Scanned from Charles A. Gross, *Power System Analysis*, New York : Wiley, c1986

2

SINUSOIDAL STEADY-STATE CIRCUIT CONCEPTS

An engineer? I had grown up among engineers, and I could remember the engineers of the twenties very well indeed: their open, shining intellects, their free and gentle humor, their agility and breadth of thought, the ease with which they shifted from one engineering field to another, and, for that matter, from technology to social concerns and art. Then, too, they personified good manners and delicacy of taste; well-bred speech that flowed evenly and was free of uncultured words; one of them might play a musical instrument, another dabble in painting; and their faces always bore a spiritual imprint.

Aleksandr I. Solzhenitsyn
The Gulag Archipelago

The normal operating mode of the electrical power system is sinusoidal steady state (alternating current). Many of the electrical phenomena of engineering interest can be analyzed by using conventional ac circuit methods. It is therefore crucial for a student of power system engineering to be familiar with such methods, and this background is assumed for readers of this book. However, because of the topic's fundamental importance, the basic concepts of ac curcuits are reviewed in here. Another objective is to explain the author's notation and symbolism. Power concepts are particularly important.

2.1 Phasor Representation

Consider the general sinusoidal function $y(t)$

$$y(t) = Y_{max} \cos(\omega t + \phi) \tag{2.1}$$

Note that the function has three important parameters

Y_{max} = amplitude, or maximum value.
ω = radian frequency.
ϕ = phase angle.

The parameter Y_{max} essentially controls the "strength" of $y(t)$; ω shows the rate at which $y(t)$ is changing; and the parameter ϕ keys the cyclic $y(t)$ to the time origin (or, more to the point, to other sinusoidal functions). The point to note is that any conceivable sinusoidal function can be represented with the proper choice of Y_{max}, ω, and ϕ. Recall Euler's identity

$$e^{j\phi} = \cos \phi + j \sin \phi \tag{2.2}$$

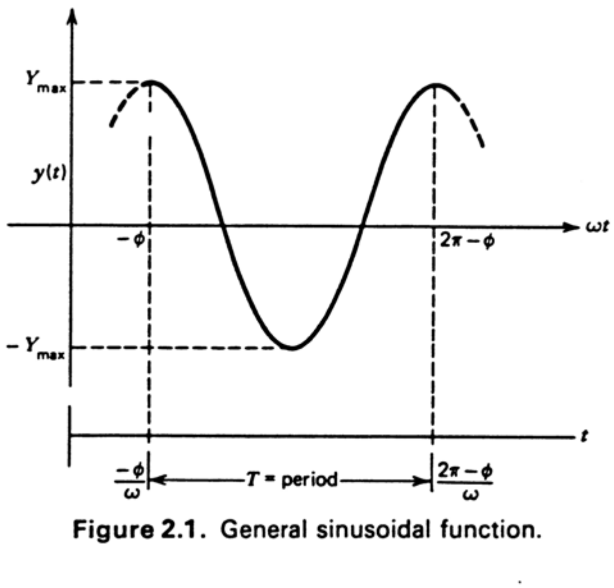


Figure 2.1. General sinusoidal function.

It follows that

$$y(t) = Y_{max}\cos(\omega t + \phi) \tag{2.3a}$$

$$= \text{Re}[Y_{max}\cos(\omega t + \phi) + j\,Y_{max}\sin(\omega t + \phi)] \tag{2.3b}$$

$$= \text{Re}[Y_{max}\,e^{j(\omega t + \phi)}] \tag{2.3c}$$

$$= \text{Re}(Y_{max}\,e^{j\phi}e^{j\omega t}) \tag{2.3d}$$

$$= \sqrt{2}\,\text{Re}\left(\frac{Y_{max}}{\sqrt{2}}\,e^{j\phi}e^{j\omega t}\right) \tag{2.3e}$$

We define

$$\bar{Y} = \frac{Y_{max}}{\sqrt{2}}\,e^{j\phi} \tag{2.4}$$

so that

$$y(t) = \sqrt{2}\,\text{Re}(\bar{Y}e^{j\omega t}) \tag{2.5}$$

The quantity $\bar{Y}$ is defined as the phasor representation of $y(t)$, and the transformation is defined in equation (2.5). Examine $\bar{Y}$ in equation (2.4) closely. Only two parameters, Y_{max} and ϕ, are involved. If we further define

$$Y = \frac{Y_{max}}{\sqrt{2}} = \text{Rms value of } y(t) \tag{2.6}$$

then

$$\bar{Y} = Ye^{j\phi} \tag{2.7}$$

or

$$\bar{Y} = Y\underline{/\phi} \tag{2.8}$$

The notation presented in equation (2.8) is simply an alternative way of writing equation (2.7). Although technically the angle ϕ should be in radians, it is frequently written in degrees because of their familiarity. The value Y is the rms value of $y(t)$, and it is useful because of power considerations (see problem 2.5 for more detail). Note that $\bar{Y}$ contains two-thirds of all the necessary information about the sinusoidal function $y(t)$ (the frequency ω is not included). If this transformation is applied to a collection of sinusoidal functions of the same frequency, which is independently known, the corresponding phasors contain *all* of the essential information. The phasor $\bar{Y}$ is a two-dimensional vector from a mathematical viewpoint. The term *phasor* was coined to aviod confusion with spatial vectors: the angular position of the phasor represents position in time, not space. Equation (2.8) is commonly referred to as polar form; rectangular form is easily produced by applying Euler's indentity.

$$\bar{Y} = Y\underline{/\phi} \tag{2.9a}$$

$$= Y\cos\phi + jY\sin\phi \tag{2.9b}$$

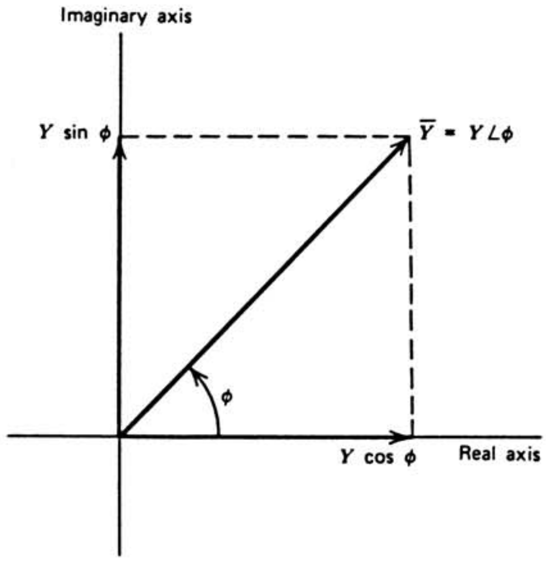


Figure 2.2. Graphical interpretation of $\bar{Y}$—the phasor diagram.

