Lecture #5
Three-Phase Induction Machines

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Summary of Dynamic Equations of Induction Machines

- machine dynamic equations in abc can be written as

$$\bar{V}_{abcs} = p\bar{\lambda}_{abcs} + r_s \bar{i}_{abcs}$$

$$\bar{V}_{abcs} = p\bar{\lambda}_{abcr} + r_r \bar{i}_{abcr}$$

- Flux linkages are

$$\begin{bmatrix} \bar{\lambda}_{abcs} \\ \lambda_{abcr} \end{bmatrix} = \begin{bmatrix} \bar{L}_s & \bar{L}_{SR} \\ (\bar{L}_{SR})^T & \bar{L}_R \end{bmatrix} \begin{bmatrix} \bar{i}_{abcs} \\ \bar{i}_{abcr} \end{bmatrix}$$

- $$i'_{abcr} = \frac{N_r}{N_s} \bar{i}_{abcr}$$

- $$V'_{abcr} = \frac{N_s}{N_r} \bar{V}_{abcr}$$

- $$\lambda'_{abcr} = \frac{N_s}{N_s} \bar{\lambda}_{abcr}$$

$$[\bar{L}_s] = \begin{bmatrix} L_{ls}+L_{ms} & -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} \\ -\frac{1}{2}L_{ms} & L_{ls}\,L_{ms} & -\frac{1}{2}L_{ms} \\ -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} & L_{ls}+L_{ms} \end{bmatrix}, \quad [\bar{L}_R] = \begin{bmatrix} L_{lr}+L_{mr} & -\frac{1}{2}L_{mr} & -\frac{1}{2}L_{mr} \\ -\frac{1}{2}L_{mr} & L_{lr}+L_{mr} & -\frac{1}{2}L_{mr} \\ -\frac{1}{2}L_{mr} & -\frac{1}{2}L_{mr} & L_{lr}+L_{mr} \end{bmatrix}$$

$$[\bar{L}_{SR}] = L_{SR} \begin{bmatrix} \cos\theta_r & \cos(\theta_r + \frac{2\pi}{3}) & \cos(\theta_r - \frac{2\pi}{3}) \\ \cos(\theta_r - \frac{2\pi}{3}) & \cos\theta_r & \cos(\theta_r + \frac{2\pi}{3}) \\ \cos(\theta_r + \frac{2\pi}{3}) & \cos(\theta_r - \frac{2\pi}{3}) & \cos\theta_r \end{bmatrix}$$

