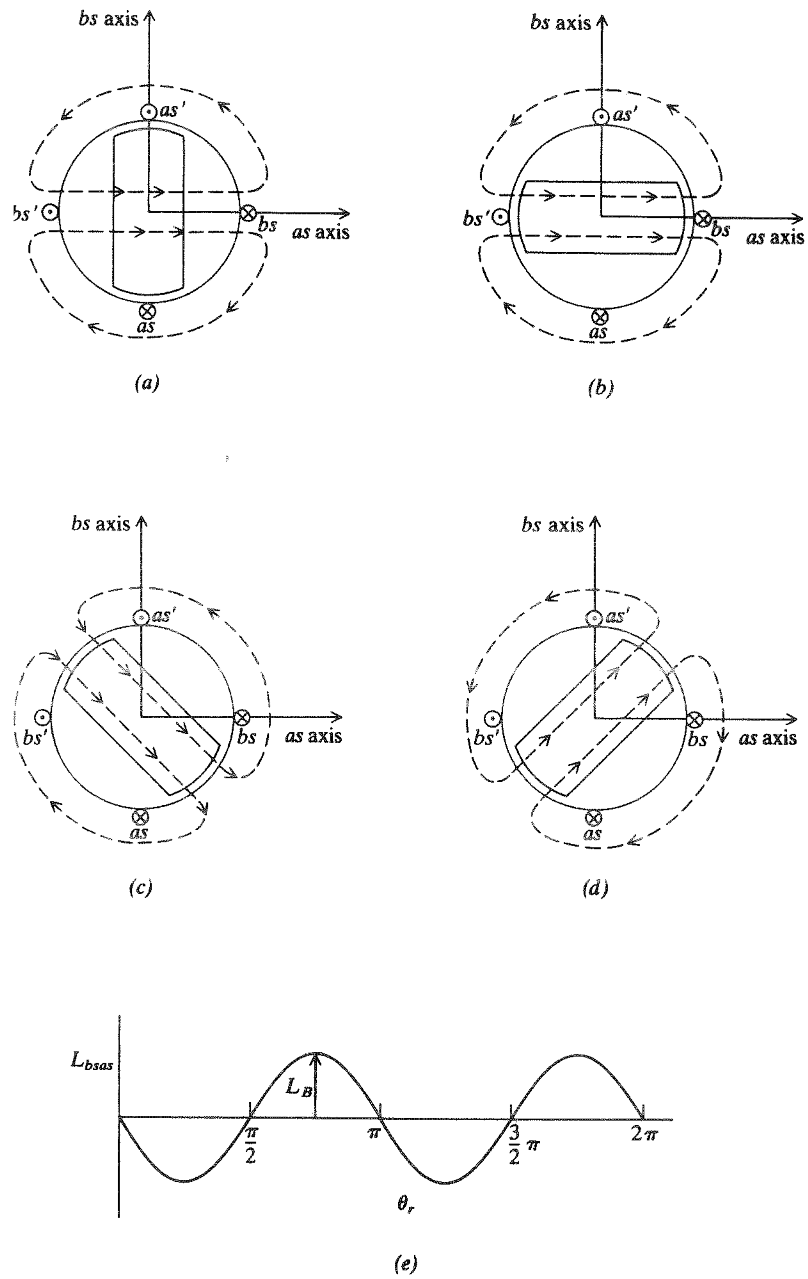
The mutual inductance $L_{asbs}(L_{bsas})$ is next. One would think that since the windings are orthogonal, the mutual coupling would always be zero. However, this is not the case due to the fact that the air gap is not uniform. Let us consider Fig. 6.3-1 where various rotor positions are shown with only the flux paths of the *as* winding illustrated. Coupling occurs when flux produced by one winding links the other winding; in particular, when the flux of the *as* winding links the *bs* winding. This will give us L_{bsas} and we know that $L_{asbs} = L_{bsas}$.

Note that, when $\theta_r = 0$, π, and 2π as shown in Fig. 6.3-1*a* or when $\theta_r = \frac{1}{2}\pi$ and $\frac{3}{2}\pi$ as shown in Fig. 6.3-1*b*, L_{bsas} is zero. In these positions, there is no channeling of the flux of one winding through the other. However, let the rotor start to turn counterclockwise from zero toward $\frac{1}{2}\pi$ and consider the flux produced by the *as* winding. As the rotor turns, the configuration of the rotor provides a low-reluctance path to the flux produced by the *as* winding and the flux is channeled through the *bs* winding with maximum coupling occurring at $\theta_r = \frac{1}{4}\pi$, as illustrated in Fig. 6.3-1*c*. We see that this same rotor position relative to the windings occurs also at $\theta_r = \frac{5}{4}\pi$. Maximum coupling will again occur at $\theta_r = \frac{3}{4}\pi$ and $\frac{7}{4}\pi$, as illustrated by Fig. 6.3-1*d*. Now, what is the sign of the mutual inductance? With the assumed direction of positive currents, the right-hand rule tells us that L_{bsas} (or L_{asbs}) is negative at $\theta_r = \frac{1}{4}\pi, \frac{5}{4}\pi, \ldots$ (the fluxes of the windings oppose each other for positive currents) and positive for $\theta_r = \frac{3}{4}\pi, \frac{7}{4}\pi, \ldots$ (the fluxes aid each other). If we sketch L_{bsas} versus θ_r using the above information, we see, from Fig. 6.3-1*e*, that, as a first approximation, L_{bsas} or L_{asbs} may be expressed as

$$L_{bsas} = L_{asbs} = -L_B \sin 2\theta_r \tag{6.3-27}$$

In order for us to prove that the coefficient is L_B, it would be necessary to become quite involved [1]. We will accept this without proving it.

Let us now go back to the flux linkage equations, (6.3-12) through (6.3-16), and write these equations in matrix form as



'IGURE 6.3-1
'lux path of *as* winding illustrating the mutual coupling between stator windings to determine L_{bsas} nd L_{asbs}. (*a*) $\theta_r = 0$, π, and 2π; (*b*) $\theta_r = \frac{1}{2}\pi$ and $\frac{3}{2}\pi$; (*c*) $\theta_r = \frac{1}{4}\pi$ and $\frac{5}{4}\pi$; (*d*) $\theta_r = \frac{3}{4}\pi$ and $\frac{7}{4}\pi$; (*e*) pproximation of L_{bsas} and L_{asbs}.

$$\begin{bmatrix} \boldsymbol{\lambda}_{abs} \\ \boldsymbol{\lambda}_{qdr} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_s & \mathbf{L}_{sr} \\ (\mathbf{L}_{sr})^T & \mathbf{L}_r \end{bmatrix} \begin{bmatrix} \mathbf{i}_{abs} \\ \mathbf{i}_{qdr} \end{bmatrix} \tag{6.3-28}$$

The matrix $\mathbf{L}_s$ can now be written as

$$\mathbf{L}_s = \begin{bmatrix} L_{asas} & L_{asbs} \\ L_{bsas} & L_{bsbs} \end{bmatrix}$$

$$= \begin{bmatrix} L_{ls} + L_A - L_B \cos 2\theta_r & -L_B \sin 2\theta_r \\ -L_B \sin 2\theta_r & L_{ls} + L_A + L_B \cos 2\theta_r \end{bmatrix} \tag{6.3-29}$$

By inspection of Fig. 6.2-1, we can write

$$\mathbf{L}_{sr} = \begin{bmatrix} L_{askq} & L_{asfd} & L_{askd} \\ L_{bskq} & L_{bsfd} & L_{bskd} \end{bmatrix}$$

$$= \begin{bmatrix} L_{skq} \cos \theta_r & L_{sfd} \sin \theta_r & L_{skd} \sin \theta_r \\ L_{skq} \sin \theta_r & -L_{sfd} \cos \theta_r & -L_{skd} \cos \theta_r \end{bmatrix} \tag{6.3-30}$$

$$\mathbf{L}_r = \begin{bmatrix} L_{kqkq} & L_{kqfd} & L_{kqkd} \\ L_{fdkq} & L_{fdfd} & L_{fdkd} \\ L_{kdkq} & L_{kdfd} & L_{kdkd} \end{bmatrix}$$

$$= \begin{bmatrix} L_{lkq} + L_{mkq} & 0 & 0 \\ 0 & L_{lfd} + L_{mfd} & L_{fdkd} \\ 0 & L_{fdkd} & L_{lkd} + L_{mkd} \end{bmatrix} \tag{6.3-31}$$

In the above inductance matrices, the leakage inductances are denoted with a l in the subscript. The skq, sfd, and skd subscripts denote the peak mutua inductances between stator and rotor windings. The following equations defin the inductances used in (6.3-30) and (6.3-31):

$$L_{skq} = \frac{N_{kq}}{N_s} L_{mq} \tag{6.3-32}$$

$$L_{sfd} = \frac{N_{fd}}{N_s} L_{md} \tag{6.3-33}$$

$$L_{skd} = \frac{N_{kd}}{N_s} L_{md} \tag{6.3-34}$$

$$L_{mkq} = \left(\frac{N_{kq}}{N_s}\right)^2 L_{mq} \tag{6.3-35}$$

$$L_{mfd} = \left(\frac{N_{fd}}{N_s}\right)^2 L_{md} \tag{6.3-36}$$

$$L_{mkd} = \left(\frac{N_{kd}}{N_s}\right)^2 L_{md} \tag{6.3-37}$$

$$L_{fdkd} = \frac{N_{kd}}{N_{fd}} L_{mfd} = \frac{N_{fd}}{N_{kd}} L_{mkd} \tag{6.3-38}$$

As in the case of the induction machine, it is convenient to refer the rotor variables to a winding with N_s turns. Thus,

$$i_j' = \frac{N_j}{N_s} i_j \tag{6.3-39}$$

$$v_j' = \frac{N_s}{N_j} v_j \tag{6.3-40}$$

$$\lambda_j' = \frac{N_s}{N_j} \lambda_j \tag{6.3-41}$$

where j may be kq, fd, or kd. The flux linkage equations given by (6.3-28) may now be written as

$$\begin{bmatrix} \boldsymbol{\lambda}_{abs} \\ \boldsymbol{\lambda}_{qdr}' \end{bmatrix} = \begin{bmatrix} \mathbf{L}_s & \mathbf{L}_{sr}' \\ (\mathbf{L}_{sr}')^T & \mathbf{L}_r' \end{bmatrix} \begin{bmatrix} \mathbf{i}_{abs} \\ \mathbf{i}_{qdr}' \end{bmatrix} \tag{6.3-42}$$

where

$$\mathbf{L}_{sr}' = \begin{bmatrix} L_{mq}\cos\theta_r & L_{md}\sin\theta_r & L_{md}\sin\theta_r \\ L_{mq}\sin\theta_r & -L_{md}\cos\theta_r & -L_{md}\cos\theta_r \end{bmatrix} \tag{6.3-43}$$

$$\mathbf{L}_r' = \begin{bmatrix} L_{lkq}' + L_{mq} & 0 & 0 \\ 0 & L_{lfd}' + L_{md} & L_{md} \\ 0 & L_{md} & L_{lkd}' + L_{md} \end{bmatrix} \tag{6.3-44}$$

The voltage equations expressed in terms of machine variables referred by a turns-ratio to the stator windings are

$$\mathbf{v}_{abs} = \mathbf{r}_s \mathbf{i}_{abs} + p\boldsymbol{\lambda}_{abs} \tag{6.3-45}$$

$$\mathbf{v}_{qdr}' = \mathbf{r}_r' \mathbf{i}_{qdr}' + p\boldsymbol{\lambda}_{qdr}' \tag{6.3-46}$$

In terms of inductances, (6.3-45) and (6.3-46) become

$$\begin{bmatrix} \mathbf{v}_{abs} \\ \mathbf{v}_{qdr}' \end{bmatrix} = \begin{bmatrix} \mathbf{r}_s + p\mathbf{L}_s & p\mathbf{L}_{sr}' \\ p(\mathbf{L}_{sr}')^T & \mathbf{r}_r' + p\mathbf{L}_r' \end{bmatrix} \begin{bmatrix} \mathbf{i}_{abs} \\ \mathbf{i}_{qdr}' \end{bmatrix} \tag{6.3-47}$$

where in the matrices $\mathbf{r}_r'$ and $\mathbf{L}_r'$

$$r_j' = \left(\frac{N_s}{N_j}\right)^2 r_j \tag{6.3-48}$$

$$L_{lj}' = \left(\frac{N_s}{N_j}\right)^2 L_{lj} \tag{6.3-49}$$

where, as before, j may be kq, fd, or kd.

Since the synchronous machine is generally operated as a generator, it is often considered more convenient to assume positive current out of the machine. This may be done in the above equations by simply placing a negative sign preceding $\mathbf{i}_{abs}$.

SP6.3-1. Express L_{asbs} for positive θ_r in the clockwise direction in Fig. 6.2-1 with (*a*) positive direction of i_{as} reversed; (*b*) positive direction of i_{bs} reversed; and (*c*) positive direction of both i_{as} and i_{bs} reversed. [(*a*) and (*b*) $L_{asbs} = (6.3\text{-}27)$; (*c*) $L_{asbs} = -(6.3\text{-}27)$]

SP6.3-2. The current i_{fd} in Fig. 6.2-1 is 1 A, $L_{sfd} = 0.1$ H, and $\theta_r = 10t$. Determine the open-circuited steady-state voltages V_{as} and V_{bs}. [$V_{as} = \cos 10t$; $V_{bs} = \sin 10t$]

SP6.3-3. The current i'_{fd} in a round-rotor synchronous machine is 1 A, $L_{mq} = 0.1$ H, $L_{asfd} = \sin\theta_r$, and $\theta_r = 10t$. Determine the open-circuited steady-state voltages V_{as} and V_{bs}. [SP6.3-2]

6.4 TORQUE

The electromagnetic torque may be evaluated from Table 2.5-1:

$$T_e = \frac{P}{2}\frac{\partial W_c(\mathbf{i}, \theta_r)}{\partial \theta_r} \tag{6.4-1}$$

For a magnetically linear system, this yields

$$\boxed{\begin{aligned} T_e = \frac{P}{2}\bigg\{ &\frac{L_{md} - L_{mq}}{2}[(i_{as}^2 - i_{bs}^2)\sin 2\theta_r - 2i_{as}i_{bs}\cos 2\theta_r] \\ &- L_{mq}i_{kq}(i_{as}\sin\theta_r - i_{bs}\cos\theta_r) \\ &+ L_{md}(i'_{fd} + i'_{kd})(i_{as}\cos\theta_r + i_{bs}\sin\theta_r)\bigg\} \end{aligned}} \tag{6.4-2}$$

The above expression for torque is positive for motor action. Obtaining (6.4-2) from (6.4-1) is a problem at the end of the chapter.

The torque and rotor speed are related by

$$\boxed{T_e = J\left(\frac{2}{P}\right)\frac{d\omega_r}{dt} + B_m\left(\frac{2}{P}\right)\omega_r + T_L} \tag{6.4-3}$$

where J is the inertia expressed in kilogram · meter^2 (kg · m^2) or joule · second^2 (J · s^2). Often, the inertia is given as WR^2 in units of pound mass · feet^2 (lbm · ft^2). As indicated in Fig. 6.2-1, T_L is positive for a torque load when the machine is operated as a motor and negative when torque is supplied to the shaft of the machine by a prime mover (generator action). The constant B_m is a damping coefficient associated with the rotational system of the machine and mechanical load. It has the units N · m · s/rad of mechanical rotation, and it generally small and often neglected in the case of the machine but may be considerable for the mechanical load.

SP6.4-1. Which of the terms on the right-hand side of (6.4-2) can be thought of as the reluctance torque? $\left\{\left(\frac{P}{2}\right)\left(\frac{L_{md}-L_{mq}}{2}\right)[\quad]\right\}$
SP6.4-2. Repeat SP6.4-1 for the damping (induction motor) torque. [Terms with i'_{kq} or i'_{kd}]
SP6.4-3. Repeat SP6.4-1 for the torque due to the interaction of mmf$_s$ and the *fd* current. [Terms with i'_{fd}]

6.5 MACHINE EQUATIONS IN THE ROTOR REFERENCE FRAME

The mutual inductances between the stator and rotor windings vary sinusoidally with θ_r. Moreover, the self-inductances of the stator windings are sinusoidal functions of $2\theta_r$. Fortunately, a change of variables is helpful. It appears that R. H. Park was the first to incorporate a change of variables in the analysis of synchronous machines [2]. He transformed the stator variables to the rotor reference frame, thereby eliminating the time-varying inductances. For a two-phase machine, Park's transformation is

$$\begin{bmatrix} f_{qs}^r \\ f_{ds}^r \end{bmatrix} = \begin{bmatrix} \cos\theta_r & \sin\theta_r \\ \sin\theta_r & -\cos\theta_r \end{bmatrix} \begin{bmatrix} f_{as} \\ f_{bs} \end{bmatrix} \tag{6.5-1}$$

or

$$\boxed{\mathbf{f}_{qds}^r = \mathbf{K}_s^r \mathbf{f}_{abs}} \tag{6.5-2}$$

where f can represent either voltage, current, or flux linkage. It follows that

$$\mathbf{f}_{abs} = (\mathbf{K}_s^r)^{-1}\mathbf{f}_{qds}^r \tag{6.5-3}$$

where it can be shown that $(\mathbf{K}_s^r)^{-1} = \mathbf{K}_s^r$. Also,

$$\theta_r = \int_0^t \omega_r(\xi)\, d\xi + \theta_r(0) \tag{6.5-4}$$

where ξ is a dummy variable of integration and $\theta_r(0)$ is the time zero position of the rotor, which is generally selected to be zero. The s subscript denotes stator variables and the r superscript indicates that the transformation is to a reference frame fixed in the rotor. This same transformation is used in Chap. 7 to analyze the permanent-magnet synchronous machine (brushless dc motor).

Although the above change of variables does not require a physical connotation, it may be helpful to visualize this transformation as trigonometric relationships between variables with directions as shown in Fig. 6.5-1. The direction of f_{as} and f_{bs} variables shown in Fig. 6.5-1 happens to be the positive direction of the magnetic axes of the associated windings (*as* winding and *bs* winding). It follows then that the f_{qs}^r and f_{ds}^r variables can be thought of as

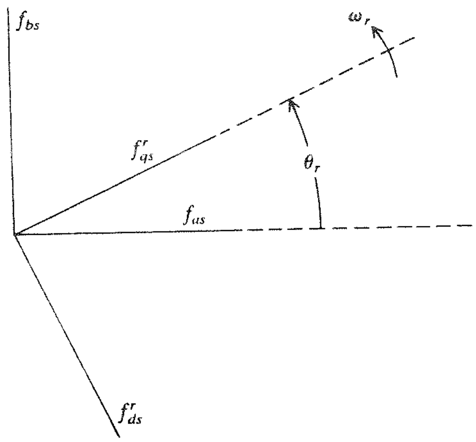


FIGURE 6.5-1
Trigonometric interpretation of the change of stator variables.

being associated with fictitious windings fixed on the rotor the positive magnetic axes of which are in the same direction as the direction of f^r_{qs} and f^r_{ds}. These fictitious windings result due to the change of variables for the stator variables. The positive magnetic axis of the winding associated with the f^r_{qs} variables coincides with the q axis of the rotor and the magnetic axis of the winding associated with the f^r_{ds} variables coincides with the d axis of the rotor.

