ANALYSIS OF ELECTRIC MACHINERY

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CHAPTER

FIVE

THEORY OF SYNCHRONOUS MACHINES

5.1 INTRODUCTION

Nearly all of the electric power used throughout the world is generated by synchronous machines driven either by hydro or steam turbines or combustion engines. Just as the induction machine is the workhorse when it comes to converting energy from electrical to mechanical, the synchronous machine is the principal means of converting energy from mechanical to electrical. The electrical and electromechanical behavior of most synchronous machines can be predicted from the equations which describe the 3-phase salient-pole synchronous machine. In particular, these equations can be used directly to predict the performance of hydro and steam turbine synchronous generators, synchronous motors, and —with only slight modifications—reluctance motors.

The rotor of a synchronous machine is equipped with a field winding and one or more damper windings and, in general, all rotor windings have different electrical characteristics. Moreover, the rotor of a salient-pole synchronous machine is magnetically unsymmetrical. Due to these rotor asymmetries a change of variables for the rotor variables offers no advantage. However, a change of variables is beneficial for the stator variables. In most cases, the stator variables are transformed to a reference frame fixed in the rotor (Park's equations) [1]; however, the stator variables may also be expressed in the arbitrary reference frame which is convenient for some computer simulations.

In this chapter, the voltage and electromagnetic torque equations are first established in machine variables. Reference-frame theory set forth in Chap. 3 is then used to establish the machine equations with the stator variables in an arbitrary reference frame and in the rotor reference frame (Park's equations). The equations which describe the steady state behavior are then derived from Park's equations using the theory established in Chap. 3. Computer traces are given to illustrate the dynamic behavior of typical hydro and steam turbine generators during sudden changes in input torque and during and following a 3-phase fault at the terminals. These dynamic responses, which are calculated using the detailed set of nonlinear differential equations, are compared to those predicted by an approximate method of calculating the transient torque-angle characteristics which was widely used before the advent of modern computers and which still offer an unequalled means of visualizing the transient behavior of synchronous machines.

The analysis given in this chapter is valid for a linear magnetic system; saturation is not considered. In some cases saturation has only a secondary effect upon the overall performance of the machine; in other cases it is very important and it must be taken into account when predicting the performance. Incorporating saturation into the machine equations which describe its dynamic behavior is quite involved. Saturation, and the method of accounting for it, is covered in a subsequent chapter on computer simulations of synchronous machines.

5.2 VOLTAGE EQUATIONS IN MACHINE VARIABLES

A 2-pole, 3-phase, wye-connected, salient-pole synchronous machine is shown in Fig. 5.2-1. The stator windings are identical sinusoidally distributed windings, displaced 120°, with N_s equivalent turns and resistance r_s. The rotor is equipped with a field winding and three damper windings. The field winding (*fd* winding) has N_{fd} equivalent turns with resistance r_{fd}. One damper winding is in the same magnetic axis as the field winding. This winding, the *kd* winding, has N_{kd} equivalent turns with resistance r_{kd}. The magnetic axis of the second and third damper windings, the *kq*1 and *kq*2 windings, is displaced 90° ahead of the magnetic axis of the *fd* and *kd* windings. The *kq*1 and *kq*2 windings have N_{kq1} and N_{kq2} equivalent turns, respectively, with resistances r_{kq1} and r_{kq2}. It is assumed that all rotor windings are sinusoidally distributed. The synchronous machine depicted in Fig. 5.2-1 differs from the one shown in Fig. 1.5-1 in that damper windings are included in Fig. 5.2-1. Also, the assumed direction of positive stator currents is out of the terminals convenient to describe generator action. Hence, with the assumed positive direction of the magnetic axes, negative flux linkages result due to positive stator currents.

In Fig. 5.2-1 the magnetic axes of the stator windings are denoted by the *as*, *bs*, and *cs* axes. This notation was also used for the stator windings of the induction machine. The quadrature axis (*q* axis) and direct axis (*d* axis) are introduced in Fig. 5.2-1. The *q* axis is the magnetic axis of the *kq*1 and *kq*2 windings while the *d* axis is the magnetic axis of the *fd* and *kd* windings. This use of the *q* and *d* axes was in existence prior to Park's work [1] and as mentioned in Chap. 3, Park used the notation of f_q, f_d, and f_0 in his transformation. Perhaps he made this choice

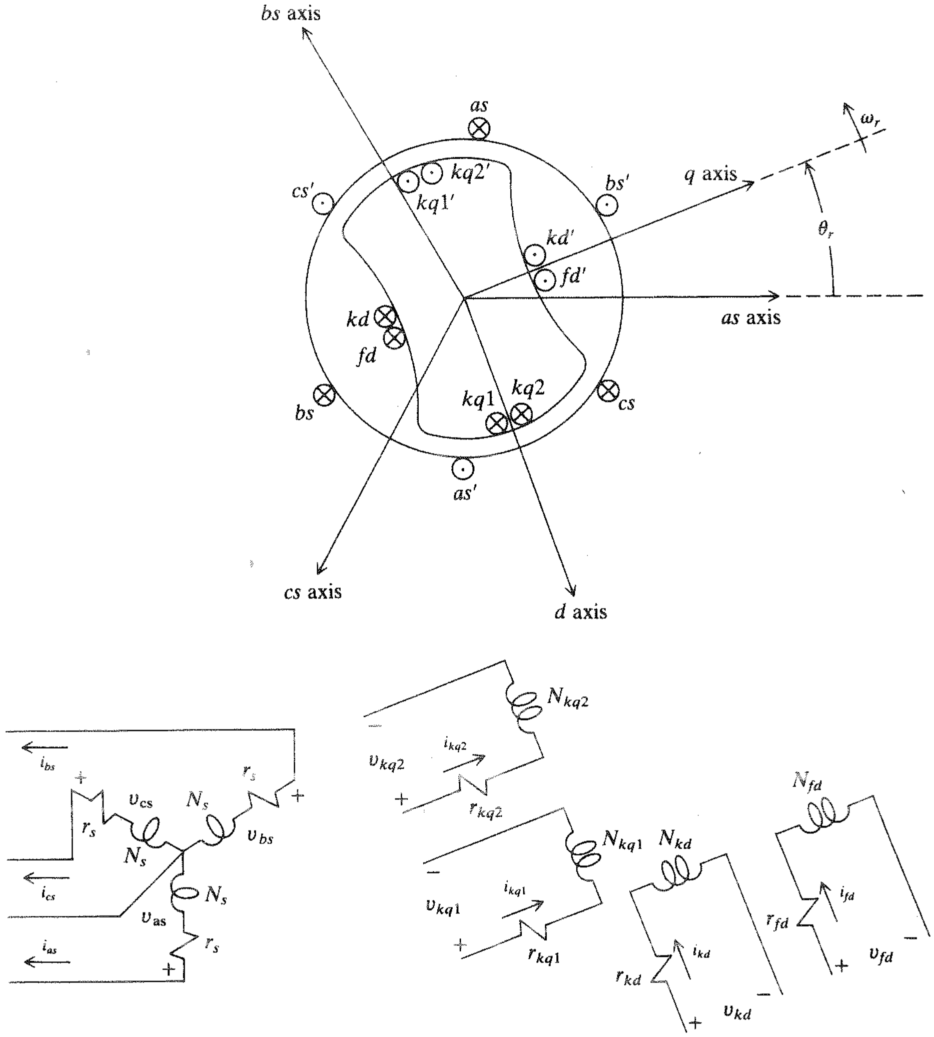

Figure 5.2-1 Two-pole, 3-phase, wye-connected salient-pole synchronous machine.

of notation since, in effect, this transformation referred the stator variables to the rotor where the traditional q and d axes are located.

We have used f_{qs}, f_{ds}, and f_{0s}, and f'_{qr}, f'_{dr}, and f'_{0r} to denote transformed induction machine variables without introducing the connotation of a q or d axis. Instead, the q and d axes have been reserved to denote the rotor magnetic axes of the synchronous machine where they have an established physical meaning quite independent of any transformation. For this reason, one may argue that the q and d subscripts should not be used to denote the transformation to the arbitrary reference frame. Indeed, this line of reasoning has merit, however, since the transformation to the arbitrary reference frame is in essence a generalization of Park's transformation, the q and d subscripts have been selected for use in the transformation to the arbitrary reference primarily out of respect for Park's work which is the basis of it all.

Although the damper windings are shown with provisions to apply a voltage they are, in fact, short-circuited windings which represent the paths for induced rotor currents. Currents may flow in either cage-type windings similar to the squirrel-cage windings of induction machines or in the actual iron of the rotor. In salient-pole machines at least the rotor surface is laminated and the damping winding currents are confined, for the most part, to the cage windings embedded in the rotor surface. In the high-speed, 2- or 4-pole machines the rotor is cylindrical, made of solid iron with a cage-type winding embedded in the rotor surface. Here currents can flow either in the cage winding or in the solid iron.

The performance of nearly all types of synchronous machines may be adequately described by straightforward modifications of the equations describing the performance of the machine shown in Fig. 5.2-1. For example, the behavior of low-speed hydro turbine generators, which are always salient-pole machines, is generally predicted sufficiently by one equivalent damper winding in the q axis. Hence, the performance of this type of machine may be described from the equations derived for the machine shown in Fig. 5.2-1 by eliminating all terms involving one of the kq windings. The reluctance machine, which has no field winding and generally only one damper winding in the q axis, may be described by eliminating the terms involving the fd winding and one of the kq windings. In solid iron rotor, steam turbine generators the magnetic characteristics of the q and d axes are identical, or nearly so, hence the inductances associated with the two axes are essentially the same. Also it is necessary, in most cases, to include all three damper windings in order to portray adequately the transient characteristics of the stator variables and the electromagnetic torque of solid iron rotor machines [2].

Since the synchronous machine is generally operated as a generator it is convenient to assume that the direction of positive stator current is out of the terminals as shown in Fig. 5.2-1. With this convention the voltage equations in machine variables may be expressed in matrix form as

$$\mathbf{v}_{abcs} = -\mathbf{r}_s\mathbf{i}_{abcs} + p\boldsymbol{\lambda}_{abcs} \tag{5.2-1}$$

$$\mathbf{v}_{qdr} = \mathbf{r}_r\mathbf{i}_{qdr} + p\boldsymbol{\lambda}_{qdr} \tag{5.2-2}$$

where

$$(\mathbf{f}_{abcs})^T = [f_{as} \quad f_{bs} \quad f_{cs}] \tag{5.2-3}$$

$$(\mathbf{f}_{qdr})^T = [f_{kq1} \quad f_{kq2} \quad f_{fd} \quad f_{kd}] \tag{5.2-4}$$

In the above equations the s and r subscripts denote variables associated with the stator and rotor windings, respectively. Both $\mathbf{r}_s$ and $\mathbf{r}_r$ are diagonal matrices, in particular

$$\mathbf{r}_s = \text{diag}\,[r_s \quad r_s \quad r_s] \tag{5.2-5}$$

$$\mathbf{r}_r = \text{diag}\,[r_{kq1} \quad r_{kq2} \quad r_{fd} \quad r_{kd}] \tag{5.2-6}$$

In Fig. 5.2-1 the positive *as*, *bs*, and *cs* axes are drawn in the direction of negative flux linkages relative to the assumed positive direction of stator currents. In this case, the flux linkage equations become

$$\begin{bmatrix} \boldsymbol{\lambda}_{abcs} \\ \boldsymbol{\lambda}_{qdr} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_s & \mathbf{L}_{sr} \\ (\mathbf{L}_{sr})^T & \mathbf{L}_r \end{bmatrix} \begin{bmatrix} -\mathbf{i}_{abcs} \\ \mathbf{i}_{qdr} \end{bmatrix} \tag{5.2-7}$$

From the work in Sec. 1.5 we can write $\mathbf{L}_s$ as

$$\mathbf{L}_s = \begin{bmatrix} L_{ls} + L_A - L_B \cos 2\theta_r & -\frac{1}{2}L_A - L_B \cos 2\left(\theta_r - \frac{\pi}{3}\right) & -\frac{1}{2}L_A - L_B \cos 2\left(\theta_r + \frac{\pi}{3}\right) \\ -\frac{1}{2}L_A - L_B \cos 2\left(\theta_r - \frac{\pi}{3}\right) & L_{ls} + L_A - L_B \cos 2\left(\theta_r - \frac{2\pi}{3}\right) & -\frac{1}{2}L_A - L_B \cos 2(\theta_r + \pi) \\ -\frac{1}{2}L_A - L_B \cos 2\left(\theta_r + \frac{\pi}{3}\right) & -\frac{1}{2}L_A - L_B \cos 2(\theta_r + \pi) & L_{ls} + L_A - L_B \cos 2\left(\theta_r + \frac{2\pi}{3}\right) \end{bmatrix} \tag{5.2-8}$$

By a straightforward extension of the work in Sec. 1.5 we can express the self- and mutual inductances of the damper windings. The inductance matrices $\mathbf{L}_{sr}$ and $\mathbf{L}_r$ may then be expressed

$$\mathbf{L}_{sr} = \begin{bmatrix} L_{skq1} \cos \theta_r & L_{skq2} \cos \theta_r & L_{sfd} \sin \theta_r & L_{skd} \sin \theta_r \\ L_{skq1} \cos\left(\theta_r - \frac{2\pi}{3}\right) & L_{skq2} \cos\left(\theta_r - \frac{2\pi}{3}\right) & L_{sfd} \sin\left(\theta_r - \frac{2\pi}{3}\right) & L_{skd} \sin\left(\theta_r - \frac{2\pi}{3}\right) \\ L_{skq1} \cos\left(\theta_r + \frac{2\pi}{3}\right) & L_{skq2} \cos\left(\theta_r + \frac{2\pi}{3}\right) & L_{sfd} \sin\left(\theta_r + \frac{2\pi}{3}\right) & L_{skd} \sin\left(\theta_r + \frac{2\pi}{3}\right) \end{bmatrix} \tag{5.2-9}$$

$$\mathbf{L}_r = \begin{bmatrix} L_{lkq1} + L_{mkq1} & L_{kq1kq2} & 0 & 0 \\ L_{kq1kq2} & L_{lkq2} + L_{mkq2} & 0 & 0 \\ 0 & 0 & L_{lfd} + L_{mfd} & L_{fdkd} \\ 0 & 0 & L_{fdkd} & L_{lkd} + L_{mkd} \end{bmatrix} \tag{5.2-10}$$

In (5.2-8) $L_A > L_B$ and L_B is zero for a round rotor machine. Also in (5.2-8) and (5.2-10), the leakage inductances are denoted with l in the subscript. The subscripts *skq*1, *skq*2, *sfd*, and *skd* in (5.2-9) denote mutual inductances between stator and rotor windings.

The magnetizing inductances are defined as

$$L_{mq} = \tfrac{3}{2}(L_A - L_B) \tag{5.2-11}$$

$$L_{md} = \tfrac{3}{2}(L_A + L_B) \tag{5.2-12}$$

whereupon it is easy to show that (Sec. 1.5)

$$L_{skq1} = \left(\frac{N_{kq1}}{N_s}\right)\left(\frac{2}{3}\right)L_{mq} \tag{5.2-13}$$

$$L_{skq2} = \left(\frac{N_{kq2}}{N_s}\right)\left(\frac{2}{3}\right)L_{mq} \tag{5.2-14}$$

$$L_{sfd} = \left(\frac{N_{fd}}{N_s}\right)\left(\frac{2}{3}\right)L_{md} \tag{5.2-15}$$

$$L_{skd} = \left(\frac{N_{kd}}{N_s}\right)\left(\frac{2}{3}\right)L_{md} \tag{5.2-16}$$

$$L_{mkq1} = \left(\frac{N_{kq1}}{N_s}\right)^2\left(\frac{2}{3}\right)L_{mq} \tag{5.2-17}$$

$$L_{mkq2} = \left(\frac{N_{kq2}}{N_s}\right)^2\left(\frac{2}{3}\right)L_{mq} \tag{5.2-18}$$

$$L_{mfd} = \left(\frac{N_{fd}}{N_s}\right)^2\left(\frac{2}{3}\right)L_{md} \tag{5.2-19}$$

$$L_{mkd} = \left(\frac{N_{kd}}{N_s}\right)^2\left(\frac{2}{3}\right)L_{md} \tag{5.2-20}$$

$$L_{kq1kq2} = \left(\frac{N_{kq2}}{N_{kq1}}\right)L_{mkq1}$$

$$= \left(\frac{N_{kq1}}{N_{kq2}}\right)L_{mkq2} \tag{5.2-21}$$

$$L_{fdkd} = \left(\frac{N_{kd}}{N_{fd}}\right)L_{mfd}$$

$$= \left(\frac{N_{fd}}{N_{kd}}\right)L_{mkd} \tag{5.2-22}$$

It is convenient to incorporate the following substitute variables which refer the rotor variables to the stator windings.

$$i'_j = \left(\frac{2}{3}\right)\left(\frac{N_j}{N_s}\right)i_j \tag{5.2-23}$$

$$v'_j = \left(\frac{N_s}{N_j}\right)v_j \tag{5.2-24}$$

$$\lambda'_j = \left(\frac{N_s}{N_j}\right)\lambda_j \tag{5.2-25}$$

where j may be $kq1$, $kq2$, fd, or kd.

The flux linkages may now be written

$$\begin{bmatrix} \boldsymbol{\lambda}_{abcs} \\ \boldsymbol{\lambda}'_{qdr} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_s & \mathbf{L}'_{sr} \\ \frac{2}{3}(\mathbf{L}'_{sr})^T & \mathbf{L}'_r \end{bmatrix} \begin{bmatrix} -\mathbf{i}_{abcs} \\ \mathbf{i}'_{qdr} \end{bmatrix} \tag{5.2-26}$$

where $\mathbf{L}_s$ is defined by (5.2-8) and

$$\mathbf{L}'_{sr} = \begin{bmatrix} L_{mq}\cos\theta_r & L_{mq}\cos\theta_r & L_{md}\sin\theta_r & L_{md}\sin\theta_r \\ L_{mq}\cos\left(\theta_r - \frac{2\pi}{3}\right) & L_{mq}\cos\left(\theta_r - \frac{2\pi}{3}\right) & L_{md}\sin\left(\theta_r - \frac{2\pi}{3}\right) & L_{md}\sin\left(\theta_r - \frac{2\pi}{3}\right) \\ L_{mq}\cos\left(\theta_r + \frac{2\pi}{3}\right) & L_{mq}\cos\left(\theta_r + \frac{2\pi}{3}\right) & L_{md}\sin\left(\theta_r + \frac{2\pi}{3}\right) & L_{md}\sin\left(\theta_r + \frac{2\pi}{3}\right) \end{bmatrix} \tag{5.2-27}$$

$$\mathbf{L}'_r = \begin{bmatrix} L'_{lkq1} + L_{mq} & L_{mq} & 0 & 0 \\ L_{mq} & L'_{lkq2} + L_{mq} & 0 & 0 \\ 0 & 0 & L'_{lfd} + L_{md} & L_{md} \\ 0 & 0 & L_{md} & L'_{lkd} + L_{md} \end{bmatrix} \tag{5.2-28}$$

The voltage equations expressed in terms of machine variables referred to the stator windings are

$$\begin{bmatrix} \mathbf{v}_{abcs} \\ \mathbf{v}'_{qdr} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_s + p\mathbf{L}_s & p\mathbf{L}'_{sr} \\ \frac{2}{3}p(\mathbf{L}'_{sr})^T & \mathbf{r}'_r + p\mathbf{L}'_r \end{bmatrix} \begin{bmatrix} -\mathbf{i}_{abcs} \\ \mathbf{i}'_{qdr} \end{bmatrix} \tag{5.2-29}$$

In (5.2-28) and (5.2-29)

$$r'_j = \left(\frac{3}{2}\right)\left(\frac{N_s}{N_j}\right)^2 r_j \tag{5.2-30}$$

$$L'_{lj} = \left(\frac{3}{2}\right)\left(\frac{N_s}{N_j}\right)^2 L_{lj} \tag{5.2-31}$$

where, again, j may be $kq1$, $kq2$, fd, or kd.

The voltage equations given in (5.2-29) are valid for the positive direction of stator current assumed out of the stator terminals. If the positive direction of stator current is reversed, making analysis of motor action somewhat more convenient, the only modification to (5.2-29) is a change in the sign preceding $\mathbf{i}_{abcs}$.

