Solution.

Problem : Assume a P-pole machine

$$T_{em}(i,\theta_r) = \frac{P}{2}\frac{\partial W_c(i,\theta_r)}{\partial \theta_r}$$

a) Derive the expression for $W_c(i_{abs}, i'_{abr}, \theta_r)$

b) Show that $T_{em} = -\frac{P}{2} L_{ms}\left[(i_{as} i'_{ar} + i_{bs} i'_{br})\sin\theta_r + (i_{as} i'_{br} - i_{bs} i'_{ar})\cos\theta_r\right]$

c) Compute i^s_{qs}, i^s_{ds}, i'^s_{dr} and i'^s_{qr} in terms of i_{as}, i_{bs}, i'_{ar} and i'_{br}

d) Show that T_{em} can be expressed as

$$T_{em} = \frac{P}{2} L_{ms}(i^s_{qs} i'^s_{dr} - i^s_{ds} i'^s_{qr})$$

or

$$T_{em} = \frac{P}{2}(\lambda'^s_{qr} i'^s_{dr} - \lambda'^s_{dr} i'^s_{qr})$$

or

$$T_{em} = \frac{P}{2}(\lambda^s_{ds} i^s_{qs} - \lambda^s_{qs} i^s_{ds})$$

a) The total field energy of a linear electromagnetic system with J electric inputs may be expressed as

$$W_f(i_1,\dots,i_J,\theta_r) = \frac{1}{2}\sum_{p=1}^{J}\sum_{q=1}^{J} L_{pq}(\theta_r)\, i_p i_q$$

For 2-pole 2-phase symmetrical induction machine, there are four electric inputs that are i_{as}, i_{bs}, i'_{ar}, and i'_{br}.

Let $i_{as} = i_1$, $i_{bs} = i_2$, $i'_{ar} = i_3$, and $i'_{br} = i_4$ $(J=4)$, then

$$\begin{aligned}
W_f(i_1,i_2,i_3,i_4,\theta_r) = {} & \tfrac{1}{2}L_{11}i_1^2 + \tfrac{1}{2}L_{12}i_1 i_2 + \tfrac{1}{2}L_{13}i_1 i_3 + \tfrac{1}{2}L_{14}i_1 i_4 \\
& + \tfrac{1}{2}L_{21}i_2 i_1 + \tfrac{1}{2}L_{22}i_2^2 + \tfrac{1}{2}L_{23}i_2 i_3 + \tfrac{1}{2}L_{24}i_2 i_4 \\
& + \tfrac{1}{2}L_{31}i_3 i_1 + \tfrac{1}{2}L_{32}i_3 i_2 + \tfrac{1}{2}L_{33}i_3^2 + \tfrac{1}{2}L_{34}i_3 i_4 \\
& + \tfrac{1}{2}L_{41}i_4 i_1 + \tfrac{1}{2}L_{42}i_4 i_2 + \tfrac{1}{2}L_{43}i_4 i_3 + \tfrac{1}{2}L_{44}i_4^2
\end{aligned}$$

$$\begin{aligned}
W_f(i_{as},i_{bs},i'_{ar},i'_{br},\theta_r) = {} & \tfrac{1}{2}L_{asas}i_{as}^2 + \tfrac{1}{2}L_{asbs}i_{as}i_{bs} + \tfrac{1}{2}L'_{asar}i_{as}i'_{ar} + \tfrac{1}{2}L'_{asbr}i_{as}i'_{br} \\
& + \tfrac{1}{2}L_{bsas}i_{bs}i_{as} + \tfrac{1}{2}L_{bsbs}i_{bs}^2 + \tfrac{1}{2}L'_{bsar}i_{bs}i'_{ar} + \tfrac{1}{2}L'_{bsbr}i_{bs}i'_{br} \\
& + \tfrac{1}{2}L'_{aras}i'_{ar}i_{as} + \tfrac{1}{2}L'_{arbs}i'_{ar}i_{bs} + \tfrac{1}{2}L'_{arar}i'^2_{ar} + \tfrac{1}{2}L'_{arbr}i'_{ar}i'_{br} \\
& + \tfrac{1}{2}L'_{bras}i'_{br}i_{as} + \tfrac{1}{2}L'_{brbs}i'_{br}i_{bs} + \tfrac{1}{2}L'_{brar}i'_{br}i'_{ar} + \tfrac{1}{2}L'_{brbr}i'^2_{br}
\end{aligned}$$