The phasor may be graphically interpreted by plotting the real and imaginary components in conventional Cartesian coordinates, with the real and x axes coinciding. Refer to Figure 2.2.

Example 2.1

(a) Transform $y(t) = 100\cos(377t - 30°)$ to the phasor form.

Solution

$$\bar{Y} = \frac{100}{\sqrt{2}}\angle -30° = 70.7\angle -30°$$

Observe that ω does not appear.

(b) Transform $\bar{Y} = 100\angle +20°$ to the instantaneous form.

Solution

$$y(t) = 100\sqrt{2}\cos(\omega t + 20°) = 141.4\cos(\omega t + 20°)$$

Observe that ω is unspecified.

(c) Add two sinusoidal functions of the same frequency using phasor methods.

Solution

$$a(t) = A\sqrt{2}\cos(\omega t + \alpha)$$
$$b(t) = B\sqrt{2}\cos(\omega t + \beta)$$
$$c(t) = a(t) + b(t)$$
$$= \sqrt{2}\{\mathrm{Re}[Ae^{j(\omega t+\alpha)} + Be^{j(\omega t+\beta)}]\}$$
$$= \sqrt{2}\,\mathrm{Re}[(Ae^{j\alpha} + Be^{j\beta})e^{j\omega t}]$$
$$= \sqrt{2}\,\mathrm{Re}[(\bar{A} + \bar{B})e^{j\omega t}]$$

or, if

$$\bar{C} = \bar{A} + \bar{B}$$
$$c(t) = \sqrt{2}\,\mathrm{Re}(\bar{C}e^{j\omega t})$$

The results of example 2.1(c) are of extreme importance. Observe that we can add sinusoidal functions of the same frequency by expressing them as phasors and then adding the phasors by the rules of vector algebra. The two basic laws of circuit theory—(Kirchhoff's voltage (KVL) and current laws (KCL)—when voltages and currents are expressed as phasors, take the following form:

KVL The sum of all *phasor* voltage drops around any path in a circuit is zero.

KCL The sum of all *phasor* currents into any node in a circuit is zero.

The summation of phasors is performed by the rules of ordinary vector addition.

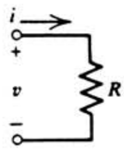
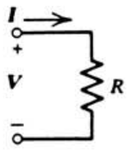
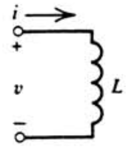
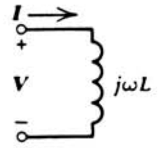
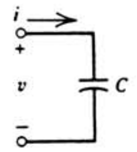
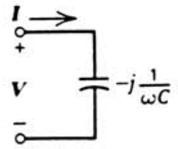
In applying Kirchhoff's laws to circuits in which all voltages and currents are expressed as phasors, it is necessary for all voltages and currents *to be assigned positive senses*, generally marked on a circuit diagram. It is important to realize that the diagram is necessary for interpreting calculated results and is therefore part of the solution.

2.2 Impedances of Passive Elements

Table 2.1 summarizes volt-ampere relationships for the three basic passive ideal circuit elements. To understand the basis for these results, consider the inductive case.

$$v = L\frac{di}{dt} \tag{2.10a}$$

Table 2.1. The Three Basic Ideal Passive-Circuit Components.

The general case		The ac special case	
	$v = iR$		$\bar{V} = R\bar{I}$
	$v = L\frac{di}{dt}$		$\bar{V} = j\omega L\bar{I}$
	$v = \frac{1}{C}\int_{-\infty}^{t} i dt$		$\bar{V} = -j\frac{1}{\omega C}\bar{I}$

If

$$i = \sqrt{2}\,\mathrm{Re}(\bar{I}\,e^{j\omega t}) \tag{2.11}$$

then

$$v = L\frac{d}{dt}[\sqrt{2}\,\mathrm{Re}(\bar{I}\,e^{j\omega t})] \tag{2.10b}$$

$$= \sqrt{2}\,\mathrm{Re}\left(L\bar{I}\frac{d}{dt}\,e^{j\omega t}\right) \tag{2.10c}$$

$$= \sqrt{2}\,\mathrm{Re}(j\omega L\bar{I}e^{j\omega t}) \tag{2.10d}$$

Examining equation (2.10d) produces

$$\bar{V} = j\omega L\bar{I} \tag{2.12}$$

If we define impedance as the ratio of phasor voltage to phasor current, for the inductor,

$$\bar{Z}_L = \frac{\bar{V}}{\bar{I}} = j\omega L \tag{2.13}$$

follows from equation (2.12). In a similar manner, for the resistor and capacitor, we can derive

$$\bar{Z}_R = R \tag{2.14}$$

$$\bar{Z}_C = -j\frac{1}{\omega C} \tag{2.15}$$

Since ω is constant, it is convenient to define reactance for the inductor and capacitor that, respectively, are

$$X_L = \omega L \tag{2.16a}$$

$$X_C = \frac{-1}{\omega C} \tag{2.16b}$$

The rules for series and parallel combinations apply to impedances just as they do to resistances in the dc case; however, such reductions require complex number arithmetic. It is assumed that the reader has the necessary computational skills; if not, consult any ac circuits text, for example, see reference 2. A more general definition of impedance is illustrated in Figure 2.3. Here,

$$\bar{Z} = \frac{\bar{V}}{\bar{I}} = R + jX \tag{2.17a}$$

so that

$$R = \text{Re}(\bar{Z}) \tag{2.17b}$$

$$X = \text{Im}(\bar{Z}) \tag{2.17c}$$

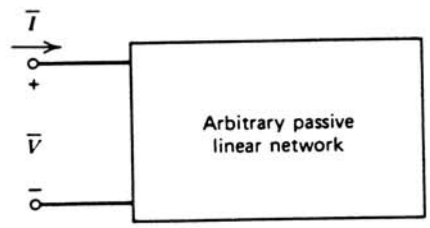


Figure 2.3. General definition of impedance.