5.3 TORQUE EQUATION IN MACHINE VARIABLES

The energy stored in the coupling field of a synchronous machine may be expressed

$$\begin{aligned} W_f = {} & \tfrac{1}{2}(\mathbf{i}_{abcs})^T(\mathbf{L}_s - L_{ls}\mathbf{I})\mathbf{i}_{abcs} - (\mathbf{i}_{abcs})^T\mathbf{L}'_{sr}\mathbf{i}'_{qdr} \\ & + \tfrac{1}{2}(\tfrac{3}{2})(\mathbf{i}'_{qdr})^T(\mathbf{L}'_r - \mathbf{L}'_{lr}\mathbf{I})\mathbf{i}'_{qdr} \end{aligned} \tag{5.3-1}$$

where $\mathbf{I}$ is the identity matrix and

$$\mathbf{L}'_{lr} = \text{diag}\,[L'_{lkq1} \quad L'_{lkq2} \quad L'_{lfd} \quad L'_{lkd}] \tag{5.3-2}$$

Since the magnetic system is assumed to be linear, $W_f = W_c$ and the second entry of Table 1.3-1 may be used with the factor $P/2$ included to account for a P-pole machine (Sec. 4.3) and a negative sign to make T_e positive for generator action. Thus

$$T_e = \left(\frac{P}{2}\right)\left\{-\frac{1}{2}(\mathbf{i}_{abcs})^T \frac{\partial}{\partial\theta_r}[\mathbf{L}_s - L_{ls}\mathbf{I}]\mathbf{i}_{abcs} + (\mathbf{i}_{abcs})^T \frac{\partial}{\partial\theta_r}[\mathbf{L}'_{sr}]\mathbf{i}'_{qdr}\right\} \tag{5.3-3}$$

In expanded form (5.3-3) becomes

$$\begin{aligned} T_e = \left(\frac{P}{2}\right)\Bigg\{-\frac{(L_{md}-L_{mq})}{3}\Bigg[&\left(i_{as}^2 - \frac{1}{2}i_{bs}^2 - \frac{1}{2}i_{cs}^2 - i_{as}i_{bs} - i_{as}i_{cs} + 2i_{bs}i_{cs}\right)\sin 2\theta_r \\ &+ \frac{\sqrt{3}}{2}(i_{bs}^2 + i_{cs}^2 - 2i_{as}i_{bs} + 2i_{as}i_{cs})\cos 2\theta_r\Bigg] \\ - L_{mq}(i'_{kq1} + i'_{kq2})&\left[\left(i_{as} - \frac{1}{2}i_{bs} - \frac{1}{2}i_{cs}\right)\sin\theta_r - \frac{\sqrt{3}}{2}(i_{bs} - i_{cs})\cos\theta_r\right] \\ + L_{md}(i'_{fd} + i'_{kd})&\left[\left(i_{as} - \frac{1}{2}i_{bs} - \frac{1}{2}i_{cs}\right)\cos\theta_r + \frac{\sqrt{3}}{2}(i_{bs} - i_{cs})\sin\theta_r\right]\Bigg\} \end{aligned} \tag{5.3-4}$$

The above expression for torque is positive for generator action with the positive direction of stator current assumed out of the stator terminals. If the positive direction of stator current is assumed into the stator terminals then the above expression for torque is positive for motor action if the sign of the coefficients of $\sin 2\theta_r$ and $\cos 2\theta_r$ are changed. This, of course, can be accomplished by changing the sign preceding the multiplier $(L_{md} - L_{mq})$.

The torque and rotor speed are related by

$$T_e = -J\left(\frac{2}{P}\right)p\omega_r + T_I \tag{5.3-5}$$

where J is the inertia expressed in kilogram meters² (kg·m²) or Joule seconds² (J·s²). Often the inertia is given as WR^2 in units of pound mass feet² (lbm·ft²). The input torque T_I is positive for a torque input to the shaft of the synchronous machine.

5.4 STATOR VOLTAGE EQUATIONS IN ARBITRARY REFERENCE-FRAME VARIABLES

The voltage equations of the stator windings of a synchronous machine can be expressed in the arbitrary reference frame. In particular, by using the results presented in Chap. 3, the voltage equations for the stator windings may be written in the arbitrary reference frame as [3]

$$\mathbf{v}_{qd0s} = -\mathbf{r}_s \mathbf{i}_{qd0s} + \omega \boldsymbol{\lambda}_{dqs} + p\boldsymbol{\lambda}_{qd0s} \tag{5.4-1}$$

where

$$(\boldsymbol{\lambda}_{dqs})^T = [\lambda_{ds} \quad -\lambda_{qs} \quad 0] \tag{5.4-2}$$

The only restriction on (5.4-1) is that the resistance of each phase is the same, otherwise, the first term on the right-hand side of (5.4-1) would be written as in (3.4-2).

The rotor windings of a synchronous machine are different; therefore, the change of variables set forth in Sec. 4.4 offers no advantage in the analysis of the rotor circuits. Since the rotor variables are not transformed, the rotor voltage equations are expressed only in the rotor reference frame. Hence, from (5.2-2) with the appropriate turns ratios included and the raised index r used to denote the rotor reference frame, the rotor voltage equations are

$$\mathbf{v}'^r_{qdr} = \mathbf{r}'_r \mathbf{i}'^r_{qdr} + p\boldsymbol{\lambda}'^r_{qdr} \tag{5.4-3}$$

For a linear magnetic system, the flux linkage equations may be expressed from (5.2-7) with the transformation of the stator variables to the arbitrary reference frame incorporated

$$\begin{bmatrix} \boldsymbol{\lambda}_{qd0s} \\ \boldsymbol{\lambda}'^r_{qdr} \end{bmatrix} = \begin{bmatrix} \mathbf{K}_s \mathbf{L}_s (\mathbf{K}_s)^{-1} & \mathbf{K}_s \mathbf{L}'_{sr} \\ \frac{2}{3}(\mathbf{L}'_{sr})^T (\mathbf{K}_s)^{-1} & \mathbf{L}'_r \end{bmatrix} \begin{bmatrix} -\mathbf{i}_{qd0s} \\ \mathbf{i}'^r_{qdr} \end{bmatrix} \tag{5.4-4}$$

It can be shown that all terms of the inductance matrix of (5.4-4) are sinusoidal in nature except $\mathbf{L}'_r$. For example, by using trigonometric identities given in App. A

$$\mathbf{K}_s \mathbf{L}'_{sr} = \begin{bmatrix} L_{mq} \cos(\theta - \theta_r) & L_{mq} \cos(\theta - \theta_r) & -L_{md} \sin(\theta - \theta_r) & -L_{md} \sin(\theta - \theta_r) \\ L_{mq} \sin(\theta - \theta_r) & L_{mq} \sin(\theta - \theta_r) & L_{md} \cos(\theta - \theta_r) & L_{md} \cos(\theta - \theta_r) \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.4-5}$$

The sinusoidal terms of (5.4-5) are constant, independent of ω and ω_r only if $\omega = \omega_r$. Similarly, $\mathbf{K}_s \mathbf{L}_s (\mathbf{K}_s)^{-1}$ and $\frac{2}{3}(\mathbf{L}'_{sr})^T (\mathbf{K}_s)^{-1}$ are constant only if $\omega = \omega_r$. Therefore, the time-varying inductances are eliminated from the voltage equations only if the reference frame is fixed in the rotor. Hence, it would appear that only the rotor reference frame is useful in the analysis of synchronous machines. Although this is essentially the case, there are situations, especially in computer simulations, where it is convenient to express the stator voltage equations in a reference frame other than the one fixed in the rotor. For these applications it is necessary to relate the arbitrary reference-frame variables to the variables in the rotor reference frame. This may be accomplished by using (3.6-1), from which

$$\mathbf{f}^r_{qd0s} = {}^r\mathbf{K}\,\mathbf{f}_{qd0s} \tag{5.4-6}$$

From (3.6-7)

$$\mathbf{K}^r = \begin{bmatrix} \cos(\theta_r - \theta) & -\sin(\theta_r - \theta) & 0 \\ \sin(\theta_r - \theta) & \cos(\theta_r - \theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.4-7}$$

Here we must again recall that the arbitrary reference does not carry a raised index.

5.5 VOLTAGE EQUATIONS IN ROTOR REFERENCE-FRAME VARIABLES—PARK'S EQUATIONS

R. H. Park was the first to incorporate a change of variables in the analysis of synchronous machines [1]. He transformed the stator variables to the rotor reference frame which eliminates the time-varying inductances in the voltage equations. Park's equations are obtained from (5.4-1) and (5.4-3) by setting the speed of the arbitrary reference frame equal to the rotor speed ($\omega = \omega_r$). Thus

$$\mathbf{v}^r_{qd0s} = -\mathbf{r}_s \mathbf{i}^r_{qd0s} + \omega_r \boldsymbol{\lambda}^r_{dqs} + p\boldsymbol{\lambda}^r_{qd0s} \tag{5.5-1}$$

$$\mathbf{v}'^r_{qdr} = \mathbf{r}'_r \mathbf{i}'^r_{qdr} + p\boldsymbol{\lambda}'^r_{qdr} \tag{5.5-2}$$

where

$$(\boldsymbol{\lambda}^r_{dqs})^T = [\lambda^r_{ds} \quad -\lambda^r_{qs} \quad 0] \tag{5.5-3}$$

For a magnetically linear system, the flux linkages may be expressed in the rotor reference frame from (5.4-4) by setting $\theta = \theta_r$ and whereupon $\mathbf{K}_s$ becomes $\mathbf{K}^r_s$. Thus,

$$\begin{bmatrix} \boldsymbol{\lambda}^r_{qd0s} \\ \boldsymbol{\lambda}'^r_{qdr} \end{bmatrix} = \begin{bmatrix} \mathbf{K}^r_s \mathbf{L}_s (\mathbf{K}^r_s)^{-1} & \mathbf{K}^r_s \mathbf{L}'_{sr} \\ \frac{2}{3}(\mathbf{L}'_{sr})^T (\mathbf{K}^r_s)^{-1} & \mathbf{L}'_r \end{bmatrix} \begin{bmatrix} -\mathbf{i}^r_{qd0s} \\ \mathbf{i}'^r_{qdr} \end{bmatrix} \tag{5.5-4}$$

Using trigonometric identities from App. A, it can be shown that

$$\mathbf{K}^r_s \mathbf{L}_s (\mathbf{K}^r_s)^{-1} = \begin{bmatrix} L_{ls} + L_{mq} & 0 & 0 \\ 0 & L_{ls} + L_{md} & 0 \\ 0 & 0 & L_{ls} \end{bmatrix} \tag{5.5-5}$$

$$\mathbf{K}^r_s \mathbf{L}'_{sr} = \begin{bmatrix} L_{mq} & L_{mq} & 0 & 0 \\ 0 & 0 & L_{md} & L_{md} \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.5-6}$$

$$\frac{2}{3}(\mathbf{L}'_{sr})^T (\mathbf{K}^r_s)^{-1} = \begin{bmatrix} L_{mq} & 0 & 0 \\ L_{mq} & 0 & 0 \\ 0 & L_{md} & 0 \\ 0 & L_{md} & 0 \end{bmatrix} \tag{5.5-7}$$

Park's voltage equations are often written in expanded form, thus from (5.5-1) and (5.5-2)

$$v^r_{qs} = -r_s i^r_{qs} + \omega_r \lambda^r_{ds} + p\lambda^r_{qs} \tag{5.5-8}$$

$$v_{ds}^r = -r_s i_{ds}^r - \omega_r \lambda_{qs}^r + p\lambda_{ds}^r \quad (5.5\text{-}9)$$

$$v_{0s} = -r_s i_{0s} + p\lambda_{0s} \quad (5.5\text{-}10)$$

$$v_{kq1}'^r = r_{kq1}' i_{kq1}'^r + p\lambda_{kq1}'^r \quad (5.5\text{-}11)$$

$$v_{kq2}'^r = r_{kq2}' i_{kq2}'^r + p\lambda_{kq2}'^r \quad (5.5\text{-}12)$$

$$v_{fd}'^r = r_{fd}' i_{fd}'^r + p\lambda_{fd}'^r \quad (5.5\text{-}13)$$

$$v_{kd}'^r = r_{kd}' i_{kd}'^r + p\lambda_{kd}'^r \quad (5.5\text{-}14)$$

Substituting (5.5-5)–(5.5-7) and (5.2-28) into (5.5-4) yields the expressions for the flux linkages. In expanded form

$$\lambda_{qs}^r = -L_{ls} i_{qs}^r + L_{mq}(-i_{qs}^r + i_{kq1}'^r + i_{kq2}'^r) \quad (5.5\text{-}15)$$

$$\lambda_{ds}^r = -L_{ls} i_{ds}^r + L_{md}(-i_{ds}^r + i_{kd}'^r + i_{fd}'^r) \quad (5.5\text{-}16)$$

$$\lambda_{0s} = -L_{ls} i_{0s} \quad (5.5\text{-}17)$$

$$\lambda_{kq1}'^r = L_{lkq1}' i_{kq1}'^r + L_{mq}(-i_{qs}^r + i_{kq1}'^r + i_{kq2}'^r) \quad (5.5\text{-}18)$$

$$\lambda_{kq2}'^r = L_{lkq2}' i_{kq2}'^r + L_{mq}(-i_{qs}^r + i_{kq1}'^r + i_{kq2}'^r) \quad (5.5\text{-}19)$$

$$\lambda_{fd}'^r = L_{lfd}' i_{fd}'^r + L_{md}(-i_{ds}^r + i_{fd}'^r + i_{kd}'^r) \quad (5.5\text{-}20)$$

$$\lambda_{kd}'^r = L_{lkd}' i_{kd}'^r + L_{md}(-i_{ds}^r + i_{fd}'^r + i_{kd}'^r) \quad (5.5\text{-}21)$$

The voltage and flux linkage equations suggest the equivalent circuits shown in Fig. 5.5-1.

As in the case of the induction machine, it is often convenient to express the voltage and flux linkage equations in terms of reactances rather than inductances. Hence, (5.5-8)–(5.5-14) are often written

$$v_{qs}^r = -r_s i_{qs}^r + \frac{\omega_r}{\omega_b} \psi_{ds}^r + \frac{p}{\omega_b} \psi_{qs}^r \quad (5.5\text{-}22)$$

$$v_{ds}^r = -r_s i_{ds}^r - \frac{\omega_r}{\omega_b} \psi_{qs}^r + \frac{p}{\omega_b} \psi_{ds}^r \quad (5.5\text{-}23)$$

$$v_{0s} = -r_s i_{0s} + \frac{p}{\omega_b} \psi_{0s} \quad (5.5\text{-}24)$$

$$v_{kq1}'^r = r_{kq1}' i_{kq1}'^r + \frac{p}{\omega_b} \psi_{kq1}'^r \quad (5.5\text{-}25)$$

$$v_{kq2}'^r = r_{kq2}' i_{kq2}'^r + \frac{p}{\omega_b} \psi_{kq2}'^r \quad (5.5\text{-}26)$$

$$v_{fd}'^r = r_{fd}' i_{fd}'^r + \frac{p}{\omega_b} \psi_{fd}'^r \quad (5.5\text{-}27)$$

$$v_{kd}'^r = r_{kd}' i_{kd}'^r + \frac{p}{\omega_b} \psi_{kd}'^r \quad (5.5\text{-}28)$$

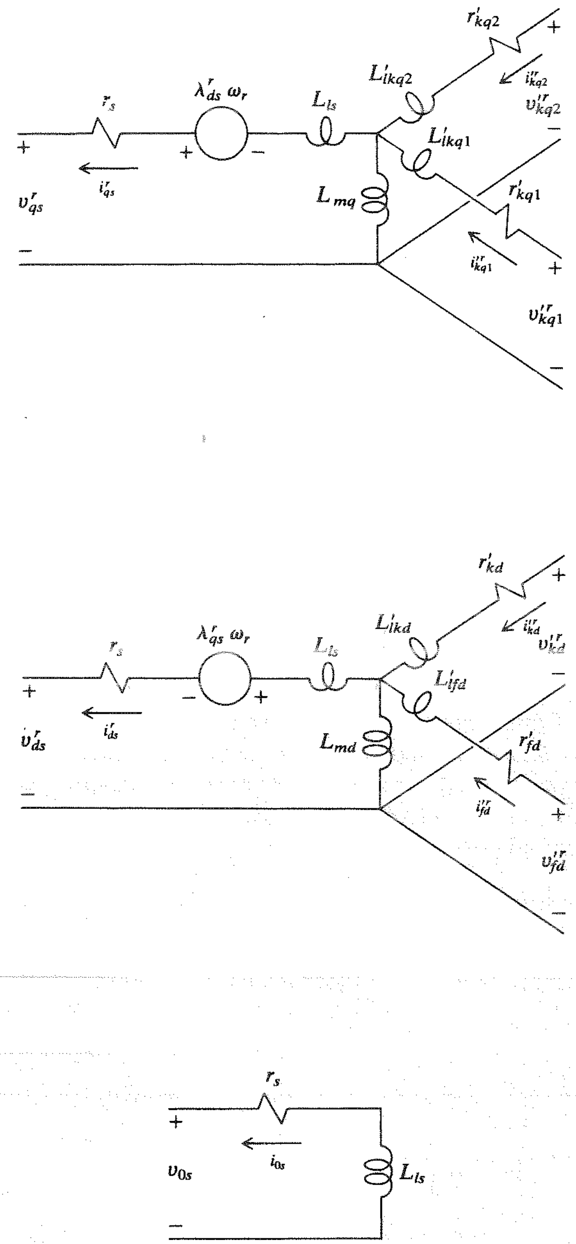

Figure 5.5-1 Equivalent circuits of a 3-phase synchronous machine with reference frame fixed in rotor—Park's equations.

where ω_b is the base electrical angular velocity used to calculate the inductive reactances. The flux linkages per second are

$$\psi_{qs}^r = -X_{ls}i_{qs}^r + X_{mq}(-i_{qs}^r + i_{kq1}'^r + i_{kq2}'^r) \tag{5.5-29}$$

$$\psi_{ds}^r = -X_{ls}i_{ds}^r + X_{md}(-i_{ds}^r + i_{fd}'^r + i_{kd}'^r) \tag{5.5-30}$$

$$\psi_{0s} = -X_{ls}i_{0s} \tag{5.5-31}$$

$$\psi_{kq1}'^r = X_{lkq1}'i_{kq1}'^r + X_{mq}(-i_{qs}^r + i_{kq1}'^r + i_{kq2}'^r) \tag{5.5-32}$$

$$\psi_{kq2}'^r = X_{lkq2}'i_{kq2}'^r + X_{mq}(-i_{qs}^r + i_{kq1}'^r + i_{kq2}'^r) \tag{5.5-33}$$

$$\psi_{fd}'^r = X_{lfd}'i_{fd}'^r + X_{md}(-i_{ds}^r + i_{fd}'^r + i_{kd}'^r) \tag{5.5-34}$$

$$\psi_{kd}'^r = X_{lkd}'i_{kd}'^r + X_{md}(-i_{ds}^r + i_{fd}'^r + i_{kd}'^r) \tag{5.5-35}$$

Park's equations are generally written without the superscript r, the subscript s, and the primes which denote referred quantities. Also, we will later find that it is convenient to define

$$e_{xfd}'^r = v_{fd}'^r \frac{X_{md}}{r_{fd}'} \tag{5.5-36}$$

and to substitute this relationship into the expression for field voltage so that (5.5-27) becomes

$$e_{xfd}'^r = \frac{X_{md}}{r_{fd}'}\left(r_{fd}'i_{fd}'^r + \frac{p}{\omega_b}\psi_{fd}'^r\right) \tag{5.5-37}$$

The voltages equations for the synchronous machine given thus far are in terms of currents and flux linkages or flux linkages per second. As we have pointed out earlier the current and flux linkages are related and both cannot be independent or state variables. As in the case of the induction machine we will need to express the voltage equations in terms of either currrents or flux linkages (flux linkages per second) when formulating transfer functions and implementing a computer simulation.