The s (r) subscript denotes association with the stator (rotor) variables. As mentioned previously, the superscript r indicates that the transformation is to the rotor reference frame. Since the rotor reference frame is generally the only reference frame used in the analysis of synchronous machines, it could be omitted; however, we will carry it along for completeness.

Substituting (6.5-3) into (6.3-45) yields

$$(\mathbf{K}^r_s)^{-1}\mathbf{v}^r_{qds} = \mathbf{r}_s(\mathbf{K}^r_s)^{-1}\mathbf{i}^r_{qds} + p[(\mathbf{K}^r_s)^{-1}\boldsymbol{\lambda}^r_{qds}] \tag{6.5-5}$$

Multiplying (6.5-5) by $\mathbf{K}^r_s$ and rewriting (6.3-46) with the superscript r incorporated for notational completeness, we obtain Park's equations.

$$\mathbf{v}^r_{qds} = \mathbf{r}_s\mathbf{i}^r_{qds} + \omega_r\boldsymbol{\lambda}^r_{dqs} + p\boldsymbol{\lambda}^r_{qds} \tag{6.5-6}$$

$$\mathbf{v}'^r_{qdr} = \mathbf{r}'_r\mathbf{i}'^r_{qdr} + p\boldsymbol{\lambda}'^r_{qdr} \tag{6.5-7}$$

where

$$(\boldsymbol{\lambda}^r_{dqs})^T = [\lambda^r_{ds} \quad -\lambda^r_{qs}] \tag{6.5-8}$$

Equation (6.5-7) is (6.3-46) with the r superscript added to emphasize the fact that the voltage equations for the rotor circuits are written in the rotor frame of reference. We will not use a change of variables for the rotor variables. The last two terms of (6.5-6) come from the last term of (6.5-5) multiplied by $\mathbf{K}^r_s$. That is,

$$\mathbf{K}^r_s p[(\mathbf{K}^r_s)^{-1}\boldsymbol{\lambda}^r_{qds}] = \mathbf{K}^r_s[p(\mathbf{K}^r_s)^{-1}]\boldsymbol{\lambda}^r_{qds} + \mathbf{K}^r_s(\mathbf{K}^r_s)^{-1}p\boldsymbol{\lambda}^r_{qds} \tag{6.5-9}$$

It is left to the reader to show that the right-hand side of (6.5-9) reduces to the last two terms of (6.5-6). As a guide, one may wish to refer to (4.8-16) where a similar procedure was carried out for the induction machine.

Equations (6.5-6) and (6.5-7) are valid for a linear or nonlinear magnetic system. For a linear magnetic system, we can express $\boldsymbol{\lambda}_{abs}$ and $\boldsymbol{\lambda}'_{qdr}$ from (6.3-42):

$$\boldsymbol{\lambda}_{abs} = \mathbf{L}_s\mathbf{i}_{abs} + \mathbf{L}'_{sr}\mathbf{i}'_{qdr} \tag{6.5-10}$$

$$\boldsymbol{\lambda}'_{qdr} = (\mathbf{L}'_{sr})^T\mathbf{i}_{abs} + \mathbf{L}'_r\mathbf{i}'_{qdr} \tag{6.5-11}$$

where $\mathbf{L}_s$, $\mathbf{L}'_{sr}$, and $\mathbf{L}'_r$ are given by (6.3-29), (6.3-43), and (6.3-44), respectively. Substituting (6.5-3) for $\boldsymbol{\lambda}_{abs}$ and $\mathbf{i}_{abs}$ and adding the r superscript to $\boldsymbol{\lambda}'_{qdr}$ and $\mathbf{i}'_{qdr}$ and then solving (6.5-10) for $\boldsymbol{\lambda}^r_{qds}$ yields

$$\boldsymbol{\lambda}^r_{qds} = \mathbf{K}^r_s\mathbf{L}_s(\mathbf{K}^r_s)^{-1}\mathbf{i}^r_{qds} + \mathbf{K}^r_s\mathbf{L}'_{sr}\mathbf{i}'^r_{qdr} \tag{6.5-12}$$

$$\boldsymbol{\lambda}'^r_{qdr} = (\mathbf{L}'_{sr})^T(\mathbf{K}^r_s)^{-1}\mathbf{i}^r_{qds} + \mathbf{L}'_r\mathbf{i}'^r_{qdr} \tag{6.5-13}$$

We can show that

$$\mathbf{K}^r_s\mathbf{L}_s(\mathbf{K}^r_s)^{-1} = \begin{bmatrix} L_{ls}+L_{mq} & 0 \\ 0 & L_{ls}+L_{md} \end{bmatrix} \tag{6.5-14}$$

$$\mathbf{K}^r_s\mathbf{L}'_{sr} = \begin{bmatrix} L_{mq} & 0 & 0 \\ 0 & L_{md} & L_{md} \end{bmatrix} \tag{6.5-15}$$

$$(\mathbf{L}'_{sr})^T(\mathbf{K}^r_s)^{-1} = \begin{bmatrix} L_{mq} & 0 \\ 0 & L_{md} \\ 0 & L_{md} \end{bmatrix} \tag{6.5-16}$$

where L_{mq} and L_{md} are defined by (6.3-21) and (6.3-22), respectively. In a problem at the end of the chapter, you are asked to obtain (6.5-14) through (6.5-16). The flux linkage equations may now be written as

$$\begin{bmatrix} \lambda^r_{qs} \\ \lambda^r_{ds} \\ \lambda'^r_{kq} \\ \lambda'^r_{fd} \\ \lambda'^r_{kd} \end{bmatrix} = \begin{bmatrix} L_{ls}+L_{mq} & 0 & L_{mq} & 0 & 0 \\ 0 & L_{ls}+L_{md} & 0 & L_{md} & L_{md} \\ L_{mq} & 0 & L'_{lkq}+L_{mq} & 0 & 0 \\ 0 & L_{md} & 0 & L'_{lfd}+L_{md} & L_{md} \\ 0 & L_{md} & 0 & L_{md} & L'_{lkd}+L_{md} \end{bmatrix} \begin{bmatrix} i^r_{qs} \\ i^r_{ds} \\ i'^r_{kq} \\ i'^r_{fd} \\ i'^r_{kd} \end{bmatrix} \tag{6.5-17}$$

We have accomplished our goal; the self- and mutual inductances in (6.5-17) are constant. Moreover, all q circuits are magnetically decoupled from d circuits. We now see that the fictitious windings are indeed fixed in the rotor reference frame. Since the mutual inductances between the fictitious windings (qs^r and ds^r windings) and the rotor windings are all constant, the fictitious windings and the rotor windings are not in relative motion. Hence, the qs^r and ds^r windings are fixed on the rotor.

The inductance $L_{ls} + L_{mq}$ is commonly called the q-axis inductance and denoted L_q. Similarly, $L_{ls} + L_{md}$ is called the d-axis inductance and denoted L_d. That is,

$$L_q = L_{ls} + L_{mq} \tag{6.5-18}$$

$$L_d = L_{ls} + L_{md} \tag{6.5-19}$$

If the air gap is uniform, $L_q = L_d$. Otherwise, $L_q < L_d$.

Park's equations are often written in expanded form. From (6.5-6) and (6.5-7),

$$v_{qs}^r = r_s i_{qs}^r + \omega_r \lambda_{ds}^r + p\lambda_{qs}^r \tag{6.5-20}$$

$$v_{ds}^r = r_s i_{ds}^r - \omega_r \lambda_{qs}^r + p\lambda_{ds}^r \tag{6.5-21}$$

$$v_{kq}'^r = r_{kq}' i_{kq}'^r + p\lambda_{kq}'^r \tag{6.5-22}$$

$$v_{fd}'^r = r_{fd}' i_{fd}'^r + p\lambda_{fd}'^r \tag{6.5-23}$$

$$v_{kd}'^r = r_{kd}' i_{kd}'^r + p\lambda_{kd}'^r \tag{6.5-24}$$

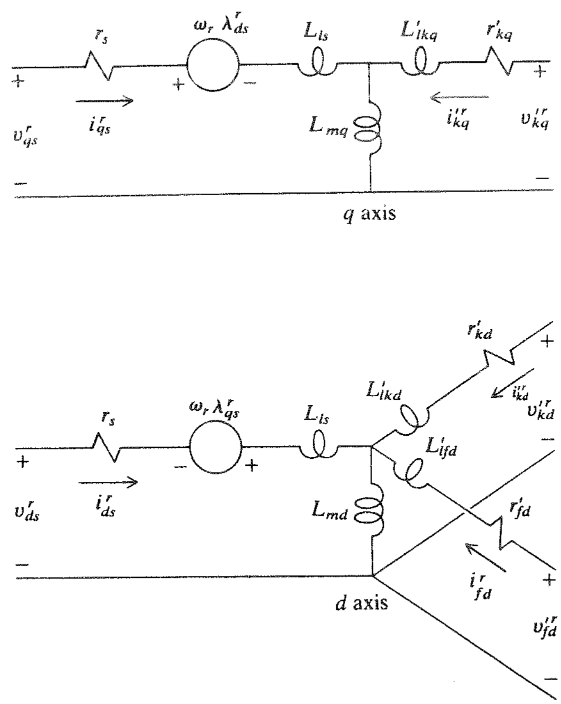


FIGURE 6.5-2
Equivalent circuits of a two-ph synchronous machine with refe frame fixed in rotor—Park's eq tions.

Although we will carry the s subscript, the r superscript, and the primes, Park's equations are generally written without these notations. Also, we realize that the damper windings are always short-circuited, hence, v'^r_{kq} and v'^r_{kd} are zero.

For a linear magnetic system, the flux linkage equations may be written from (6.5-17) as

$$\begin{aligned}\lambda^r_{qs} &= L_{ls}i^r_{qs} + L_{mq}(i^r_{qs} + i'^r_{kq}) \\ &= L_q i^r_{qs} + L_{mq} i'^r_{kq} \qquad (6.5\text{-}25)\end{aligned}$$

$$\begin{aligned}\lambda^r_{ds} &= L_{ls}i^r_{ds} + L_{md}(i^r_{ds} + i'^r_{fd} + i'^r_{kd}) \\ &= L_d i^r_{ds} + L_{md}(i'^r_{fd} + i'^r_{kd}) \qquad (6.5\text{-}26)\end{aligned}$$

$$\begin{aligned}\lambda'^r_{kq} &= L'_{lkq}i'^r_{kq} + L_{mq}(i^r_{qs} + i'^r_{kq}) \\ &= L'_{kq} i'^r_{kq} + L_{mq} i^r_{qs} \qquad (6.5\text{-}27)\end{aligned}$$

$$\begin{aligned}\lambda'^r_{fd} &= L'_{lfd}i'^r_{fd} + L_{md}(i^r_{ds} + i'^r_{fd} + i'^r_{kd}) \\ &= L'_{fd} i'^r_{fd} + L_{md}(i^r_{ds} + i'^r_{kd}) \qquad (6.5\text{-}28)\end{aligned}$$

$$\begin{aligned}\lambda'^r_{kd} &= L'_{lkd}i'^r_{kd} + L_{md}(i^r_{ds} + i'^r_{fd} + i'^r_{kd}) \\ &= L'_{kd} i'^r_{kd} + L_{md}(i^r_{ds} + i'^r_{fd}) \qquad (6.5\text{-}29)\end{aligned}$$

/here L_q and L_d are defined by (6.5-18) and (6.5-19), respectively, and

$$L'_{kq} = L'_{lkq} + L_{mq} \qquad (6.5\text{-}30)$$

$$L'_{fd} = L'_{lfd} + L_{md} \qquad (6.5\text{-}31)$$

$$L'_{kd} = L'_{lkd} + L_{md} \qquad (6.5\text{-}32)$$

he voltage and flux linkage equations suggest the equivalent circuits shown in ig. 6.5-2. Substituting (6.5-25) through (6.5-29) into (6.5-20) through (6.5-24) elds the voltage equations in terms of currents.

$$\begin{bmatrix} v^r_{qs} \\ v^r_{ds} \\ v'^r_{kq} \\ v'^r_{fd} \\ v'^r_{kd} \end{bmatrix} = \begin{bmatrix} r_s + pL_q & \omega_r L_d & pL_{mq} & \omega_r L_{md} & \omega_r L_{md} \\ -\omega_r L_q & r_s + pL_d & -\omega_r L_{mq} & pL_{md} & pL_{md} \\ pL_{mq} & 0 & r'_{kq} + pL'_{kq} & 0 & 0 \\ 0 & pL_{md} & 0 & r'_{fd} + pL'_{fd} & pL_{md} \\ 0 & pL_{md} & 0 & pL_{md} & r'_{kd} + pL'_{kd} \end{bmatrix} \begin{bmatrix} i^r_{qs} \\ i^r_{ds} \\ i'^r_{kq} \\ i'^r_{fd} \\ i'^r_{kd} \end{bmatrix} \qquad (6.5\text{-}33)$$

The expression for the electromagnetic torque in rotor reference frame riables may be obtained by substituting the equation of transformation into 4-2). After considerable work,

$$T_e = \frac{P}{2}\left[L_{md}(i'_{ds} + i'^{r}_{fd} + i'^{r}_{kd})i'_{qr} - L_{mq}(i'_{qs} + i'^{r}_{kq})i'_{ds}\right] \tag{6.5-34}$$

which may also be written as

$$T_e = \frac{P}{2}(\lambda^r_{ds}i^r_{qs} - \lambda^r_{qs}i^r_{ds}) \tag{6.5-35}$$

It is helpful to review what has been done thus far in this chapter. Since the stator and rotor windings of a synchronous machine are in relative motion, it is necessary to implement a change of variables which, in effect, eliminates the relative motion between circuits. However, the synchronous machine is not too cooperative. Not only are there circuits in relative motion, but also the rotor of the salient-pole synchronous machine gives rise to sinusoidal variations of $2\theta_r$ in the self-inductances of the stator windings. To make matters worse, the rotor windings are not symmetrical.

As we think about this situation, we conclude that Park really had no choice but to devise a change of variables which would transform the stator variables to fictitious circuits fixed on the rotor. First, the air gap of a salient-pole synchronous machine is not uniform, hence, only circuits fixed on the rotor could experience a constant self-inductance. Yes, that is right, but what about a round-rotor synchronous machine? Why, in the case of a round-rotor machine, would it be necessary to transform the stator variables to fictitious rotor circuits? The windings must be symmetrical to benefit from a change of variables which is a function of an angular displacement. (You may wish to refer to Example 5A if you have questions about this statement.) The rotor windings of a synchronous, salient-pole, or round-rotor machine, are, in general, asymmetrical. We are unaware of a change of variables which provide an advantage in transforming the variables of asymmetrical windings to reference frame other than that where the windings physically exist. Perhaps another Park will come along and help us out. Until then, we shall have to be content to use the rotor reference frame to analyze synchronous machines.

Although we are probably getting too involved in reference frame theory you might have thought by this time that, if the rotor were round and if the rotor windings were made symmetrical, we could use any reference frame. This is indeed the case. If the air gap were uniform (round rotor) and if the *kd* and *kq* windings were identical, and if an *fq* winding identical to the *fd* winding were added in the *q* axis, the rotor windings would form two symmetrical sets. The machine would then have the same configuration as an induction machine with two sets of symmetrical rotor windings and we would not be restricted by the configuration of the machine to any reference frame.

Example 6A. A two-pole two-phase reluctance machine is identical in configuration to the synchronous machine shown in Fig. 6.2-1 with the field winding

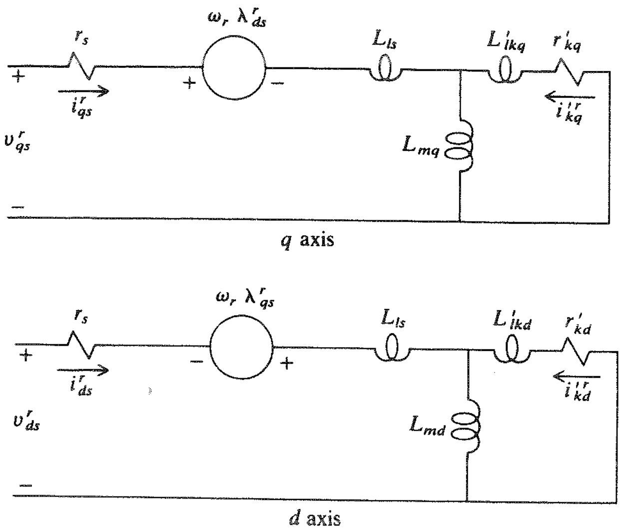


FIGURE 6A-1
Rotor reference frame equivalent circuits for reluctance machine.

winding) absent. It is our job to derive the equivalent circuits for the reluctance machine in the rotor reference frame.

Actually we need not do any derivation; it has already been done. We need only to eliminate the field winding from the equivalent circuits shown in Fig. 6.5-2. In particular, Fig. 6A-1 shows equivalent circuits in the rotor reference frame for a two-phase reluctance machine wherein the damper windings are shown short-circuited as they always are in real life.

SP6.5-1. $f_{as} = \cos \omega_e t$ and $f_{bs} = \sin \omega_e t$. Determine f_{qs}^r and f_{ds}^r if $\theta_r = \omega_r t$ and $\omega_r = \omega_e$. [$f_{qs}^r = 1$; $f_{ds}^r = 0$]

SP6.5-2. Determine $\theta_r(0)$ so that $f_{qs}^r = f_{ds}^r$ in SP6.5-1, where here $\theta_r = \omega_r t + \theta_r(0)$. [$\theta_r(0) = \frac{1}{4}\pi$ and $-\frac{3}{4}\pi$]

SP6.5-3. Determine $\theta_{ef}(0)$ so that $f_{qs}^r = f_{ds}^r$ in SP6.5-1, where $\theta_{ef} = \omega_e t + \theta_{ef}(0)$. [$\theta_{ef}(0) = -\frac{1}{4}\pi$ and $\frac{3}{4}\pi$]

SP6.5-4. Repeat SP6.5-1 for $\omega_r = -\omega_e$. [$f_{qs}^r = \cos 2\omega_e t$; $f_{ds}^r = -\sin 2\omega_e t$]

6.6 ROTOR ANGLE

It would seem that we have already done our share of defining concepts and terms which we have not seen or used before; the sinusoidally distributed winding, self- and mutual inductances that vary, changes of variables that give rise to fictitious windings, and a machine that develops "three types of torque." We cannot help but wonder when these "new concepts" are going to stop. Unfortunately, we are now faced with another definition which has evolved over the years to become deeply ingrained in synchronous-machine theory. The rotor angle is the case in point.