where
$$
\left.
\begin{aligned}
L_{asas} &= L_{bsbs} = L_{ss} = L_{ls} + L_{ms} \\
L_{arar} &= L_{brbr} = L_{rr} = L_{lr} + L_{mr} \\
L'_{arar} &= L'_{brbr} = L'_{rr} = L'_{lr} + L_{ms}
\end{aligned}
\right\}
\begin{array}{l}\text{self-inductances}\\ \text{(referred to stator)}\end{array}
$$

$$L_{asar} = L_{aras} = L_{sr}\cos\theta_r$$

$$L_{asbr} = L_{bras} = -L_{sr}\sin\theta_r$$

$$L_{bsar} = L_{arbs} = L_{sr}\sin\theta_r$$

$$L_{bsbr} = L_{brbs} = L_{sr}\cos\theta_r$$

when $\frac{N_s}{N_r} L_{sr} = L_{ms}$

$$
\left.
\begin{aligned}
\frac{N_s}{N_r} L_{asar} &= L'_{asar} = \frac{N_s}{N_r} L_{aras} = L'_{aras} = L_{ms}\cos\theta_r \\
\frac{N_s}{N_r} L_{asbr} &= L'_{asbr} = \frac{N_s}{N_r} L_{bras} = L'_{bras} = -L_{ms}\sin\theta_r \\
\frac{N_s}{N_r} L_{bsar} &= L'_{bsar} = \frac{N_s}{N_r} L_{arbs} = L'_{arbs} = L_{ms}\sin\theta_r \\
\frac{N_s}{N_r} L_{bsbr} &= L'_{bsbr} = \frac{N_s}{N_r} L_{brbs} = L'_{brbs} = L_{ms}\cos\theta_r
\end{aligned}
\right\}
\begin{array}{l}\text{mutual inductances}\\ \text{(referred to stator)}\end{array}
$$

$$L_{asbs} = L_{bsas} = 0 \quad \text{(as-bs windings are orthogonal)}$$

$$L'_{arbr} = L'_{brar} = 0 \quad \text{(ar-br windings are orthogonal)}$$

Then,

$$
\begin{aligned}
W_f(i_{as}, i_{bs}, i'_{ar}, i'_{br}, \theta_r) = {} & \tfrac{1}{2} L_{ss} i_{as}^2 + \tfrac{1}{2} L_{ms}\cos\theta_r\, i_{as} i'_{ar} - \tfrac{1}{2} L_{ms}\sin\theta_r\, i_{as} i'_{br} \\
& + \tfrac{1}{2} L_{ss} i_{bs}^2 + \tfrac{1}{2} L_{ms}\sin\theta_r\, i_{bs} i'_{ar} + \tfrac{1}{2} L_{ms}\cos\theta_r\, i_{bs} i'_{br} \\
& + \tfrac{1}{2} L_{ms}\cos\theta_r\, i'_{ar} i_{as} + \tfrac{1}{2} L_{ms}\sin\theta_r\, i'_{ar} i_{bs} + \tfrac{1}{2} L'_{rr} i'^2_{ar} \\
& - \tfrac{1}{2} L_{ms}\sin\theta_r\, i'_{br} i_{as} + \tfrac{1}{2} L_{ms}\cos\theta_r\, i'_{br} i_{bs} + \tfrac{1}{2} L'_{rr} i'^2_{br}
\end{aligned}
$$

$$
\begin{aligned}
\therefore W_f(i_{as}, i_{bs}, i'_{ar}, i'_{br}, \theta_r) = {} & \tfrac{1}{2} L_{ss} i_{as}^2 + \tfrac{1}{2} L_{ss} i_{bs}^2 + \tfrac{1}{2} L'_{rr} i'^2_{ar} + \tfrac{1}{2} L'_{rr} i'^2_{br} \\
& + L_{ms} i_{as} i'_{ar}\cos\theta_r - L_{ms} i_{as} i'_{br}\sin\theta_r \\
& + L_{ms} i_{bs} i'_{ar}\sin\theta_r + L_{ms} i_{bs} i'_{br}\cos\theta_r
\end{aligned}
$$

For a linear magnetic system, $W_f = W_c$

$$
\begin{aligned}
\therefore W_c(i_{abs}, i'_{abr}, \theta_r) = {} & \tfrac{1}{2} L_{ss} i_{as}^2 + \tfrac{1}{2} L_{ss} i_{bs}^2 + \tfrac{1}{2} L'_{rr} i'^2_{ar} + \tfrac{1}{2} L'_{rr} i'^2_{br} \\
& + L_{ms} i_{as} i'_{ar}\cos\theta_r - L_{ms} i_{as} i'_{br}\sin\theta_r \\
& + L_{ms} i_{bs} i'_{ar}\sin\theta_r + L_{ms} i_{bs} i'_{br}\cos\theta_r
\end{aligned}
$$