If we select the currents as independent variables the flux linkages (flux linkages per second) are replaced by currents and the voltage equations given by (5.5-22)–(5.5-28) with (5.5-37) used instead of (5.5-27) become (5.5-38) where

$$X_q = X_{ls} + X_{mq} \tag{5.5-39}$$

$$X_d = X_{ls} + X_{md} \tag{5.5-40}$$

$$X_{kq1}' = X_{lkq1}' + X_{mq} \tag{5.5-41}$$

$$X_{kq2}' = X_{lkq2}' + X_{mq} \tag{5.5-42}$$

$$X_{fd}' = X_{lfd}' + X_{md} \tag{5.5-43}$$

$$X_{kd}' = X_{lkd}' + X_{md} \tag{5.5-44}$$

$$
\begin{bmatrix} v_{qs}^r \\ v_{ds}^r \\ v_{0s} \\ v_{kq1}'^r \\ v_{kq2}'^r \\ e_{xkd}'^r \\ v_{kd}'^r \end{bmatrix}
=
\begin{bmatrix}
-r_s - \frac{p}{\omega_b} X_q & -\frac{\omega_r}{\omega_b} X_d & 0 & \frac{p}{\omega_b} X_{mq} & \frac{p}{\omega_b} X_{mq} & \frac{\omega_r}{\omega_b} X_{md} & \frac{\omega_r}{\omega_b} X_{md} \\
\frac{\omega_r}{\omega_b} X_q & -r_s - \frac{p}{\omega_b} X_d & 0 & -\frac{\omega_r}{\omega_b} X_{mq} & -\frac{\omega_r}{\omega_b} X_{mq} & \frac{p}{\omega_b} X_{md} & \frac{p}{\omega_b} X_{md} \\
0 & 0 & -r_s - \frac{p}{\omega_b} X_{ls} & 0 & 0 & 0 & 0 \\
-\frac{p}{\omega_b} X_{mq} & 0 & 0 & r_{kq1}' + \frac{p}{\omega_b} X_{kq1}' & \frac{p}{\omega_b} X_{mq} & 0 & 0 \\
-\frac{p}{\omega_b} X_{mq} & 0 & 0 & \frac{p}{\omega_b} X_{mq} & r_{kq2}' + \frac{p}{\omega_b} X_{kq2}' & 0 & 0 \\
0 & -\frac{X_{md}}{r_{fd}'}\left(\frac{p}{\omega_b} X_{md}\right) & 0 & 0 & 0 & \frac{X_{md}}{r_{fd}'}\left(r_{fd}' + \frac{p}{\omega_b} X_{fd}'\right) & \frac{X_{md}}{r_{fd}'}\left(\frac{p}{\omega_b} X_{md}\right) \\
0 & -\frac{p}{\omega_b} X_{md} & 0 & 0 & 0 & \frac{p}{\omega_b} X_{md} & r_{kd}' + \frac{p}{\omega_b} X_{kd}'
\end{bmatrix}
\begin{bmatrix} i_{qs}^r \\ i_{ds}^r \\ i_{0s} \\ i_{kq1}'^r \\ i_{kq2}'^r \\ i_{fd}'^r \\ i_{kd}'^r \end{bmatrix}
\tag{5.5-38}
$$

The reactances X_q and X_d are generally referred to as q- and d-axis reactances, respectively. The flux linkages per second may be expressed from (5.5-29)–(5.5-35) as

$$\begin{bmatrix} \psi_{qs}^r \\ \psi_{ds}^r \\ \psi_{0s} \\ \psi_{kq1}'^r \\ \psi_{kq2}'^r \\ \psi_{fd}'^r \\ \psi_{kd}'^r \end{bmatrix} = \begin{bmatrix} -X_q & 0 & 0 & X_{mq} & X_{mq} & 0 & 0 \\ 0 & -X_d & 0 & 0 & 0 & X_{md} & X_{md} \\ 0 & 0 & X_{ls} & 0 & 0 & 0 & 0 \\ -X_{mq} & 0 & 0 & X_{kq1}' & X_{mq} & 0 & 0 \\ -X_{mq} & 0 & 0 & X_{mq} & X_{kq2}' & 0 & 0 \\ 0 & -X_{md} & 0 & 0 & 0 & X_{fd}' & X_{md} \\ 0 & -X_{md} & 0 & 0 & 0 & X_{md} & X_{kd}' \end{bmatrix} \begin{bmatrix} i_{qs}^r \\ i_{ds}^r \\ i_{0s} \\ i_{kq1}'^r \\ i_{kq2}'^r \\ i_{fd}'^r \\ i_{kd}'^r \end{bmatrix} \tag{5.5-45}$$

If flux linkages or flux linkages per second are selected as independent variables it is convenient to first express (5.5-45) as

$$\begin{bmatrix} \psi_{qs}^r \\ \psi_{kq1}'^r \\ \psi_{kq2}'^r \end{bmatrix} = \begin{bmatrix} -X_q & X_{mq} & X_{mq} \\ -X_{mq} & X_{kq1}' & X_{mq} \\ -X_{mq} & X_{mq} & X_{kq2}' \end{bmatrix} \begin{bmatrix} i_{qs}^r \\ i_{kq1}'^r \\ i_{kq2}'^r \end{bmatrix} \tag{5.5-46}$$

$$\begin{bmatrix} \psi_{ds}^r \\ \psi_{fd}'^r \\ \psi_{kd}'^r \end{bmatrix} = \begin{bmatrix} -X_d & X_{md} & X_{md} \\ -X_{md} & X_{fd}' & X_{md} \\ -X_{md} & X_{md} & X_{kd}' \end{bmatrix} \begin{bmatrix} i_{ds}^r \\ i_{fd}'^r \\ i_{kd}'^r \end{bmatrix} \tag{5.5-47}$$

$$\psi_{0s} = X_{ls} i_{0s} \tag{5.5-48}$$

Solving the above equations for currents yields

$$\begin{bmatrix} i_{qs}^r \\ i_{kq1}'^r \\ i_{kq2}'^r \end{bmatrix} = \frac{1}{D_q} \begin{bmatrix} X_{kq1}'X_{kq2}' - X_{mq}^2 & -X_{mq}X_{kq2}' + X_{mq}^2 & -X_{mq}X_{kq1}' + X_{mq}^2 \\ X_{mq}X_{kq2}' - X_{mq}^2 & -X_qX_{kq2}' + X_{mq}^2 & X_qX_{mq} - X_{mq}^2 \\ X_{mq}X_{kq1}' - X_{mq}^2 & X_qX_{mq} - X_{mq}^2 & -X_qX_{kq1}' + X_{mq}^2 \end{bmatrix} \begin{bmatrix} \psi_{qs}^r \\ \psi_{kq1}'^r \\ \psi_{kq2}'^r \end{bmatrix} \tag{5.5-49}$$

$$\begin{bmatrix} i_{ds}^r \\ i_{fd}'^r \\ i_{kd}'^r \end{bmatrix} = \frac{1}{D_d} \begin{bmatrix} X_{fd}'X_{kd}' - X_{md}^2 & -X_{md}X_{kd}' + X_{md}^2 & -X_{md}X_{fd}' + X_{md}^2 \\ X_{md}X_{kd}' - X_{md}^2 & -X_dX_{kd}' + X_{md}^2 & X_dX_{md} - X_{md}^2 \\ X_{md}X_{fd}' - X_{md}^2 & X_dX_{md} - X_{md}^2 & -X_dX_{fd}' + X_{md}^2 \end{bmatrix} \begin{bmatrix} \psi_{ds}^r \\ \psi_{fd}'^r \\ \psi_{kd}'^r \end{bmatrix} \tag{5.5-50}$$

$$i_{0s} = \frac{1}{X_{ls}} \psi_{0s} \tag{5.5-51}$$

where

$$D_q = X_{mq}^2(X_q - 2X_{mq} + X_{kq1}' + X_{kq2}') - X_qX_{kq1}'X_{kq2}' \tag{5.5-52}$$

$$D_d = X_{md}^2(X_d - 2X_{md} + X_{fd}' + X_{kd}') - X_dX_{fd}'X_{kd}' \tag{5.5-53}$$

Substituting (5.5-49)–(5.5-51) for the currents into the voltage equations (5.5-22)–(5.5-26), (5.5-37), and (5.5-38) yields (5.5-54). In this equation a_{ij} and b_{ij}

$$
\begin{bmatrix} v_{qs}^r \\ v_{ds}^r \\ v_{0s} \\ v_{kq1}'^r \\ v_{kq2}'^r \\ e_{xfd}'^r \\ v_{kd}'^r \end{bmatrix} = \begin{bmatrix} -r_s a_{11} + \frac{p}{\omega_b} & \frac{\omega_r}{\omega_b} & 0 & -r_s a_{12} & -r_s a_{13} & 0 & 0 \\ -\frac{\omega_r}{\omega_b} & -r_s b_{11} + \frac{p}{\omega_b} & 0 & 0 & 0 & -r_s b_{12} & -r_s b_{13} \\ 0 & 0 & -\frac{r_s}{X_{ls}} + \frac{p}{\omega_b} & 0 & 0 & 0 & 0 \\ r_{kq1}' a_{21} & 0 & 0 & r_{kq1}' a_{22} + \frac{p}{\omega_b} & r_{kq1}' a_{23} & 0 & 0 \\ r_{kq2}' a_{31} & 0 & 0 & r_{kq2}' a_{32} & r_{kq2}' a_{33} + \frac{p}{\omega_b} & 0 & 0 \\ 0 & X_{md} b_{21} & 0 & 0 & 0 & X_{md} b_{22} + \frac{X_{md}}{r_{fd}'}\frac{p}{\omega_b} & X_{md} b_{23} \\ 0 & r_{kd}' b_{31} & 0 & 0 & 0 & r_{kd}' b_{32} & r_{kd}' b_{33} + \frac{p}{\omega_b} \end{bmatrix} \begin{bmatrix} \psi_{qs}^r \\ \psi_{ds}^r \\ \psi_{0s} \\ \psi_{kq1}'^r \\ \psi_{kq2}'^r \\ \psi_{fd}'^r \\ \psi_{kd}'^r \end{bmatrix} \qquad (5.5\text{-}54)
$$

are the elements of the 3×3 matrices given in (5.5-49) and (5.5-50), respectively. The statements at the end of Sec. 4.5 regarding the computer simulation of induction machines also apply in the case of synchronous machines.

In some applications, especially in variable-frequency drive systems, the synchronous machine is operated as a motor. When analyzing motor operation it may be desirable to write the equations with the direction of positive stator current into the stator terminals. The equations are modified to accommodate this change in assumed direction of positive current by simply changing the sign of i_{qs}^r, i_{ds}^r, and i_{0s} in all voltage and flux linkage equations.

5.6 TORQUE EQUATION IN SUBSTITUTE VARIABLES

The expression for the electromagnetic torque in terms of rotor reference-frame variables may be obtained by substituting the equation of transformation into (5.3-3). Hence

$$T_e = \left(\frac{P}{2}\right)[(\mathbf{K}_s^r)^{-1}\mathbf{i}_{qd0s}^r]^T\left\{-\frac{1}{2}\frac{\partial}{\partial\theta_r}[\mathbf{L}_s - L_{ls}\mathbf{I}](\mathbf{K}_s^r)^{-1}\mathbf{i}_{qd0s}^r + \frac{\partial}{\partial\theta_r}[\mathbf{L}_{sr}']\mathbf{i}_{qdr}'^r\right\} \quad (5.6\text{-}1)$$

After considerable work the above equation reduces to

$$T_e = \left(\frac{3}{2}\right)\left(\frac{P}{2}\right)[L_{md}(-i_{ds}^r + i_{fd}'^r + i_{kd}'^r)i_{qs}^r - L_{mq}(-i_{qs}^r + i_{kq1}'^r + i_{kq2}'^r)i_{ds}^r] \quad (5.6\text{-}2)$$

Equation (5.6-2) is equivalent to

$$T_e = \left(\frac{3}{2}\right)\left(\frac{P}{2}\right)(\lambda_{ds}^r i_{qs}^r - \lambda_{qs}^r i_{ds}^r) \quad (5.6\text{-}3)$$

In terms of flux linkages per second and currents

$$T_e = \left(\frac{3}{2}\right)\left(\frac{P}{2}\right)\left(\frac{1}{\omega_b}\right)(\psi_{ds}^r i_{qs}^r - \psi_{qs}^r i_{ds}^r) \quad (5.6\text{-}4)$$

It is left to the reader to show that in terms of flux linkages per second the electromagnetic torque may be expressed

$$T_e = \left(\frac{3}{2}\right)\left(\frac{P}{2}\right)\left(\frac{1}{\omega_b}\right)[(a_{11} - b_{11})\psi_{qs}^r\psi_{ds}^r + \psi_{ds}^r(a_{12}\psi_{kq1}'^r + a_{13}\psi_{kq2}'^r)$$
$$- \psi_{qs}^r(b_{12}\psi_{fd}'^r + b_{13}\psi_{kd}'^r)] \quad (5.6\text{-}5)$$

where a_{ij} and b_{ij} are elements of the 3×3 matrices given in (5.5-49) and (5.5-50), respectively.

The above equations yield positive torque for generator action with the positive direction of stator current assumed out of the stator terminals. If, for the analysis of motor action, the positive direction of stator current is assumed into the stator terminals then the voltage and flux linkage equations are modified as discussed previously. The expression for torque given by (5.6-3), (5.6-4), and

(5.6-5) is then positive for motor action without change; however, (5.6-2) must be modified by changing the sign i_{ds}^r in the first term on the right-hand side and i_{qs}^r in the second term.

The electromagnetic torque expressed with the stator variables in the arbitrary reference frame may be obtained by employing the transformation of the stator variables from the rotor reference frame to the arbitrary reference frame. From (3.6-7)

$$ {}^r\mathbf{K} = \begin{bmatrix} \cos(\theta - \theta_r) & -\sin(\theta - \theta_r) & 0 \\ \sin(\theta - \theta_r) & \cos(\theta - \theta_r) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.6-6} $$

Since only the qs and ds variables are involved in the transformation

$$ \begin{bmatrix} f_{qs} \\ f_{ds} \end{bmatrix} = \begin{bmatrix} \cos(\theta - \theta_r) & -\sin(\theta - \theta_r) \\ \sin(\theta - \theta_r) & \cos(\theta - \theta_r) \end{bmatrix} \begin{bmatrix} f_{qs}^r \\ f_{ds}^r \end{bmatrix} \tag{5.6-7} $$

The inverse is

$$ \begin{bmatrix} f_{qs}^r \\ f_{ds}^r \end{bmatrix} = \begin{bmatrix} \cos(\theta - \theta_r) & \sin(\theta - \theta_r) \\ -\sin(\theta - \theta_r) & \cos(\theta - \theta_r) \end{bmatrix} \begin{bmatrix} f_{qs} \\ f_{ds} \end{bmatrix} \tag{5.6-8} $$

Appropriate substitution of (5.6-8) into the above torque equations yields the torque expressed with the stator variables in the arbitrary reference frame. For example, (5.6-3) becomes

$$ T_e = \left(\frac{3}{2}\right)\left(\frac{P}{2}\right)(\lambda_{ds} i_{qs} - \lambda_{qs} i_{ds}) \tag{5.6-9} $$

5.7 ROTOR ANGLE AND ANGLE BETWEEN ROTORS

Except for isolated operation, it is convenient for analysis and interpretation purposes to relate the position of the rotor of a synchronous machine to a voltage or to the rotor of another machine. The electrical angular displacement of the rotor relative to its terminal voltage is defined as the rotor angle while in a multiple-machine power system it is customary to express the angle between machine rotors.

The rotor angle is the displacement of the rotor generally referenced to the maximum positive value of the fundamental component of the terminal voltage of phase a. Therefore, the rotor angle expressed in radians is

$$ \begin{aligned} \delta &= \theta_r - \theta_{ev} \\ &= \int_0^t [\omega_r(\xi) - \omega_e(\xi)]\, d\xi + \theta_r(0) - \theta_{ev}(0) \end{aligned} \tag{5.7-1} $$

where ξ is a dummy variable of integration. Also, $\omega_r(\xi)$ and $\omega_e(\xi)$ are the electrical angular velocity of the rotor and the terminal voltages, respectively, and the time zero position is selected so that the fundamental component of v_{as} is maxi-

Once the base quantities are established the corresponding base current and base impedance may be calculated. Park's equations written in terms of flux linkages per second and reactances are readily per unitized by dividing each term by the peak of the base voltage (or the peak value of the base current times base impedance). The form of these equations remains unchanged as a result of per unitizing. When per unitizing the voltage equation of the field winding (*fd* winding) it is convenient to use the form given by (5.5-37) involving e'^{r}_{xfd}. The reason for this choice is established later.

Base torque is the base power divided by the synchronous speed of the rotor. Thus

$$T_B = \frac{P_B}{(2/P)\omega_b}$$

$$= \frac{(\frac{3}{2})V_{B(qd0)}I_{B(qd0)}}{(2/P)\omega_b} \tag{5.8-1}$$

where ω_b corresponds to rated or base frequency, P_B is the base power, $V_{B(qd0)}$ is the peak of the base phase voltage and $I_{B(qd0)}$ is the peak value of the base phase current. Dividing the torque equations by (5.8-1) yields the torque expressed in per unit. For example, (5.6-4) with all quantities expressed in per unit becomes

$$T_e = (\psi^r_{ds} i^r_{qs} - \psi^r_{qs} i^r_{ds}) \tag{5.8-2}$$

Equation (5.3-5), which relates torque and speed, is expressed in per unit as

$$T_e = -2Hp\frac{\omega_r}{\omega_b} + T_I \tag{5.8-3}$$

If ω_e is constant then this relationship becomes

$$T_e = -\frac{2H}{\omega_b}p^2\delta + T_I \tag{5.8-4}$$

where δ is in electrical radians. The inertia constant H is in seconds. It is defined as

$$H = \left(\frac{1}{2}\right)\left(\frac{2}{P}\right)\frac{J\omega_b}{T_B}$$

$$= \left(\frac{1}{2}\right)\left(\frac{2}{P}\right)^2\frac{J\omega_b^2}{P_B} \tag{5.8-5}$$

where J is often the combined inertia of the rotor and prime mover expressed in $kg{\cdot}m^2$ or given as the quantity WR^2 in $lb{\cdot}ft^2$.

5.9 ANALYSIS OF STEADY STATE OPERATION

Although the voltage equations which describe balanced steady state operation of synchronous machines may be derived using several approaches, it is convenient to use Park's equations in this derivation. For balanced conditions the 0*s* quantities are zero. For balanced steady state conditions the electrical angular velocity of the rotor is constant and equal to ω_e whereupon the electrical angular velocity of the rotor reference frame becomes the electrical angular velocity of the synchronously rotating reference frame. In this mode of operation the rotor windings do not experience a change of flux linkages, hence current is not flowing in the short-circuited damper windings. Thus, with ω_r set equal to ω_e and the time rate of change of all flux linkages neglected, the steady state versions of (5.5-22), (5.5-23), and (5.5-27) become

$$V_{qs}^r = -r_s I_{qs}^r - \frac{\omega_e}{\omega_b} X_d I_{ds}^r + \frac{\omega_e}{\omega_b} X_{md} I_{fd}'^r \tag{5.9-1}$$

$$V_{ds}^r = -r_s I_{ds}^r + \frac{\omega_e}{\omega_b} X_q I_{qs}^r \tag{5.9-2}$$

$$V_{fd}'^r = r_{fd}' I_{fd}'^r \tag{5.9-3}$$

Here the ω_e to ω_b ratio is again included to accommodate analysis when the operating frequency is other than rated. It is recalled that all reactances used in this text are calculated using base or rated frequency.

The reactances X_q and X_d are defined by (5.5-39) and (5.5-40), that is, $X_q = X_{ls} + X_{mq}$ and $X_d = X_{ls} + X_{md}$. As mentioned previously, Park's equations are generally written with the primes and the s and r indexes omitted. The uppercase letters are used here to denote steady state quantities.

Equations (3.7-5) and (3.7-6) express the instantaneous variables in the arbitrary reference frame for balanced conditions. In the rotor reference frame these expressions become

$$f_{qs}^r = \sqrt{2} f_s \cos(\theta_{ef} - \theta_r) \tag{5.9-4}$$

$$f_{ds}^r = -\sqrt{2} f_s \sin(\theta_{ef} - \theta_r) \tag{5.9-5}$$

For steady state balanced conditions (5.9-4) and (5.9-5) may be expressed

$$F_{qs}^r = \mathrm{Re}\,[\sqrt{2} F_s e^{j(\theta_{ef} - \theta_r)}] \tag{5.9-6}$$

$$F_{ds}^r = \mathrm{Re}\,[j\sqrt{2} F_s e^{j(\theta_{ef} - \theta_r)}] \tag{5.9-7}$$

It is to our advantage to express (5.9-6) and (5.9-7) in terms of δ, (5.7-1). Hence, if we multiply each equation by $e^{j\theta_{ev}(1-1)}$ and since θ_{ef} and θ_{ev} are both functions of ω_e the above equations may be written

$$F_{qs}^r = \mathrm{Re}\,[\sqrt{2} F_s e^{j[\theta_{ef}(0) - \theta_{ev}(0)]} e^{-j\delta}] \tag{5.9-8}$$

$$F_{ds}^r = \mathrm{Re}\,[j\sqrt{2} F_s e^{j[\theta_{ef}(0) - \theta_{ev}(0)]} e^{-j\delta}] \tag{5.9-9}$$

It is important to note that

$$\tilde{F}_{as} = F_s e^{j[\theta_{ef}(0) - \theta_{ev}(0)]} \tag{5.9-10}$$

is a phasor which represents the *as* variables referenced to the time zero position of θ_{ev} which we will select so that maximum v_{as} occurs at $t = 0$.

From (5.9-8) and (5.9-9)

$$F^r_{qs} = \sqrt{2}F_s \cos[\theta_{ef}(0) - \theta_{ev}(0) - \delta] \tag{5.9-11}$$

$$F^r_{ds} = -\sqrt{2}F_s \sin[\theta_{ef}(0) - \theta_{ev}(0) - \delta] \tag{5.9-12}$$

From which

$$\sqrt{2}\tilde{F}_{as}e^{-j\delta} = F^r_{qs} - jF^r_{ds} \tag{5.9-13}$$

where $\tilde{F}_{as}$ is defined by (5.9-10). Hence

$$\sqrt{2}\tilde{V}_{as}e^{-j\delta} = V^r_{qs} - jV^r_{ds} \tag{5.9-14}$$

Substituting (5.9-1) and (5.9-2) into (5.9-14) yields

$$\sqrt{2}\tilde{V}_{as}e^{-j\delta} = -r_s I^r_{qs} - \frac{\omega_e}{\omega_b}X_d I^r_{ds} + \frac{\omega_e}{\omega_b}X_{md}I'^r_{fd} + j\left(r_s I^r_{ds} - \frac{\omega_e}{\omega_b}X_q I^r_{qs}\right) \tag{5.9-15}$$

If $(\omega_e/\omega_b)X_q I^r_{ds}$ is added to and subtracted from the right-hand side of (5.9-15) and if it is noted that

$$j\sqrt{2}\tilde{I}_{as}e^{-j\delta} = I^r_{ds} + jI^r_{qs} \tag{5.9-16}$$

then (5.9-15) may be written

$$\tilde{V}_{as} = -\left(r_s + j\frac{\omega_e}{\omega_b}X_q\right)\tilde{I}_{as} + \frac{1}{\sqrt{2}}\left[-\frac{\omega_e}{\omega_b}(X_d - X_q)I^r_{ds} + \frac{\omega_e}{\omega_b}X_{md}I'^r_{fd}\right]e^{j\delta} \tag{5.9-17}$$

It is convenient to define the last term on the right-hand side of (5.9-17) as

$$\tilde{E}_a = \frac{1}{\sqrt{2}}\left[-\left(\frac{\omega_e}{\omega_b}\right)(X_d - X_q)I^r_{ds} + \left(\frac{\omega_e}{\omega_b}\right)X_{md}I'^r_{fd}\right]e^{j\delta} \tag{5.9-18}$$

which is sometimes referred to as the excitation voltage. Thus, (5.9-17) becomes

$$\tilde{V}_{as} = -\left(r_s + j\frac{\omega_e}{\omega_b}X_q\right)\tilde{I}_{as} + \tilde{E}_a \tag{5.9-19}$$

Equations (5.9-18) and (5.9-19) are written for the positive direction of stator current assumed out of the machine, convenient for generator action. The ω_e to ω_b ratio is included so that the equations are valid for the analysis of balanced steady state operation at a frequency other than rated.