Example 2.2

Given a 100-V sinusoidal source in series with a 3-Ω resistor, a 8-Ω inductor, and a 4-Ω capacitor,

(a) draw the circuit diagram.

Solution

The solution is shown in Figure 2.4. First, the source. 100 V is understood to be rms. The angle 0° is assigned arbitrarily and is a logical choice. We selected the source voltage as phase reference for convenience. The 8-Ω value assigned to the inductor is its reactance. (Its impedance is $+j8$; its inductance is measured in henries, not ohms.)

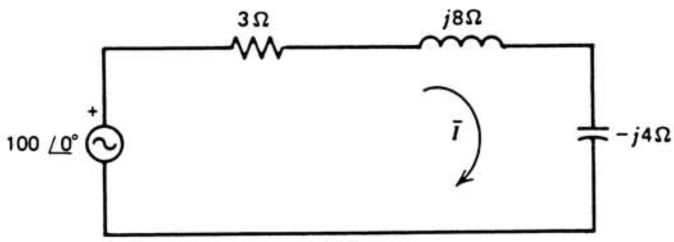


Figure 2.4. Circuit for example 2.2.

(b) compute the series impedance

Solution

$$\begin{aligned}\bar{Z} &= 3 + j8 - j4 \\ &= 3 + j4 \\ &= 5\angle{+53.1^\circ}\quad \Omega \text{ (Note the impedance is a complex number.)}\end{aligned}$$

(c) Compute the current $\bar{I}$

Solution

$$\bar{I} = \frac{100\angle 0^\circ}{5\angle 53.1^\circ} = 20\angle{-53.1^\circ}\ \text{A}$$

The 20 A is the rms value of the current. The phase angle -53.1° tells us the current is 53.1° in phase *behind* the source voltage. It is common to say that the current *lags* the voltage by 53.1°.

For some applications, an equivalent parameter, admittance, is sometimes used. From equation (2.17a),

$$\bar{Y} = \text{complex admittance} = \frac{\bar{I}}{\bar{V}} = \frac{1}{\bar{Z}} = G + jB. \tag{2.18a}$$

$$G = \text{conductance} = \text{Re}(\bar{Y}). \tag{2.18b}$$

$$B = \text{susceptance} = \text{Im}(\bar{Y}). \tag{2.18c}$$

The units are ohms^{-1} (mhos or siemens).

2.3 Power in Sinusodial Steady-State Circuits

Consider the situation shown in Figure 2.5. The instantaneous power flowing from network A to network B is

$$p(t) = vi \tag{2.19a}$$

$$= [V\sqrt{2}\cos(\omega t - \alpha)][I\sqrt{2}\cos(\omega t - \beta)]$$

$$= VI[\cos(\alpha - \beta) + \cos(2\omega t - \alpha - \beta)] \tag{2.19b}$$

Note that $p(t)$ consists of a double-frequency term combined with a dc term, as shown in Figure 2.6. Since sinusoidal functions average to zero, the average power is

$$P_{av} = \frac{1}{T}\int_0^T vi\,dt = VI\cos(\alpha - \beta) \tag{2.20a}$$

$$= VI\cos\psi \tag{2.20b}$$

where

$$\psi = \alpha - \beta \tag{2.20c}$$

Note that P_{av} has an important phsical meaning. It represents the average rate of energy flow from network A to network B. Note that the angle $\psi = \alpha - \beta$ is the angle by which the current lags the voltage in time phase.

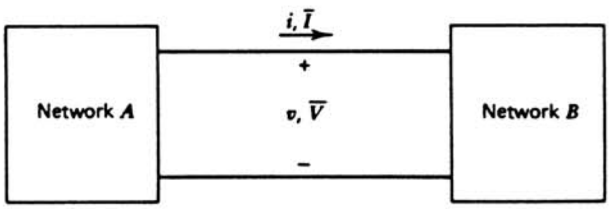


Figure 2.5. General situation for power flow in an ac circuit.

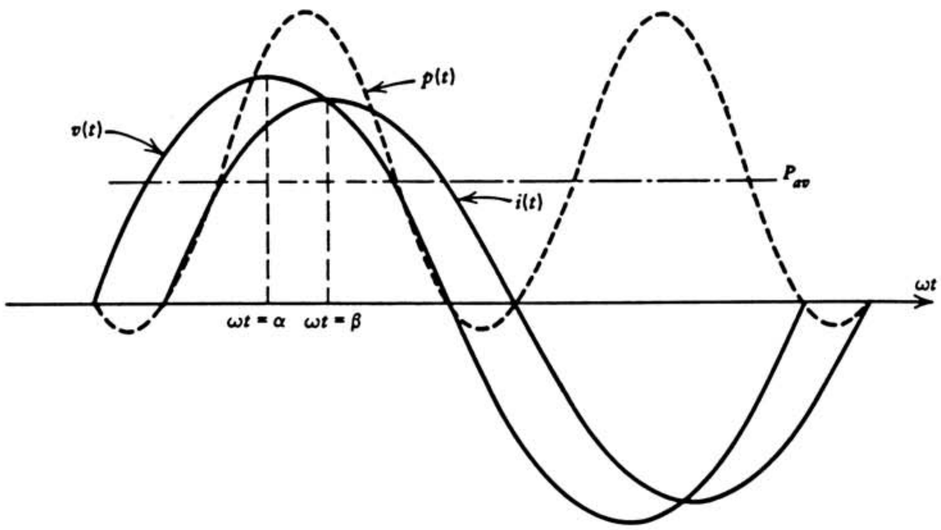


Figure 2.6. Voltage, current, and power variation with time.