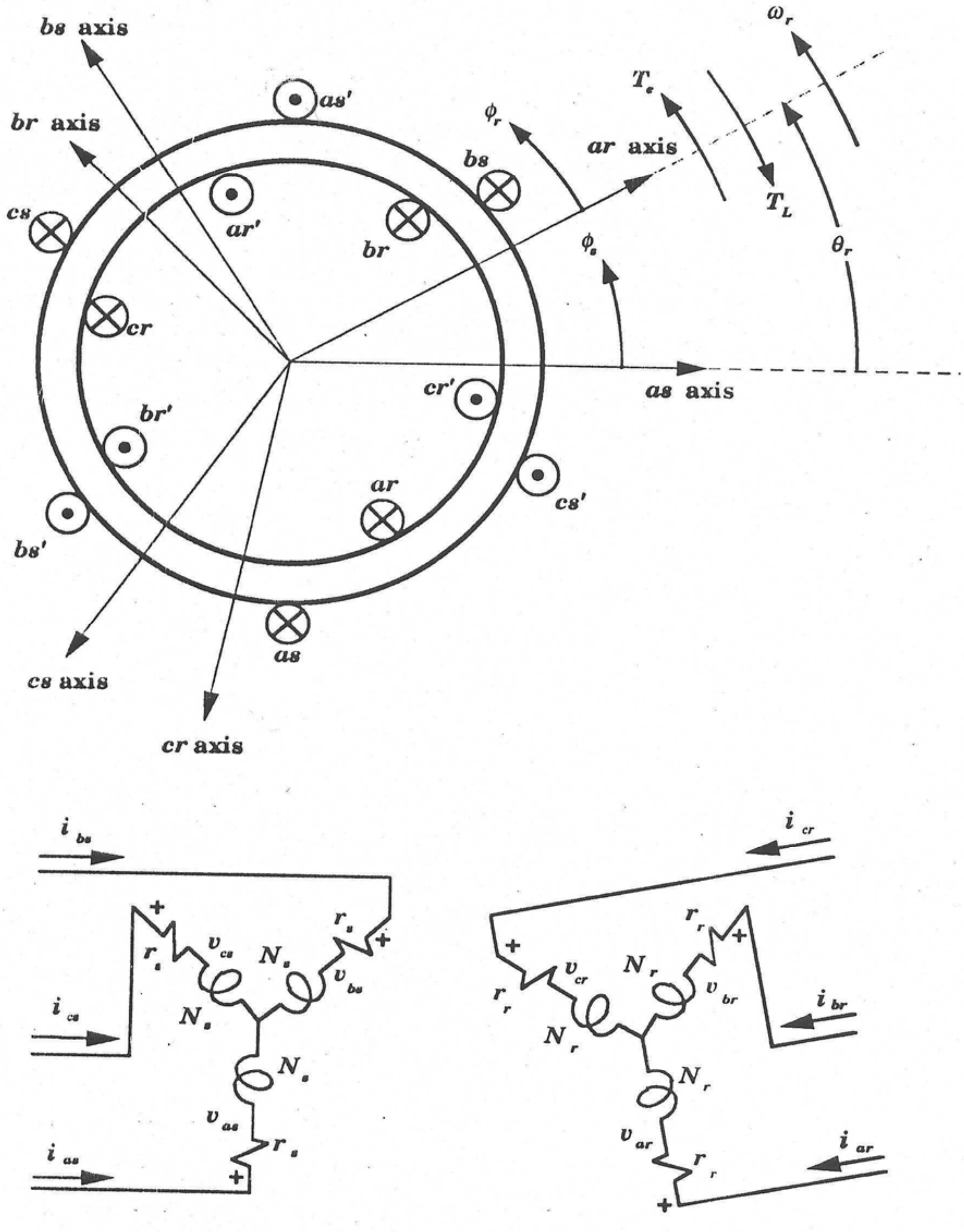


FIGURE 1
A two-pole three-phase symmetrical induction machine.