In its broadest definition, the rotor angle δ is

$$\boxed{\begin{aligned}\delta &= \theta_r - \theta_{esv} \\ &= \int_0^t [\omega_r(\xi) - \omega_e(\xi)]\, d\xi + \theta_r(0) - \theta_{esv}(0)\end{aligned}} \tag{6.6-1}$$

where θ_{esv} is the angular displacement of a stator phase voltage, generally v_{as}. In (6.6-1) ξ is a dummy variable of integration and $\omega_r(\xi)$ and $\omega_e(\xi)$ are the electrical angular velocity of the rotor and the terminal voltages, respectively. The time zero position is generally selected so that the fundamental component of v_{as} is maximum at $t = 0$, for example, a cosine with $\theta_{esv}(0) = 0$. Although the above definition of δ is valid regardless of the mode of operation (either or both ω_r and ω_e may vary), a physical interpretation is most easily visualized during balanced steady-state operation. It is, however, important to mention in passing that here we see the mixing of a variable associated with the electric system, θ_e (ω_e), and a variable associated with the mechanical system, θ_r (ω_r). Fortunately, good will come from this, even though it can be somewhat confusing when, in the next section, we superimpose a phasor diagram which rotates at ω_e upon the rotor, which also rotates at ω_e during steady-state operation, and then note that the angle between $\tilde{V}_{as}$ and the q axis is δ, the rotor angle.

SP6.6-1. Calculate δ for SP6.5-1, SP6.5-2, and SP6.5-4. Assume f_{as} and f_{bs} are v_{as} and v_{bs}, respectively. [$\delta = 0$; $\delta = \frac{1}{4}\pi$ and $-\frac{3}{4}\pi$; $\delta = -2\omega_e t$]
SP6.6-2. Why did we not ask you to calculate δ for SP6.5-3? [We could have, but $\theta_{esv}(0)$ is generally selected to be zero.]

6.7 ANALYSIS OF STEADY-STATE OPERATION

In the case of the synchronous machine, we have found it necessary to refer the stator variables to the rotor reference frame. What will be the frequency of the variables in the rotor reference frame during balanced steady-state operation? Actually, we have our answer from SP6.5-1, but let us think more about this before doing anything analytical. First, we know that, during steady-state operation, the electrical angular velocity of the rotor, ω_r, is equal to ω_e. Hence, the circuits that physically exist on the rotor (kq, fd, and kd windings) or fictitious windings put there because of a change of variables ($\overset{r}{qs}$ and $\overset{r}{ds}$ windings) do not experience a change of flux linkages. How do we know this? Well, the air gap mmf established by the constant (dc) field current is, of course, constant relative to the windings on the rotor. Now what about mmf_s, the rotating air gap mmf established by the balanced, sinusoidal stator currents? It rotates at ω_e which is also the speed ω_r of the rotor. Hence, since neither mmf_s nor mmf_r is changing relative to the rotor, the windings on the rotor (physical or fictitiously) will not experience a change in flux linkages. At this point, we can forget about the short-circuited damper windings since, without a change of flux linkages, there can be no induced voltage and, thus, i'^r_{kq} and i'^r_{kd} must be zero for balanced steady-state operation where $\omega_r = \omega_e$. Actually, if we accept the fact that there is not a change of flux linkages relative to the rotor circuits, then there can be no induced voltage due to transformer action in any of the circuits on the rotor. One would then guess that the currents and voltages associated with all rotor windings, actual or fictitious, would have to be constant (i'^r_{kq} and i'^r_{kd} are constant at zero). This seems logical, but what has happened to the balanced, sinusoidal stator variables? Remember the balanced, steady-state sinusoidal stator currents give rise to a constant mmf$_s$ rotating at ω_e. But, if mmf$_s$, which rotates at ω_e, is now to be produced by currents flowing in fictitious windings (i^r_{qs} and i^r_{ds}), which are mathematically fixed on the rotor which is rotating at ω_e during steady-state operation, what must be the frequency of these currents flowing in the fictitious windings? They must be constant (dc).

Now that we know what to expect, let us proceed. During balanced steady-state operation, the stator variables may be expressed as

$$F_{as} = \sqrt{2}F_s \cos\left[\omega_e t + \theta_{esf}(0)\right] \tag{6.7-1}$$

$$F_{bs} = \sqrt{2}F_s \sin\left[\omega_e t + \theta_{esf}(0)\right] \tag{6.7-2}$$

Substituting F_{as} and F_{bs} into the equations of transformation, (6.4-2), with $\omega_r = \omega_e$, yields

$$F^r_{qs} = \sqrt{2}F_s \cos\left[\theta_{esf}(0) - \theta_r(0)\right] \tag{6.7-3}$$

$$F^r_{ds} = -\sqrt{2}F_s \sin\left[\theta_{esf}(0) - \theta_r(0)\right] \tag{6.7-4}$$

Clearly, $\theta_{esf}(0)$ and $\theta_r(0)$ are constants and, thus, F^r_{qs} and F^r_{ds} are constants. In other words, a balanced set of sinusoidal stator variables become constants in the rotor reference frame during steady-state conditions where $\omega_r = \omega_e$.

Let us now go back to the voltage equations in the rotor reference frame, (6.5-20) through (6.5-24). As we have agreed, for balanced steady-state operation we can forget about the damper windings. Hence, (6.5-22) and (6.5-24) play no role in the analysis of steady-state operation of a synchronous machine. Moreover, since the voltages and currents in the rotor reference frame are constants during balanced steady-state operation, we can apply dc circuit theory to the $\overset{r}{qs}$, $\overset{r}{ds}$, and $\overset{r}{fd}$ voltage equations. In particular, since all variables are constant during steady-state operation, all variables multiplied by the operator p (d/dt) are zero. Hence, (6.5-20), (6.5-21), and (6.5-23) may be written as

$$V^r_{qs} = r_s I^r_{qs} + \omega_r \lambda^r_{ds} \tag{6.7-5}$$

$$V^r_{ds} = r_s I^r_{ds} - \omega_r \lambda^r_{qs} \tag{6.7-6}$$

$$V'^r_{fd} = r'_{fd} I'^r_{fd} \tag{6.7-7}$$

where capital letters have been used to denote steady-state quantities. Now λ_{qs}^{r} and λ_{ds}^{r} are (6.5-25) and (6.5-26), respectively, wherein the damper winding currents are set equal to zero. Appropriate substitution of (6.5-25) and (6.5-26) into (6.7-5) and (6.7-6), wherein ω_r is set equal to ω_e, yields

$$V_{qs}^{r} = r_s I_{qs}^{r} + X_d I_{ds}^{r} + X_{md} I_{fd}^{\prime r} \tag{6.7-8}$$

$$V_{ds}^{r} = r_s I_{ds}^{r} - X_q I_{qs}^{r} \tag{6.7-9}$$

Note these are dc voltage equations and, yet, we have reactances X_d, X_q, and X_{md} without j's. We are not dealing with phasors, yet X times I is a voltage, regardless. Reactances in dc voltage equations? Add another "new" concept to the list at the beginning of Sec. 6.6.

The above voltage equations can be used in their present form to analyze the synchronous machine; however, it is convenient and customary to relate the F_{qs}^{r} and F_{ds}^{r} quantities, which are constants, to $\tilde{F}_{as}$, which is a phasor representing a sinusoidal voltage. Once we have done this, we will have another one of these "new" concepts. They seem to be coming one right after another. Sorry! To accomplish this goal, let us first look at δ for steady-state operation. In particular, if we arbitrarily select or "call" time zero with $\omega_r = \omega_e$, the steady-state rotor angle from (6.6-1) becomes

$$\delta = \theta_r(0) - \theta_{esv}(0) \tag{6.7-10}$$

Later, we will set $\theta_{esv}(0) = 0$, but for now we shall let it be. If (6.7-10) is solved for $\theta_r(0)$ and the result substituted into (6.7-3) and (6.7-4), we obtain

$$F_{qs}^{r} = \sqrt{2} F_s \cos\left[\theta_{esf}(0) - \theta_{esv}(0) - \delta\right] \tag{6.7-11}$$

$$F_{ds}^{r} = -\sqrt{2} F_s \sin\left[\theta_{esf}(0) - \theta_{esv}(0) - \delta\right] \tag{6.7-12}$$

Now let us leave these equations for just a moment. From (6.7-1) and (6.7-2), F_{as} and F_{bs} may be written as

$$F_{as} = \mathrm{Re}\left[\sqrt{2}\tilde{F}_{as} e^{j\omega_e t}\right] \tag{6.7-13}$$

$$F_{bs} = \mathrm{Re}\left[\sqrt{2}\tilde{F}_{bs} e^{j\omega_e t}\right] \tag{6.7-14}$$

where

$$\tilde{F}_{as} = F_s e^{j\theta_{esf}(0)} \tag{6.7-15}$$

and $\tilde{F}_{bs} = -j\tilde{F}_{as}$. If each side of (6.7-15) is multiplied by $\sqrt{2}e^{-j\delta}$, we will obtain

$$\sqrt{2}\tilde{F}_{as} e^{-j\delta} = \sqrt{2} F_s \cos\left[\theta_{esf}(0) - \delta\right] + j\sqrt{2} F_s \sin\left[\theta_{esf}(0) - \delta\right] \tag{6.7-16}$$

We will now do what we promised. We will select or call time zero at the maximum positive value of V_{as}. That is, $\theta_{esv}(0) = 0$, whereupon

$$V_{as} = \sqrt{2} V_s \cos \omega_e t \tag{6.7-17}$$

$$V_{bs} = \sqrt{2} V_s \sin \omega_e t \tag{6.7-18}$$

and $\tilde{V}_{as}$ is at zero degrees. Let us remember that from now on whenever we conduct an analysis of steady-state operation of synchronous machines, $\theta_{esv}(0) = 0$. With this restriction, compare the right-hand terms of (6.7-16) with (6.7-11) and (6.7-12). Here is the "new" concept coming at us head on. From his comparison we can write

$$\boxed{\sqrt{2}\tilde{F}_{as}e^{-j\delta} = F^r_{qs} - jF^r_{ds}} \tag{6.7-19}$$

Can this be correct? Here, we are equating a phasor which represents a inusoidal quantity to F^r_{qs} and F^r_{ds}, which are constants. Yes, but, in its naked orm, a phasor is nothing more than a complex number. The $e^{j\omega_e t}$ gives it otation. (See Appendix B.) So if we forget about what a phasor is used to epresent and think only that we are equating complex numbers, we can go long with it, at least for now.

We have only a few more steps. From (6.7-19) we can write

$$\sqrt{2}\tilde{V}_{as}e^{-j\delta} = V^r_{qs} - jV^r_{ds} \tag{6.7-20}$$

ubstituting (6.7-8) and (6.7-9) into (6.7-20) yields

$$\sqrt{2}\tilde{V}_{as}e^{-j\delta} = r_s I^r_{qs} + X_d I^r_{ds} + X_{md} I'^r_{fd} + j(-r_s I^r_{ds} + X_q I^r_{qs}) \tag{6.7-21}$$

$X_q I^r_{ds}$ is added to and subtracted from the right-hand side of (6.7-21) and if it noted from (6.7-19) that

$$j\sqrt{2}\tilde{I}_{as}e^{-j\delta} = I^r_{ds} + jI^r_{qs} \tag{6.7-22}$$

en (6.7-21) may be written as

$$\tilde{V}_{as} = (r_s + jX_q)\tilde{I}_{as} + \frac{1}{\sqrt{2}}[(X_d - X_q)I^r_{ds} + X_{md}I'^r_{fd}]e^{j\delta} \tag{6.7-23}$$

is convenient to define the last term of (6.7-23) as

$$\boxed{\tilde{E}_a = \frac{1}{\sqrt{2}}[(X_d - X_q)I^r_{ds} + X_{md}I'^r_{fd}]e^{j\delta}} \tag{6.7-24}$$

ich is often referred to as the *excitation voltage*. Thus, (6.7-23) becomes

$$\boxed{\tilde{V}_{as} = (r_s + jX_q)\tilde{I}_{as} + \tilde{E}_a} \tag{6.7-25}$$

Equation (6.7-25) is used widely in the analysis of steady-state operation synchronous machines. It is very compact and easy to use, far more reduced n one would have expected at the outset of this development. However, re is something more involved here than first meets the eye. Note that the le of the phasor $\tilde{E}_a$ is δ, but δ has to do with the rotor. In particular, from 7-10) with $\theta_{esv}(0) = 0$ ($\tilde{V}_{as}$ is at zero degrees) the steady-state value of δ is). In other words, if we look back to Fig. 6.2-1, we see that δ or $\theta_r(0)$ is the position of the q axis at the instant we called time zero, which was at the positive maximum value of V_{as}. (We will not mention the fact that another one of these "new" concepts is appearing on the horizon.) In the end, we will have superimposed the phasor diagram upon the rotor. To start, consider Fig. 6.7-1. In the upper left-hand corner of Fig. 6.7-1a we see the position of the rotor at $t = 0$. However, we know that the rotor is rotating at ω_r (ω_e). Now, because at time zero $\theta_{esv}(0) = 0$, $\tilde{V}_{as}$ is at zero degrees, and the phase angle of $\tilde{E}_a$ is equal to δ, as shown in the upper right-hand corner of Fig. 6.7-1b, where we have assumed $\tilde{I}_{as}$ leading $\tilde{V}_{as}$ to illustrate generator operation. How can a and b of Fig. 6.7-1 be superimposed? Well, the rotor is rotating counterclockwise at ω_e and, although we generally do not think of it in this way, the phasors are also rotating counterclockwise at ω_e (Appendix B). Perhaps, this is enough of an explanation to allow us to superimpose the two as shown in c of Fig. 6.7-1. If not, consider this. With your eyes closed, keep your left eye ready to look at (a) and your right eye ready to look at (b). Blink them open at time zero and you see what is shown in Fig. 6.7-1a and b. Now close them immediately and

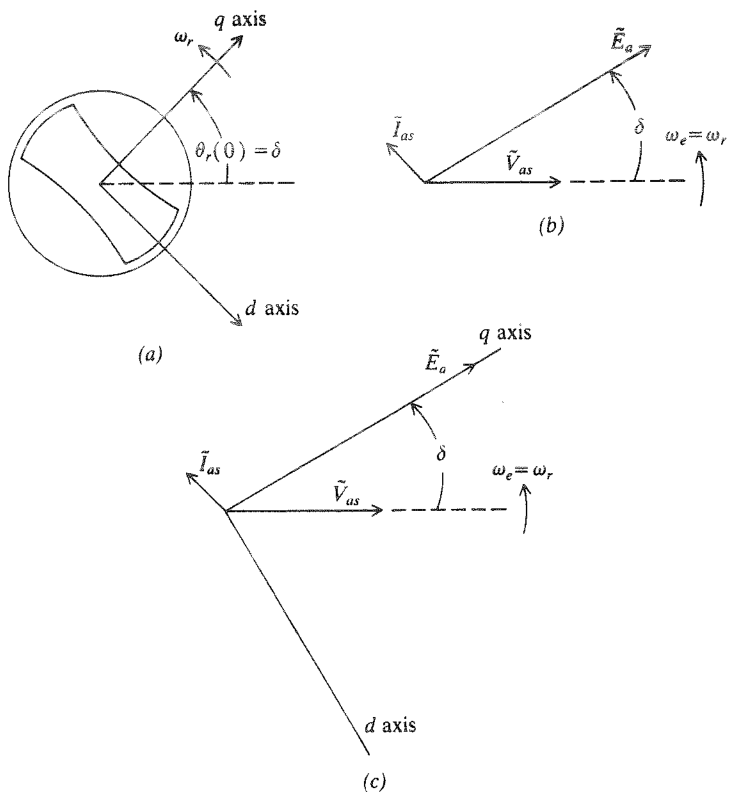


FIGURE 6.7-1
Superimposing the phasor diagram on the rotor of a synchronous machine [$\theta_{esv}(0) = 0$]. (a) R at $t = 0$; (b) phasor at $t = 0$; (c) together at $t = 0$.

keep them closed until the V_{as} voltage is again a positive maximum. At that instant, blink them open again. What do you see? Exactly what you saw at ime zero. Keep doing this and each time let the "focus" of each eye come loser and closer. Did you notice that we could superimpose these two and onsider them to be stationary as in Fig. 6.7-1*c*?

We can now express the electromagnetic torque in terms of the rotor ngle. If (6.7-8) and (6.7-9) are solved for I^r_{qs} and I^r_{ds} and the results ubstituted into (6.5-34), we obtain

$$T_e = -\frac{P}{2}\frac{1}{\omega_e}\left\{\frac{r_s X_{md} I'^r_{fd}}{r_s^2 + X_q X_d}\left(V^r_{qs} - X_{md} I'^r_{fd} - \frac{X_d}{r_s} V^r_{ds}\right) + \frac{X_d - X_q}{(r_s^2 - X_q X_d)^2}\left[r_s X_q (V^r_{qs} - X_{md} I'^r_{fd})^2 + (r_s^2 - X_q X_d) V^r_{ds}(V^r_{qs} - X_{md} I'^r_{fd}) - r_s X_d (V^r_{ds})^2\right]\right\} \tag{6.7-26}$$

ote that, if we want to use (6.7-11) and (6.7-12), respectively, to express V^r_{qs} nd V^r_{ds}, then $\theta_{esv} = \theta_{esf}$ and

$$V^r_{qs} = \sqrt{2}V_s \cos[\theta_{esv}(0) - \theta_{esv}(0) - \delta] = \sqrt{2}V_s \cos\delta \tag{6.7-27}$$

$$V^r_{ds} = -\sqrt{2}V_s \sin[\theta_{esv}(0) - \theta_{esv}(0) - \delta] = \sqrt{2}V_s \sin\delta \tag{6.7-28}$$

e should mention in passing that (6.7-27) and (6.7-28) are valid for transient well as steady-state operation. Although this fact is interesting, we will use e equations only for steady-state operation in this section. For compactness, will define

$$E'^r_{xfd} = X_{md} I'^r_{fd} \tag{6.7-29}$$

(6.7-27) through (6.7-29) are substituted into (6.7-26) and if the resistance r_s the stator windings is neglected, the steady-state electromagnetic torque may written as

$$\boxed{T_e = -\frac{P}{2}\frac{1}{\omega_e}\left[\frac{E'^r_{xfd}\sqrt{2}V_s}{X_d}\sin\delta + \frac{1}{2}\left(\frac{1}{X_q} - \frac{1}{X_d}\right)(\sqrt{2}V_s)^2 \sin 2\delta\right]} \tag{6.7-30}$$

glecting r_s is justified if its ohmic value is small relative to the magnetizing ctances (X_{mq} and X_{md}) of the machine. In variable-frequency drive systems, is not the case at low frequencies, whereupon (6.7-26) must be used to ulate the steady-state torque rather than (6.7-30). The electromagnetic que evaluated by (6.7-26) or (6.7-30) is positive for motor action (torque d) and negative for generator action (torque input).