When the positive direction of stator current is assumed into the machine, convenient for motor action, then

$$\tilde{V}_{as} = \left(r_s + j\frac{\omega_e}{\omega_b}X_q\right)\tilde{I}_{as} + \tilde{E}_a \tag{5.9-20}$$

where

$$\tilde{E}_a = \frac{1}{\sqrt{2}}\left[\frac{\omega_e}{\omega_b}(X_d - X_q)I^r_{ds} + \frac{\omega_e}{\omega_b}X_{md}I'^r_{fd}\right]e^{j\delta} \tag{5.9-21}$$

If (5.9-1) and (5.9-2) are solved for I_{qs}^r and I_{ds}^r and the results substituted into (5.6-2) the expression for the balanced steady state electromagnetic torque can be written

$$T_e = -\left(\frac{3}{2}\right)\left(\frac{P}{2}\right)\left(\frac{1}{\omega_b}\right)$$
$$\times \left\{\frac{r_s X_{md} I_{fd}'^r}{r_s^2 + (\omega_e/\omega_b)^2 X_q X_d}\left(V_{qs}^r - \frac{\omega_e}{\omega_b} X_{md} I_{fd}'^r - \frac{\omega_e}{\omega_b}\frac{X_d}{r_s} V_{ds}^r\right)\right.$$
$$+ \frac{X_d - X_q}{[r_s^2 + (\omega_e/\omega_b)^2 X_q X_d]^2}\left[r_s \frac{\omega_e}{\omega_b} X_q\left(V_{qs}^r - \frac{\omega_e}{\omega_b} X_{md} I_{fd}'^r\right)^2\right.$$
$$\left.\left. + \left[r_s^2 - \left(\frac{\omega_e}{\omega_b}\right)^2 X_q X_d\right] V_{ds}^r\left(V_{qs}^r - \frac{\omega_e}{\omega_b} X_{md} I_{fd}'^r\right) - r_s \frac{\omega_e}{\omega_b} X_d (V_{ds}^r)^2\right]\right\} \tag{5.9-22}$$

where P is the number of poles and ω_b is the base electrical angular velocity used to calculate the reactances and ω_e corresponds to the operating frequency.

For balanced operation the stator voltages may be expressed in the form given by (3.7-1)–(3.7-3). Thus

$$v_{as} = \sqrt{2} v_s \cos \theta_{ev} \tag{5.9-23}$$

$$v_{bs} = \sqrt{2} v_s \cos\left(\theta_{ev} - \frac{2\pi}{3}\right) \tag{5.9-24}$$

$$v_{cs} = \sqrt{2} v_s \cos\left(\theta_{ev} + \frac{2\pi}{3}\right) \tag{5.9-25}$$

where

$$\theta_{ev} = \int_0^t \omega_e(\xi)\, d\xi + \theta_{ev}(0) \tag{5.9-26}$$

where ξ is a dummy variable of integration. These voltages may be expressed in the rotor reference frame by replacing θ with θ_r in (3.7-5) and (3.7-6).

$$v_{qs}^r = \sqrt{2} v_s \cos(\theta_{ev} - \theta_r) \tag{5.9-27}$$

$$v_{ds}^r = -\sqrt{2} v_s \sin(\theta_{ev} - \theta_r) \tag{5.9-28}$$

If the rotor angle from (5.7-1) is substituted into (5.9-27) and (5.9-28) we obtain

$$v_{qs}^r = \sqrt{2} v_s \cos \delta \tag{5.9-29}$$

$$v_{ds}^r = \sqrt{2} v_s \sin \delta \tag{5.9-30}$$

The only restrictions on (5.9-29) and (5.9-30) are that the voltages form a balanced set. These equations are valid for transient and steady state operation, that is, v_s and δ may both be functions of time with $\theta_{ev}(0)$ generally set equal to zero.

The torque given by (5.9-22) is for balanced steady state conditions. In this mode of operation (5.9-29) and (5.9-30) are constants since v_s and δ are both constants. Before proceeding, it is noted from (5.5-36) that for balanced steady state operation

$$E'^{r}_{xfd} = X_{md} I'^{r}_{fd} \tag{5.9-31}$$

Although this expression is sometimes substituted into the above steady state voltage equations it is most often used in the expression for torque. In particular, if (5.9-31) and the steady state versions of (5.9-29) and (5.9-30) are substituted in (5.9-22) and if r_s is neglected the torque may be expressed

$$T_e = \left(\frac{3}{2}\right)\left(\frac{P}{2}\right)\left(\frac{1}{\omega_b}\right)\left[\frac{E'^{r}_{xfd}\sqrt{2}V_s}{(\omega_e/\omega_b)X_d}\sin\delta + \left(\frac{1}{2}\right)\left(\frac{\omega_e}{\omega_b}\right)^{-2}\left(\frac{1}{X_q}-\frac{1}{X_d}\right)(\sqrt{2}V_s)^2\sin 2\delta\right] \tag{5.9-32}$$

In per unit, (5.9-32) becomes

$$T_e = \frac{E'^{r}_{xfd}V_s}{(\omega_e/\omega_b)X_d}\sin\delta + \left(\frac{1}{2}\right)\left(\frac{\omega_e}{\omega_b}\right)^{-2}\left(\frac{1}{X_q}-\frac{1}{X_d}\right)V_s^2\sin 2\delta \tag{5.9-33}$$

Neglecting r_s is justified if r_s is small relative to the reactances of the machine. In variable-frequency drive systems this may not be the case at low frequencies, whereupon (5.9-22) must be used to calculate torque rather than (5.9-32). With the stator resistance neglected, steady state power and torque are related by rotor speed, and if torque and power are expressed in per unit, they are equal during steady state operation. The above expressions are positive for generator action. If one wishes positive torque for motor action then (5.9-22), (5.9-32), and (5.9-33) must be multiplied by -1.

Although (5.9-32) is valid only for balanced steady state operation and if the stator resistance is small relative to the magnetizing reactances (X_{mq} and X_{md}) of the machine, it permits a quantitative description of the nature of the steady state electromagnetic torque of a synchronous machine. The first term on the right-hand side of (5.9-32) is due to the interaction of the magnetic system produced by the currents flowing in the stator windings and the magnetic system produced by the current flowing in the field winding. The second term is due to the saliency of the rotor. This component is commonly referred to as the reluctance torque. The predominate torque is the torque due to the interaction of the stator and field currents. The amplitude of this component is proportional to the magnitudes of the stator voltage V_s, and the voltage applied to the field, E'^{r}_{xfd}. In power systems it is desirable to maintain the stator voltage near rated. This is achieved by automatically adjusting the voltage applied to the field winding. Hence, the amplitude of this torque component varies as E'^{r}_{xfd} is varied to maintain the terminal voltage at or near rated and/or to control reactive power flow. The reluctance torque component is generally a relatively small part of the total torque of a synchronous generator. In power systems where the terminal voltage is maintained nearly constant, the amplitude of the reluctance torque would also be nearly constant, a function only of the parameters of the machine. A steady state reluctance torque does not exist in round or cylindrical rotor synchronous machines since $X_q = X_d$. On the other hand, a reluctance machine is a device which is not equipped with a field winding, hence, the only torque produced is reluctance torque. Reluctance machines are used widely as motors especially in variable-frequency drive systems.

Let us return for a moment to the steady state voltage equation for generator action, (5.9-19). With $\theta_{ev}(0) = 0$, $\tilde{V}_{as}$ lies along the positive real axis of a phasor diagram. Since δ is the angle associated with $\tilde{E}_a$, (5.9-18), its position relative to $\tilde{V}_{as}$ is also the position of the q axis of the machine relative to $\tilde{V}_{as}$. Therefore, we can superimpose the q and d axes of the synchronous machine upon the phasor diagram.

If T_I is assumed zero and if we neglect friction and windage losses along with the stator resistance then T_e and δ are also zero and the machine will theoretically run at synchronous speed without absorbing energy from either the electrical or mechanical system. Although this mode of operation is not feasible in practice since the machine will actually absorb some small amount of energy to satisfy the ohmic and friction and windage losses, it is convenient for purposes of explanation. With the machine "floating on the line" the field voltage can be adjusted to establish the desired terminal conditions. Three situations may exist: (1) $|\tilde{E}_a| = |\tilde{V}_{as}|$, whereupon $\tilde{I}_{as} = 0$; (2) $|\tilde{E}_a| > |\tilde{V}_{as}|$, whereupon $\tilde{I}_{as}$ lags $\tilde{V}_{as}$ and since $\tilde{I}_{as}$ is positive out of the machine the synchronous machine appears as a capacitor supplying reactive power to the system; or (3) $|\tilde{E}_a| < |\tilde{V}_{as}|$ with $\tilde{I}_{as}$ leading $\tilde{V}_{as}$, whereupon the machine is absorbing reactive power appearing as an inductor to the system.

In order to maintain the voltage in a power system at rated value the synchronous generators are normally operated in the overexcited mode with $|\tilde{E}_a| > |\tilde{V}_{as}|$ since they are the main source of reactive power for the inductive loads throughout the system. In fact, some synchronous machines are placed in the power system for the sole purpose of supplying reactive power without any provision to provide real power. During peak load conditions when the system voltage is depressed these so-called "synchronous condensers" are brought on line and the field voltage adjusted to help increase the system voltage. In this mode of operation the synchronous machine behaves like an adjustable capacitor. On the other hand, it may be necessary for a generator to absorb reactive power in order to regulate voltage in a high-voltage transmission system during light load conditions. This mode of operation is, however, not desirable and should be avoided since machine oscillations become less damped as the reactive power required is decreased. This will be shown in a later chapter when we calculate eigenvalues.

The manner in which torque is produced in a synchronous machine may now be further explained with a somewhat more detailed consideration of the interaction of the resulting air-gap MMF established by the stator currents and the field current with (1) the MMF established by the field current and with (2) the minimum reluctance path of the rotor. With the machine operating with T_I equal zero and $|\tilde{E}_a| > |\tilde{V}_{as}|$ the stator currents are

$$i_{as} = \sqrt{2} I_s \cos\left(\omega_e t - \frac{\pi}{2}\right) \qquad \text{etc.} \tag{5.9-34}$$

The rotor angle δ is zero and the q axis of the machine coincides with the real axis of a phasor diagram with the d axis with the negative imaginary axis as shown in Fig. 5.9-1. Electromagnetic torque is developed so as to align the poles or the MMF created by the field current with the resultant air-gap MMF produced by the field and stator currents. In this mode of operation, the MMF due to the field current is downward in the direction of the positive d axis at the instant v_{as} is maximum. At this time i_{as} is zero while i_{bs} and i_{cs} are equal and opposite. Hence, the MMF produced by the stator currents is directed upward in the direction of the negative d axis. The resultant of these two MMFs must be in the direction of the positive d axis since it was the increasing of the field MMF, by increasing the field current, which caused the stator current to lag the voltage (positive current out of machine) thus causing the MMF produced by the stator currents to oppose the MMF produced by the field current. Therefore the resultant air-gap MMF and the field MMF are aligned. Moreover, the resultant air-gap MMF and the minimum reluctance path of the rotor (d axis) are also aligned. It follows that zero torque is produced and the rotor and MMFs will rotate while maintaining this alignment. If, however, the rotor tries to move from this alignment by either speeding up or slowing down ever so slightly, there will be both a torque due to the interaction of stator and field currents and a reluctance torque to bring the rotor back into alignment.

Let us now consider the procedure by which generator action is established. A prime mover is mechanically connected to the shaft of the synchronous generator. This prime mover can be either a steam turbine, a hydro turbine, or a combustion engine. If, initially, the torque input on the shaft due to the prime mover is zero then T_e is very slightly negative due to losses. The synchronous

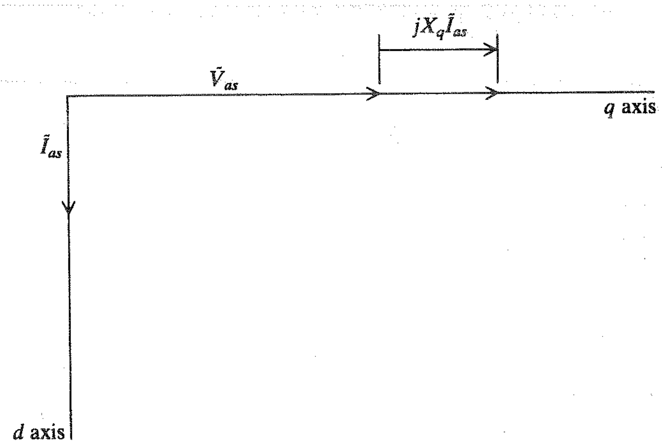

Figure 5.9-1 Phasor diagram with all losses neglected and zero power transfer.

machine is essentially floating on the line. If now the input torque is increased to some positive value by supplying steam to the turbine blades, for example, a torque imbalance occurs since T_e must remain at its original value until δ changes. Hence the rotor will temporarily accelerate slightly above synchronous speed, whereupon δ will increase in accordance with (5.7-1). Thus, T_e increases and a new operating point will be established with a positive δ where T_I is equal to T_e plus the losses. The rotor will again rotate at synchronous speed with a torque exerted on it in an attempt to align the field MMF with the resultant air-gap MMF. The actual dynamic response of the electrical and mechanical systems during this loading process is illustrated by computer traces in the following section. If, during generator operation, the torque input from the prime mover is increased to a value greater than the maximum possible value of T_e, the machine will be unable to maintain steady state operation since it cannot transmit the power supplied to the shaft. In this case, the device will accelerate above synchronous speed theoretically without bound, however, protection is normally provided which disconnects the machine from the system and reduces the input torque to zero by closing the steam valves of the steam turbine, for example, when it exceeds synchronous speed by generally 3 to 5 percent.

Normal steady state generator operation is depicted by the phasor diagram shown in Fig. 5.9-2. Here $\theta_{ei}(0)$ is the angle between the voltage and the current since the time zero position is $\theta_{ev}(0) = 0$ after steady state operation is established. Since the phasor diagram and the q and d axes of the machine may be superimposed the rotor reference-frame voltages and currents are also shown in Fig. 5.9-2. For example, V_{qs}^r and I_{qs}^r are shown directed along the positive q axis. If we wish to show each component of V_{qs}^r it can be broken up according to (5.9-1) and each term added algebraically along the q axis. Care must be taken, however, when interpreting this diagram. $\tilde{V}_{as}$, $\tilde{I}_{as}$, and $\tilde{E}_a$ are phasors representing

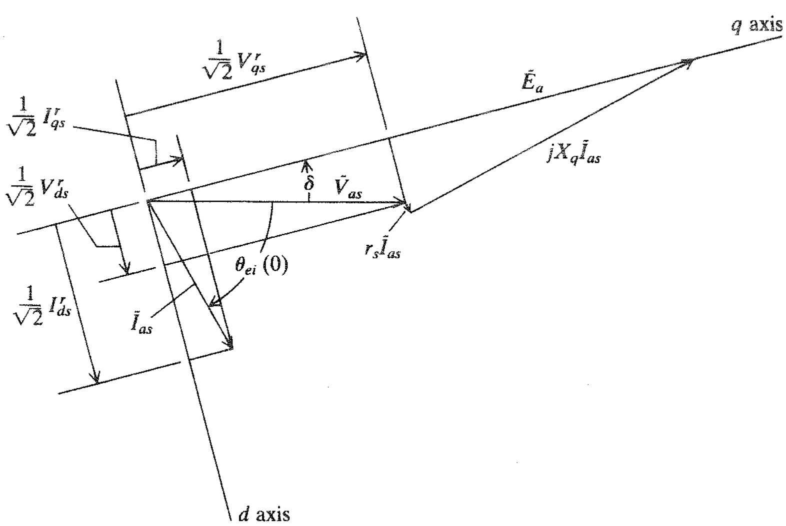

Figure 5.9-2 Phasor diagram for generator operation.

sinusoidal quantities. On the other hand, all rotor reference-frame quantities are constants. They do not represent phasors in the rotor reference frame even though they are displayed on a phasor diagram.

There is one last detail to clear up. In (5.5-36) we defined $e_{xfd}'^r$ ($E_{xfd}'^r$ for steady state operation) and indicated we would later find a convenient use for this term. If we assume that the stator of the synchronous machine is open-circuited and the rotor is being driven at synchronous speed then, from (5.9-19)

$$\tilde{V}_{as} = \tilde{E}_a \tag{5.9-35}$$

Substituting (5.9-18) for $\tilde{E}_a$ with I_{ds}^r equal to zero yields

$$\sqrt{2}|\tilde{V}_{as}| = \frac{\omega_e}{\omega_b} X_{md} I_{fd}'^r \tag{5.9-36}$$

However, as given by (5.9-31) for balanced steady state operation

$$E_{xfd}'^r = X_{md} I_{fd}'^r \tag{5.9-37}$$

hence

$$\sqrt{2}|\tilde{V}_{as}| = \frac{\omega_e}{\omega_b} E_{xfd}'^r \tag{5.9-38}$$

Now let us per unitize the above equation. To do so we must divide each side of (5.9-38) by $V_{B(qd0)}$ or $\sqrt{2}\,V_{B(abc)}$ since $E_{xfd}'^r$ is a rotor reference frame quantity. Thus

$$\frac{\sqrt{2}|\tilde{V}_{as}|}{\sqrt{2}\,V_{B(abc)}} = \frac{(\omega_e/\omega_b)E_{xfd}'^r}{V_{B(qd0)}} \tag{5.9-39}$$

Therefore, when $|\tilde{V}_{as}|$ is one per unit, $(\omega_e/\omega_b)E_{xfd}'^r$ is one per unit. During steady state rated speed operation, (ω_e/ω_b) is unity and therefore one per unit $E_{xfd}'^r$ produces one per unit open-circuit terminal voltage. Since this provides a convenient relationship, $E_{xfd}'^r$ is used extensively to define the field voltage rather than the actual voltage applied to the field winding.

Example 5A A 3-phase, 64-pole, hydro turbine generator is rated at 325 MVA, with 20 kV line-to-line voltage and a power factor of 0.85. The machine parameters in ohms at 60 Hz are: $r_s = 0.00234$, $X_q = 0.5911$, and $X_d = 1.0467$. For balanced, steady state rated conditions calculate (*a*) $\tilde{E}_a$, (*b*) $E_{xfd}'^r$, and (*c*) T_e.

The apparent power $|S|$ is

$$|S| = 3|\tilde{V}_{as}||\tilde{I}_{as}| \tag{5A-1}$$

Thus

$$\begin{aligned} |\tilde{I}_{as}| &= \frac{|S|}{3|V_{as}|} \\ &= \frac{325 \times 10^6}{(3 \times 20 \times 10^3)/\sqrt{3}} \\ &= 9.37\ \text{kA} \end{aligned} \tag{5A-2}$$

The power factor angle is $\cos^{-1} 0.85 = 31.8°$. Since current is positive out of the terminals of the generator, reactive power is delivered by the generator when the current is lagging the terminal voltage. Thus, $\tilde{I}_{as} = 9.37\underline{/-31.8°}$ kA since the machine would be delivering reactive power during normal operating conditions. Therefore from (5.9-19) we can obtain the answer to part *a*

$$\begin{aligned}\tilde{E}_a &= \tilde{V}_{as} + \left(r_s + j\frac{\omega_e}{\omega_b}X_q\right)\tilde{I}_{as}\\ &= \frac{20 \times 10^3}{\sqrt{3}}\underline{/0°} + [0.00234 + j(1)(0.5911)]9.37 \times 10^3\underline{/-31.8°}\\ &= 15.2\underline{/18°}\text{ kV}\end{aligned} \tag{5A-3}$$

Hence $\delta = 18°$.