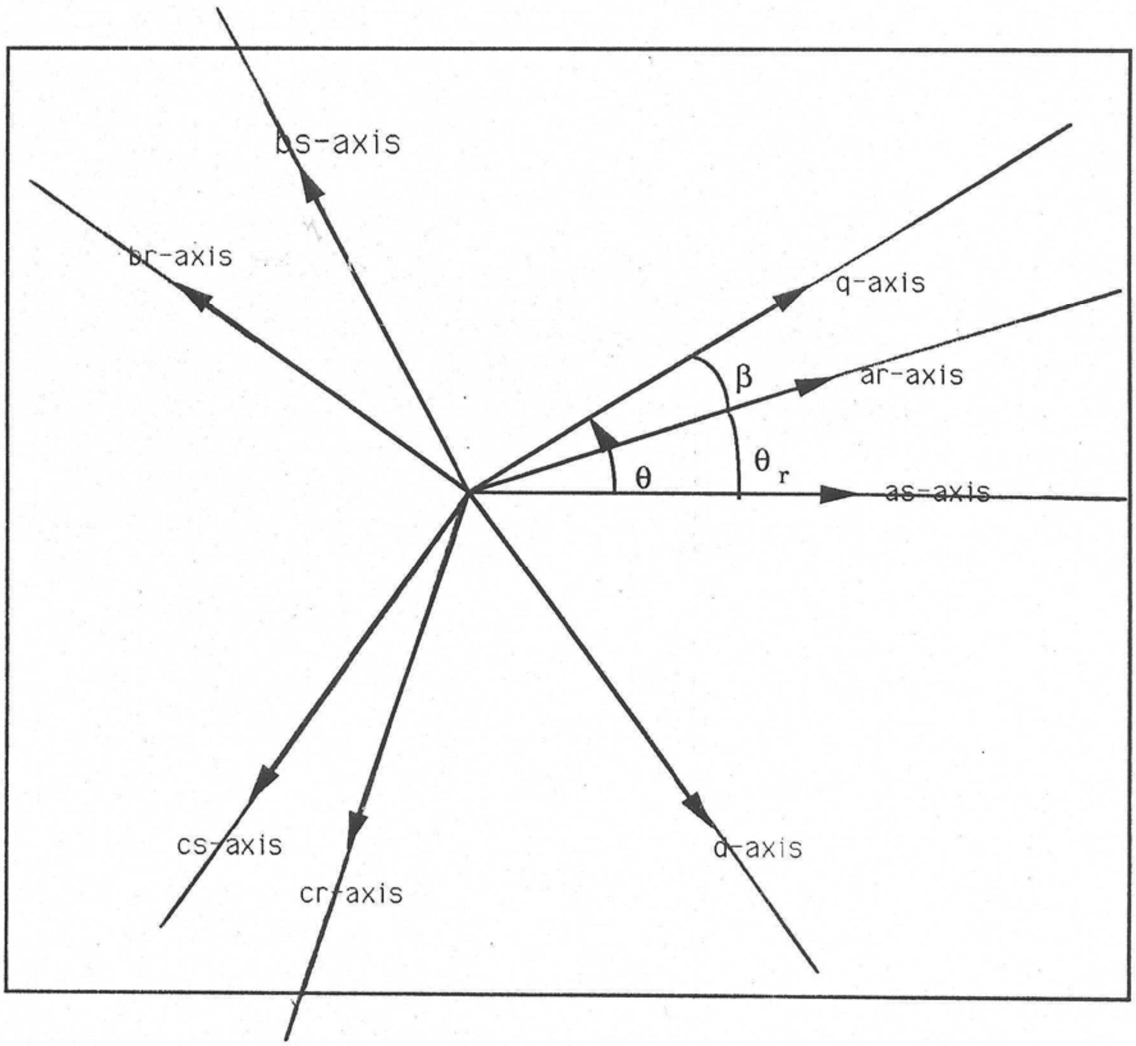


Fig. 1 Axis of 2-pole, 3-phase symmetrical machine

• Transformation equations are:

$$\bar{f}_{qdos} = T_s(\theta)\,\bar{f}_{abcs}$$

$$T_s(\theta) = \frac{2}{3}\begin{bmatrix} \cos\theta & \cos(\theta - \frac{2\pi}{3}) & \cos(\theta + \frac{2\pi}{3}) \\ \sin\theta & \sin(\theta - \frac{2\pi}{3}) & \sin(\theta + \frac{2\pi}{3}) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\bar{f}_{qdor} = T_r(\beta)\,\bar{f}_{abcr}$$

$$T_r(\beta) = \frac{2}{3}\begin{bmatrix} \cos\beta & \cos(\beta - \frac{2\pi}{3}) & \cos(\beta + \frac{2\pi}{3}) \\ \sin\beta & \sin(\beta - \frac{2\pi}{3}) & \sin(\beta + \frac{2\pi}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$[T_s(\theta)]^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & 1 \\ \cos(\theta - \frac{2\pi}{3}) & \sin(\theta - \frac{2\pi}{3}) & 1 \\ \cos(\theta + \frac{2\pi}{3}) & \sin(\theta + \frac{2\pi}{3}) & 1 \end{bmatrix}$$

$$[T_r(\beta)]^{-1} = \begin{bmatrix} \cos\beta & \sin\beta & 1 \\ \cos(\beta - \frac{2\pi}{3}) & \sin(\beta - \frac{2\pi}{3}) & 1 \\ \cos(\beta + \frac{2\pi}{3} & \sin(\beta + \frac{2\pi}{3}) & 1 \end{bmatrix}$$

$$\beta = \theta - \theta_r$$

• Machine dynamic equations from a reference frame with an angular speed ~~ω~~ of rotating d-q axes can be written as

$$V_{qs} = r_s i_{qs} + P\lambda_{qs} + \omega \lambda_{ds}$$
$$V_{ds} = r_s i_{ds} + P\lambda_{ds} - \omega \lambda_{qs}$$