There is an alternative way of calculating P_{av}. The complex power $\bar{S}$ flowing from network A to network B in Figure 2.5 is defined as

$$\bar{S} = \bar{V}\bar{I}^* \tag{2.21a}$$

$$= (V\angle\alpha)(I\angle\beta)^*$$

$$= VI\angle\alpha - \beta = VI\angle\psi \tag{2.21b}$$

The magnitude of the complex power is defined as the apparent power and is

$$S = |\bar{S}| = VI \tag{2.21c}$$

The angle ψ is related to the sense of power flow direction and can be understood more readily if the complex power is considered in rectangular form

$$\bar{S} = VI/\psi$$

$$= VI \cos \psi + jVI \sin \psi$$

$$= P + jQ \tag{2.22a}$$

where

$$P = \text{Re}(\bar{S}) = VI \cos \psi \tag{2.22b}$$

$$Q = \text{Im}(\bar{S}) = VI \sin \psi \tag{2.22c}$$

Now observe that P is identical to P_{av}. Hence, the real part of $\bar{S}$, (P), *is* P_{av}, and represents the average rate of energy transfer from A to B. P is sometimes called the real power.

Note that the sign on P is determined by the value of ψ. Specifically, consider $90° < \psi < 270°$. Observe that $P = VI \cos \psi$ is negative. But what does negative P mean physically? As with any directed quantity, a negative value simply means opposite to the assigned positive value. Thus, the real question is, what does *positive* P represent? We return to equation (2.22) and Figure 2.5 and recall that the assigned positive flow sense of $\bar{S}$ (and likewise P and Q) was from A to B. It is clear then that negative P implies average energy flow from B to A.

It is more difficult to assign a physical meaning to the quantity Q.† In a sense, Q is simply the other component of $\bar{S}$. However, we shall find that it is an extremely useful concept, despite its ethereal qualities. Q is called the reactive power because of its close association with reactive (i.e., inductive or capacitive) elements.

Observe that the sign on Q is also determined by the value of ψ. The significance attached to positive and negative Q can be reasonably described the same as for P (e.g., for $0° < \psi < 180°$, Q is interpreted as flowing from A to B). The P, Q, and ψ interrelationships are summarized in Figure 2.7.

It is clear that the SI units of S, P, and Q must all be the same, namely joule/s. It is conventional to assign different names to the units of S, P, and Q and occasionally use these unit names synonymously with the quantities themselves. They are

S = voltamperes (VA), kilovoltamperes (kVA), megavolt amperes (MVA).
P = watts (W), kilowatts (kW), megawatts (MW).
Q = voltamperes reactive (var), kilovoltamperes reactive (kvar), megavolt-amperes reactive (Mvar).

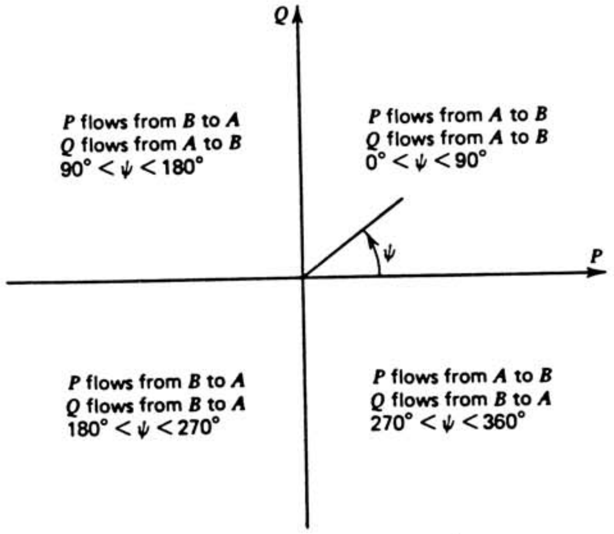


Figure 2.7. Flow direction of real and reactive power correlated to ψ.

† In a simple R-L or R-C situation, Q can be shown to be the radian frequency times the maximum stored (field) energy.

Now, consider network A to be active and network B to be passive. If network B is passive, it is appropriate to view it as an impedance. Observe that

$$\bar{Z} = \frac{\bar{V}}{\bar{I}} = \frac{V}{I}\underline{/\alpha - \beta} \qquad (2.23)$$

$$= Z\underline{/\psi}$$

Hence, ψ is also the angle of the impedance $\bar{Z}$.

Consider network B to contain pure R, L, or C elements, one at a time. The $\bar{V}$, $\bar{I}$, P, Q, and ψ relationships are summarized in Figure 2.8.

It is common practice to refer to the angle $(\alpha - \beta)$ as the power factor angle (ψ) and $\cos(\alpha - \beta)$ as the power factor (pf).

$$\psi = \text{power factor angle } = \alpha - \beta. \qquad (2.24a)$$

$$\text{pf} = \text{power factor } = \cos\psi \qquad (2.24b)$$

A *lagging* power factor indicates an *inductive* impedance and therefore a *positive* value for ψ.