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b) $$T_{em}(i,\theta_r) = \frac{P}{2}\frac{\partial W_c(i,\theta_r)}{\partial \theta_r}$$

$$= \frac{P}{2}L_{ms}\left[i_{as}i'_{ar}(-\sin\theta_r) - i_{as}i'_{br}\cos\theta_r + i_{bs}i'_{ar}\cos\theta_r + i_{bs}i'_{br}(-\sin\theta_r)\right]$$

$$\therefore\ T_{em}(i,\theta_r) = -\frac{P}{2}L_{ms}\left[(i_{as}i'_{ar} + i_{bs}i'_{br})\sin\theta_r + (i_{as}i'_{br} - i_{bs}i'_{ar})\cos\theta_r \right]$$

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c) $$f^s_{qds} = K^s_s f_{abs} \quad \text{and} \quad f'^s_{qdr} = K^s_r f'_{abr}$$

$$\begin{bmatrix} i^s_{qs} \\ i^s_{ds} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} i_{as} \\ i_{bs} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} i'^s_{qr} \\ i'^s_{dr} \end{bmatrix} = \begin{bmatrix} \cos\theta_r & -\sin\theta_r \\ -\sin\theta_r & -\cos\theta_r \end{bmatrix}\begin{bmatrix} i'_{ar} \\ i'_{br} \end{bmatrix}$$

then,

$$i^s_{qs} = i_{as}$$

$$i^s_{ds} = -i_{bs}$$

$$i'^s_{qr} = i'_{ar}\cos\theta_r - i'_{br}\sin\theta_r$$

$$i'^s_{dr} = -i'_{ar}\sin\theta_r - i'_{br}\cos\theta_r$$

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d) $$f_{abs} = (K^s_s)^{-1} f^s_{qds} \quad \text{and} \quad f'_{abr} = (K^s_r)^{-1} f'^s_{qdr}$$

$$\begin{bmatrix} i_{as} \\ i_{bs} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} i^s_{qs} \\ i^s_{ds} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} i'_{ar} \\ i'_{br} \end{bmatrix} = \begin{bmatrix} \cos\theta_r & -\sin\theta_r \\ -\sin\theta_r & -\cos\theta_r \end{bmatrix}\begin{bmatrix} i'^s_{qr} \\ i'^s_{dr} \end{bmatrix}$$

$$T_{em}(i^s_{qds}, i'^s_{qdr}, \theta_r) = -\frac{P}{2}L_{ms}\Big[(i^s_{qs}i'^s_{qr}\cos\theta_r - i^s_{qs}i'^s_{dr}\sin\theta_r + i^s_{ds}i'^s_{qr}\sin\theta_r + i^s_{ds}i'^s_{dr}\cos\theta_r)\sin\theta_r$$
$$+ (-i^s_{qs}i'^s_{qr}\sin\theta_r - i^s_{qs}i'^s_{dr}\cos\theta_r + i^s_{ds}i'^s_{qr}\cos\theta_r - i^s_{ds}i'^s_{dr}\sin\theta_r)\cos\theta_r \Big]$$

$$= -\frac{P}{2}L_{ms}\Big[\cancel{i^s_{qs}i'^s_{qr}\cos\theta_r\sin\theta_r} - i^s_{qs}i'^s_{dr}\sin^2\theta_r + i^s_{ds}i'^s_{qr}\sin^2\theta_r + \cancel{i^s_{ds}i'^s_{dr}\cos\theta_r\sin\theta_r}$$
$$- \cancel{i^s_{qs}i'^s_{qr}\sin\theta_r\cos\theta_r} - i^s_{qs}i'^s_{dr}\cos^2\theta_r + i^s_{ds}i'^s_{qr}\cos^2\theta_r - \cancel{i^s_{ds}i'^s_{dr}\sin\theta_r\cos\theta_r} \Big]$$

$$= \frac{P}{2}L_{ms}\left[i^s_{qs}i'^s_{dr}(\sin^2\theta_r + \cos^2\theta_r) - i^s_{ds}i'^s_{qr}(\sin^2\theta_r + \cos^2\theta_r) \right]$$

$$\Longrightarrow \ \therefore\ T_{em}(i^s_{qds}, i'^s_{qdr}) = \frac{P}{2}L_{ms}(i^s_{qs}i'^s_{dr} - i^s_{ds}i'^s_{qr})$$