$$v_{os} = r_s i_{os} + P\lambda_{os}$$

$$V_{qr} = r_r i'_{qr} + P\lambda'_{qr} + (\omega - \omega_r)\lambda'_{dr}$$

$$V_{dr} = r_r i'_{dr} + P\lambda'_{dr} - (\omega - \omega_r)\lambda'_{qr}$$

$$V_{or} = r_r i'_{or} + P\lambda'_{or}$$

$$\omega = \frac{d\theta}{dt} \qquad \omega - \omega_r = \frac{d}{dt}(\beta) = \frac{d}{dt}(\theta - \theta_r)$$

• Flux linkage equations can be written as

$$\lambda_{qs} = L_{ls} i_{qs} + L_{mq}(i_{qs} + i'_{qr})$$
$$\lambda_{ds} = L_{ls} i_{ds} + L_{md}(i_{ds} + i'_{dr})$$
$$\lambda_{os} = L_{ls} i_{os}$$

$$\lambda'_{qr} = L'_{lr} i'_{qr} + L_{mq}(i_{qs} + i'_{qr})$$
$$\lambda'_{dr} = L'_{lr} i'_{dr} + L_{md}(\lambda_{ds} + i'_{dr})$$
$$\lambda'_{or} = L'_{lr} i'_{or}$$