Although (6.7-30) is a valid expression for the electromagnetic torque during balanced steady-state operation only if the stator winding resistance is small relative to the magnetizing reactances, it permits a quantitative description of two of the three torques produced by a salient-pole synchronous machine. Since $\omega_r = \omega_e$, the damper winding currents are zero and, hence, induction motor torque is not present. The first term on the right-hand side of (6.7-30) is due to the interaction of the mmf produced by the stator currents and the mmf produced by the field current. The second term is the reluctance torque which occurs owing to the forces set up to attempt to align the minimum-reluctance path of the rotor with the resultant air gap mmf.

A word of caution seems appropriate. With the advent of controlled electronic switching devices, electric machines are often operated in systems where the frequency and amplitude of applied stator voltages can be varied. In the above steady-state voltage, (6.7-24), (6.7-25), and (6.7-29), and torque equations, (6.7-26) and (6.7-30), inductive reactances are used. There reactances are calculated by using the frequency of the applied stator voltages. Therefore, the ω_e in the reactances as well as the ω_e which appears in the torque equations must be changed as frequency changes.

Let us return to the expression for steady-state electromagnetic torque given by (6.7-30). Remember that it is valid only if r_s is small relative to X_{mq} and X_{md}. The first term on the right side of (6.7-30) is plotted in Fig. 6.7-2*a*, the second in Fig 6.7-2*b*, and the total or sum of the two components is plotted in Fig. 6.7-2*c*. (We shall talk about the points 1 and 1′ appearing in Fig. 6.7-2*c* a little later.) It is noted that, for a given frequency of the applied stator voltages and for a given machine design, the amplitude of the first term (denoted as A in Fig. 6.7-2*a*) is proportional to the product of the amplitude of the stator voltages ($\sqrt{2}V_s$) and the field voltage $V_{fd}'^r$ since, from (6.7-29), during steady-state operation

$$E_{xfd}'^r = \frac{X_{md}}{r'_{fd}} V_{fd}'^r \tag{6.7-31}$$

When synchronous machines are used as motors in variable-frequency drive systems, both V_s and $E_{xfd}'^r$ can be varied. However, in applications where the synchronous machine is used as a generator to produce electric power such as in a power system, the stator voltage (V_s) is generally regulated and consequently, it is not allowed to vary more than, say, 1 to 3 percent during normal steady-state operation. In this case, the amplitude of the first term which is the main torque component in generators, is changed by changing the field voltage $V_{fd}'^r$.

The amplitude of the second term, the reluctance torque, is denoted as B in Fig. 6.7-2*b*. For a fixed frequency of operation and a given machine design the reluctance torque varies as the square of the amplitude of the applied stator voltages ($\sqrt{2}V_s$). In variable-frequency drive systems, the reluctance torque may be changed by changing V_s for a given frequency of operation. However in a power system where both the frequency and amplitude of the stator

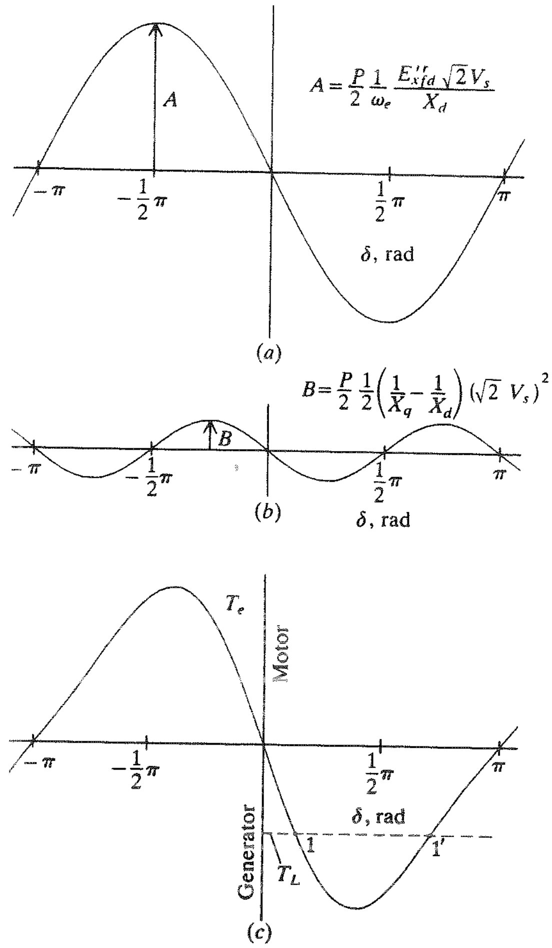


FIGURE 6.7-2
Steady-state electromagnetic torque of synchronous machine.

voltages are essentially constant, the amplitude of the reluctance torque is also essentially constant, depending upon the design of the machine.

Recall that the rotor angle δ is the angle between the phasor $\tilde{E}_a$ which lays along the q axis and the phasor $\tilde{V}_{as}$. Also, for a given load or input torque, the rotor angle is constant during steady-state operation and, since T_e versus δ is periodic, we need only consider the plot for $-\pi < \delta < \pi$. Now, the load torque and the electromagnetic torque are related by (6.4-3) from which it is clear that during steady-state operation $T_e = T_L$. For illustrative purposes, let the synchronous machine be connected to an electric system and let the load torque be negative. That is, torque is applied to the shaft by some external means—a steam or hydroturbine or a combustion engine, or, perhaps a wind turbine. Regardless of how the applied torque is developed, it is an input torque to the generator shaft and, if friction and windage losses are neglected, the steady-state T_e must equal T_L, thus T_e is negative also. Let us think about this for a minute. The machine is connected to an electric system. If a torque is applied to the shaft and since torque times rotor speed is equal to power, the synchronous machine must deliver an equal amount of power (neglecting friction and windage losses and ohmic losses) to the electric system. Otherwise, there would be a torque or power imbalance and, if T_e is not equal to T_L, the synchronous machine would accelerate for $T_e - T_L > 0$ and decelerate for $T_e - T_L < 0$.

In the next section, we will discuss the dynamic performance of the synchronous machine. It is interesting, however, to continue our example of steady-state operation to determine the stable operating region. We can do as we did in the case of the relay. With T_L negative, there are two possible operating points between $-\pi < \delta < \pi$. They are denoted 1 and 1′ in Fig. 6.7-2*c*. Now we must remember that T_L is constant. First, assume that steady-state operation occurs at point 1. This is a valid operating point if the system will return to this point when distributed from it. To test this, let δ decrease ever so slightly. In this case $T_e - T_L > 0$ and the rotor will accelerate, thereby increasing δ, (6.6-1); hence, a torque is developed to move the system back to operating point 1. If now δ increases ever so slightly, $T_e - T_L < 0$, and the system will again move back to point 1. Hence, point 1 is a stable operating point.

Although we all suspect what is going to happen at point 1′, let us go through the exercise anyway. If we assume that the system is operating stably at point 1′, then a displacement in δ from this operating point should cause a torque to restore the system to this operating point. If δ decreases slightly from point 1′, $T_e - T_L < 0$ and the machine would decelerate which would further decrease δ. The system would move away from point 1′ and, after all transients have subsided, the system would operate stably at point 1. If, instead, δ increases from point 1′, $T_e - T_L > 0$, whereupon the machine will accelerate moving away from point 1′. We can conclude that, although point 1′ satisfies the "torque balance equation" (6.4-3), it is not a stable operating condition. Let us go back a step. If δ increases from point 1′ and the rotor accelerates where will it end up? At point $1 + 2\pi$ if it does not go unstable dynamically. We really cannot appreciate the meaning of "unstable dynamically" until we can study the dynamic characteristics in the next section. It is sufficient here to say that we are dealing with steady-state characteristics, and one can get into trouble using steady-state characteristics to explain the large-excursion dynamic (transient) characteristics of a system.

It is apparent that the only way that the steady-state electric power output or input (or T_e) can be changed is to change the torque T_L. However, the electrical characteristics of a synchronous machine connected to a system can be changed by changing the field voltage V'^r_{fd}. Although the following is, perhaps, of most interest to power system engineers, it is worth a passing consideration by all. To explain the influence of E'^r_{xfd} (V'^r_{fd}), we will assume that the machine is connected to a large system so that, regardless what we do we the synchronous machine, it will not change the magnitude or phase of the system voltage, i.e., $\tilde{V}_{as}$. This is commonly called an *infinite bus* in power system language. If we now assume that the load torque T_L is zero and if we neglect friction and windage losses along with the stator resistance, then T_e and δ are also zero and the machine will run at synchronous speed without absorbing energy from either the electric or mechanical systems.

Although this mode of operation is not feasible in practice since the machine will actually absorb a small amount of power to satisfy the ohmic, friction, and windage losses and thus a small δ would exist at no load, it is convenient for purposes of explanation. With the machine "floating on the line" the field voltage can be adjusted to establish the desired terminal conditions. Three situations may exist:

1. $|\tilde{E}_a| = |\tilde{V}_{as}|$, whereupon $\tilde{I}_{as} = 0$.
2. $|\tilde{E}_a| > |\tilde{V}_{as}|$, whereupon $\tilde{I}_{as}$ leads $\tilde{V}_{as}$ and the synchronous machine appears as a capacitor supplying reactive power to the system.
3. $|\tilde{E}_a| < |\tilde{V}_{as}|$ with $\tilde{I}_{as}$ lagging $\tilde{V}_{as}$, whereupon the machine is absorbing reactive power appearing as an inductor to the system.

We should define reactive power which is generally denoted as Q. In particular, the reactive power per phase is (Appendix B)

$$\begin{aligned} Q &= |\tilde{V}_{as}||\tilde{I}_{as}| \sin [\theta_{esv}(0) - \theta_{esi}(0)] \\ &= |\tilde{V}_{as}||\tilde{I}_{as}| \sin \phi_{\text{pf}} \qquad (6.7\text{-}32) \end{aligned}$$

where ϕ_{pf} is the power factor angle and the units of Q are var (voltampere eactive). An inductance is said to absorb reactive power and thus, by lefinition, Q is positive for an inductor and negative for a capacitor. Actually, Q is a measure of the exchange of energy stored in the electric (capacitor) and nagnetic (inductance) fields; however, there is no average power interchanged etween these energy storage devices.

Now, to maintain the voltage in a power system at rated value, the ynchronous generators are normally operated in the overexcited mode, $|\tilde{E}_a| > |\tilde{V}_{as}|$, since the generators are the main source of reactive power for the nductive loads throughout the system. In the past, synchronous machines have een placed in the power system for the sole purpose of supplying reactive ower without any provision to absorb or provide real power. During peak ad conditions when the system voltage is depressed, these so-called *synhronous condensers* are brought on line and the field voltage is adjusted to elp increase the system voltage. In this mode of operation, the synchronous achine behaves like an adjustable capacitor. On the other hand, it may be ecessary for a generator to absorb reactive power in order to regulate voltage a high-voltage transmission system during light load conditions. This mode operation is, however, not desirable and should be avoided since machine cillations become less damped as the reactive power required is decreased.

The influence of the field voltage during motor operation is illustrated in Example 6B.

As a finale to the analysis of steady-state operation of synchronous machines, let us consider the procedure by which generator action is established and then look at the phasor diagram for this mode of operation. A prime mover is mechanically connected to the shaft of the synchronous generator. As mentioned, this prime mover can be either a steam turbine, a hydroturbine, or a combustion engine. If initially the torque input to the shaft due to the prime mover is zero, the synchronous machine is essentially floating on the line. If now the input torque is increased to some value (T_L negative), for example, by supplying steam to the turbine blades, a torque imbalance occurs since T_e must remain at its original value (zero) until δ changes. Hence, the rotor will temporarily accelerate slightly above synchronous speed, whereupon δ will increase in accordance with (6.6-1). Thus, T_e increases negatively and a new operating point will be established with a positive δ where T_L is equal to T_e. The rotor will again rotate at synchronous speed. The actual dynamic response of the electric and mechanical systems during this loading process is illustrated by computer traces in the following section. If, during generator operation, the torque input from the prime mover is increased (T_L negative) to a value greater than the maximum possible value of T_e, the machine will be unable to maintain steady-state operation since it cannot electrically transmit the power supplied to the shaft. In this case, the device will accelerate above synchronous speed theoretically without bound. However, protection is normally provided in power systems which disconnects the machine from the system and reduces the input torque to zero by closing the steam valves of the steam turbine, for example, when it exceeds synchronous speed by 3 to 5 percent.

Normally, steady-state generator operation is depicted by the phasor diagram shown in Fig. 6.7-3. Here $\theta_{esi}(0)$ is the angle between the voltage and

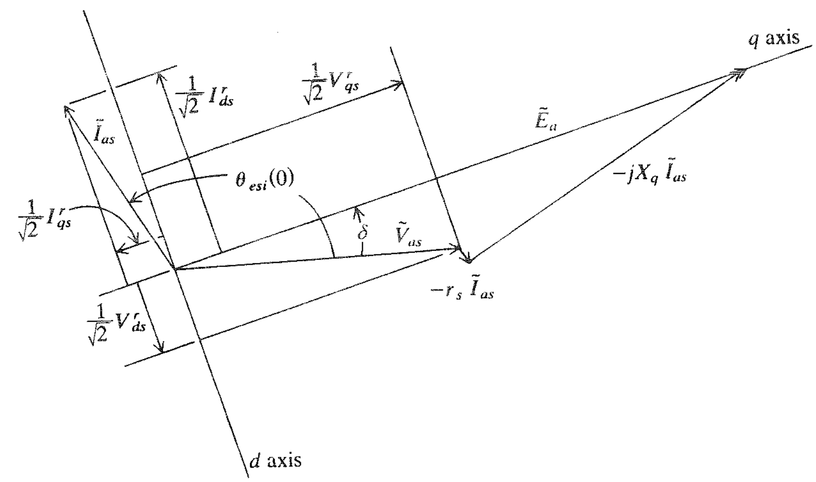


FIGURE 6.7-3
Phasor diagram for generator operation.

the current since the time zero position is $\theta_{esv}(0) = 0$. Since the phasor diagram and the q and d axes of the machine may be superimposed, the rotor reference frame voltages and currents are also shown. If we wish to show each component of V_{qs}^r and V_{ds}^r, they can be broken up according to (6.7-8) and (6.7-9), and each term added algebraically along the appropriate axes. However, care must be taken when interpreting this diagram. $\tilde{V}_{as}$, $\tilde{I}_{as}$, and $\tilde{E}_a$ are phasors representing sinusoidal quantities. On the other hand, all rotor reference frame quantities are constants. They do not represent phasors in the rotor reference frame even though we have displayed them on a phasor diagram.

Example 6B. A six-pole two-phase salient-pole synchronous machine is supplied from a 440-V 60-Hz source. The machine is operated as a motor with a total power input of 40 kW at the terminals. The parameters of the machine are $r_s = 0.3\,\Omega$, $L_{ls} = 0.001$ H, $L_{md} = 0.015$ H, $L_{mq} = 0.008$ H, $r'_{fd} = 0.03\,\Omega$, $L'_{lfd} = 0.001$ H. (*a*) The excitation is adjusted so that $\tilde{I}_{as}$ lags $\tilde{V}_{as}$ by 30°. Calculate $\tilde{E}_a$. (*b*) Repeat (*a*) with the excitation adjusted so that $\tilde{I}_{as}$ is in phase with $\tilde{V}_{as}$. (*c*) Repeat (*a*) with the excitation adjusted so that $\tilde{I}_{as}$ leads $\tilde{V}_{as}$ by 30°.

(*a*) The phase current may be calculated from the power as

$$|\tilde{I}_{as}| = \frac{\frac{1}{2}(40 \times 10^3)}{440 \cos 30°} = 52.5 \text{ A} \tag{6B-1}$$

Note that the 40×10^3 W is the total power which is the sum of the two phases. With $\tilde{V}_{as} = 440\underline{/0°}$,

$$\tilde{I}_{as} = 52.5\underline{/-30°} \text{ A} \tag{6B-2}$$

From (6.7-25),

$$\begin{aligned}\tilde{E}_a &= \tilde{V}_{as} - (r_s + jX_q)\tilde{I}_{as} \\ &= 440\underline{/0°} - [0.3 + j377(0.001 + 0.008)]52.5\underline{/-30°} \\ &= 368\underline{/-23.4°} \text{ V}\end{aligned} \tag{6B-3}$$

(*b*) The phase current is

$$|\tilde{I}_{as}| = \frac{20 \times 10^3}{440} = 45.4 \text{ A} \tag{6B-4}$$

From (6.7-25),

$$\begin{aligned}\tilde{E}_a &= 440\underline{/0°} - (0.3 + j3.39)45.4\underline{/0°} \\ &= 453\underline{/-19.9°} \text{ V}\end{aligned} \tag{6B-5}$$

(*c*) The phase current is

$$|\tilde{I}_{as}| = \frac{20 \times 10^3}{440 \cos 30°} = 52.5 \text{ A} \tag{6B-6}$$

From (6.7-25),

$$\begin{aligned}\tilde{E}_a &= 440\underline{/0°} - (0.3 + j3.39)52.5\underline{/30°} \\ &= 540\underline{/-17.4°} \text{ V}\end{aligned} \tag{6B-7}$$

It is important to note that the characteristics of the reactive component of the input power of the machine may be changed by changing the magnitude of $\tilde{E}_a$. If r_s is negligibly small, the power output is determined entirely by the input torque and, therefore, it is the same in (*a*), (*b*), and (*c*). It is left to the reader to construct the phasor diagram for each case.

Example 6C. A two-pole 60-Hz 110-V $\frac{3}{4}$-hp two-phase reluctance machine has the following parameters: $r_s = 1\,\Omega$, $L_{ls} = 0.005$ H, $L_{md} = 0.10$ H, $L_{mq} = 0.02$ H. The machine is operating at rated torque output. Calculate δ and $\tilde{I}_{as}$.

With the machine operating at rated conditions, the power output is

$$P_{out} = (0.75)(746) = 559.5 \text{ W} \tag{6C-1}$$

Therefore, the electromagnetic torque is

$$T_e = \frac{P_{out}}{(2/P)\omega_r} = \frac{559.5}{(\frac{2}{2})377} = 1.484 \text{ N}\cdot\text{m} \tag{6C-2}$$

Substituting into (6.7-30) and noting that I'^r_{fd} is zero, we can solve for δ. In particular,

$$\sin 2\delta = \frac{-(2/P)\omega_e T_e(2)(1/X_q - 1/X_d)^{-1}}{(\sqrt{2}V_s)^2}$$

$$= \frac{-(\frac{2}{2})(377)(1.484)(2)[1/(377)(0.025) - 1/(377)(0.105)]^{-1}}{(2)(110)^2}$$

$$= \frac{-(377)(1.484)(0.0808)^{-1}}{(110)^2} = -0.572 \tag{6C-3}$$

Therefore $\delta = -17.4°$.