$$\begin{bmatrix} \lambda'^s_{qr} \\ \lambda'^s_{dr} \end{bmatrix} = \begin{bmatrix} L_{ms} & 0 \\ 0 & L_{ms} \end{bmatrix}\begin{bmatrix} i^s_{qs} \\ i^s_{ds} \end{bmatrix} + \begin{bmatrix} L'_{rr} & 0 \\ 0 & L'_{rr} \end{bmatrix}\begin{bmatrix} i'^s_{qr} \\ i'^s_{dr} \end{bmatrix} = \begin{bmatrix} L_{ms}i^s_{qs} + L'_{rr}i'^s_{qr} \\ L_{ms}i^s_{ds} + L'_{rr}i'^s_{dr} \end{bmatrix}$$

$$L_{ms}i^s_{qs} = \lambda'^s_{qr} - L'_{rr}i'^s_{qr}$$

$$L_{ms}i^s_{ds} = \lambda'^s_{dr} - L'_{rr}i'^s_{dr}$$

$$T_{em}(i^{s}_{qds}, i'^{s}_{qdr}) = \frac{P}{2}\left[(L_{ms} i^{s}_{qs}) i'^{s}_{dr} - (L_{ms} i^{s}_{ds}) i'^{s}_{qr} \right]$$

$$= \frac{P}{2}\left[(\lambda'^{s}_{qr} - L'_{rr} i'^{s}_{qr}) i'^{s}_{dr} - (\lambda'^{s}_{dr} - L'_{rr} i'^{s}_{dr}) i'^{s}_{qr} \right]$$

$$= \frac{P}{2}\left[\lambda'^{s}_{qr} i'^{s}_{dr} - \cancel{L'_{rr} i'^{s}_{qr}} i'^{s}_{dr} - \lambda'^{s}_{dr} i'^{s}_{qr} + \cancel{L'_{rr} i'^{s}_{dr}} i'^{s}_{qr} \right]$$

$$\Longrightarrow \therefore T_{em}(\lambda'^{s}_{qdr}, i'^{s}_{qdr}) = \frac{P}{2}(\lambda'^{s}_{qr} i'^{s}_{dr} - \lambda'^{s}_{dr} i'^{s}_{qr})$$

$$\begin{bmatrix} \lambda^{s}_{qs} \\ \lambda^{s}_{ds} \end{bmatrix} = \begin{bmatrix} L_{ss} & 0 \\ 0 & L_{ss} \end{bmatrix} \begin{bmatrix} i^{s}_{qs} \\ i^{s}_{ds} \end{bmatrix} + \begin{bmatrix} L_{ms} & 0 \\ 0 & L_{ms} \end{bmatrix} \begin{bmatrix} i'^{s}_{qr} \\ i'^{s}_{dr} \end{bmatrix} = \begin{bmatrix} L_{ss} i^{s}_{qs} + L_{ms} i'^{s}_{qr} \\ L_{ss} i^{s}_{ds} + L_{ms} i'^{s}_{dr} \end{bmatrix}$$

$$L_{ms} i'^{s}_{qr} = \lambda^{s}_{qs} - L_{ss} i^{s}_{qs}$$

$$L_{ms} i'^{s}_{dr} = \lambda^{s}_{ds} - L_{ss} i^{s}_{ds}$$

$$T_{em}(i^{s}_{qds}, i'^{s}_{qdr}) = \frac{P}{2}\left[(L_{ms} i'^{s}_{dr}) i^{s}_{qs} - (L_{ms} i'^{s}_{qr}) i^{s}_{ds} \right]$$

$$= \frac{P}{2}\left[(\lambda^{s}_{ds} - L_{ss} i^{s}_{ds}) i^{s}_{qs} - (\lambda^{s}_{qs} - L_{ss} i^{s}_{qs}) i^{s}_{ds} \right]$$

$$= \frac{P}{2}\left[\lambda^{s}_{ds} i^{s}_{qs} - \cancel{L_{ss} i^{s}_{ds}} i^{s}_{qs} - \lambda^{s}_{qs} i^{s}_{ds} + \cancel{L_{ss} i^{s}_{qs}} i^{s}_{ds} \right]$$

$$\Longrightarrow \therefore T_{em}(\lambda^{s}_{qds}, i^{s}_{qds}) = \frac{P}{2}(\lambda^{s}_{ds} i^{s}_{qs} - \lambda^{s}_{qs} i^{s}_{ds})$$

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