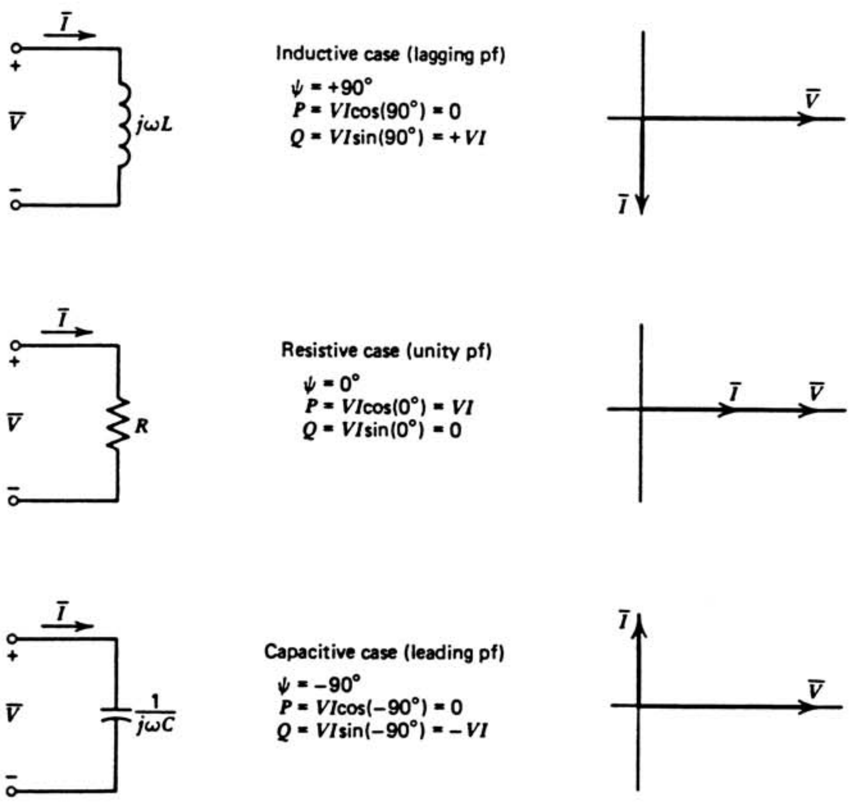


Figure 2.8. *P* and *Q* relationships in pure *R*, *L*, *C* situations.

Example 2.3

Calculate the complex power delivered to each of the four elements in the circuit in example 2.2.

Solution

The resistor

$$\bar{S}_R = \bar{V}_R\bar{I}^* = R\bar{I}\bar{I}^* = I^2R$$
$$= (20)^2 3 = 1200 + j0$$

The inductor

$$\bar{S}_L = \bar{V}_L\bar{I}^* = (jX_L\bar{I})\bar{I}^* = jX_LI^2$$
$$= j(8)(20)^2 = 0 + j3200$$

The capacitor

$$\bar{S}_C = \bar{V}_C\bar{I}^* = (-j\bar{I}X_C)\bar{I}^* = -jX_CI^2$$
$$= -j(4)(20)^2 = 0 - j1600$$

The total load† complex power

$$\bar{S}_{\text{Load}} = \bar{S}_R + \bar{S}_L + \bar{S}_C$$
$$= 1200 + j3200 - j1600 = 2000\underline{/+53.1^\circ}$$

The total source complex power

$$\bar{S}_{\text{Source}} = \bar{V}\bar{I}^*$$
$$= (100\underline{/0^\circ})(20\underline{/-53.1^\circ})^*$$
$$= 2000\underline{/+53.1^\circ}$$

Note that $\bar{S}_{\text{Source}} = \bar{S}_{\text{Load}}$. It is true in general in a given circuit that the net source complex power will equal the net load complex power.

Energy delivered to network B in Figure 2.5 in the time interval $t_2 - t_1$ is

$$W = \int_{t_1}^{t_2} p\,dt \qquad (2.25a)$$

† When used in electrical power applications, the term *load* does not have a precise technical definition. It can mean current, power, or impedance, depending on the context. The general idea refers to components that absorb electrical energy from the system.

For sinusoidal steady-state operation

$$W = P(t_2 - t_1) \tag{2.25b}$$

where P is defined in equation (2.22b). If P is in watts and t is in seconds, the energy W is in joules. Sometimes power is given in kilowatts and time in hours, forcing W into kilowatt hours. Observe that the kilowatt hour is not an SI unit and will require conversion factors in other equations.

2.4 The General *N*-Phase Situation

Consider two general networks A and B interconnected by $N + 1$ ideal conductors, as shown in Figure 2.9. Label the conductors in a special way: Starting from the top, label them $a, b, \ldots, z$, skipping the letters n and p (the first N) and the last conductor n. Likewise, refer to the top N conductors as *phases* and the bottom conductor as the *neutral*. The neutral will play a special role, serving as our voltage reference point. The voltages indicated in Figure 2.9 will be referred to collectively as the phase voltages; specifically,

$\bar{V}_{an} = \bar{V}_a = V_a\underline{/\alpha_a}$ = phase a to neutral voltage.

$\bar{V}_{bn} = \bar{V}_b = V_b\underline{/\alpha_b}$ = phase b to neutral voltage.

$\vdots$

$\bar{V}_{zn} = \bar{V}_z = V_z\underline{/\alpha_z}$ = phase z to neutral voltage.

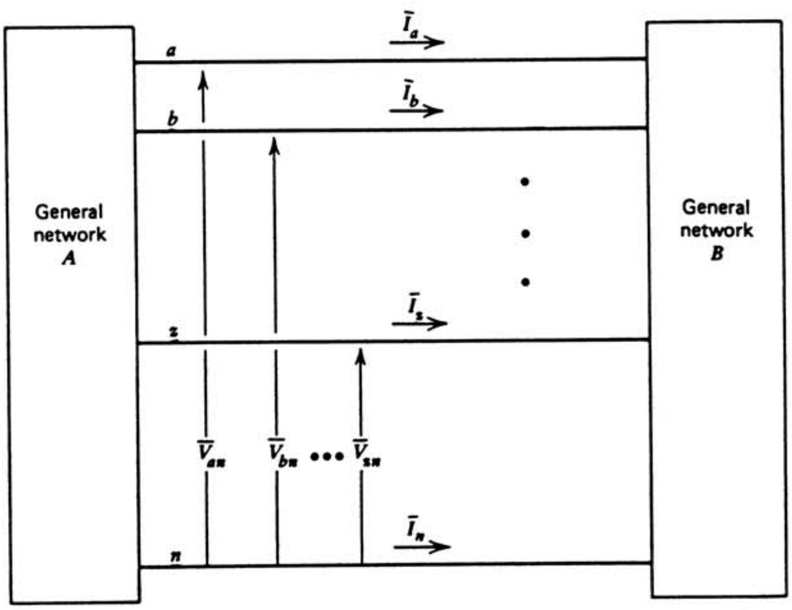


Figure 2.9. Two general networks interconnected by $N + 1$ conductors.

The currents indicated in Figure 2.9 are defined as

$\bar{I}_a = I_a \underline{/\beta_a}$ = phase *a* current.

$\bar{I}_b = I_b \underline{/\beta_b}$ = phase *b* current.