We can solve for E'^r_{xfd} by first substituting (5.9-31) into (5.9-18), however, I'^r_{ds} is required before E'^r_{xfd} can be evaluated. Thus from (5.9-12)

$$\begin{aligned}I^r_{ds} &= -\sqrt{2}I_s \sin[\theta_{ei}(0) - \theta_{ev}(0) - \delta]\\ &= -\sqrt{2}|\tilde{I}_{as}| \sin[-31.8° - 0 - 18°]\\ &= -\sqrt{2}(9.37 \times 10^3)\sin(-49.8°)\\ &= 10.12\text{ kA}\end{aligned} \tag{5A-4}$$

From (5.9-18) and (5.9-31)

$$\begin{aligned}E'^r_{xfd} &= \frac{\omega_e}{\omega_b}\left[\sqrt{2}|\tilde{E}_a| + \frac{\omega_e}{\omega_b}(X_d - X_q)I^r_{ds}\right]\\ &= \sqrt{2}(15.2 \times 10^3) + (1.0467 - 0.5911)10.12 \times 10^3\\ &= 26.1\text{ kV}\end{aligned} \tag{5A-5}$$

Since r_s is small, T_e may be calculated by substitution into (5.9-32)

$$\begin{aligned}T_e &= \left(\frac{3}{2}\right)\left(\frac{P}{2}\right)\left(\frac{1}{\omega_b}\right)\left[\frac{E'^r_{xfd}\sqrt{2}|\tilde{V}_{as}|}{(\omega_e/\omega_b)X_d}\sin\delta\right.\\ &\qquad\left. + \left(\frac{1}{2}\right)\left(\frac{\omega_e}{\omega_b}\right)^{-2}\left(\frac{1}{X_q} - \frac{1}{X_d}\right)(\sqrt{2}|\tilde{V}_{as}|)^2 \sin 2\delta\right]\\ &= \left(\frac{3}{2}\right)\left(\frac{64}{2}\right)\left(\frac{1}{377}\right)\left[\frac{(26.1 \times 10^3)(\sqrt{2})[(20 \times 10^3)/\sqrt{2}]}{1.0467}\sin 18°\right.\\ &\qquad\left. + \left(\frac{1}{2}\right)\left(\frac{1}{0.5911} - \frac{1}{1.0467}\right)\left[\sqrt{2}\left(\frac{20 \times 10^3}{\sqrt{3}}\right)^2\right]\sin 36°\right]\\ &= 23.4 \times 10^6\text{ N}\cdot\text{m}\end{aligned} \tag{5A-6}$$

5.10 DYNAMIC PERFORMANCE DURING A SUDDEN CHANGE IN INPUT TORQUE

It is instructive to observe the dynamic performance of a synchronous machine during a step change in input torque. For this purpose, the differential equations which describe the synchronous machine were programmed on a computer and a study was performed. Two large machines are considered, a low-speed hydro turbine generator and a high-speed steam turbine generator. Information regarding each machine is given in Tables 5.10-1 and 5.10-2. In the case of hydro

Table 5.10-1 Hydro turbine generator

Rating: 325 MVA

Line to line voltage: 20 kV

Power factor: 0.85

Poles: 64

Speed: 112.5 r/min

Combined inertia of generator and turbine:

$J = 35.1 \times 10^6$ J·s², or $WR^2 = 833.1 \times 10^6$ lbm·ft² $\quad H = 7.5$ s

Parameters in ohms and per unit:

$r_s = 0.00234\ \Omega,\ 0.0019$ pu

$X_{ls} = 0.1478\ \Omega,\ 0.120$ pu

$X_q = 0.5911\ \Omega,\ 0.480$ pu	$X_d = 1.0467\ \Omega,\ 0.850$ pu
	$r'_{fd} = 0.00050\ \Omega,\ 0.00041$ pu
	$X'_{lfd} = 0.2523\ \Omega,\ 0.2049$ pu
$r'_{kq2} = 0.01675\ \Omega,\ 0.0136$ pu	$r'_{kd} = 0.01736\ \Omega,\ 0.0141$ pu
$X'_{lkq2} = 0.1267\ \Omega,\ 0.1029$ pu	$X'_{lkd} = 0.1970\ \Omega,\ 0.160$ pu

Table 5.10-2 Steam turbine generator

Rating: 835 MVA

Line to line voltage: 26 kV

Power factor: 0.85

Poles: 2

Speed: 3600 r/min

Combined inertia of generator and turbine:

$J = 0.0658 \times 10^6$ J·s², or $WR^2 = 1.56 \times 10^6$ lbm·ft² $\quad H = 5.6$ s

Parameters in ohms and per unit:

$r_s = 0.00243\ \Omega,\ 0.003$ pu

$X_{ls} = 0.1538\ \Omega,\ 0.19$ pu

$X_q = 1.457\ \Omega,\ 1.8$ pu	$X_d = 1.457\ \Omega,\ 1.8$ pu
$r'_{kq1} = 0.00144\ \Omega,\ 0.00178$ pu	$r'_{fd} = 0.00075\ \Omega,\ 0.000929$ pu
$X'_{lkq1} = 0.6578\ \Omega,\ 0.8125$ pu	$X'_{lfd} = 0.1145\ \Omega,\ 0.1414$ pu
$r'_{kq2} = 0.00681\ \Omega,\ 0.00841$ pu	$r'_{kd} = 0.01080\ \Omega.\ 0.01334$ pu
$X'_{lkg2} = 0.07602\ \Omega,\ 0.0939$ pu	$X'_{lkd} = 0.06577\ \Omega,\ 0.08125$ pu

turbine generators parameters are given for only one damper winding in the q axis. The reason for denoting this winding as the $kq2$ winding rather than the $kq1$ winding will become clear in Chap. 6.

The computer traces shown in Figs. 5.10-1 and 5.10-2 illustrate the dynamic behavior of the hydro turbine generator following a step change in input torque from zero to 27.6×10^6 N·m (rated). The dynamic behavior of the steam turbine generator is depicted in Figs. 5.10-3 and 5.10-4. In this case the step change in input torque is from zero to 1.11×10^6 N·m (50 percent rated). In Figs. 5.10-1

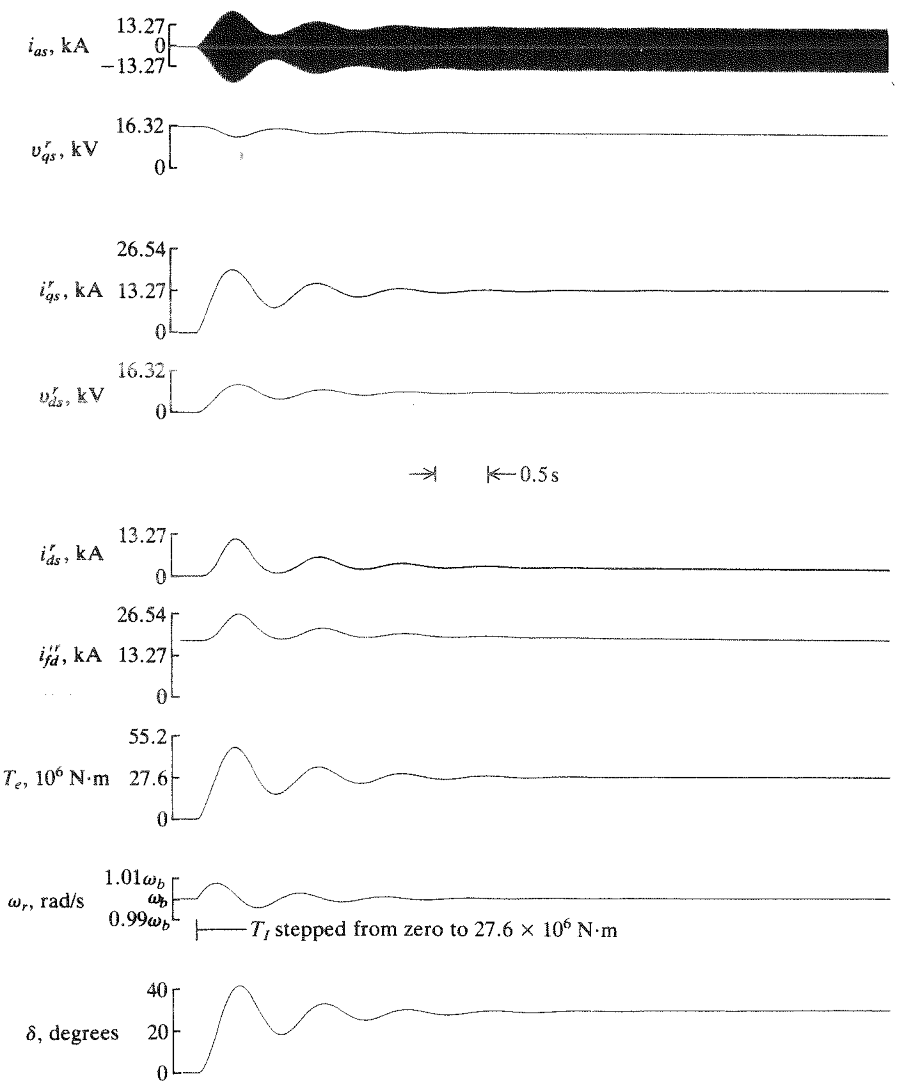

Figure 5.10-1 Dynamic performance of a hydro turbine generator during a step increase in input torque from zero to rated.

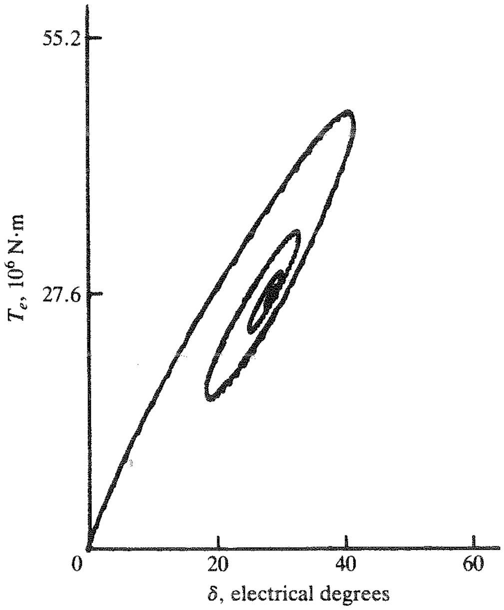

Figure 5.10-2 Torque versus rotor angle characteristics for the study shown in Fig. 5.10-1.

and 5.10-3 the following variables are plotted: i_{as}, v_{qs}^r, i_{qs}^r, v_{ds}^r, i_{ds}^r, $i_{fd}'^r$, T_e, ω_r, and δ; where ω_r is in electrical radians per second and δ in electrical degrees. Figures 5.10-2 and 5.10-4 illustrate the dynamic torque versus rotor angle characteristics. In all figures, the scales of the voltages and currents are given in multiples of peak rated values.

In each study, it is assumed that the machine is connected to a bus whose voltage and frequency remain constant, at the rated values, regardless of the stator current. This is commonly referred to as an infinite bus, since its characteristics do not change regardless of the power supplied or consumed by any device connected to it. Although an infinite bus cannot be realized in practice, its characteristics are approached if the power delivery capability of the system, at the point where the machine is connected, is much larger than the rating of the machine.

Initially each machine is operating with zero input torque with the excitation held fixed at the value which gives rated open-circuit terminal voltage at synchronous speed. It is instructive to observe the plots of T_e, ω_r, and δ following the step change input torque. In particular, consider the response of the hydro turbine generator (Fig. 5.10-1) where the machine is subjected to a step increase in input torque from zero to 27.6×10^6 N·m. The rotor speed begins to increase immediately following the step increase in input torque as predicted by (5.3-8) whereupon the rotor angle increases in accordance with (5.7-1). The rotor speeds up until the accelerating torque on the rotor is zero. As noted in Fig. 5.10-1, the speed increases to approximately 380 electrical radians per second at which time T_e is equal to T_I since the change of ω_r is zero and hence the inertial torque (T_{IT})

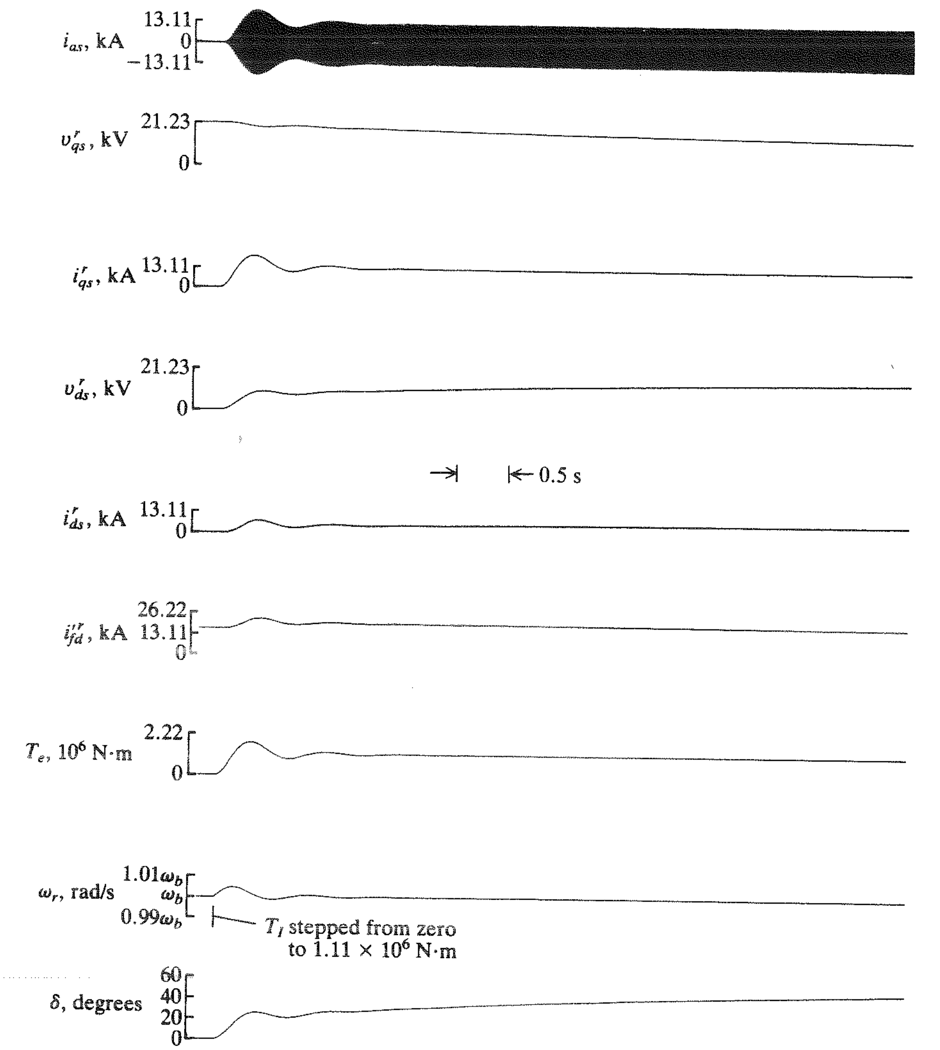

Figure 5.10-3 Dynamic performance of a steam turbine generator during a step increase in input torque from zero to 50 percent rated.

is zero. Even though the accelerating torque is zero at this time, the rotor is running above synchronous speed, hence δ, and thus T_e, will continue to increase. The increase in T_e, which is an increase in the power output of the machine, causes the rotor to decelerate toward synchronous speed. However, when synchronous speed is reached the magnitude of δ has become larger than necessary to satisfy the input torque. Note that at the first synchronous speed crossing of ω_r after the change in input torque, δ is approximately 42 electrical degrees and T_e approximately 47×10^6 N·m. Hence, the rotor continues to decelerate below synchronous speed and consequently δ begins to decrease which in turn decreases

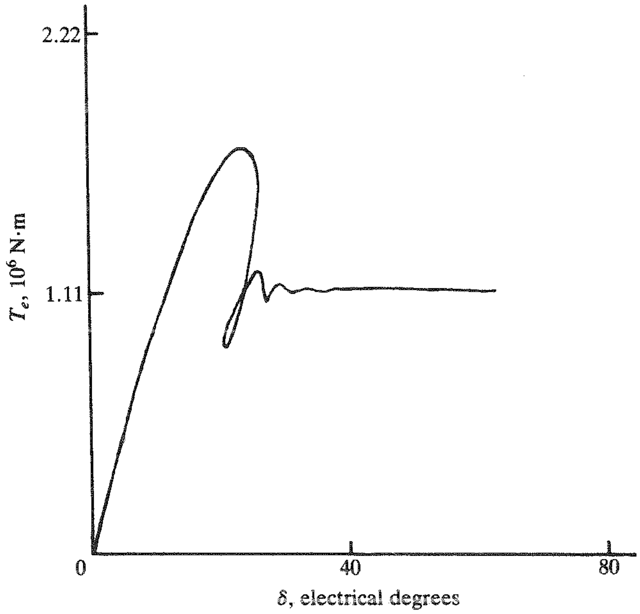

Figure 5.10-4 Torque versus rotor angle characteristics for the study shown in Fig. 5.10-3.

T_e. Damped oscillations of the machine variables continue and a new steady state operating point is finally attained.

In the case of the hydro turbine generator (Fig. 5.10-1) the oscillations in machine variables subside in a matter of two or three seconds and the machine establishes the new steady state operating point within eight or ten seconds. In the case of the steam turbine generator (Fig. 5.10-3) the oscillations subside rapidly but the new steady state operating point is slowly approached. The damping is of course a function of the damper windings and can be determined from an eigenvalue analysis which will be discussed later. The point of interest here is the time required for the machine variables to reestablish steady state operation after the torque disturbance. This rather slow approach to the new steady state operating point in the case of the steam turbine generator is also apparent from the plot of T_e versus δ (Fig. 5.10-4).

Let us consider, for a moment, the expression for steady state torque, (5.9-32). For the hydro turbine generator with $E'^r_{xfd} = \sqrt{\frac{2}{3}}\,20\,\text{kV}$

$$T_e = (32.5 \sin\delta + 12.5 \sin 2\delta) \times 10^6\ \text{N·m} \tag{5.10-1}$$

and for the steam turbine generator with $E'^r_{xfd} = \sqrt{\frac{2}{3}}26\,\text{kV}$

$$T_e = 1.23 \times 10^6 \sin\delta\ \text{N·m} \tag{5.10-2}$$

If (5.10-1) and (5.10-2) are plotted on Figs. 5.10-2 and 5.10-4 respectively, the steady state T_e versus δ curves will pass through the final value of the dynamic T_e versus δ plots. However, the dynamic torque-angle characteristics immediately following the input torque disturbance yields a much larger T_e for a given value of δ than does the steady state characteristic. In other words, the dynamic or transient torque-angle characteristic is considerably different from the steady state characteristic and the steady state T_e versus δ curve applies only after all transients have subsided. Although the computation of the transient torque during speed variations requires the solution of nonlinear differential equations, it can be approximated quite simply. This is the subject of a following section.

The studies shown in this section are for generator action. Motor action, wherein a load torque is applied to the machine, would essentially yield the mirror image of the T_e versus δ plots differing only by the ohmic losses.

5.11 DYNAMIC PERFORMANCE DURING A THREE-PHASE FAULT AT THE MACHINE TERMINALS

The stability of synchronous machines throughout a power system following a fault is of major concern. A 3-phase fault or short-circuit rarely occurs and a 3-phase fault at the machine terminals is even more uncommon; nevertheless it is instructive to observe the dynamic performance of a synchronous machine during this type of a fault.

The computer traces shown in Figs. 5.11-1 and 5.11-2 illustrate the dynamic behavior of the hydro turbine generator during and following a 3-phase fault at the terminals. The dynamic behavior of the steam turbine generator as a result of a 3-phase terminal fault is shown in Figs. 5.11-3 and 5.11-4. The parameters of the machines are those given in the previous section. In Figs. 5.11-1 and 5.11-3 the following variables are plotted: i_{as}, v_{qs}^r, i_{qs}^r, v_{ds}^r, i_{ds}^r, $i_{fd}'^r$, T_e, ω_r, and δ. Figures 5.11-2 and 5.11-4 illustrate the dynamic torque-angle characteristics during and following the 3-phase fault.

In each case the machine is initially connected to an infinite bus delivering rated MVA at rated power factor. In the case of the hydro turbine generator the input torque is held constant at $(0.85)27.6 \times 10^6$ N·m with $E_{xfd}'^r$ fixed at $(1.6)\sqrt{\frac{2}{3}}20$ kV; for the steam turbine generator $T_I = (0.85)2.22 \times 10^6$ N·m and $E_{xfd}'^r = (2.48)\sqrt{\frac{2}{3}}26$ kV. (Rated operating conditions for the hydro turbine generator are calculated in Example 5A.) With the machines operating in this steady state condition, a 3-phase terminal fault is simulated by setting v_{as}, v_{bs}, and v_{cs} to zero, in the simulation, at the instant v_{as} passes through zero going positive. The transient offset in the phase currents is reflected into the rotor reference-frame variables and the instantaneous torque as a decaying 60 Hz pulsation. Since the terminal voltage is zero during the 3-phase fault the machine is unable to transmit power to the system. Hence, all of the input torque, with the exception of the ohmic losses, accelerates the rotor.