$$T_e = \frac{3}{2}(P/2)(\lambda_{ds} i_{qs} - \lambda_{qs} i_{ds})$$

N-m

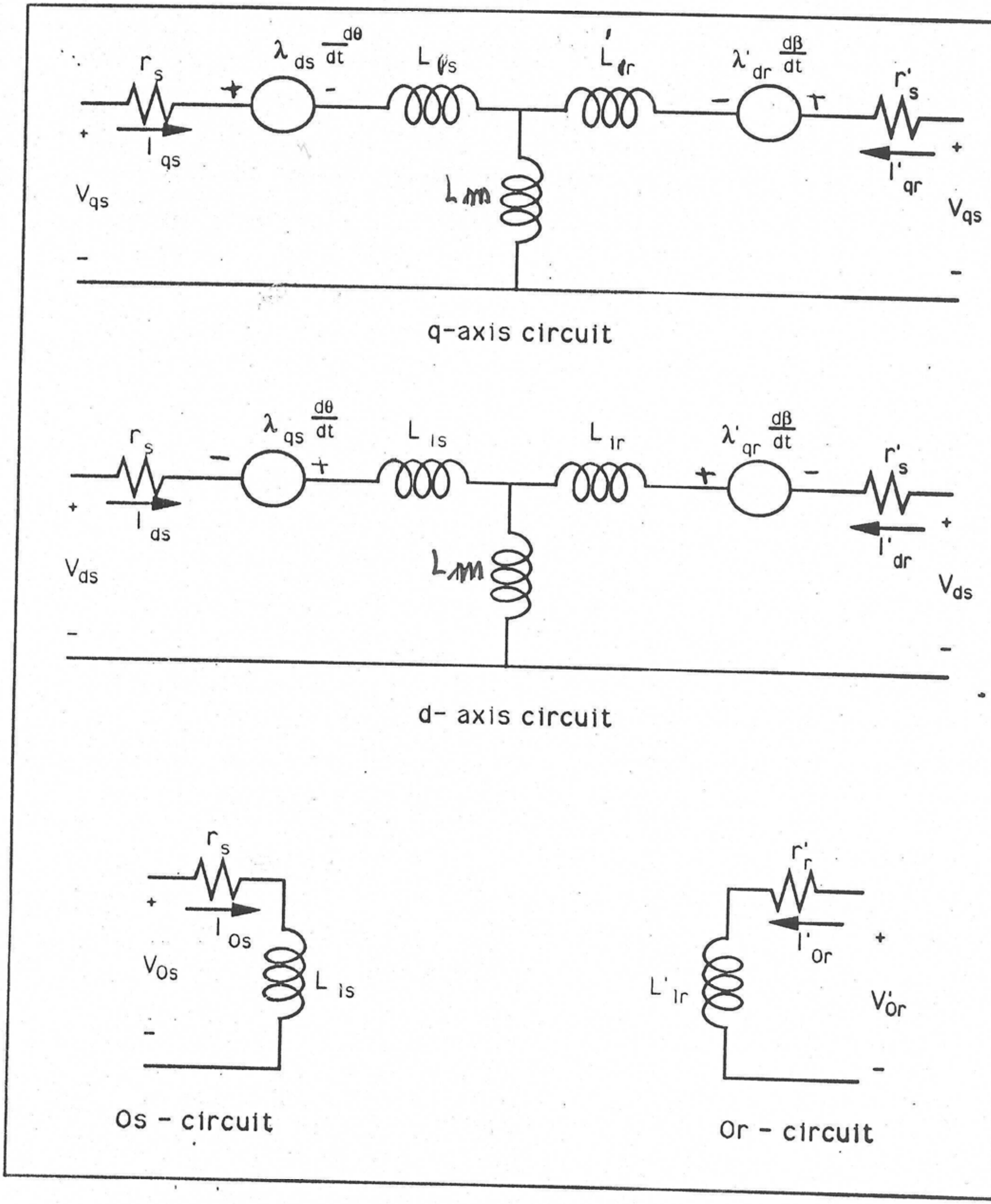


Fig. 3 Equivalent circuits of a 3-phase, symmetrical induction machine with rotating d-q axis at speed of ω

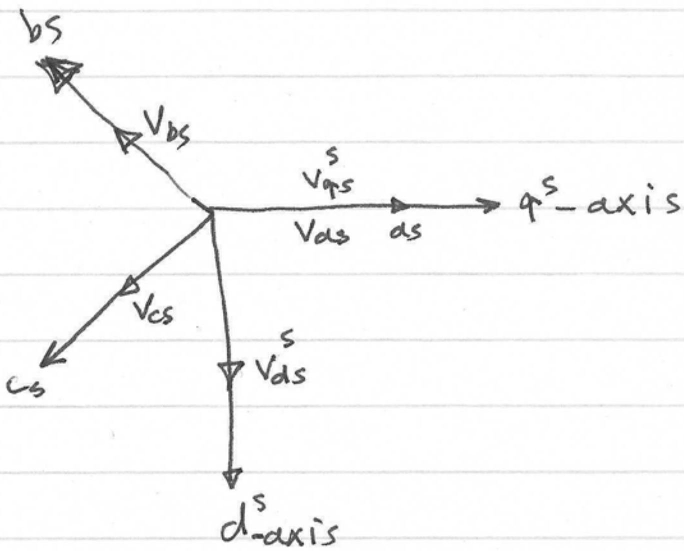
- when ω ~~is~~ equal to zero, the reference frame is fixed in the stator ($d^s - q^s$ ref. frame)

- when ω is equal to ω_e, the reference frame is fixed on the synchronously rotating reference frame.

- when ω is equal to ω_r, the reference frame is fixed in the rotor. That is, the reference frame is rotating at speed of ω_r.

Axes transformation.

- consider the stationary reference where the q^s-axis is coincident with the as-axis. That is $\theta = 0$



stationary as-bs-cs to ~~qs~~ $d^s - q^s$ axes transformation.