$\vdots$

$\bar{I}_z = I_z \underline{/\beta_z}$ = phase *z* current.

$\bar{I}_n = I_n \underline{/\beta_n}$ = neutral current.

Some general constraints can be written for this situation. First, the currents must satisfy KCL

$$\bar{I}_a + \bar{I}_b + \cdots + \bar{I}_z + \bar{I}_n = 0 \qquad (2.26)$$

The total complex power flowing from network *A* to network *B* is:†

$$S_{N\phi} = \bar{V}_a \bar{I}_a^* + \bar{V}_b \bar{I}_b^* + \cdots + \bar{V}_z \bar{I}_z^* \qquad (2.27a)$$

$$= \sum_{i=a}^{z} \bar{S}_i \qquad (2.27b)$$

where

$$\bar{S}_i = \bar{V}_i \bar{I}_i^* \qquad (2.27c)$$

$$= V_i I_i \underline{/\alpha_i - \beta_i} \qquad (2.27d)$$

$$= V_i I_i \underline{/\psi_i}$$

$$i = a, b, \ldots, z \qquad (N \text{ of these}) \qquad (2.27e)$$

Note that

$$P_{N\phi} = \sum_{i=a}^{z} P_i \qquad (2.28a)$$

where

$$P_i = V_i I_i \cos \psi_i \qquad (2.28b)$$

and

$$Q_{N\phi} = \sum_{i=a}^{z} Q_i \qquad (2.29a)$$

$$Q_i = V_i I_i \sin \psi_i \qquad (2.29b)$$

† The letter *i* (and later *j*) is used to symbolize both specifically the ninth (tenth) phase and any one of the *N* phases. The reader's indulgence is requested for this ambiguity of notation. Blessed are the rigorous for they shall inhibit the earth.

We define the term *balanced* to mean as follows, when applied to phase voltages

$$V_a = V_b = \ \dots \ = V_z = V_p = \text{the phase voltage} \qquad (2.30a)$$

and

$$\begin{aligned} \alpha_a &= 0°\dagger \\ \alpha_b &= -\theta \\ \alpha_c &= -2\theta \\ &\vdots \\ \alpha_z &= -(N-1)\theta \end{aligned} \qquad (2.30b)$$

where

$$\theta = \frac{360°}{N}$$

In words, balanced phase voltages have equal magnitudes and are $360°/N$ phase separated.

In a similar fashion, define balanced phase currents to have the following properties:

$$I_a = I_b = \cdots = I_z = I_p = \text{the phase current} \qquad (2.31)$$

$$\psi_a = \psi_b = \cdots = \psi_z = \psi = \text{the pf angle} \qquad (2.32a)$$

where

$$\psi_i = \alpha_i - \beta_i \qquad i = a, b, \dots, z \qquad (2.32b)$$

Hence, balanced phase currents are also equal in magnitude and are $360°/N$ phase separated.

The concept of phase sequence, phase order, or phase rotation is related to the possible orderings of the N equal phase quantities. The number of possible permutations of N entities taken N at a time is $N!$. However, considering phase sequence, there is N-fold redundancy in these permutations, since each of the N phases may be taken as reference. Hence, the number of possible phase sequences in a N-phase system is

$$N_{seq} = \frac{N!}{N} = (N-1)! \qquad (2.33)$$

However, to some extent, this is simply a labeling problem: When a given system is operating in its normal balanced condition, we shall label the system phases in such a way that the phase sequence is *abc* ... *z* (alphabetical order) or more simply, *abc*. Thus, *positive* phase-sequence by definition will be *abc*. One other specific

† Using $\bar{V}_a$ as the phase reference is not essential but is convenient and causes no loss of generality. Whenever possible in this book, we shall take $\bar{V}_a$ as the phase reference quantity.

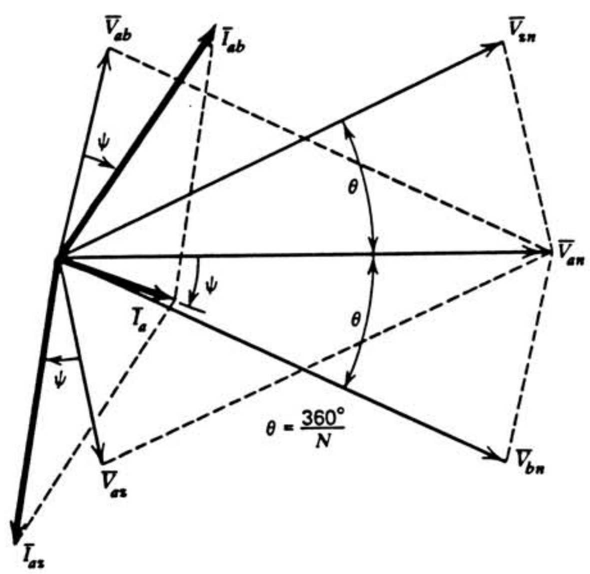


Figure 2.10. Balanced N-phase current and voltage relationships.

phase sequence will be meaningful to us: $z \ldots cba$ (reverse alphabetical order) or, more concisely, cba, which we define as *Negative* phase-sequence.

Also of interest are the phase-to-phase voltages, which by KVL are

$$\bar{V}_{ij} = \bar{V}_i - V_j \qquad i, j = a, b, c, \ldots, z \tag{2.34a}$$

$$= V_i \angle \alpha_i - V_j \angle \alpha_j \tag{2.34b}$$

For the balanced case,

$$|\bar{V}_i| = |\bar{V}_j| = V_p \tag{2.30a}$$

and

$$V_{ij} = V_p \sqrt{2[1 - \cos(\alpha_i - \alpha_j)]} \tag{2.35}$$

See Figure 2.10. Phase-to-phase voltages normalized to phase-to-neutral voltages for $2 \leq N \leq 12$ are presented in Table 2.2.