In the case of the hydro turbine generator the fault is removed in 0.466 seconds; 0.362 seconds in the case of the steam turbine generator. If the fault had been allowed to remain on the system slightly longer the machines would have

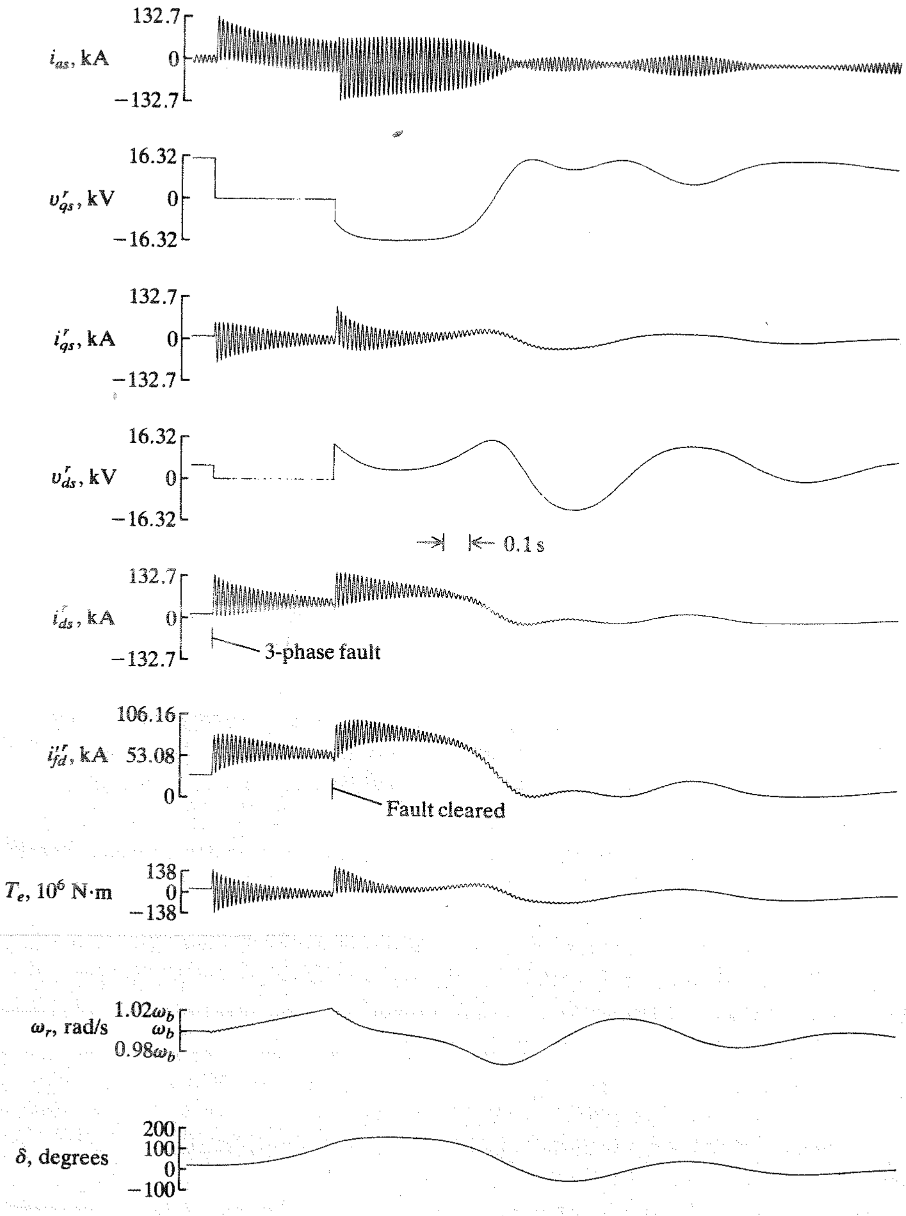

Figure 5.11-1 Dynamic performance of a hydro turbine generator during a 3-phase fault at the terminals.

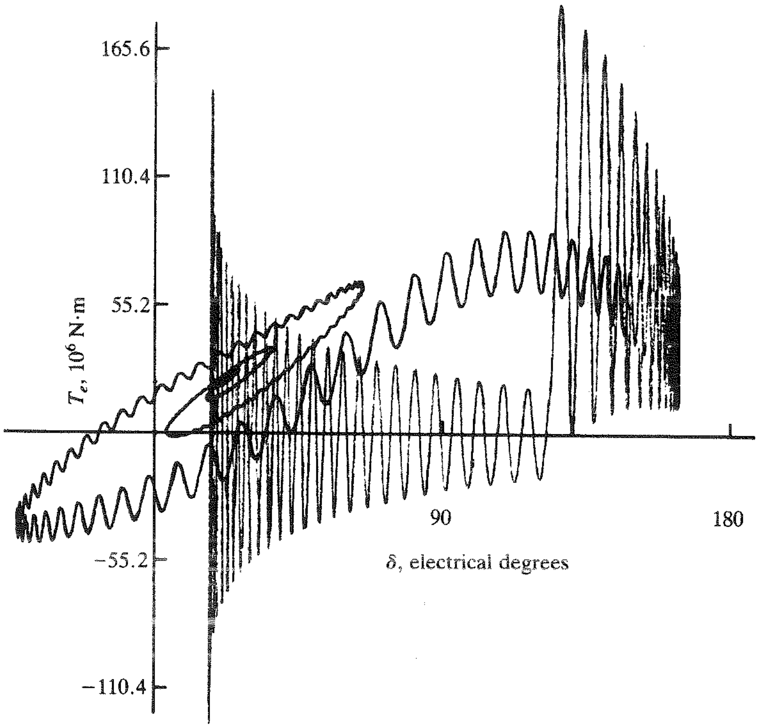

Figure 5.11-2 Torque versus rotor angle characteristics for the study shown in Fig. 5.11-1.

become unstable, that is, they would either not have returned to synchronous speed after removal of the fault or slipped poles before returning to synchronous speed. Asynchronous operation (pole slipping) is discussed in Chap. 10.

When the fault is cleared the system voltages are reapplied to the machine; offsets again occur in the phase currents giving rise to the decaying 60 Hz oscillations in the rotor reference variables and the instantaneous torque. The dynamic torque-angle charcteristics shown in Figs. 5.11-2 and 5.11-4 yield a very lucid illustration of the fault and switching sequence and the return of the machine to its original operating condition after the fault is cleared. These torque-angle plots are shown in Figs. 8.4-2 and 8.4-4, respectively, with the stator electric transients neglected, which eliminates the 60 Hz pulsating electromagnetic torque and permits the average torque to be more clearly depicted.

The expression for the steady state torque-angle characteristic for the hydro turbine generator is

$$T_e = (52.1 \sin \delta + 12.5 \sin 2\delta) \times 10^6 \text{ N}\cdot\text{m} \qquad (5.11\text{-}1)$$

For the steam turbine generator

$$T_e = 3.05 \times 10^6 \sin \delta \text{ N}\cdot\text{m} \qquad (5.11\text{-}2)$$

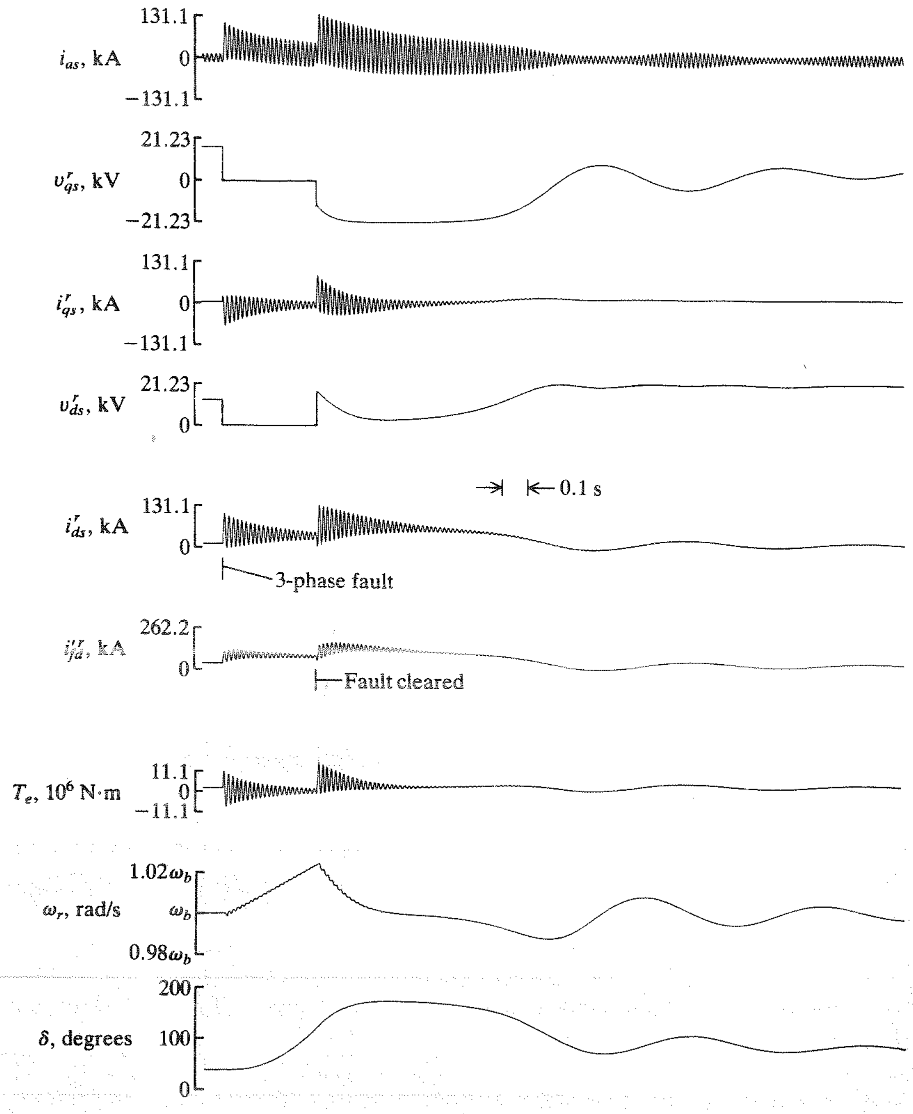

Figure 5.11-3 Dynamic performance of a steam turbine generator during a 3-phase fault at the terminals.

If these steady state torque-angle characteristics are plotted on Figs. 5.11-2 and 5.11-4 respectively, they would pass through only the initial (or final) steady state operating point. As in the case of a sudden change in input torque, the instantaneous and/or average value of the dynamic or transient torque-angle characteristic differs markedly from the steady state torque-angle characteristics.

It is perhaps appropriate to mention that this example is somewhat impractical. In the case of a 3-phase fault close to a fully loaded machine, the circuit

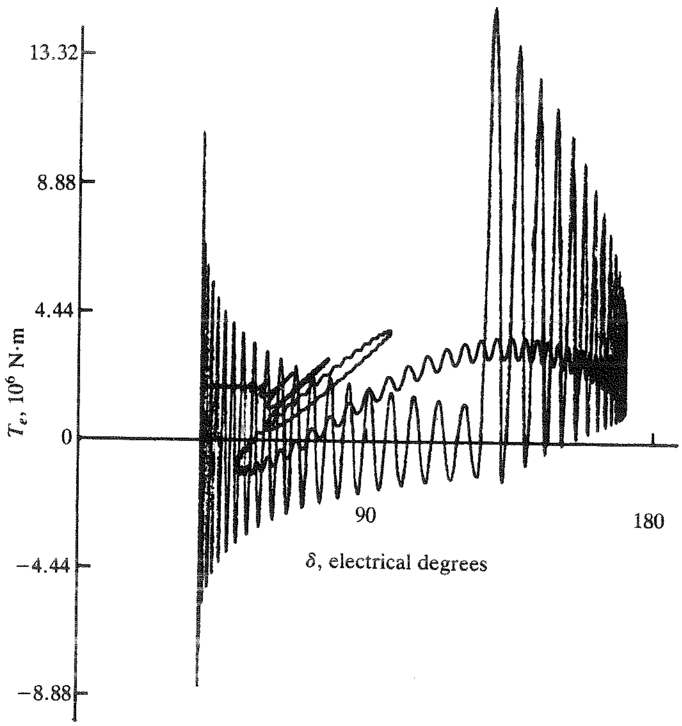

Figure 5.11-4 Torque versus rotor angle characteristics for the study shown in Fig. 5.11-3.

breakers would probably remove the machine from the system and reclosing would be prohibited since the machine would accelerate beyond speed limits before it would be physically possible to reclose the circuit breakers. A practical situation which is approximated by the example might be a 3-phase fault on a large radial transmission line close to the machine terminals. Clearing or "switching out" of this line would then remove the fault from the system.

5.12 APPROXIMATE TRANSIENT TORQUE VERSUS ROTOR ANGLE CHARACTERISTICS

As pointed out in the previous sections, the transient and steady state torque versus rotor angle characteristics are quite different. Since the transient characteristics will determine if the machine remains in synchronism after a disturbance, it is necessary to calculate these characteristics accurately whenever determining the transient stability of a synchronous machine. With present-day computers the calculation of the transient electromagnetic torque is a straightforward procedure. Consequently, it is difficult to appreciate the complex computational problems faced by machine and power system analysts before the advent of the computer and the techniques which they devised to simplify these problems.

In the late 1920s, R. E. Doherty and C. A. Nickle [4] described a simple method of approximating the transient torque-angle characteristics. This method, combined with the concept of equal-area criterion—which is discussed in a following section—formed the basis for transient stability studies of power systems until the 1960s. Although it is not the purpose here to dwell on techniques which have long been replaced, it is interesting to look back for a moment, not only to gain some appreciation of this early work, but to become acquainted with an approximate method which still remains invaluable in visualizing machine stability.

The method of approximating the transient torque-angle characteristics set forth by Doherty and Nickle [4] is based on the fact that the flux linkages will tend to remain constant in circuits which are largely inductive with a relative small resistance. Therefore, since the field winding has a small resistance, it is generally assumed that the field flux linkages will remain constant during the early part of the transient period. Moreover, it is assumed that all electric transients can be neglected and the action of the damper windings ignored, whereupon the steady state voltage and flux linkage equations apply. With these assumptions, the steady state versions of (5.5-30) and (5.5-34) may be written

$$\Psi^r_{ds} = -X_d I^r_{ds} + X_{md} I'^r_{fd} \tag{5.12-1}$$

$$\Psi'^r_{fd} = X'_{fd} I'^r_{fd} - X_{md} I^r_{ds} \tag{5.12-2}$$

where X_d is defined by (5.5-40) and X'_{fd} is

$$X'_{fd} = X'_{lfd} + X_{md} \tag{5.12-3}$$

Solving (5.12-1) for I'^r_{fd} and substituting the result into (5.12-2) yields

$$\frac{X_{md}}{X'_{fd}} \Psi'^r_{fd} = \Psi^r_{ds} + \left(X_d - \frac{X^2_{md}}{X'_{fd}}\right) I^r_{ds} \tag{5.12-4}$$

Let us now define

$$X'_d = X_d - \frac{X^2_{md}}{X'_{fd}} \tag{5.12-5}$$

and

$$E'_q = \frac{X_{md}}{X'_{fd}} \Psi'^r_{fd} \tag{5.12-6}$$

Equation (5.12-4) may now be written

$$\Psi^r_{ds} = -X'_d I^r_{ds} + E'_q \tag{5.12-7}$$

The reactance X'_d is referred to as the d axis transient reactance. If the field flux linkages are assumed constant then E'_q, which is commonly referred to as the voltage behind transient reactance, is also constant.

The quantities X'_d and E'_q, which are specifically related to the transient period, are each denoted by a prime. Heretofore, we have used the prime to denote rotor variables and rotor parameters referred to the stator windings by a turns ratio. As mentioned, the prime or any other distinguishing notation is seldom used in literature to denote referred quantities; on the other hand, the primes are always used to denote transient quantities. We will use the prime to denote both; the double meaning should not be confusing since the primed quantities which pertain to the transient period are few in number and readily recognized.

Let us now return to the method used to obtain the steady state voltage and torque equations. The steady state voltage equations in the rotor reference frame, (5.9-1) and (5.9-2), were obtained from (5.5-22) and (5.5-23) with the time rate of change of all flux linkages neglected and ω_r set equal to ω_e. These equations could have also been written in the form

$$V^r_{qs} = -r_s I^r_{qs} + \frac{\omega_e}{\omega_b}\Psi^r_{ds} \tag{5.12-8}$$

$$V^r_{ds} = -r_s I^r_{ds} - \frac{\omega_e}{\omega_b}\Psi^r_{qs} \tag{5.12-9}$$

where

$$\Psi^r_{ds} = -X_d I^r_{ds} + X_{md} I'^r_{fd} \tag{5.12-10}$$

$$\Psi^r_{qs} = -X_q I^r_{qs} \tag{5.12-11}$$

If we compare (5.12-10) and (5.12-7) we see that the two equations have the same form. Therefore if, in our previous derivation, we replace X_d with X'_d and $X_{md} I'^r_{fd}$ or E'^r_{xfd} with E'_q we will obtain voltage and torque expressions which should approximate the behavior of the synchronous machine during the early part of transient period assuming the field flux linkages remain constant. In particular, the so-called transient torque-angle characteristic is expressed

$$T_e = \left(\frac{3}{2}\right)\left(\frac{P}{2}\right)\left(\frac{1}{\omega_b}\right)\left[\frac{E'_q\sqrt{2}V_s}{(\omega_e/\omega_b)X'_d}\sin\delta + \left(\frac{1}{2}\right)\left(\frac{\omega_e}{\omega_b}\right)^{-2}\left(\frac{1}{X_q} - \frac{1}{X'_d}\right)(\sqrt{2}V_s)^2 \sin 2\delta\right] \tag{5.12-12}$$

In per unit

$$T_e = \frac{E'_q V_s}{(\omega_e/\omega_b)X'_d}\sin\delta + \left(\frac{V_s^2}{2}\right)\left(\frac{\omega_e}{\omega_b}\right)^{-2}\left(\frac{1}{X_q} - \frac{1}{X'_d}\right)\sin 2\delta \tag{5.12-13}$$

As in the case of (5.9-32) and (5.9-33), (5.12-12) and (5.12-13) are valid only if r_s can be neglected. Also, it is interesting to note that the coefficient of $\sin 2\delta$ is zero if $X_q = X'_d$, which is seldom if ever the case. Actually $X_q > X'_d$ and the coefficient of $\sin 2\delta$ is negative. In other words, the transient electromagnetic torque-angle curve is a function of $\sin 2\delta$ even if $X_q = X_d$.

We have yet to determine E'_q from a readily available quantity. If in the expression for $\tilde{E}_a$, (5.9-18), X_d is replaced with X'_d and $X_{md}I'^r_{fd}$ with E'_q, then

$$\tilde{E}_a = \frac{1}{\sqrt{2}}\left(\frac{\omega_e}{\omega_b}\right)[-(X'_d - X_q)I^r_{ds} + E'_q]e^{j\delta} \tag{5.12-14}$$

Hence, the familiar phasor voltage equation given by (5.9-19) can be used to calculate the pre-disturbance $\tilde{E}_a$, whereupon we can use (5.12-14) to determine E'_q which is assumed to remain constant during the early part of the transient period.

5.13 COMPARISON OF ACTUAL AND APPROXIMATE TRANSIENT TORQUE-ANGLE CHARACTERISTICS DURING A SUDDEN CHANGE IN INPUT TORQUE—FIRST SWING TRANSIENT STABILITY LIMIT

In the studies involving a step increase in input torque, which were reported in a previous section, the machines were initially operating with essentially zero stator current and zero input torque. Hence, $\tilde{E}_a = \tilde{V}_{as}$ and $E'_q = \sqrt{2}V_s$. The steady state torque-angle curve is given by (5.10-1) for the hydro unit and (5.10-2) for the

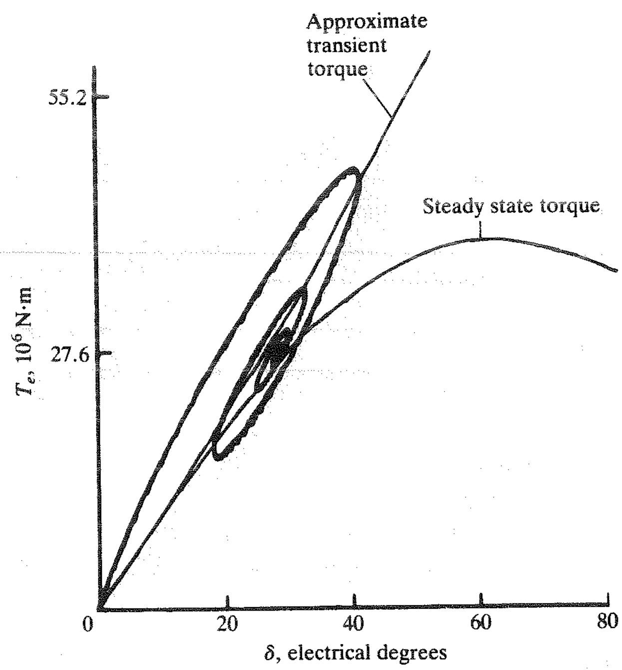

Figure 5.13-1 Comparison of the dynamic torque-angle characteristic during a step increase in input torque from zero to rated with the calculated steady state and approximate transient torque-angle characteristics—hydro turbine generator.

steam unit. For the hydro turbine generator, $X'_d = 0.3448$ ohms and the transient torque-angle characteristic is

$$T_e = (98.5 \sin \delta - 20.5 \sin 2\delta) \times 10^6 \text{ N·m} \tag{5.13-1}$$

For the steam turbine generator, where $X'_d = 0.2591$ ohms

$$T_e = (6.92 \sin \delta - 2.84 \sin 2\delta) \times 10^6 \text{ N·m} \tag{5.13-2}$$

Figures 5.13-1 and 5.13-2 show the approximate transient and steady state torque-angle curves plotted along with the actual dynamic torque-angle characteristics obtained from the studies involving a step increase in input torque (Figs. 5.10-2 and 5.10-4). During the initial swing of the rotor the dynamic torque-angle characteristic follows the approximate transient torque-angle curve more closely than the steady state curve even though the approximation is rather crude especially in the case of the steam turbine generator. As the transients subside, the actual torque-angle characteristic moves toward the steady state torque-angle curve.