Although we could use (6.7-25) to obtain $\tilde{I}_{as}$, it is more straightforward to use (6.7-8) and (6.7-9). We know, from (6.7-27) and (6.7-28) that

$$V^r_{qs} = \sqrt{2}V_s \cos\delta$$

$$= \sqrt{2}\,110\cos(-17.4°) = 148.4 \text{ V} \tag{6C-4}$$

$$V^r_{ds} = \sqrt{2}\,110\sin(-17.4°) = -46.5 \text{ V} \tag{6C-5}$$

Therefore, we can write (6.7-8) and (6.7-9) as

$$\begin{bmatrix} V^r_{qs} \\ V^r_{ds} \end{bmatrix} = \begin{bmatrix} r_s & X_d \\ -X_q & r_s \end{bmatrix} \begin{bmatrix} I^r_{qs} \\ I^r_{ds} \end{bmatrix} \tag{6C-6}$$

which may be written as

$$\begin{bmatrix} 148.4 \\ -46.5 \end{bmatrix} = \begin{bmatrix} 1 & (377)(0.105) \\ -(377)(0.025) & 1 \end{bmatrix} \begin{bmatrix} I^r_{qs} \\ I^r_{ds} \end{bmatrix} \tag{6C-7}$$

Solving for I^r_{qs} and I^r_{ds} yields

$$I^r_{qs} = 5.32 \text{ A} \tag{6C-8}$$

$$I^r_{ds} = 3.61 \text{ A} \tag{6C-9}$$

From (6.7-19),

$$\tilde{I}_{as} = \frac{1}{\sqrt{2}}(I_{qs}^r - jI_{ds}^r)e^{j\delta}$$

$$= \frac{1}{\sqrt{2}}(5.32 - j3.61)e^{-j17.4^\circ} = 4.55\underline{/-51.6^\circ}\text{ A} \qquad (6C\text{-}10)$$

If we calculate the input power from the voltage and current, we obtain approximately 620 W. If we add the output power to the ohmic losses, we obtain approximately 601 W. Why the discrepancy? [*Hint:* What are the restrictions on (6.7-30)?]

SP6.7-1. A two-pole two-phase synchronous machine is operated as a generator with $\tilde{V}_{as} = 110\underline{/0^\circ}$ and $\tilde{I}_{as} = 5\underline{/150^\circ}$. Calculate (*a*) total real power and (*b*) total reactive power. [(*a*) $P = -952.6$ W; (*b*) $Q = -550$ var]

SP6.7-2. The machine in SP6.7-1 is a round-rotor device with $\omega_r = 377$ rad/s. $L_{ls} = 4$ mH, $L_{md} = 50$ mH, and $r_s \cong 0$. Calculate δ. [$\delta = 28.7°$]

SP6.7-3. Calculate $I_{fd}^{\prime r}$ for SP6.7-2. [$I_{fd}^{\prime r} = 13.76$ A]

SP6.7-4. The reluctance machine in Example 6C is operating as a motor with $\delta = -30°$. Calculate T_e. Neglect r_s. [$T_e = -2.25$ N·m]

SP6.7-5. Determine the approach in SP6.7-4 if we are to take the stator resistance into account. [(6.7-26) with $I_{fd}^{\prime r} = 0$]

6.8 DYNAMIC AND STEADY-STATE PERFORMANCE

It is instructive to observe the variables of the synchronous machine during dynamic and steady-state operation. In this section, generator operation of a synchronous machine is illustrated by computer traces as well as motor operation of a reluctance machine. Although a two-phase reluctance machine is often used in practice, a two-phase synchronous generator would not normally be used. Instead, the three-phase synchronous machine is the device normally used for generating electric power. Nevertheless, our purpose is to understand the theory and principles of operation of a synchronous machine. A two-phase machine is just as applicable in this regard as a three-phase machine. The following section on the three-phase synchronous machine provides the information necessary for the power system engineer to make the straightforward "transformation" from a two-phase to a three-phase machine.

Two-Phase Synchronous Machine

The two-phase synchronous machine which we will consider is a four-pole 50-hp 440-V (rms) 60-Hz machine with the following parameters: $r_s = 0.26\ \Omega$, $L_{ls} = 1.14$ mH, $r'_{kq} = 0.02\ \Omega$, $L'_{lkq} = 1$ mH, $L_{mq} = 11$ mH, $L_{md} = 13.7$ mH, $r'_{fd} = 0.13\ \Omega$, $L'_{lfd} = 2.1$ mH, $r'_{kd} = 0.0224\ \Omega$, $L'_{lkd} = 1.4$ mH. The inertia of the rotor and connected mechanical load is $J = 16.6$ kg·m^2 and B_m is assumed to be zero.

The dynamic performance of this synchronous machine during a step decrease in load torque from zero to -400 N·m is illustrated in Fig. 6.8-1.

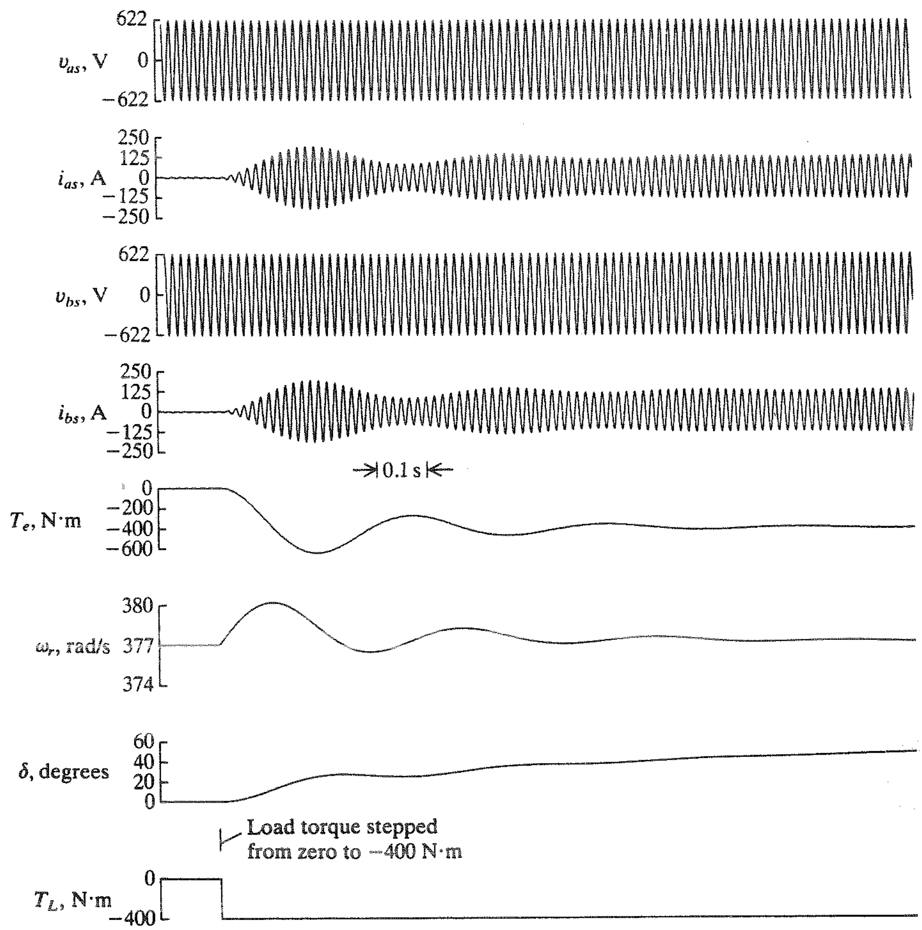
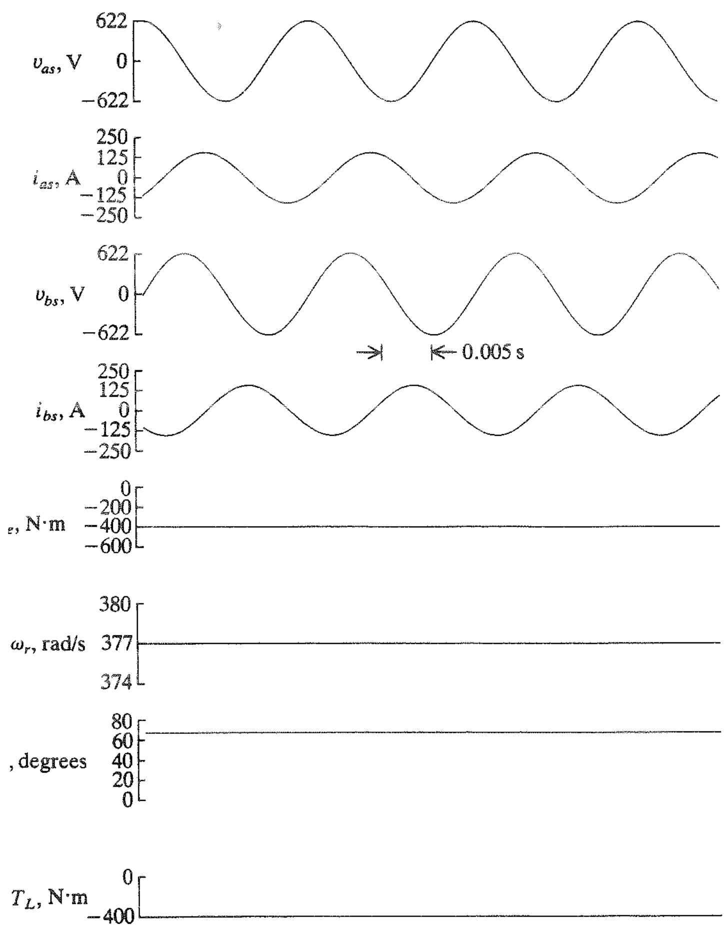


FIGURE 6.8-1
Dynamic performance of a two-phase synchronous generator during a step increase in input torque.

Since we are talking about generator operation, perhaps it is more appropriate to consider this as a step increase in input torque from zero to 400 N · m. In any event, the machine is initially operating at synchronous speed with the field voltage adjusted so that the open-circuit voltage of the stator windings is equal to the rated voltage of the machine (440 V). Therefore, the stator currents are very small since $T_L = 0$. Plotted are v_{as}, i_{as}, v_{bs}, i_{bs}, T_e, ω_r (electrical angular velocity), δ, and T_L.

Immediately upon the application of the input torque ($-T_L$), the machine accelerates above synchronous speed as predicted by (6.4-3) and the rotor angle increases in accordance with (6.6-1). The rotor continues to speed up until the accelerating torque on the rotor is zero. This occurs when T_e is equal

magnitude to the input torque. As noted in Fig. 6.8-1, the speed increases to
pproximately 380 rad/s (electrical angular velocity). Even though the ac-
lerating torque on the rotor is zero at this time, the rotor is still running
bove synchronous speed. Hence, δ will continue to increase and, consequent-
, T_e will continue to decrease (increase negatively). The decrease in T_e causes
e rotor to decelerate and the speed of the rotor decreases toward synchron-
us speed. Note at the first synchronous speed crossing of ω_r after the torque
sturbance, the rotor angle is approximately 28 electrical degrees and T_e is
pproximately $-600\,\mathrm{N\cdot m}$. The rotor speed decreases below synchronous
peed, whereupon the integrand of (6.6-1) becomes negative and the rotor



IGURE 6.8-2
teady-state operation of the two-phase synchronous generator with an input torque of $400\,\mathrm{N\cdot m}$
$T_L = 400\,\mathrm{N\cdot m}$).

angle will begin to decrease. Damped oscillations of the rotor about synchronous speed continue until the new steady-state operating point is attained. We might wish to think of the instantaneous electromagnetic torque during this disturbance resulting from the interaction between (1) the stator and field currents, (2) the stator currents and saliency of the rotor, and (3) the stator and damper winding currents. Although this line of thinking may be helpful in visualizing what is going on, we must be careful since we cannot actually separate the expression for T_e given by (6.4-2) or by (6.5-35) into these three different torques during this transient period.

Steady-state operation at the new operating point is depicted in Fig. 6.8-2 (p. 253). Note from the phase relationship between v_{as} and i_{as} or v_{bs} and i_{bs} that the synchronous machine "looks like" a negative resistance (generator action) in series with an inductor. [$\pi < \theta_{esi}(0) < \frac{3}{2}\pi$; $\tilde{I}_{as}$ lags $\tilde{V}_{as}$ by more than 90° but less than 180°.]

The dynamic torque versus rotor angle characteristics during and following the step change in input torque is shown in Fig. 6.8-3. It is interesting to note that it requires considerable time before the machine establishes steady-state operation at $T_L = -400\ \text{N}\cdot\text{m}$. The steady-state torque-angle curve which is also shown, in part, in Fig. 6.8-3 will pass through $T_e = 0$ at $\delta \cong 0$ and $T_L = -400\ \text{N}\cdot\text{m}$ at $\delta = 68°$; however, it is much different from the T_e versus δ during the transient period.

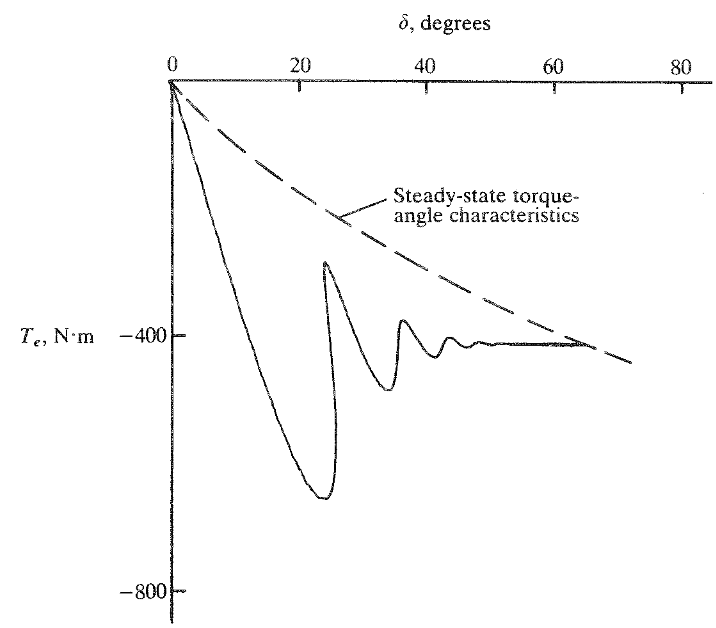


FIGURE 6.8-3
Dynamic torque versus rotor angle characteristic.

Recall that, if we slowly increase the input torque in small increments, theoretically we could reach the maximum value of T_e shown in Fig. 6.7-2c before the machine would fall out of synchronism. The machine is generally rated at 50 to 70 percent of maximum torque capability. It is interesting to mention in passing that the maximum value of input torque (or load torque) that can be applied, with T_L initially zero and with the machine still being able to return to synchronous speed, is referred to as the *transient stability limit*.

It is necessary to employ a computer to predict the dynamic torque-angle characteristics as shown in Fig. 6.8-3 and to determine the transient stability limit. However, before computers, the dynamic torque-angle characteristics were approximated for the "first swing" of the rotor by replacing X_d in (6.7-30) with X'_d, the so-called *transient reactance*, and E'^r_{xfd} with a voltage behind this

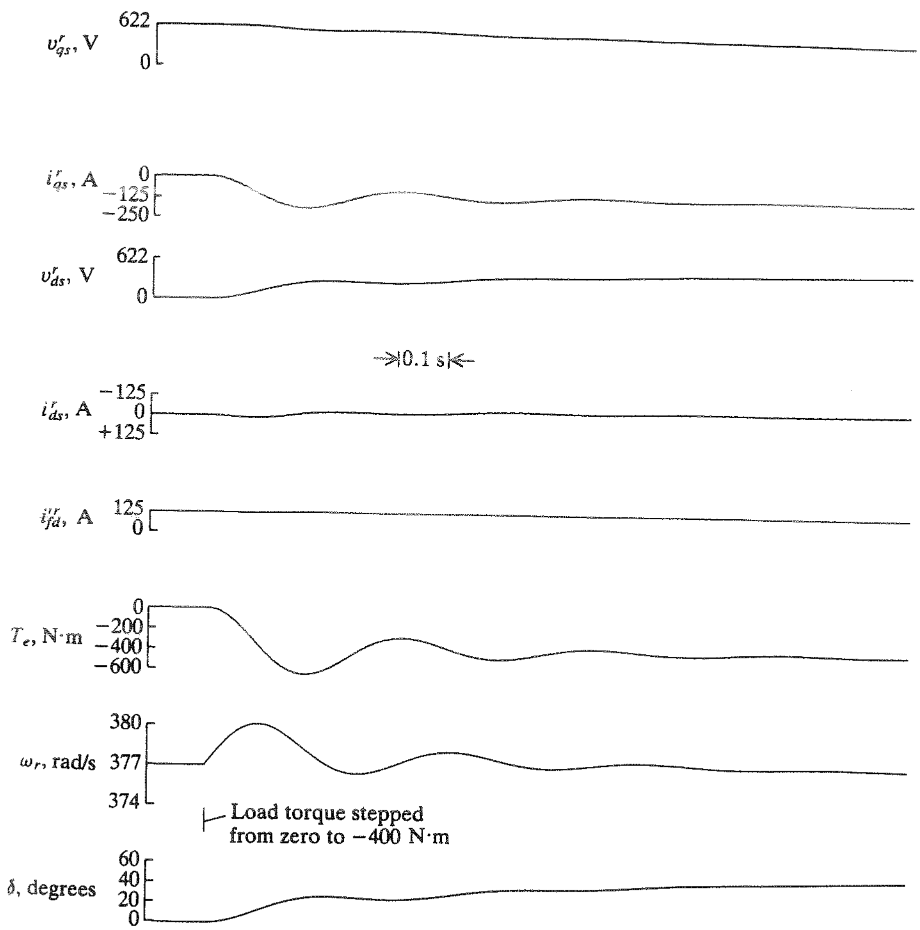


FIGURE 6.8-4
Same as Fig. 6.8-1 with rotor reference frame variables plotted.

transient reactance [1]. The transient reactance X'_d is always smaller than X_d and approximately equal to the sum of X_{ls} and X'_{lfd}. Also, the voltage behind this reactance which replaces E'^r_{xfd} is always larger than E'^r_{xfd}. It is shown in [1] that the resulting approximate, dynamic torque-angle characteristic is quite accurate during the first swing of the rotor. We shall leave this all to the power system engineer because it is, indeed, a topic which should be studied by one working in the area of power system stability. However, our purpose here is to make the first-time reader aware that the steady-state and dynamic torque versus rotor angle characteristics are different, sometimes markedly different as illustrated here.