- From the transformation equations with $\theta = 0$, we will have,

$$V_{as} = V_{qs}^{s}$$

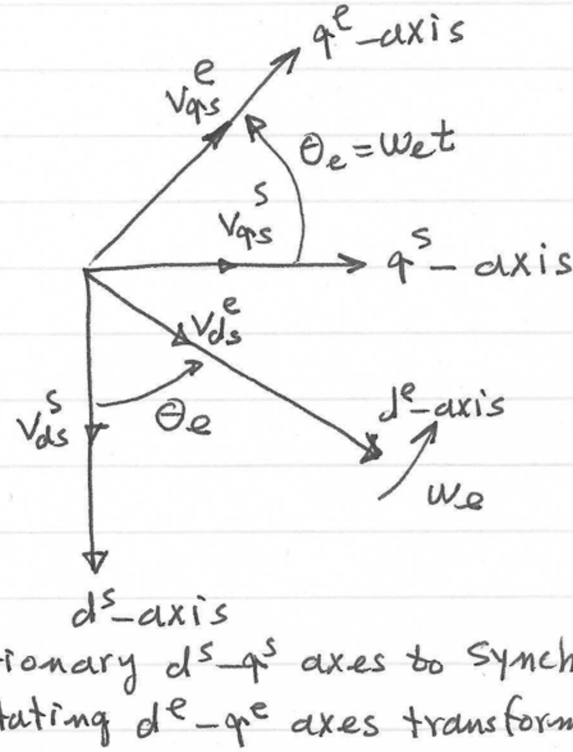
$$V_{bs} = -\frac{1}{2} V_{qs}^{s} - \frac{\sqrt{3}}{2} V_{ds}^{s}$$

$$V_{cs} = -\frac{1}{2} V_{qs}^{s} + \frac{\sqrt{3}}{2} V_{ds}^{s}$$

$$V_{qs}^{s} = \frac{2}{3} V_{as} - \frac{1}{3} V_{bs} - \frac{1}{3} V_{cs} = V_{as}$$

$$V_{ds}^{s} = -\frac{1}{\sqrt{3}} V_{bs} + \frac{1}{\sqrt{3}} V_{cs}$$

- Voltages in the stationary $d^s - q^s$ frame can be converted to the synchronously rotating $d^e - q^e$ frame using the fig. below:



Stationary d^s-q^s axes to synchronously rotating $d^e - q^e$ axes transformation.

$$V_{qs}^{e} = V_{qs}^{s} \cos w_e t - V_{ds}^{s} \sin w_e t$$

$$V_{ds}^{e} = V_{qs}^{s} \sin w_e t + V_{ds}^{s} \cos w_e t$$

or

$$V_{qs}^{s} = V_{qs}^{e} \cos w_e t + V_{ds}^{e} \sin w_e t$$

$$V_{ds}^{s} = -V_{qs}^{e} \sin w_e t + V_{ds}^{e} \cos w_e t$$

- For example, assume

$$V_{as} = V_m \cos w_e t$$
$$V_{bs} = V_m \cos (w_e t - 120^\circ)$$
$$V_{cs} = V_m \cos (w_e t + 120^\circ)$$

- $V_{qs}^{s} = \frac{2}{3} V_{as} - \frac{1}{3} V_{bs} - \frac{1}{3} V_{cs} = V_{as} = V_m \cos w_e t$

- $V_{ds}^{s} = -\frac{1}{\sqrt{3}} V_{bs} + \frac{1}{\sqrt{3}} V_{cs} = -V_m \sin w_e t$

- $V_{qs}^{e} = V_{qs}^{s} \cos w_e t - V_{ds}^{s} \cdot \sin w_e t$

$$= V_m \cos w_e t \cos w_e t - (-V_m \sin w_e t)(\sin w_e t)$$
$$= V_m$$

[redacted]

- The above relations verify that the sinusoidal variables appear as dc quantities in a synchronously rotating reference frame.

Induction Machine Steady - State
Equivalent circuit.

- consider stationary reference frame
  $\omega = 0$

$$V_{qs}^{s} = r_s i_{qs}^{s} + p \lambda_{qs}^{s}$$

$$\lambda_{qs}^{s} = L_{ss} i_{qs}^{s} + L_{ms} i_{qr}^{\prime s} \quad , L_{ss} = L_{ls} + L_{mq}$$

$$V_{ds}^{s} = r_s i_{ds}^{s} + p \lambda_{ds}^{s}$$

$$\lambda_{ds}^{s} = L_{ss} i_{ds}^{s} + L_{ms} i_{dr}^{\prime s} \quad , L_{ss} = L_{ls} + L_{md}$$