The instantaneous power in the balanced N-phase case is particularly simple for $N \geq 3$

$$p_{N\psi} = \text{total instantaneous power flow from } A \text{ to } B \tag{2.36a}$$

$$= \sum_{i=a}^{z} v_i i_i \tag{2.36b}$$

$$= \sum_{i=a}^{z} (V_p\sqrt{2})(I_p\sqrt{2})\cos(\omega t + i\theta - \alpha_i)\cos(\omega t + i\theta - \beta_i) \tag{2.36c}$$

$$= NV_pI_p \cos \psi. \tag{2.36d}$$

Table 2.2. Normalized Phase-to-Phase Voltages for two- to 12-Phase Systems.

$N =$	1	2	3	4	5	6
Vab/Va	—	2.000	1.732	1.414	1.176	1.000
Vac/Va	—	—	1.732	2.000	1.902	1.732
Vad/Va	—	—	—	1.414	1.902	2.000
Vae/Va	—	—	—	—	1.176	1.732
Vaf/Va	—	—	—	—	—	1.000
$N =$	7	8	9	10	11	12
Vab/Va	0.868	0.765	0.684	0.618	0.563	0.518
Vac/Va	1.564	1.414	1.286	1.176	1.081	1.000
Vad/Va	1.950	1.848	1.732	1.618	1.511	1.414
Vae/Va	1.950	2.000	1.970	1.902	1.819	1.732
Vaf/Va	1.564	1.848	1.970	2.000	1.980	1.932
Vag/Va	0.868	1.414	1.732	1.902	1.980	2.000
Vah/Va	—	0.765	1.286	1.618	1.819	1.932
Vai/Va	—	—	0.684	1.176	1.511	1.732
Vaj/Va	—	—	—	0.618	1.081	1.414
Vak/Va	—	—	—	—	0.563	1.000
Val/Va	—	—	—	—	—	0.518

which is an extraordinary result, because it is constant! That is, ac voltages and currents produce dc power! This is the major reason why large ac power systems are polyphase and operate in the balanced mode. Furthermore,

$$S_{N\phi} = NV_pI_p \tag{2.37a}$$

$$P_{N\phi} = NV_pI_p \cos\psi \tag{2.37b}$$

$$Q_{N\phi} = NV_pI_p \sin\psi \tag{2.37c}$$

2.4.1 General Balanced *N*-Phase Impedance Connections

For the moment, think of network B in Figure 2.9 as passive. Now consider N single port elements with equal impedances. How many possible interconnections can we make that would result in balanced-current flow for balanced voltage applied? Only two! They are the general balanced-star-and-mesh connections that are shown in Figure 2.11.

In some situations, it is convenient to transform general balanced-mesh connections to general balanced-star connections. Consider Figure 2.11a. For the star case,

$$\bar{I}_{aY} = \frac{\bar{V}_{an}}{\bar{Z}_Y} = \frac{V_p\underline{/0^\circ}}{\bar{Z}_Y} \tag{2.38}$$

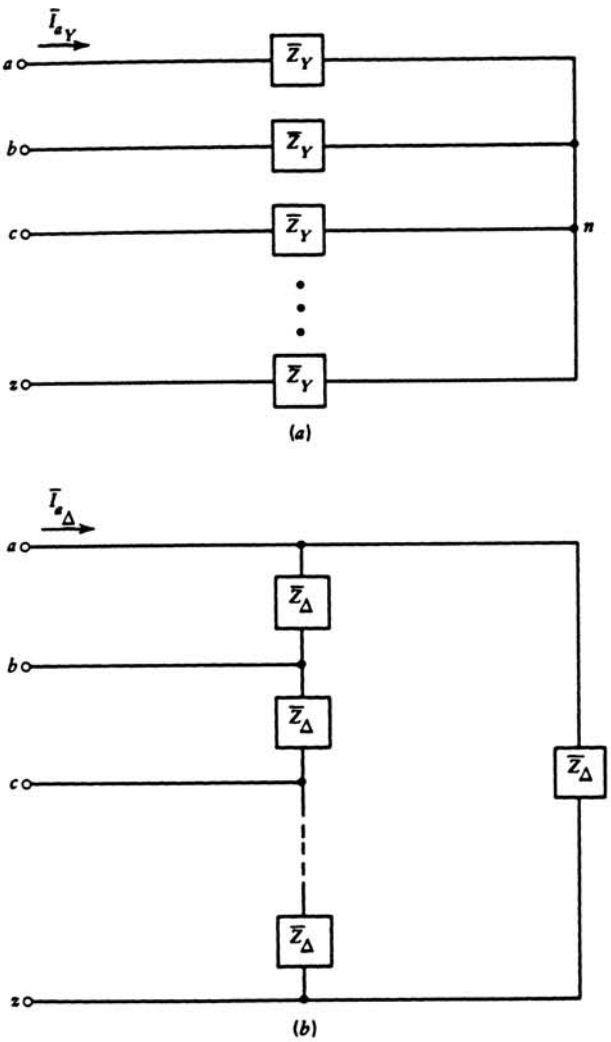


Figure 2.11. General balanced connections. (*a*) The general balanced-star connection. (*b*) The general balanced-mesh connection.

For the mesh case,

$$\bar{I}_{a\Delta} = \frac{\bar{V}_{ab}}{\bar{Z}_\Delta} + \frac{\bar{V}_{az}}{\bar{Z}_\Delta} = \frac{1}{\bar{Z}_\Delta}(\bar{V}_{an} - \bar{V}_{bn} + \bar{V}_{an} - \bar{V}_{zn}) \tag{2.39a}$$

$$= \frac{V_p}{\bar{Z}_\Delta}(1\underline{/0^\circ} - 1\underline{/-\theta} + 1\underline{/0^\circ} - 1\underline{/\theta}) = \frac{2V_p}{\bar{Z}_\Delta}(1 - \cos\theta) \tag{2.39b}$$

To force equivalence, set

$$\bar{I}_{a\Delta} = \bar{I}_{aY} \tag{2.40a}$$

$$\frac{2\bar{V}_p}{\bar{Z}_\Delta}(1 - \cos\theta) = \frac{\bar{V}_p}{\bar{Z}_Y} \tag{2.40b}$$

or

$$\bar{Z}_\Delta = 2(1 - \cos\theta)\bar{Z}_Y \tag{2.40c}$$

A tabulation of results for various phase orders follow.