Figures 6.8-4 and 6.8-5 are repeats of Figs. 6.8-1 and 6.8-2, respectively, with the rotor reference frame variables plotted rather than the stator or

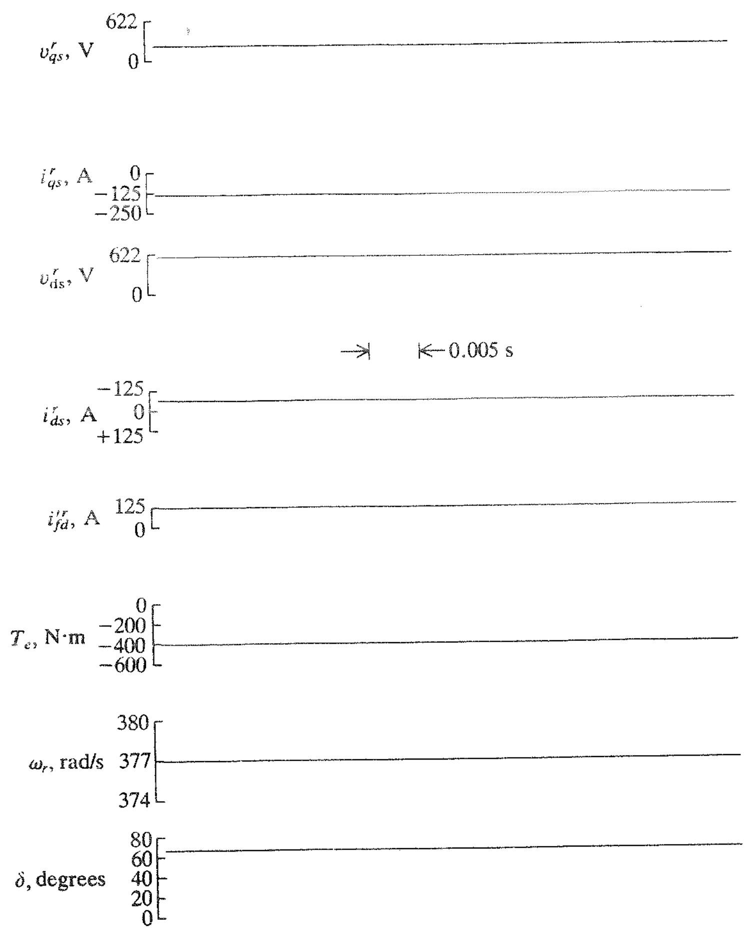


FIGURE 6.8-5
Same as Fig. 6.8-2 with rotor reference frame variables plotted.

machine variables. Also plotted is the field current i'^r_{fd}. Although for this machine the field current changed only slightly owing to a change in flux linkages, this is not typical of all machines. In some cases, depending upon the parameters and the type of the disturbance, a considerable voltage may be induced in the field winding resulting in a marked change in field current during the transient period [1].

Two-Phase Reluctance Machine

As we have mentioned, the two-phase reluctance machine is a two-phase salient-pole synchronous machine without a field winding. It is instructive to observe the steady-state and dynamic performance of a low-power two-phase reluctance motor. The parameters of a 115-V (rms) 60-Hz two-pole two-phase

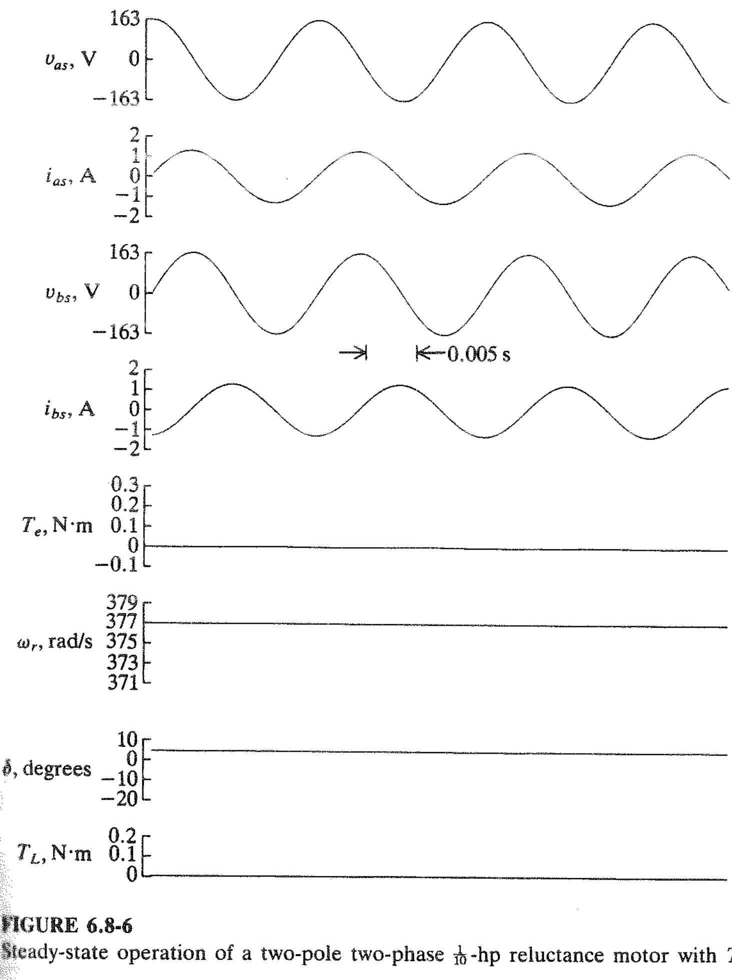


FIGURE 6.8-6
Steady-state operation of a two-pole two-phase $\frac{1}{10}$-hp reluctance motor with $T_L = 0$.

$\frac{1}{10}$-hp reluctance motor are $r_s = 10\,\Omega$, $L_{ls} = 26.5$ mH, $r'_{kq} = 2\,\Omega$, $L'_{lkq} = 26.5$ mH, $J = 1 \times 10^{-3}$ kg · m^2, $L_{mq} = 132.6$ mH, $L_{md} = 318.3$ mH, $r'_{kd} = 4\,\Omega$, $L'_{lkd} = 26.5$ mH, $B_m = 0$.

Steady-state operation with rated stator voltages and no-load torque ($T_L = 0$) is shown in Fig. 6.8-6 (p. 257). The following variables are plotted: v_{as}, i_{as}, v_{bs}, i_{bs}, T_e, ω_r, δ, and T_L. The dynamic performance when T_L is stepped from zero to 0.2 N · m is shown in Fig. 6.8-7. Steady-state operation with $T_L = 0.2$ N · m is depicted in Fig. 6.8-8. Figure 6.8-9 illustrates the dynamic performance when T_L is stepped back to zero from 0.2 N · m. The

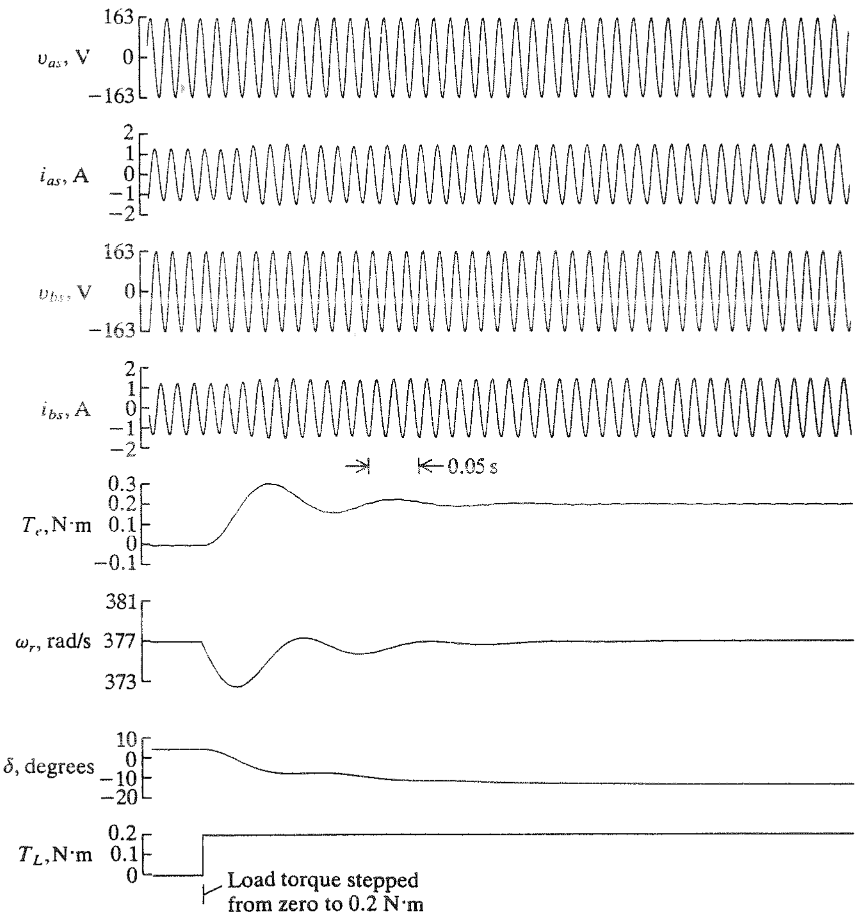


FIGURE 6.8-7
Dynamic performance of a two-pole two-phase $\frac{1}{10}$-hp reluctance motor when T_L is stepped from zero to 0.2 N · m.

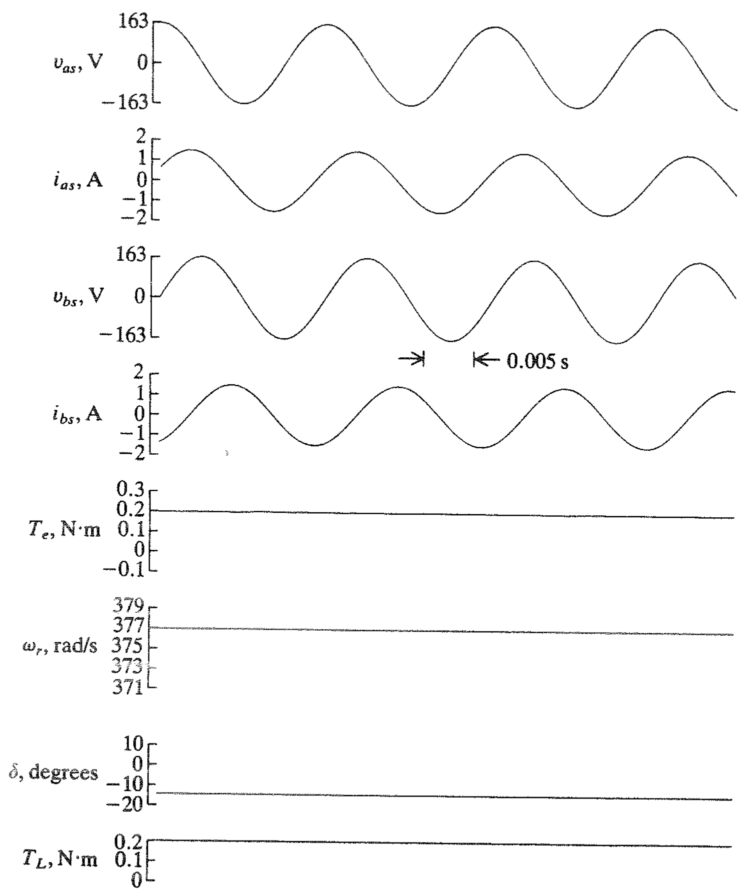


FIGURE 6.8-8
Same as Fig. 6.8-6 with $T_L = 0.2\ \text{N} \cdot \text{m}$.

steady-state torque versus rotor angle characteristic is shown in Fig. 6.8-10. It is interesting to note that this characteristic does not pass through the origin as does the reluctance component of the steady-state torque portrayed in Fig. 6.7-2*b*. Recall that the characteristics plotted in Fig. 6.7-2 were calculated by using (6.7-30) wherein the stator resistance is neglected. The stator resistance of this small reluctance motor is relatively large. The characteristics shown in Fig. 6.8-10 are calculated without neglecting r_s.

We understand that a reluctance or synchronous machine is equipped with short-circuited rotor windings for the purpose of damping rotor oscillations about synchronous speed and that is why we call them damper windings. Also, we understand that this damping torque occurs because of currents induced in the rotor circuits whenever $\omega_r \neq \omega_e$. As we have mentioned, this is

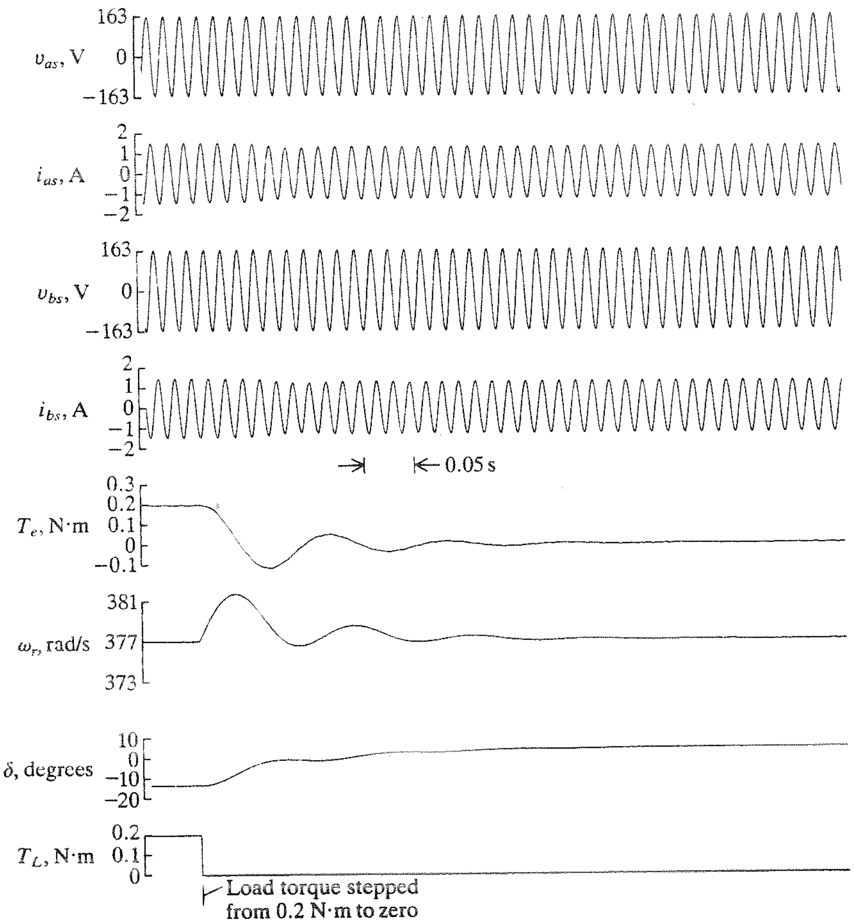


FIGURE 6.8-9
Same as Fig. 6.8-7 with T_L stepped from 0.2 N · m back to zero.

often called induction motor action since, with the short-circuited rotor wind ings, the reluctance machine produces an average torque-speed characteristi similar to an induction machine. If the rotor should slow down ever so slightl from synchronous speed, the induction motor torque would become positiv which would tend to accelerate the rotor back to synchronous speed. Likewis if the rotor should speed up from synchronous speed, the induction mot torque would be negative, which would tend to slow the rotor speed. On wonders then if it is possible for the reluctance machine to produce a startin torque by induction motor action. Yes; in most cases the machine is design so that it can accelerate from stall and "pull in" to synchronous-spe operation often under 50 to 70 percent of rated load. Actually, we are not surprised at this since we expected that somehow the reluctance motors of t

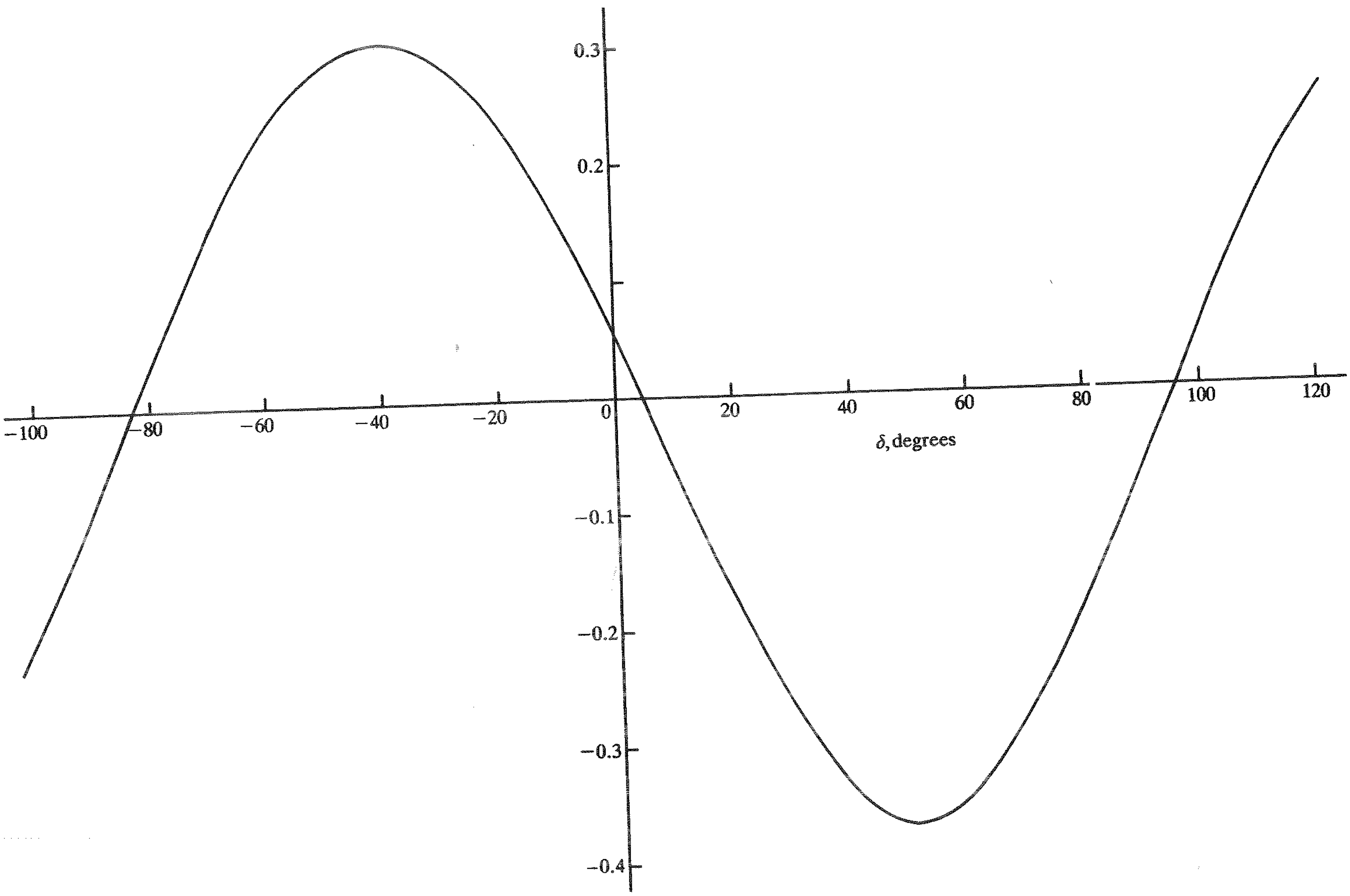


FIGURE 6.8-10
Steady-state electromagnetic torque versus rotor angle for a two-pole two-phase $\frac{1}{10}$-hp reluctance motor.

world would be started by some means other than physically twisting each by its shaft in order to get it up to synchronous speed.