$L_{ms} = L_{md} = L_{mq}$

$$V_{qr}' = r_r' i_{qr}' - \omega_r \lambda_{dr}^{\prime s} + p \lambda_{qr}^{\prime s}$$

$$V_{dr}' = r_r' i_{dr}' + \omega_r \lambda_{qr}^{\prime s} + p \lambda_{dr}^{\prime s}$$

$$\lambda_{dr}^{\prime s} = L_{rr}' i_{dr}^{\prime s} + L_{ms} i_{ds}^{s}$$

$L_{rr}' = L_{lr}' + L_{ms}$

$$\lambda_{qr}^{\prime s} = L_{rr}' i_{qr}^{\prime s} + L_m i_{qs}^{s}$$

- Assume V_{as}, V_{bs} and V_{cs} are sinusoidal voltages, then
  $p = j\omega_e$.

$$\tilde{V}_{qs}^{s} = (r_s + jW_e L_{ss})\tilde{I}_{qs}^{s} + jW_e L_{ms}\tilde{I}_{qr}^{\prime s}$$

$$\hat{V}_{ds}^{s} = (r_s + jW_e L_{ss})\hat{I}_{ds}^{s} + jW_e L_{ms}\hat{I}_{dr}^{\prime s}$$

$$\tilde{V}_{qr}^{\prime s} = (jW_e L_{ms})\hat{I}_{qs}^{s} - \omega_r L_{ms}\hat{I}_{ds}^{s} + (r_r' + jW_e L_{rr}')I_{qr}^{\prime s} - \omega_r L_{rr}' I_{qr}^{\prime s}$$

$$V_{dr}^{\prime s} = \omega_r L_{ms}\tilde{I}_{qs}^{s} + jW_e L_{ms}\tilde{I}_{ds}^{s} + \omega_r L_{rr}' I_{qr}^{\prime s} + (r_r' + jW_e L_{rr}')\hat{I}_{dr}^{\prime s}$$

Recall $\quad \tilde{F}_{qs} = -j\tilde{F}_{ds}^{s}$

$$\tilde{F}_{qr}^{\prime s} = -j\hat{F}_{dr}^{\prime s}$$

- We can use either ~~the~~ q_s^s and q_r^s or d_s^s and d_r^s equations ~~too~~ obtain the steady-state phase equation and equivalent circuit of induction machine. We will use q_s^s and q_r^s voltage equations

- $V_{qs}^{s} = [r_s + j(X_{ls} + X_{ms})]\tilde{I}_{qs}^{s} + jX_{ms}\tilde{I}_{qr}^{\prime s}$

$$L_{ss} = L_{ls} + L_{ms} \qquad X_{ms} = W_e L_{ms},\ X_{ls} = W_e L_{ls}$$

- Rewrite

$$\omega_r L_{ms} = \frac{\omega_r}{W_e} W_e L_{ms} = \frac{\omega_r}{W_e} X_{ms}$$

- $L'_{rr} = L'_{lr} + L_{ms}$

- $\omega_r L'_{rr} = \frac{\omega_r}{\omega_e} \omega_e (L'_{lr} + L_{ms}) = \frac{\omega_r}{\omega_e} (X'_{ls} + X_{ms})$

- we can rewrite V'^{s}_{qr} as

$$V'^{s}_{qr} = jX_{ms} I^{s}_{qs} \overbrace{- \frac{\omega_r}{\omega_e} X_{ms} \tilde{I}^{s}_{ds}}^{A} +$$

$$[r'_r + j(X'_{lr} + X_{ms})] I'^{s}_{qr}$$

$$\underbrace{- \frac{\omega_r}{\omega_e} (X'_{lr} + X_{ms}) \tilde{I}'^{s}_{dr}}_{B}$$

Recall

$$\tilde{F}^{s}_{qs} = -j F^{s}_{ds} \qquad \tilde{F}^{s}_{ds} = j \tilde{F}^{s}_{qs}$$

$$F'^{s}_{qr} = -j \tilde{F}^{s}_{dr} \qquad \tilde{F}^{s}_{dr} = j F'^{s}_{qr}$$

- replace I^{s}_{ds} and $\tilde{I}'^{s}_{dr}$ with $\tilde{I}^{s}_{qs}$ and I'^{s}_{qr}

- Term A

$$- \frac{\omega_r}{\omega_e} X_{ms} I^{s}_{ds} = - \frac{\omega_r}{\omega_e} j X_{ms} \tilde{I}^{s}_{qs}$$