The approximate transient torque-angle characteristics is most often used along with equal-area criterion to predict the maximum change in input torque possible without the machine becoming unstable (transient stability limit) rather than the dynamic performance during a relatively small step increase in input torque as portrayed in Figs. 5.13-1 and 5.13-2. In order to compare the results

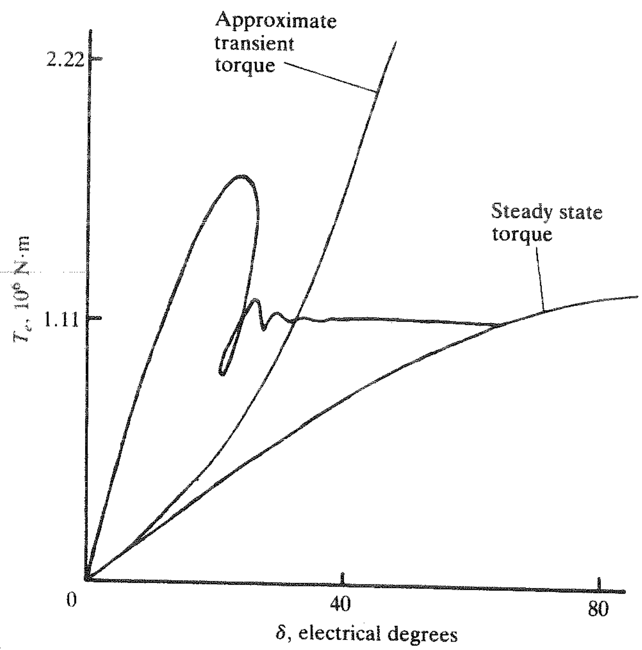

Figure 5.13-2 Same as Fig. 5.13-1 for a steam turbine generator with input torque stepped from zero to 50 percent rated.

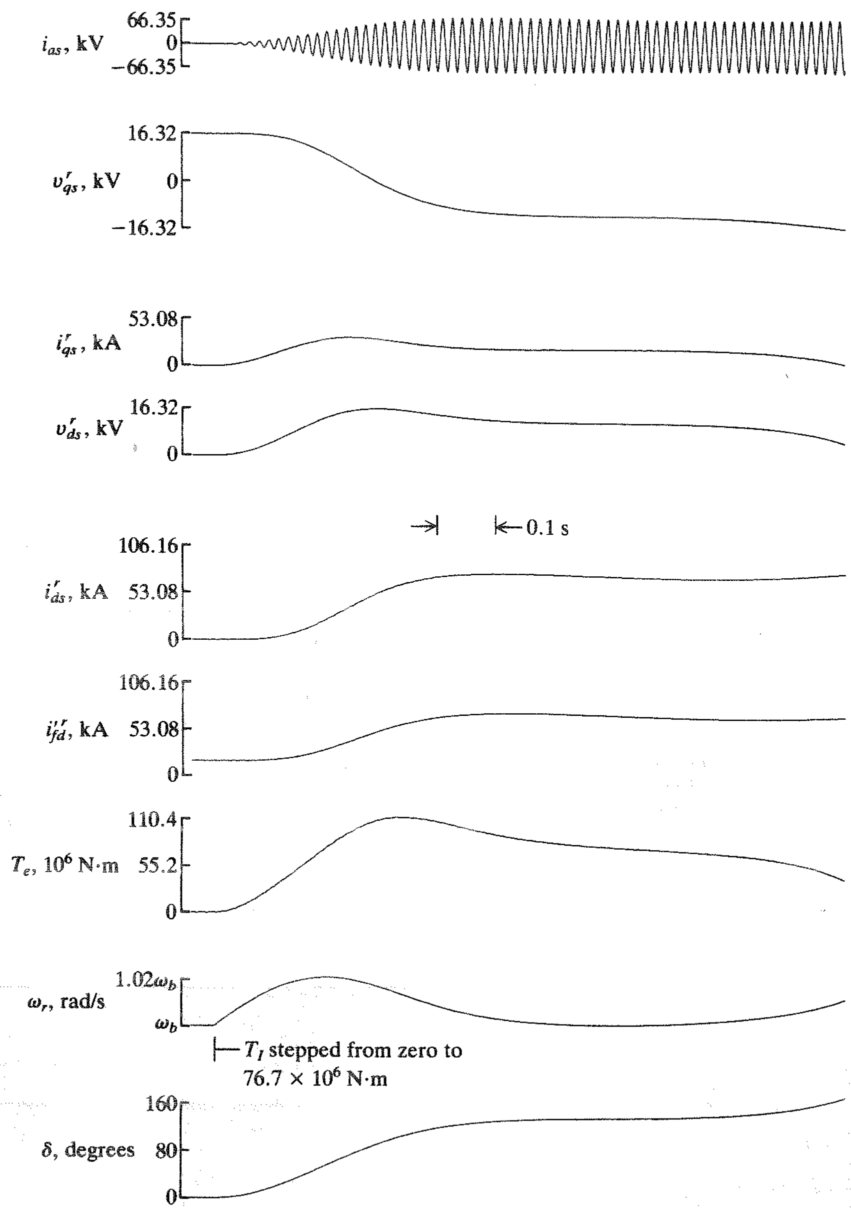

Figure 5.13-3 Dynamic performance of a hydro turbine generator at the "first swing" transient stability limit.

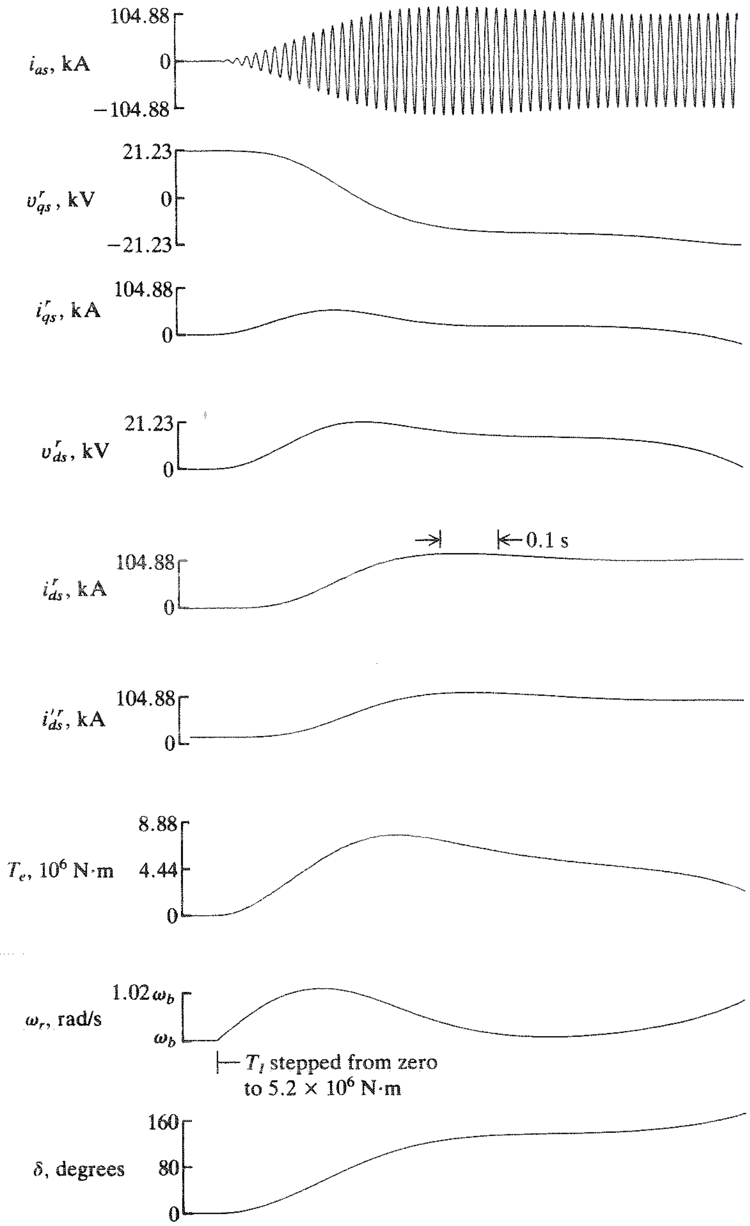

Figure 5.13-4 Same as Fig. 5.13-3 for a steam turbine generator.

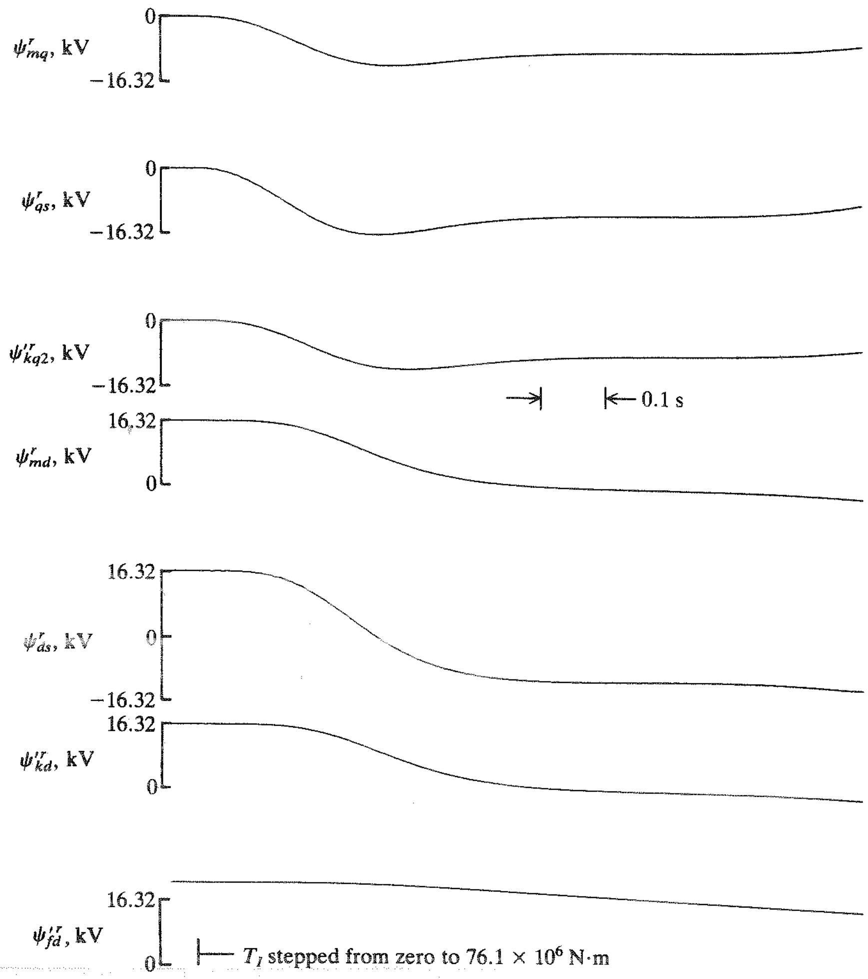

Figure 5.13-5 Traces of flux linkages per second for a hydro turbine generator at the "first swing" transient stability limit.

obtained here with the transient stability limit which we will calculate in a later section using equal-area criterion, it is necessary to define the "first swing" transient stability limit as the maximum value of input torque which can be suddenly applied and the rotor just returns to synchronous speed at the end of the first acceleration above synchronous speed. By trial and error, the transient stability limit for the hydro turbine generator was found to be 76.7 × 10^6 N·m and 5.2 × 10^6 N·m for the steam turbine generator. The computer traces shown in Figs 5.13-3 through 5.13-6 show the transient response of the machine variables with the step input torque equal to the value at the "first swing" transient stability limit. Figures 5.13-3 and 5.13-4 are for the hydro turbine generator and steam turbine generator, respectively, with the same variables shown as in Figs. 5.10-1

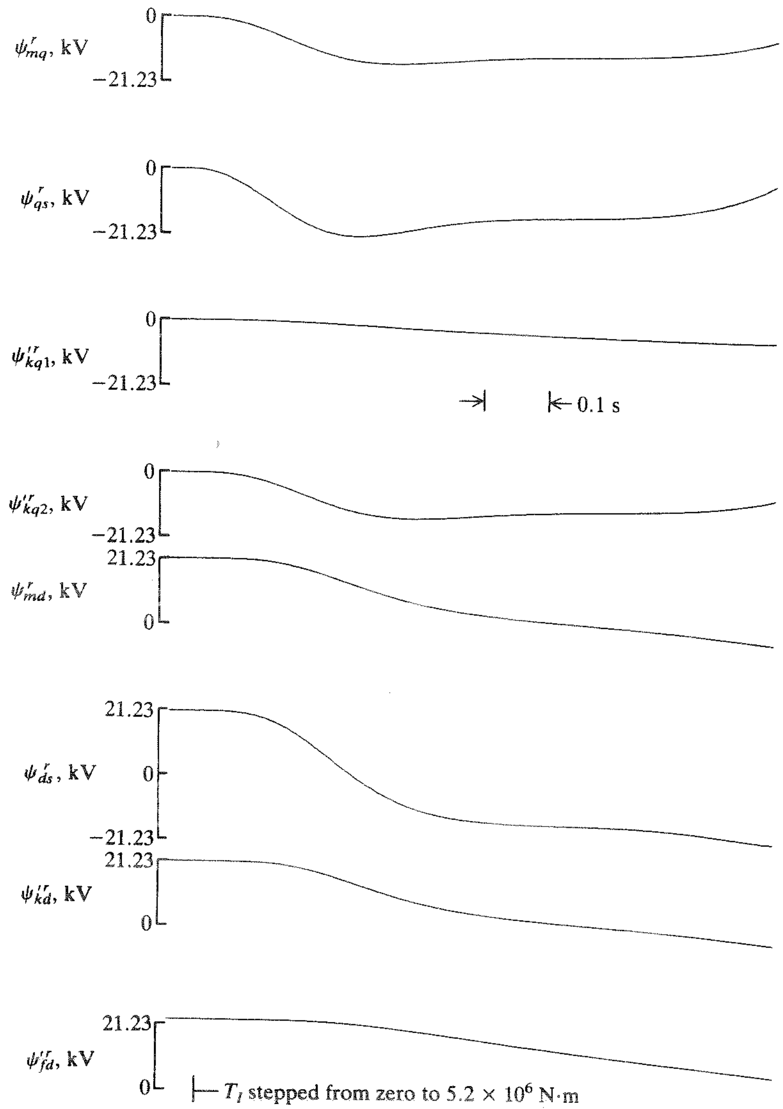

Figure 5.13-6 Same as Fig. 5.13-5 for a steam turbine generator.

and 5.10-3. The machine flux linkages per second are shown in Fig. 5.13-5 for the hydro turbine generator and in Fig. 5.13-6 for the steam turbine generator.

In the case of the hydro unit the field flux linkages are relatively constant during the first swing of the machine varying approximately 17 percent from the original value (Fig. 5.13-5), however, the field flux linkages for the steam unit (Fig. 5.13-6) vary approximately 40 percent during the first swing. This is due to the fact that the field circuit of the hydro unit has a higher reactance to resistance ratio than the field circuit of the steam unit. This observation casts doubt on the

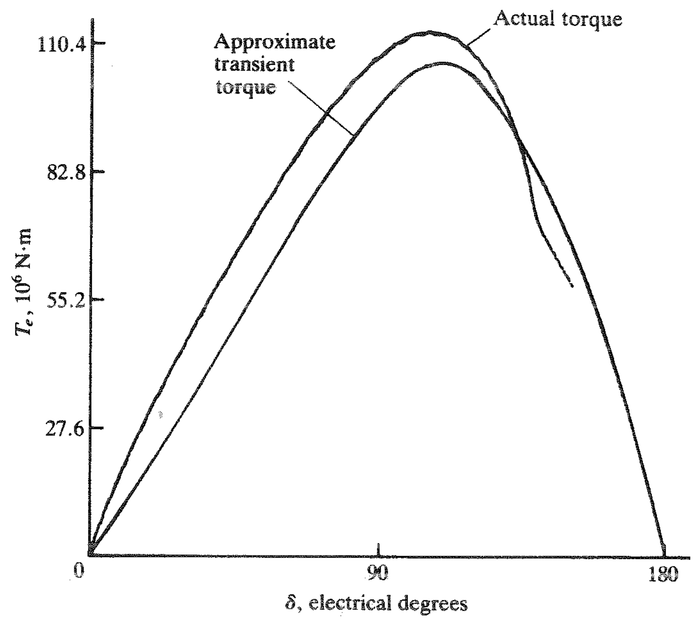

Figure 5.13-7 Comparison of the dynamic torque-angle characteristic at the "first swing" transient stability limit with the approximate transient torque-angle curve—hydro turbine generator.

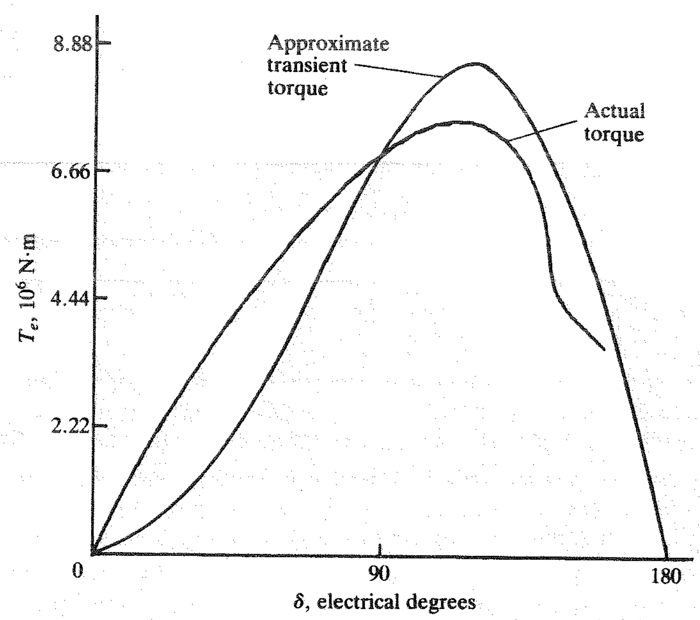

Figure 5.13-8 Same as Fig. 5.13-7 for a steam turbine generator.

accuracy of the approximate transient torque-angle characteristics, especially for the steam unit, as we have already noted in Fig. 5.13-2. It is also important to note from Fig. 5.13-6 that the change in ψ'^{r}_{kq1} is much less than in the case of ψ'^{r}_{fd}. The reactance to resistance ratios of the two circuits gives an indication of this result. In particular, the reactance to resistance ratio of the field circuit of the steam unit is 1333 while this ratio for the $kq1$ winding is 1468.

The actual dynamic torque-angle characteristics obtained from the computer study are shown in Figs. 5.13-7 and 5.13-8 with the approximate transient torque-angle curve calculated using constant flux linkages superimposed thereon. In the computer study the machines went unstable at the transient stability limit. Nevertheless, the transient stability limit has been defined here as the "first swing" transient stability limit in an attempt to duplicate the conditions for which the equal-area criterion is to predict. The facility of the approximate transient torque-angle curve in predicting this transient stability must be deferred until we have described this criterion.

5.14 COMPARISON OF ACTUAL AND APPROXIMATE TRANSIENT TORQUE-ANGLE CHARACTERISTICS DURING A THREE-PHASE FAULT AT THE TERMINALS—CRITICAL CLEARING TIME

In the studies involving a 3-phase fault at the terminals (Figs. 5.11-1 through 5.11-4), each machine is initially operating at rated conditions. In the case of the hydro turbine generator the steady state torque-angle curve is given by (5.11-1) and for the steam turbine generator by (5.11-2). For rated conditions, $E'_q = (1.16)\sqrt{\tfrac{2}{3}}20$ kV for the hydro unit and the approximate transient torque-angle characteristic is

$$T_e = (114.3 \sin \delta - 20.5 \sin 2\delta) \times 10^6 \text{ N}\cdot\text{m} \qquad (5.14\text{-}1)$$

For the steam turbine generator, $E'_q = (1.09)\sqrt{\tfrac{2}{3}}26$ kV and the approximate transient torque-angle characteristic is

$$T_e = (7.53 \sin \delta - 2.84 \sin 2\delta) \times 10^6 \text{ N}\cdot\text{m} \qquad (5.14\text{-}2)$$

Figures 5.14-1 and 5.14-2 show the approximate transient and steady state torque-angle curves for the hydro and steam units, respectively, plotted along with the actual dynamic torque-angle characteristics shown previously in Figs. 5.11-2 and 5.11-4. It is important to note that the approximate transient and steady state torque-angle curves both pass through the steady state operating point. The flux linkages per second during a 3-phase fault and subsequent clearing are shown in Fig. 5.14-3 for the hydro turbine generator and in Fig. 5.14-4 for the steam turbine generator. The corresponding plots of voltages, currents, torque, speed, and rotor angle are shown in Figs. 5.11-1 and 5.11-3.

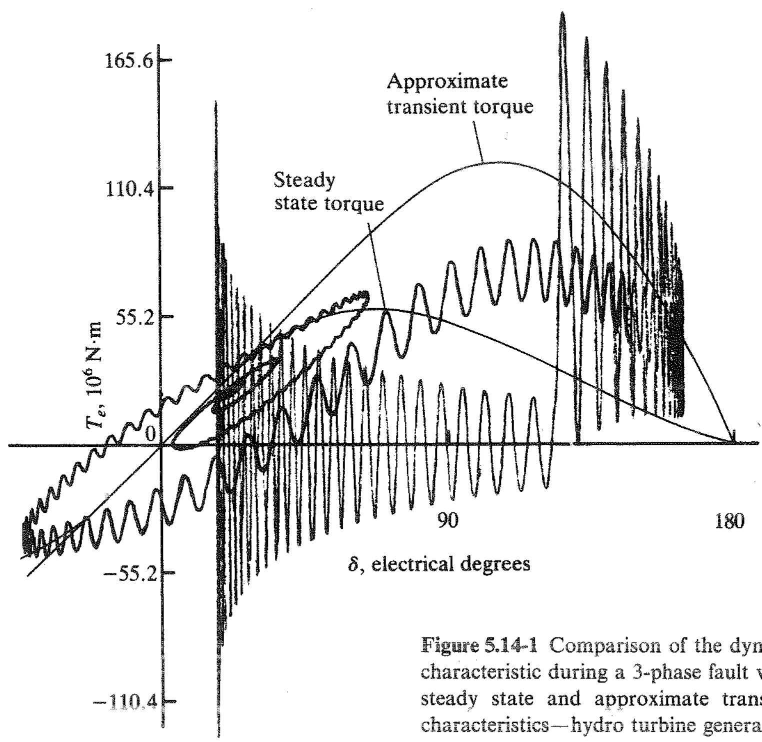

Figure 5.14-1 Comparison of the dynamic torque-angle characteristic during a 3-phase fault with the calculated steady state and approximate transient torque-angle characteristics—hydro turbine generator.