The acceleration characteristics of the reluctance motor from stall to synchronous speed are shown in Fig. 6.8-11 and the torque versus speed characteristics are shown in Fig. 6.8-12. The load torque during acceleration is $T_L = K\omega_{rm}^2$, where $K = 0.2(377)^{-2}$ N · m · s^2/rad^2. The electromagnetic torque pulsates until synchronous speed is reached, whereupon the device operates as a reluctance motor producing a constant torque. But what causes the pulsating torque? Well, immediately following the application of the stator voltages, the pulsation in torque appears to be 60-Hz. This is caused by the transient dc offset in the stator currents; however, as this 60-Hz pulsation decays in

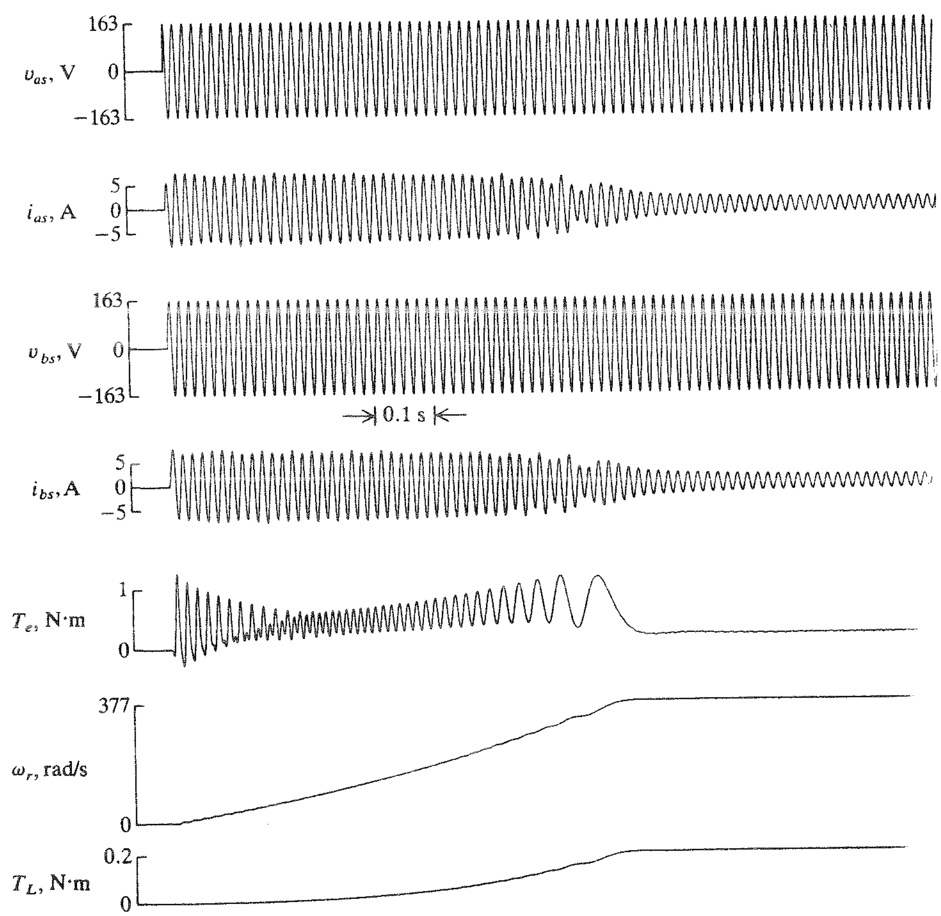


FIGURE 6.8-11
Acceleration from stall of the $\frac{1}{10}$-hp reluctance machine with $T_L = K\omega_{rm}^2$, where $K = 0.2(377)^{-2}$ N · m · s^2/rad^2.

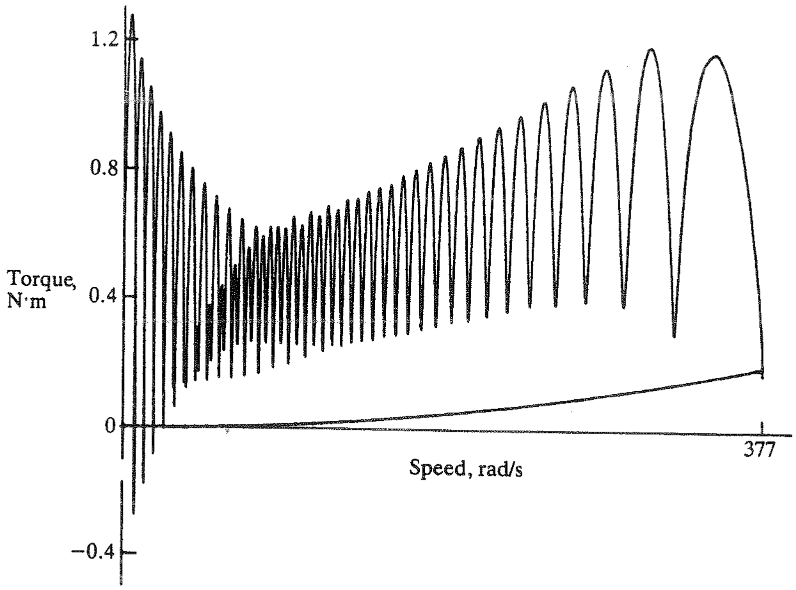


FIGURE 6.8-12
Torque versus speed during acceleration shown in Fig. 6.8-11.

amplitude, a higher-frequency pulsation starts to appear. As the rotor accelerates, this pulsation in torque increases in amplitude while decreasing in frequency. Actually, the frequency of this pulsating component is $2(\omega_e - \omega_r)$ and it is due primarily to the saliency of the rotor (instantaneous reluctance torque) and, to a much less degree, due to the fact that the kq and kd windings have different resistances. The "pulsating reluctance torque" is understandable; however, the fact that a pulsating torque with a frequency of $2(\omega_e - \omega_r)$ occurs because of unequal r'_{kq} and r'_{kd} is not at all apparent. It requires considerable work to prove that it exists; however, the interested reader will find this proof in [1].

The reluctance motor is often used in variable-speed drive applications. In this type of operation, the reluctance motor is supplied from an inverter which can be made to vary rapidly the frequency and/or the amplitude of the fundamental component of the stator voltages. The dynamic behavior of the reluctance motor following a step decrease in frequency from 60 to 50 Hz with an accompanying step decrease in the amplitude of the applied voltages from 110 V (rms) to $(\frac{5}{6})$110 V (rms) is shown in Fig. 6.8-13. Once the machine reaches steady-state operation, the frequency and voltage amplitude are stepped back to their original values. Although the stator voltages supplied from an inverter would contain harmonics, the assumption of a sinusoidal variable-frequency source allows us a first look at variable-frequency operation without getting involved with the operation of the inverter.

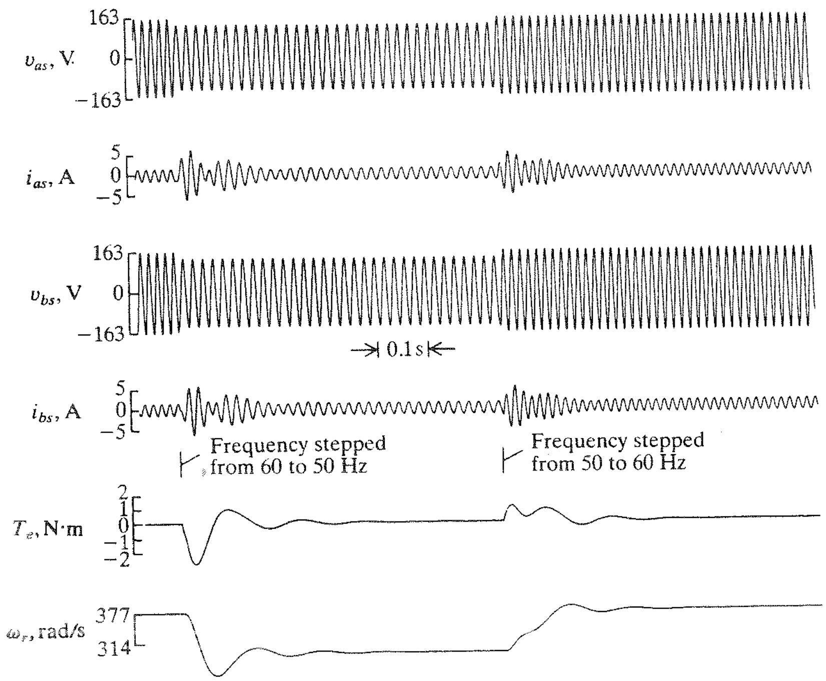


FIGURE 6.8-13
Step changes in frequency of the stator voltages of a $\frac{1}{10}$-hp reluctance motor.

SP6.8-1. From the steady-state voltage and current waveforms given in Fig. 6.8-2 approximate the power input from the electric system and compare this value to the shaft power ($T_L \omega_{rm}$). [$\tilde{V}_{as} = 440\underline{/0°}$, $\tilde{I}_{as} \cong 111\underline{/-135°}$, $P_{in} \cong -69$ kW, $P_{shaft} \cong -75$ kW]

SP6.8-2. Why is there a difference between P_{in} and P_{shaft} in SP6.8-1? Check your answer. [Ohmic loss]

SP6.8-3. Repeat SP6.8-1 by using the voltage and current waveforms shown in Fig. 6.8-5

SP6.8-4. Use either Fig. 6.8-1 or Fig. 6.8-4 to determine each term of (6.4-3) at the first time $\omega_r = \omega_e$ after the step in T_L from zero to -400 N·m. [-400 N·m, -57 N·m, -170 N·m]

SP6.8-5. From Fig. 6.8-8, approximate the per-phase input impedance of the reluctance motor with $T_L = 0.2$ N·m. Compare the imaginary part of this impedance to X_q and X_d. Show that the real part is r_s plus a resistance which represents the power output at the shaft. [$Z \cong 42 + j100\ \Omega$]

6.9 THREE-PHASE SYNCHRONOUS MACHINE

A two-pole three-phase salient-pole synchronous machine is shown in Fig. 6.9-1. The stator windings are identical and sinusoidally distributed with their magnetic axes displaced 120° from each other. The extension from the analy

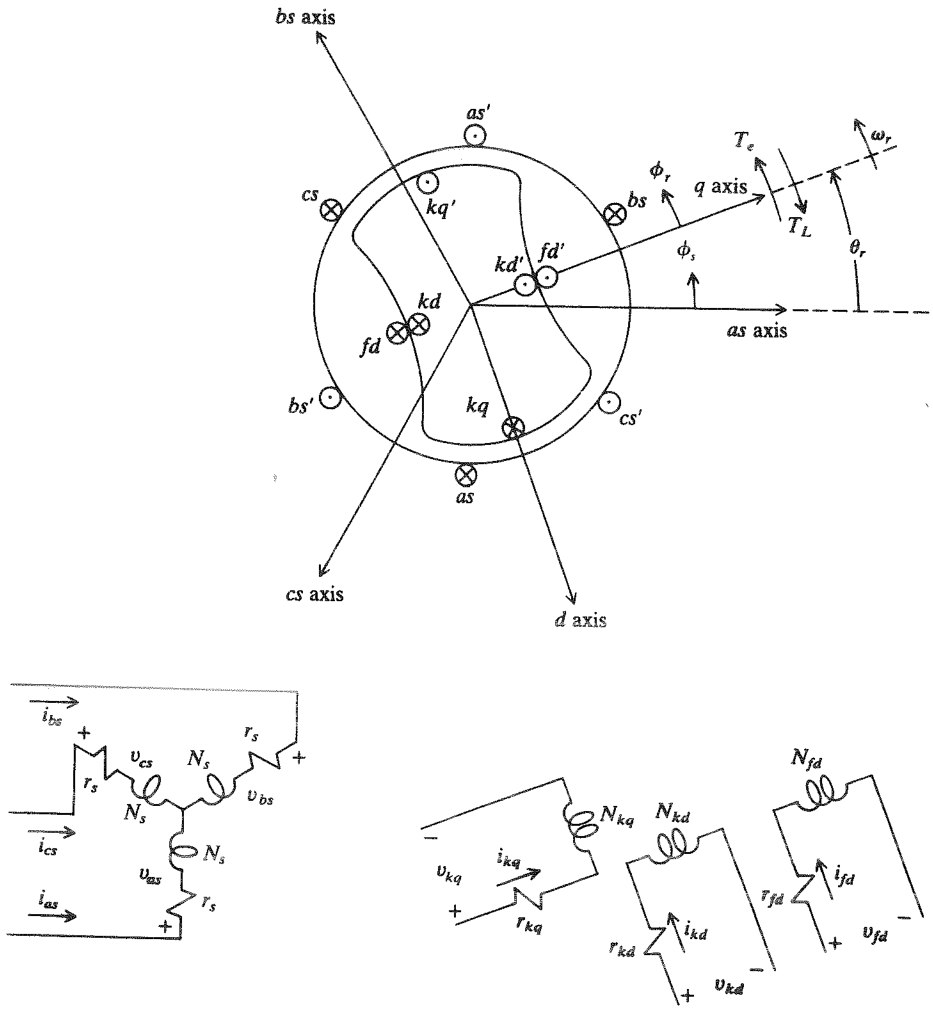


FIGURE 6.9-1
Two-pole three-phase salient-pole synchronous machine.

of a two-phase machine to a three-phase machine is straightforward. However, it is worthwhile to note the expressions of the mutual inductances of the stator windings. Also, the addition of a third substitute variable, the zero variable, is necessary since we have three stator variables.

Voltage Equations and Winding Inductances

The voltage equations for the three-phase synchronous machine are those given by (6.3-1) through (6.3-5) for the two-phase machine, with the voltage equation for the c phase added. In matrix form,

$$\mathbf{v}_{abcs} = \mathbf{r}_s \mathbf{i}_{abcs} + p\boldsymbol{\lambda}_{abcs} \tag{6.9-1}$$

$$\mathbf{v}_{qdr} = \mathbf{r}_r \mathbf{i}_{qdr} + p\boldsymbol{\lambda}_{qdr} \tag{6.9-2}$$

where

$$(\mathbf{f}_{abcs})^T = [f_{as} \quad f_{bs} \quad f_{cs}] \tag{6.9-3}$$

$$(\mathbf{f}_{qdr})^T = [f_{kq} \quad f_{fd} \quad f_{kd}] \tag{6.9-4}$$

The matrix $\mathbf{r}_s$ is an equal-element diagonal matrix and $\mathbf{r}_r$ is defined by (6.3-11).

The flux linkage equations may be written as

$$\begin{bmatrix} \boldsymbol{\lambda}_{abcs} \\ \boldsymbol{\lambda}_{qdr} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_s & \mathbf{L}_{sr} \\ (\mathbf{L}_{sr})^T & \mathbf{L}_r \end{bmatrix} \begin{bmatrix} \mathbf{i}_{abcs} \\ \mathbf{i}_{qdr} \end{bmatrix} \tag{6.9-5}$$

where

$$\mathbf{L}_s = \begin{bmatrix} L_{ls} + L_A - L_B \cos 2\theta_r & -\frac{1}{2}L_A - L_B \cos 2(\theta_r - \frac{1}{3}\pi) & -\frac{1}{2}L_A - L_B \cos 2(\theta_r + \frac{1}{3}\pi) \\ -\frac{1}{2}L_A - L_B \cos 2(\theta_r - \frac{1}{3}\pi) & L_{ls} + L_A - L_B \cos 2(\theta_r - \frac{2}{3}\pi) & -\frac{1}{2}L_A - L_B \cos 2(\theta_r + \pi) \\ -\frac{1}{2}L_A - L_B \cos 2(\theta_r + \frac{1}{3}\pi) & -\frac{1}{2}L_A - L_B \cos 2(\theta_r + \pi) & L_{ls} + L_A - L_B \cos 2(\theta_r + \frac{2}{3}\pi) \end{bmatrix} \tag{6.9-6}$$

where L_{ls} is the leakage inductance, and L_A and L_B are defined by (6.3-23) and (6.3-24), respectively. The matrix $\mathbf{L}_{sr}$ is an extension of (6.3-30) to account for a three-phase stator.

$$\mathbf{L}_{sr} = \begin{bmatrix} L_{skq} \cos \theta_r & L_{sfd} \sin \theta_r & L_{skd} \sin \theta_r \\ L_{skq} \cos (\theta_r - \frac{2}{3}\pi) & L_{sfd} \sin (\theta_r - \frac{2}{3}\pi) & L_{skd} \sin (\theta_r - \frac{2}{3}\pi) \\ L_{skq} \cos (\theta_r + \frac{2}{3}\pi) & L_{sfd} \sin (\theta_r + \frac{2}{3}\pi) & L_{skd} \sin (\theta_r + \frac{2}{3}\pi) \end{bmatrix} \tag{6.9-7}$$

The matrix $\mathbf{L}_r$ is (6.3-31).

The $\mathbf{L}_s$ given by (6.9-6) requires some discussion. The expressions for the self-inductances, which are the diagonal terms in (6.9-6), are apparent from our explanation for the self-inductances of a two-phase machine. Also the $-\frac{1}{2}L_A$ factor in the off-diagonal terms of (6.9-6) seems logical since the mutual inductance between two sinusoidally distributed windings of the stator can be adequately portrayed by the cosine of the angle between their magnetic axes (120°). Therefore, if the air gap were uniform, as in the case of the round-rotor synchronous machine, the off-diagonal terms (mutual inductances) would be $-\frac{1}{2}L_A$. However, as in the case of the two-phase synchronous machine, it is not clear that the variation of the mutual inductance would be L_B, nor is it immediately obvious that maximum coupling between the *as* and *bs* phase, for example, would occur at $\theta_r = \frac{1}{3}\pi$ and $\frac{4}{3}\pi$ and minimum coupling at $\theta_r = \frac{5}{6}\pi$ and $\frac{11}{6}\pi$. We will accept (6.9-6) without proof. A derivation is given in [1] for those who wish additional information.