- Term B

$$- \frac{\omega_r}{\omega_e} (X'_{lr} + X_{ms}) \tilde{I}'^{s}_{dr} = -j \frac{\omega_r}{\omega_e} (X'_{lr} + X_{ms}) \bar{I}'^{s}_{qr}$$

Then

$$V'^{s}_{qr} = jX_{ms}\hat{I}^{s}_{qs} - \frac{\omega_r}{\omega_e} jX_{ms}\tilde{I}^{s}_{qs} +$$

$$[r'_r + j(X'_{lr} + X_{ms})]\, I'^{s}_{qr}$$

$$-j\frac{\omega_r}{\omega_e}(X'_{lr} + X_{ms})\,\tilde{I}'^{s}_{qr}$$

or

$$V'^{s}_{qr} = jX_{ms}\left(1 - \frac{\omega_r}{\omega_e}\right) I^{s}_{qs} + \left[r'_r + j\left(1 - \frac{\omega_r}{\omega_e}\right)(X'_{lr} + X_{ms})\right] \tilde{I}'^{s}_{qr}$$

$$V'^{s}_{qr} = jX_{ms}\left(\frac{\omega_e - \omega_r}{\omega_e}\right)\hat{I}^{s}_{qs} + \left[r'_r + j\,(X'_{lr} + X_m) \times \left(\frac{\omega_e - \omega_r}{\omega_e}\right)\right]\tilde{I}'^{s}_{qr}$$

Let

$$s = \frac{\omega_e - \omega_r}{\omega_e}$$

Recall $\hat{F}^{s}_{qs} = \hat{F}_{as}$

$$\tilde{F}'^{s}_{qr} = \tilde{F}'_{ar}$$

Then V^{s}_{qs} can be written as

$$\tilde{V}_{as} = [r_s + j(X_{ls} + X_{ms})]\,\tilde{I}_{as} + jX_{ms}\, I'_{ar}$$

or

$$\hat{V}_{as} = (r_s + jX_{\ell s})\tilde{I}_{as} + jX_{ms}(\tilde{I}_{as} + \tilde{I}'_{ar})$$

Divide $\hat{V}'^{s}_{qr}$ by s

$$\frac{V'_{qr}}{s} = jX_{ms}(\tilde{I}^{s}_{qs} + \tilde{I}'^{s}_{qr}) + [(\frac{r'_r}{s} + jX'_{\ell r})]\tilde{I}'^{s}_{qr}$$

Recall

$$\tilde{I}^{s}_{qr} = \tilde{I}_{ar} \qquad \tilde{I}^{s}_{qs} = \tilde{I}_{as}$$

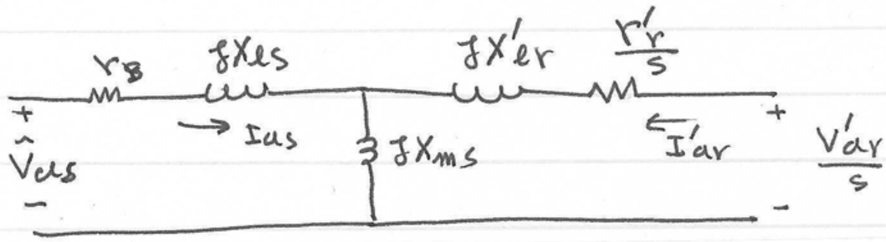
$$\tilde{V}'^{s}_{qr} = \tilde{V}'_{ar}$$

$$\frac{\tilde{V}'_{ar}}{s} = (\frac{r'_r}{s} + jX'_{\ell r})\tilde{I}'_{ar} + jX_{ms}(\tilde{I}_{as} + \tilde{I}'_{ar})$$

where

$$X_{\ell s} + X_{ms} = \omega_e L_{ss} = \omega_e(L_{\ell s} + L_{ms})$$

$$X'_{\ell r} + X_{ms} = \omega_e L'_{rr} = \omega_e(L'_{\ell r} + L_{ms})$$



Equivalent circuit of an induction machine

- For short-circuited rotor $V'_{ar} = 0$