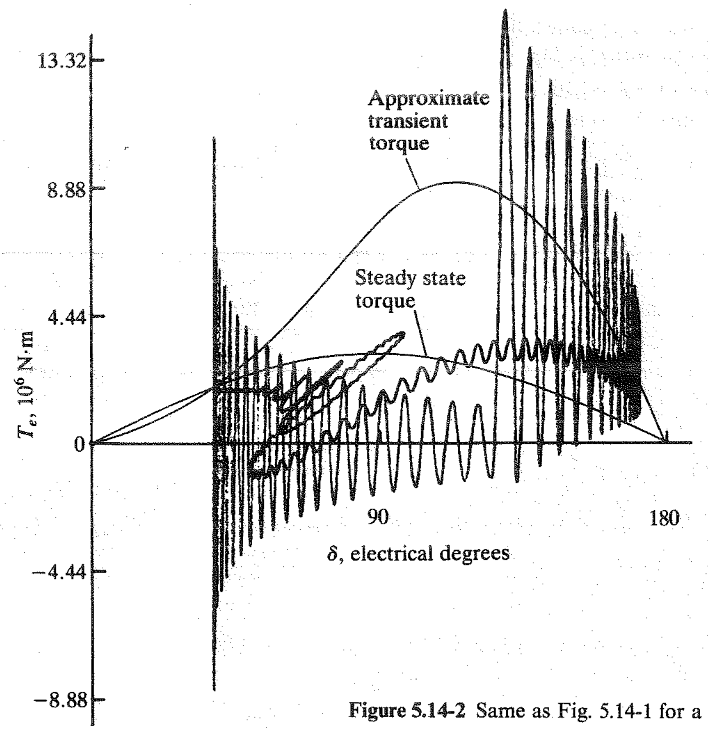

Figure 5.14-2 Same as Fig. 5.14-1 for a steam turbine generator.

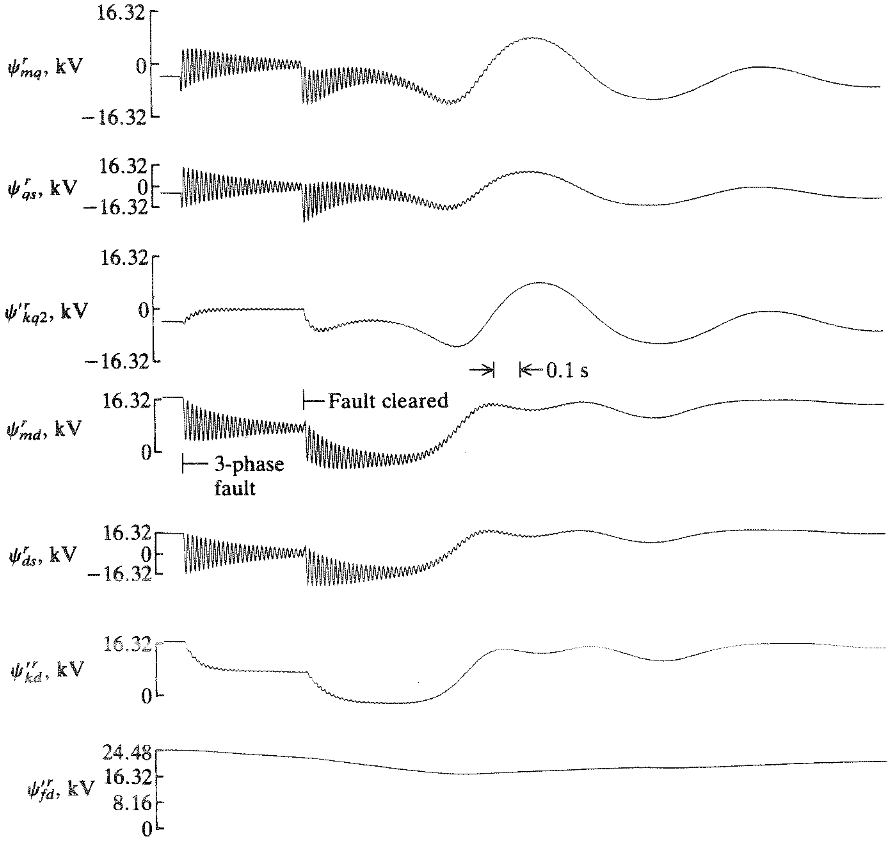

Figure 5.14-3 Traces of flux linkages per second for a hydro turbine generator during a 3-phase fault at the terminals

As mentioned earlier the situation portrayed in this study (Figs. 5.11-1 through 5.11-4 and Figs. 5.14-3 and 5.14-4) is one where only a slight increase in the fault time would cause the machines to slip poles before returning to synchronous speed (see Chap. 10). This limiting condition is commonly referred to as the *critical clearing time* and the corresponding maximum rotor angle attained is called the *critical clearing angle*. In the case of the hydro turbine generator the critical clearing time and angle were found by trial and error to be 0.466 seconds and 123 degrees. For the steam turbine generator these values were found to be 0.362 seconds and 128 degrees.

During the fault, the average value of the electromagnetic torque is essentially zero since the ohmic losses are small. The approximate transient torque-angle curve during this period is zero since the ohmic losses are neglected. The approximate transient torque-angle curve plotted in Figs. 5.14-1 and 5.14-2 applies only before the 3-phase fault and after it is cleared. If the approximate

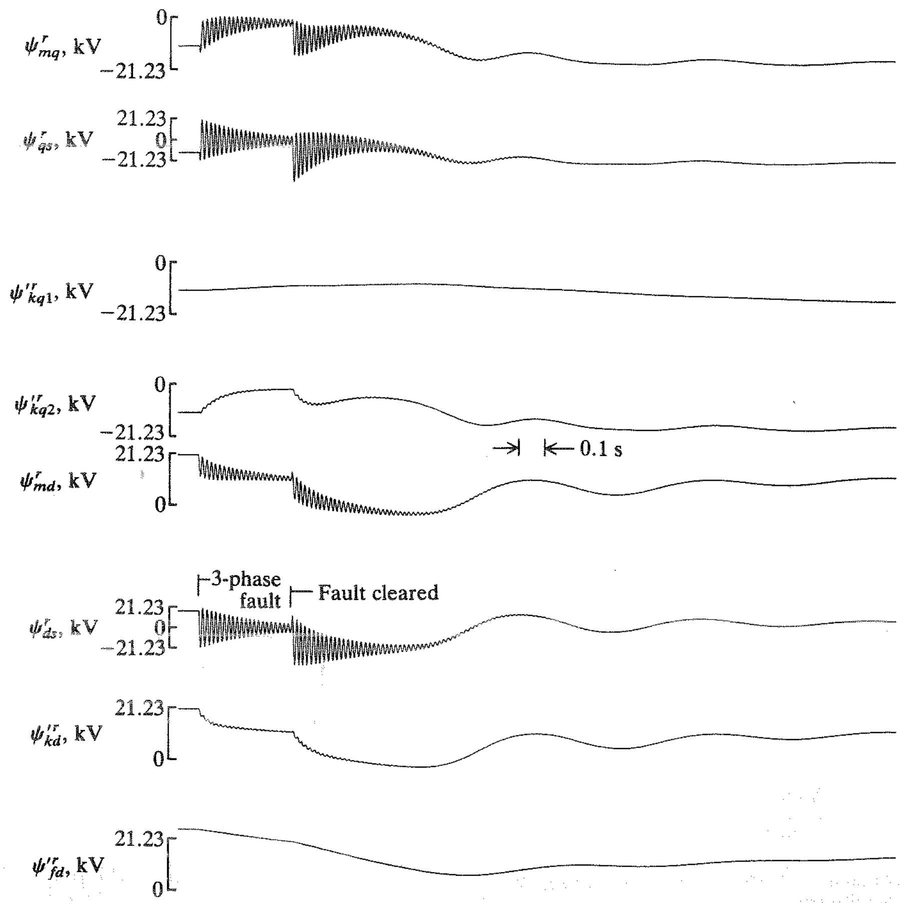

Figure 5.14-4 Traces of flux linkages per second for a hydro turbine generator during a 3-phase fault at the terminals.

transient torque-angle curve accurately portrayed the electromagnetic torque following the fault the average of the instantaneous torque would traverse back and forth on this torque-angle characteristic until the initial operating condition was reestablished. The approximate characteristic appears to be adequate immediately following the fault when the rotor has reached the maximum angle. Thereafter, the approximation is quite inadequate especially in the case of the steam turbine generator (Fig. 5.14-2). This inaccuracy can of course be contributed primarily to the fact that the average value of the field flux linkages do not remain constant during and following the fault. The change in field flux linkages is less in the case of the hydro unit and the approximation is more accurate than in the case of the steam unit.

As in the case of a step increase in input torque the damper winding flux linkages, ψ'^r_{kq1}, of the steam unit remains more nearly constant than any of the other machine flux linkages. Although the assumption of constant field flux linkages is by far the most common, there are refinements which can be made to yield the approximate transient torque-angle characteristic more accurate. In particular, it is sometimes assumed that the flux linkages of a q axis damper winding also remains constant along with the field flux linkages. A voltage, different from E'_q, is then calculated behind a transient impedance. We will not consider this or other refinements in this text.

5.15 EQUAL-AREA CRITERION

As mentioned previously, the approximate transient torque-angle characteristics along with the equal-area method were used extensively to predict the transient response of synchronous machines. In most cases, these concepts were used to determine the transient stability limit and the critical clearing time. The theory underlying the equal-area method as applied to an input torque disturbance or a system fault can be readily established. Regardless of discrepancies or questions which might arise as to the validity of the results, the approximate method of determining the transient torque and the application of the equal-area criterion to predict stability are very useful in understanding the overall dynamic behavior of synchronous machines.

Input Torque Change

For this development, let us consider the approximate transient torque-angle curve for the hydro turbine generator shown in Fig. 5.13-7 and given again in Fig. 5.15-1. Consider a sudden step increases in input torque of T_I from an initial value of zero so as to correspond with our earlier work. This torque level of T_I is identified by a horizontal line in Fig. 5.15-1. At the instant the input torque is applied the accelerating torque is T_I since T_e is initially zero and the losses are neglected. We see that the accelerating torque on the rotor is positive when $T_I > T_e$, where here T_e is the approximate transient torque-angle curve.

Work or energy is the integral of force times a differential distance or, in the case of a rotational system, the integral of torque times a differential angular displacement. Hence, the energy stored in the rotor during the initial acceleration is

$$\int_{\delta_0}^{\delta_1} (T_I - T_e)\, d\delta = \text{area } OABO \tag{5.15-1}$$

The energy given up by the rotor as it decelerates back to synchronous speed is

$$\int_{\delta_1}^{\delta_2} (T_I - T_e)\, d\delta = \text{area } BDCB \tag{5.15-2}$$

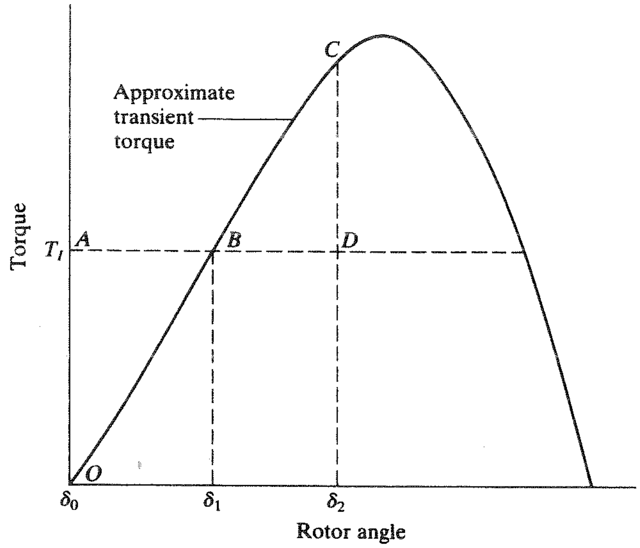

Figure 5.15-1 Equal-area criterion for sudden change in input torque.

The maximum angle is reached and the rotor will return to synchronous speed after the application of the input torque when

$$|\text{Area } OABO| = |\text{area } BDCB| \tag{5.15-3}$$

It follows that the rotor angle of the synchronous machine would then oscillate back and forth between δ_0 and δ_2. We know, however, that the action of the damper windings will cause these oscillations to subside. One might then be led to believe that the new steady state operating point would be established at point B. We, of course, know that this is not the case; instead, the new operating point is reestablished on the steady state torque-angle curve as illustrated in Figs. 5.10-2 and 5.10-4 or Figs. 5.13-1 and 5.13-2. We are aware of the limitations of the approximate transient torque-angle curve and thus any method utilizing this approximation will, at best, be adequate only for the first swing of the rotor from synchronous speed. It is left to the reader to show that the application of equal-area criterion, which involves a graphically solution, gives a transient stability limit of $T_I = 68.1 \times 10^6$ N·m for the hydro turbine generator and $T_I = 4.65 \times 10^6$ N·m for the steam turbine generator.

It is instructive to compare the "first swing" transient stability limit obtained by means of a computer study for the two machines under consideration with the limits obtained by applying equal-area criterion. It is recalled that the actual dynamic torque-angle characteristics, which are shown in Figs. 5.13-7 and 5.13-8 with the transient torque-angle curve calculated using constant flux linkages superimposed thereon, gave a "first swing" transient stability limit for the hydro turbine generator of 76.7×10^6 N·m and 5.2×10^6 N·m for the steam turbine generator. The maximum torques obtained using the equal-area criterion are approximately 10 percent less than those obtained from the computer study.

It should also be mentioned that the term involving the $\sin 2\delta$ is often ignored and only the $\sin \delta$ term of (5.12-12) is used to calculate the approximate torque-angle curve. This enables a trial-and-error analytical solution and in this case it is a closer approximation. In particular, this method yields a transient stability limit of 71.2×10^6 N·m for the hydro turbine generator and 5.03×10^6 N·m for the steam turbine generator.

Three-Phase Fault

The approximate transient torque-angle curve along with the equal-area criterion is most often used to predict the large excursion dynamic behavior of a synchronous machine during a system fault. The application of the equal-area method during a 3-phase system fault can be described by considering the approximate transient torque-angle curve of the hydro unit given in Fig. 5.15-2. Assume that the input torque T_I is constant and the machine is operating steadily, delivering power to the system with a rotor angle δ_0. When the 3-phase fault occurs at the terminals the power output drops to zero and thus the approximate T_e is zero since the resistances are neglected. The machine accelerates with the total input torque as the accelerating torque. The fault is cleared at δ_1 and, in this case, the torque immediately becomes the value of the approximate transient torque (point

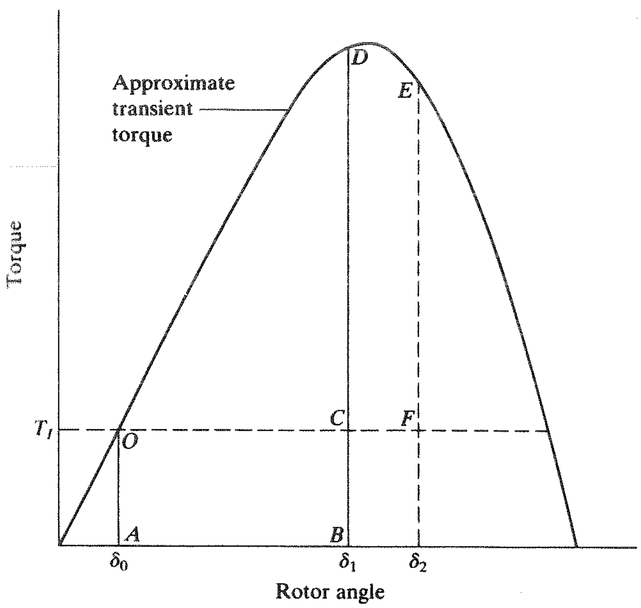

Figure 5.15-2 Equal-area criterion for a 3-phase fault at the terminals.

D in Fig. 5.15-2). The energy stored in the rotor during the acceleration or advance in angle from δ_0 to δ_1 is

$$\int_{\delta_0}^{\delta_1} (T_I - T_e)\, d\delta = \text{area } OABCO \tag{5.15-4}$$

where T_e is zero and T_I is constant.

After the clearing of the fault the rotor decelerates back to synchronous speed. The energy given up by the rotor during this time is

$$\int_{\delta_1}^{\delta_2} (T_I - T_e)\, d\delta = \text{area } CDEFC \tag{5.15-5}$$

The maximum rotor angle is reached when

$$|\text{Area } OABCO| = |\text{area } CDEFC| \tag{5.15-6}$$

The critical clearing angle is reached when any future increase in δ_1 causes the total area representing decelerating energy to become less than the area representing the accelerating energy. This occurs when δ_2, or point F, is at the intersection of T_I and T_e.

The clearing time may be calculated from the clearing angle by using (5.8-4) which may be written

$$\delta_1 - \delta_0 = \frac{\omega_b}{2H} \int_{t_0}^{t_1} \int_{t_0}^{\tau} (T_I - T_e)\, dt\, d\tau \tag{5.15-7}$$

If, as in this example, $T_e = 0$ and T_I a constant during the fault and with $t_0 = 0$, the clearing time t_1 becomes

$$t_1 = \sqrt{\frac{(\delta_1 - \delta_0)4H}{\omega_b T_I}} \tag{5.15-8}$$

It is recalled that the critical clearing times and angles obtained by computer study were 0.466 seconds and 123 degrees for the hydro unit and 0.362 seconds and 128 degrees for the steam unit. These values compare to 0.41 seconds and 122 degrees for the hydro unit and 0.33 seconds and 128 degrees for the steam unit, obtained by a graphical application of the method of equal-area. Since the critical clearing angles were found to be essentially the same by both methods, the larger in critical clearing times can be attributed to the power lost in the resistance during the fault. The accuracy of the approximate method is indeed surprising; a result which should not be taken as an indication that the approximate method is highly accurate in general.

As mentioned previously, the $\sin 2\delta$ terms are often ignored in the approximate transient torque-angle characteristics. With the machine operating under load, ignoring the $\sin 2\delta$ term yields a fictitious initial rotor angle. Although this leads to a simplified method of solution and even though it was used widely in the past, we will not consider it in this text. Actually, sufficient background has been established so that the reader can readily develop this technique if the need arises.

5.16 REFERENCES

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[2] D. R. Brown and P. C. Krause, "Modeling of Transient Electrical Torques in Solid Iron Rotor Turbogenerators," *IEEE Trans. Power Apparatus and Systems*, Vol. 98, September/October 1979, pp. 1502–1508.

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5.17 PROBLEMS

1 A 2-pole, 2-phase, salient-pole synchronous machine is shown in Fig. 5.17-1. In the case of a 2-phase machine the magnetizing inductances are defined

$$L_{mq} = L_A - L_B$$

$$L_{md} = L_A + L_B$$

Derive the voltage equations in machine variables similar in form to (5.2-29). Express all resistance and inductance matrices.

2 In the case of a 3-phase synchronous machine L_{mq} and L_{md} are defined with a 3/2 factor. This factor is unity in the case of the 2-pole machine. Why?

3 Modify the voltage equations (5.2-29) to describe a 3-phase round rotor synchronous machine.

4 Modify the voltage equations (5.2-29) to describe a 3-phase reluctance machine. Assume the *fd* and *kq*1 windings are not present and denote the *kq*2 winding as simply the *kq* winding. Also, assume positive stator currents flow into the machine.

5 Justify the statement following (5.3-4) regarding the fact that this torque expression will be positive for positive currents assumed into the machine if the sign of the coefficients of $\sin 2\theta_r$ and $\cos 2\theta_r$ are changed.

6 Derive the expression for electromagnetic torque in machine variables for the 2-phase synchronous machine shown in Fig. 5.17-1.

7 Determine $\mathbf{K}_s\mathbf{L}_s(\mathbf{K}_s)^{-1}$ given in (5.4-4). Show that it is independent of θ and θ_r only if $\theta = \theta_r$.

8 Repeat Prob. 7 for $(\mathbf{L}'_{sr})^T(\mathbf{K}_s)^{-1}$.

9 Show that if the arbitrary reference-frame variables given in (5.4-4) are transformed to the rotor reference frame, (5.5-4) results if $\theta = \theta_r$. Is this also true if only $\omega = \omega_r$? Why?

10 Verify (5.5-5)–(5.5-7).

11 Derive Park's equations for the 2-pole, 2-phase, salient-pole synchronous machine shown in Fig. 5.17-1. Express the final equations in terms of flux linkages per second similar in form to (5.5-22)–(5.5-35).