In the case of the three-phase synchronous machine, the stator magnetizing inductances are defined as one and one-half times the magnetizing inductances of a two-phase machine. In particular,

$$L_{mq} = \tfrac{3}{2}(L_A - L_B) \tag{6.9-8}$$

$$L_{md} = \tfrac{3}{2}(L_A + L_B) \tag{6.9-9}$$

With the above definition of L_{mq} and L_{md}, the right-hand sides of (6.3-32) through (6.3-37) must be multiplied by $\frac{2}{3}$ in order to define the amplitudes of the stator-to-rotor mutual inductances used in $\mathbf{L}_{sr}$ and the rotor magnetizing inductances used in $\mathbf{L}_r$. With the selection of (6.9-8) and (6.9-9) as the stator magnetizing inductances, the turns-ratio for the rotor currents is two-thirds that given by (6.3-39); however, the turns-ratio for the rotor voltages and flux linkages are unchanged from (6.3-40) and (6.3-41). The flux linkage equations can now be written as

$$\begin{bmatrix} \boldsymbol{\lambda}_{abcs} \\ \boldsymbol{\lambda}'_{qdr} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_s & \mathbf{L}'_{sr} \\ \frac{2}{3}(\mathbf{L}'_{sr})^T & \mathbf{L}'_r \end{bmatrix} \begin{bmatrix} \mathbf{i}_{abcs} \\ \mathbf{i}'_{qdr} \end{bmatrix} \tag{6.9-10}$$

where

$$\mathbf{L}'_{sr} = \begin{bmatrix} L_{mq}\cos\theta_r & L_{md}\sin\theta_r & L_{md}\sin\theta_r \\ L_{mq}\cos(\theta_r - \frac{2}{3}\pi) & L_{md}\sin(\theta_r - \frac{2}{3}\pi) & L_{md}\sin(\theta_r - \frac{2}{3}\pi) \\ L_{mq}\cos(\theta_r + \frac{2}{3}\pi) & L_{md}\sin(\theta_r + \frac{2}{3}\pi) & L_{md}\sin(\theta_r + \frac{2}{3}\pi) \end{bmatrix} \tag{6.9-11}$$

The matrix $\mathbf{L}'_r$ is (6.3-44) with our new definition of L_{mq} and L_{md}.

The voltage equations become

$$\mathbf{v}_{abcs} = \mathbf{r}_s\mathbf{i}_{abcs} + p\boldsymbol{\lambda}_{abcs} \tag{6.9-12}$$

$$\mathbf{v}'_{qdr} = \mathbf{r}'_r\mathbf{i}'_{qdr} + p\boldsymbol{\lambda}'_{qdr} \tag{6.9-13}$$

In terms of inductances,

$$\begin{bmatrix} \mathbf{v}_{abcs} \\ \mathbf{v}'_{qdr} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_s + p\mathbf{L}_s & p\mathbf{L}'_{sr} \\ \frac{2}{3}p(\mathbf{L}'_{sr})^T & \mathbf{r}'_r + p\mathbf{L}'_r \end{bmatrix} \begin{bmatrix} \mathbf{i}_{abcs} \\ \mathbf{i}'_{qdr} \end{bmatrix} \tag{6.9-14}$$

where r'_j and L'_{lj} are one and one-half times (6.3-48) and (6.3-49), respectively.

Torque

The electromagnetic torque, positive for motor action, may be expressed by using the second entry in Table 2.5-1. In particular,

$$T_e = \frac{P}{2}\left\{\frac{L_{md} - L_{mq}}{3}\left[(i_{as}^2 - \tfrac{1}{2}i_{bs}^2 - \tfrac{1}{2}i_{cs}^2 - i_{as}i_{bs} - i_{as}i_{cs} + 2i_{bs}i_{cs})\sin 2\theta_r \right.\right.$$
$$\left. + \frac{\sqrt{3}}{2}(i_{bs}^2 + i_{cs}^2 - 2i_{as}i_{bs} + 2i_{as}i_{cs})\cos 2\theta_r\right]$$
$$- L_{mq}i_{kq}\left[(i_{as} - \tfrac{1}{2}i_{bs} - \tfrac{1}{2}i_{cs})\sin\theta_r - \frac{\sqrt{3}}{2}(i_{bs} - i_{cs})\cos\theta_r\right]$$
$$\left. + L_{md}(i'_{fd} + i'_{kd})\left[(i_{as} - \tfrac{1}{2}i_{bs} - \tfrac{1}{2}i_{cs})\cos\theta_r + \frac{\sqrt{3}}{2}(i_{bs} - i_{cs})\sin\theta_r\right]\right\}$$

(6.9-15)

The torque and rotor speed are related by (6.4-3), which is repeated here for convenience:

$$T_e = J\left(\frac{2}{P}\right)\frac{d\omega_r}{dt} + B_m\left(\frac{2}{P}\right)\omega_r + T_L \qquad (6.9\text{-}16)$$

where J is the inertia and B_m is the damping coefficient the units of which ar discussed following (6.4-3). The load torque T_L is positive for a torque loa (motor action) and negative for a torque input (generator action), as shown i Fig. 6.9-1.

Machine Equations in the Rotor Reference Frame

Since there are three stator variables (f_{as}, f_{bs}, f_{cs}), we must use three substitu variables in the transformation of the stator variables to the rotor referen frame. In particular,

$$\mathbf{f}_{qdos}^r = \mathbf{K}_s^r \mathbf{f}_{abcs} \qquad (6.9\text{-}1$$

$$(\mathbf{f}_{qd0s}^r)^T = [f_{qs}^r \quad f_{ds}^r \quad f_{0s}] \qquad (6.9\text{-}1$$

$$\mathbf{K}_s^r = \frac{2}{3}\begin{bmatrix} \cos\theta_r & \cos(\theta_r - \frac{2}{3}\pi) & \cos(\theta_r + \frac{2}{3}\pi) \\ \sin\theta_r & \sin(\theta_r - \frac{2}{3}\pi) & \sin(\theta_r + \frac{2}{3}\pi) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad (6.9\text{-}($$

where the rotor displacement θ_r is defined by (6.5-4). The inverse of $\mathbf{K}_s^r$ i

$$(\mathbf{K}_s^r)^{-1} = \begin{bmatrix} \cos\theta_r & \sin\theta_r & 1 \\ \cos(\theta_r - \frac{2}{3}\pi) & \sin(\theta_r - \frac{2}{3}\pi) & 1 \\ \cos(\theta_r + \frac{2}{3}\pi) & \sin(\theta_r + \frac{2}{3}\pi) & 1 \end{bmatrix} \qquad (6.9\text{-}$$

It is important to note that the same notation ($\mathbf{K}_s^r$) is used for the transfor tion for both the two- and three-phase change of variables. A trigonometric interpretation of the above change of variables is shown in Fig. 6.9-2.

As in the case of the induction machine, the zero variable ($0s$ variable) is the third substitute variable. We note that f_{0s} is zero for balanced conditions and that f_{0s} is not a function of θ_r and, therefore, the $0s$ quantities (voltage, current, and flux linkage) are associated with stationary circuits. For this reason, a raised index is not incorporated with the zero variables.

If the above change of variables is substituted into the stator voltage equations and the flux linkage equations, the resulting q and d voltage equations are identical to (6.5-20) through (6.5-24). The voltage equation for the $0s$ variables must be added. In particular,

$$\boxed{v_{0s} = r_s i_{0s} + p\lambda_{0s}} \tag{6.9-21}$$

For a linear magnetic system, the q and d flux linkage equations for a three-phase synchronous machine are identical to those given by (6.5-25) through (6.5-29) for the two-phase machine. One must remember, however, that for a three-phase machine L_{mq} and L_{md} are defined by (6.9-8) and (6.9-9), respectively. The expression for λ_{0s} is

$$\lambda_{0s} = L_{ls} i_{0s} \tag{6.9-22}$$

Hence, the voltage equation in the rotor reference frame may be written in terms of inductances as

$$\boxed{\begin{bmatrix} v^r_{qs} \\ v^r_{ds} \\ v_{0s} \\ v'^r_{kq} \\ v'^r_{fd} \\ v'^r_{kd} \end{bmatrix} = \begin{bmatrix} r_s + pL_q & \omega_r L_d & 0 & pL_{mq} & \omega_r L_{md} & \omega_r L_{md} \\ -\omega_r L_q & r_s + pL_d & 0 & -\omega_r L_{mq} & pL_{md} & pL_{md} \\ 0 & 0 & r_s + pL_{ls} & 0 & 0 & 0 \\ pL_{mq} & 0 & 0 & r'_{kq} + pL'_{kq} & 0 & 0 \\ 0 & pL_{md} & 0 & 0 & r'_{fd} + pL'_{fd} & pL_{md} \\ 0 & pL_{md} & 0 & 0 & pL_{md} & r'_{kd} + pL'_{kd} \end{bmatrix} \begin{bmatrix} i^r_{qs} \\ i^r_{ds} \\ i_{0s} \\ i'^r_{kq} \\ i'^r_{fd} \\ i'^r_{kd} \end{bmatrix}} \tag{6.9-23}$$

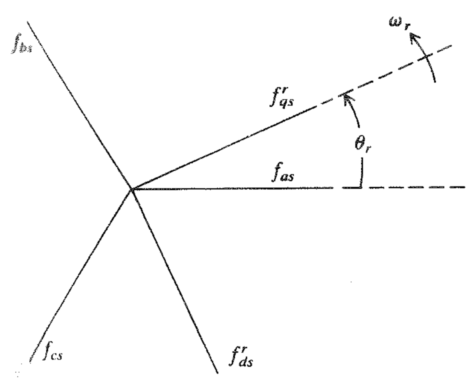


FIGURE 6.9-2
Trigonometric interpretation of the change of stator variables for a three-phase synchronous machine.

where L_q and L_d are defined by (6.5-18) and (6.5-19) while L'_{kq}, L'_{fd}, and L'_{kd} are defined by (6.5-30) through (6.5-32), respectively. We must realize that L_{mq} and L_{md} in these equations are defined by (6.9-8) and (6.9-9), respectively, for the three-phase machine.

The q and d equivalent circuits given in Fig. 6.5-2 for a two-phase machine are valid for the three-phase machine if L_{mq} and L_{md} are defined by (6.9-8) and (6.9-9), respectively, and the appropriate turns-ratio is used for the rotor currents [one and one-half times (6.3-39)]. The equivalent circuit for the $0s$ quantities is a series rL circuit.

The expression for the electromagnetic torque for a three-phase synchronous machine in terms of q and d variables is identical to (6.5-34) and (6.5-35) if each expression is multiplied by $\frac{3}{2}$ and, of course, with the appropriate expression for L_{mq} and L_{md}. It follows that the steady-state voltage and torque equations given for the two-phase machine are also valid for the three-phase machine with the $\frac{3}{2}$ factor properly taken into account in the torque equation and in L_{mq} and L_{md}.

SP6.9-1. The parameters of a three-phase synchronous machine are identical to those for the two-phase synchronous machine given in Sec. 6.8. What must J and T_L be to make the dynamic and steady-state response of v^r_{qs}, i^r_{qs}, v^r_{ds}, and i^r_{ds} shown in Figs. 6.8-4 and 6.8-5 the same for the two- and three-phase machines? [$J_3 = \frac{3}{2}J_2$; $T_{L3} = \frac{3}{2}T_{L2}$]

SP6.9-2. The values of L_A and L_B are identical for two machines. One is a two-phase reluctance motor, the other a three-phase reluctance motor; otherwise, the parameters are identical. Will v^r_{qs}, i^r_{qs}, v^r_{ds}, and i^r_{ds} be identical for steady-state operation with $T_{L3} = \frac{3}{2}T_{L2}$? Why? [No; $L_{mq3} = \frac{3}{2}L_{mq2}$, $L_{md3} = \frac{3}{2}L_{md2}$]

6.10 RECAPPING

To conduct a rigorous analysis of a synchronous machine, it was necessary t[o] incorporate a change of variables. In effect, this change of variables repla[ces] the stator variables (voltages, currents, and flux linkages) with variables th[at] are associated with fictitious windings fixed in the rotor. In this way, th[e] time-varying stator self-inductances as well as the time-varying stator-to-ro[tor] mutual inductances are eliminated and all inductances are constant. Althou[gh] this analysis is rather involved, the resulting equations form the basis for t[he] analysis and the computer simulation of synchronous machines. Also, we fou[nd] that, for steady-state operation, the voltage equations reduce to a single pha[sor] equation, making steady-state motor or generator operation readily analyzab[le].

Since the equations which describe the dynamic behavior of synchrono[us] machines are nonlinear, it is necessary to use a computer to solve th[e] equations. Computer traces are given to illustrate the dynamic and steady-s[tate] performance of a synchronous generator and a reluctance motor. Altho[ugh] implementation of a computer simulation which can be used for these calc[ula]tions is beyond the goals of this text, the advantage of portraying mach[ine] variables by computer simulation is vividly illustrated by the dynamic resp[onse] of the synchronous generator resulting from a step change in input torque. These traces not only give meaning to the dynamic behavior of a single synchronous machine but also they set the stage for studying the dynamic characteristics of a power system containing hundreds of such machines.

Our purpose has been to establish a method of analysis which is valid for synchronous machines ranging from a large synchronous generator to a small, low-power reluctance motor and to illustrate, by computer traces, the dynamic and steady-state behavior of these devices. Hopefully, we have been successful.

6.11 REFERENCES

1. P. C. Krause, *Analysis of Electric Machinery*, McGraw-Hill Book Company, New York, 1986.
2. R. H. Park, "Two Reaction Theory of Synchronous Machines—Generalized Method of Analysis—Part I," *AIEE Trans.*, vol. 48, July 1929, pp. 716–727.

6.12 PROBLEMS

1. Express all self- and mutual inductances for the synchronous machine shown in Fig. 6.12-1. Note that θ_r is referenced to the d axis.
*2. Obtain (6.4-2) from (6.4-1).
3. Write Park's transformation ($\mathbf{K}_s^r$) with θ_r referenced to the d axis as in Fig. 6.12-1.

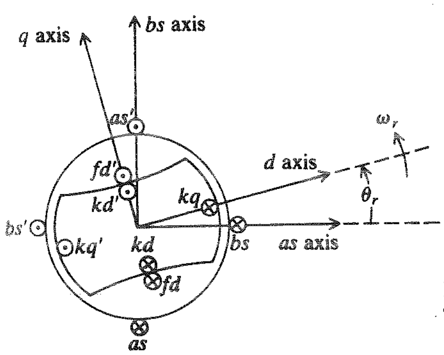


FIGURE 6.12-1
Two-pole two-phase salient-pole synchronous machine.

*4. Derive the inductance matrices given by (6.5-14) through (6.5-16).
5. Derive the expression for torque given by (6.5-34) and show that (6.5-34) may be written as (6.5-35).
6. It is often convenient to express the flux linkage equations given by (6.5-25) through (6.5-29) in terms of flux linkages per second rather than flux linkages. This is accomplished by multiplying the flux linkage equations by a base electrical angular velocity ω_b. For example, ω_b for a 60-Hz machine would be 377 rad/s. In terms of flux linkages per second (6.5-25) would become

$$\psi_{qs}^r = X_{ls} i_{qs}^r + X_{mq}(i_{qs}^r + i_{kq}'^r)$$

where $X_{ls} = \omega_b L_{ls}$ and $X_{mq} = \omega_b L_{mq}$. Rewrite the voltage equations (6.5-20) through (6.5-24) and the expression for torque given by (6.5-35) in terms of flux linkages per second.

7. Rewrite the steady-state voltage equations, (6.7-24) and (6.7-25), and the steady-state torque equations, (6.7-26) and (6.7-30), for the reluctance machine.
8. A four-pole 2-hp two-phase round-rotor synchronous machine is connected to a 110-V 60-Hz source. The machine is operating as a generator with a total steady-state power output of 1 kW at the terminals. The phase current lags the phase voltage by 160°. The parameters are $r_s = 0.5\ \Omega$, $L_{ls} = 0.005$ H, $L_{mq} = L_{md} = 0.05$ H. Calculate $\tilde{E}_a$ and draw the phasor diagram showing $\tilde{V}_{as}$, $\tilde{I}_{as}$, $\tilde{E}_a$, and $(r_s + jX_q)\tilde{I}_{as}$.
9. Refer to Example 6B. Calculate I'_{ds}, I'^r_{fd}, and the torque output in N · m for each mode of operation described.
10. Refer to Example 6C. Calculate $\tilde{E}_a$ and draw the phasor diagram showing $\tilde{V}_{as}$, $\tilde{I}_{as}$, $\tilde{E}_a$, and $(r_s + jX_q)\tilde{I}_{as}$.
11. In Fig. 6.8-1, the field voltage is adjusted so that the open-circuit stator phase voltage would be equal to rated voltage if the rotor were driven at synchronous speed. Calculate V'^r_{fd} and E'^r_{xfd} for this condition. Check your answer by noting the value of i'^r_{fd} in Fig. 6.8-4.
12. Approximate $\tilde{V}_{as}$, $\tilde{I}_{as}$, and δ from Fig. 6.8-2. From these values calculate V^r_{qs}, I^r_{qs}, V^r_{ds}, and I^r_{ds} and compare with the values of these quantities shown in Fig. 6.8-5.
*13. Assume that the rotor angle is correct in Figs. 6.8-6 and 6.8-8. Calculate T_e, $\tilde{I}_{as}$, and $\tilde{I}_{bs}$ for both operating conditions of the two-phase reluctance motor and verify these calculations from the waveforms shown in Figs. 6.8-6 and 6.8-8.
14. Calculate the efficiency (output power divided by input power) of the two-phase reluctance motor considered in Sec. 6.8 when operating as depicted in Fig. 6.8-8. If you have worked Prob. 13 you may use these calculated results; if not, you may obtain the values of the variables from the plots given in Fig. 6.8-8.
*15. A three-phase sixty-four-pole hydroturbine generator is rated at 325 MVA, with a 20-kV line-to-line voltage, and the generator delivers reactive power with a power factor of 0.85. In this case, the power factor is determined by the cosine of the angle between the voltage and the current with the positive direction of current out of the machine. In our work we have assumed positive current into the machine, thus the power factor would be −0.85. The machine parameters in ohms at 60 Hz are $r_s = 0.0023$, $X_q = 0.591$, and $X_d = 1.047$. For balanced steady-state rated conditions calculate (a) $\tilde{E}_a$, (b) E'^r_{xfd}, and (c) T_e.
*16. A two-pole 220-V (rms, line-to-line) 5-hp three-phase reluctance machine has the following parameters: $r_s = 1\ \Omega$, $L_{ls} = 0.005$ H, $L_{md} = 0.10$ H, $L_{mq} = 0.02$ H.
   (a) The machine is supplied from a 60-Hz 220-V source with zero load torque. Calculate δ and $\tilde{I}_{as}$.
   (b) Repeat (a) with the machine connected to a 6-Hz 22-V source.