N	θ (degrees)	Z_Δ/Z_Y
2	180	4.0000
3	120	3.0000
4	90	2.0000
5	72	1.3820
6	60	1.0000
7	51.4	0.7530
8	45	0.5858
9	40	0.4679
10	36	0.3820
11	32.7	0.3175
12	30	0.2679
24	15	0.0681

Finally, observe that for the balanced case

$$\bar{I}_n = -(\bar{I}_a + \bar{I}_b + \cdots + \bar{I}_z) = 0 \tag{2.41}$$

Thus, considering the neutral as the "return path" for the phase currents, conductors of modest physical size can be used for this purpose. Consider a comparison of the six-phase mode versus the single-phase mode, as illustrated in Figure 2.12. In the six-phase case, a total conductor cross-sectional area of $6A$ is required, neglecting the neutral, compared with $12A$ for the single-phase case, to achieve the same total current supplied. Balanced-polyphase-current transmission will have an inherent advantage of a factor of two when compared with single-phase or two-wire direct current.

Example 2.4

A balanced 6 ϕ source with $\bar{V}_a = 1000\underline{/0^\circ}$ V, phase sequence *abc*, supplies a balanced 6 ϕ load such that $S_{6\phi} = 900$ kVA, pf = 0.8 lagging.

(a) Calculate the phase current.
(b) Calculate the phase-to-phase voltage $\bar{V}_{ae}$.
(c) Calculate the equivalent star impedance $\bar{Z}_Y$.
(d) Calculate the equivalent mesh impedance $\bar{Z}_\Delta$.

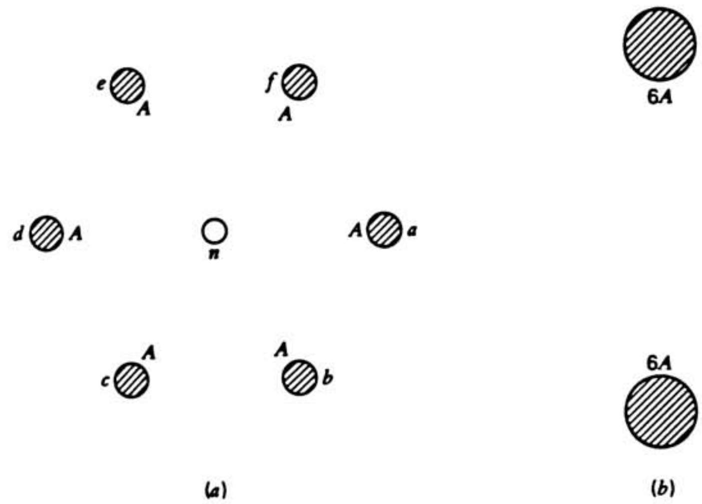


Figure 2.12. Comparison of required conducting areas for six and single-phase power transmission. (*a*) Six-phase mode. (*b*) Single-phase mode.

Solution

(a) $I_P = \dfrac{S_{6\phi}/6}{V_P} = \dfrac{900/6}{1.0} = 150 \text{ A}$

(b) $\bar{V}_{ae} = \bar{V}_a - \bar{V}_e = 1000\underline{/0^\circ} - 1000\underline{/-240^\circ}$
$= 1732\underline{/-30^\circ} \text{ V}$

(c) $Z_Y = \dfrac{V_P}{I_p} = \dfrac{1000}{150} = 6.67\ \Omega$

$\psi = \cos^{-1}(0.8) = \pm 36.9^\circ$

$\bar{Z}_Y = 6.67\underline{/+36.9^\circ}\ \Omega$

(d) $\bar{Z}_\Delta = 2(1 - \cos 60^\circ)\bar{Z}_Y$
$= \bar{Z}_Y = 6.67\underline{/+36.9^\circ}\ \Omega$

2.5 The General Balanced Three-Phase System

Most of the bulk electrical power throughout the world is transmitted in the three-phase mode. This mode represents the minimum number of phases for which all of the advantages of polyphase transmission are realized. Therefore, the three-phase case is of special interest and importance to us.

Recall the basic situation to be considered, as shown in Figure 2.13, for $N = 3$. There are four conductors to be considered: a, b, c, and the neutral n. The phase voltages for the balanced 3ϕ case, phase sequence abc, are

$$\bar{V}_{an} = \bar{V}_a = V_p\underline{/0^\circ} \tag{2.42a}$$

$$\bar{V}_{bn} = \bar{V}_b = V_p\underline{/-120^\circ} \tag{2.42b}$$

$$\bar{V}_{cn} = \bar{V}_c = V_p\underline{/+120^\circ} \tag{2.42c}$$

The phase-to-phase† voltages are

$$\bar{V}_{ab} = \bar{V}_a - \bar{V}_b = V_p\sqrt{3}\underline{/30^\circ} = V_L\underline{/30^\circ} \tag{2.43a}$$

$$\bar{V}_{bc} = \bar{V}_b - \bar{V}_c = V_p\sqrt{3}\underline{/-90^\circ} = V_L\underline{/-90^\circ} \tag{2.43b}$$

$$\bar{V}_{ca} = \bar{V}_c - \bar{V}_c = V_p\sqrt{3}\underline{/150^\circ} = V_L\underline{/150^\circ} \tag{2.43c}$$

Consider that network A is active and network B is passive. Recall that there are two balanced configurations of impedance connections within network B that result in balanced-current flow: the wye connection and the delta connection, as shown in Figure 2.14. For the wye case, observe that

$$\bar{I}_a = \frac{\bar{V}_a}{\bar{Z}_Y} = \frac{V_p}{Z_Y}\underline{/0^\circ - \psi} = I_p\underline{/0^\circ - \psi} \qquad -90^\circ \le \psi \le +90^\circ \tag{2.44a}$$

$$-90^\circ \le \psi < 0^\circ \qquad \bar{Z}_Y \text{ is capacitive } (\bar{I}_a \text{ leads } \bar{V}_a) \tag{2.44b}$$

$$0 < \psi \le 90^\circ \qquad \bar{Z}_Y \text{ is inductive } (\bar{I}_a \text{ lags } \bar{V}_a) \tag{2.44c}$$

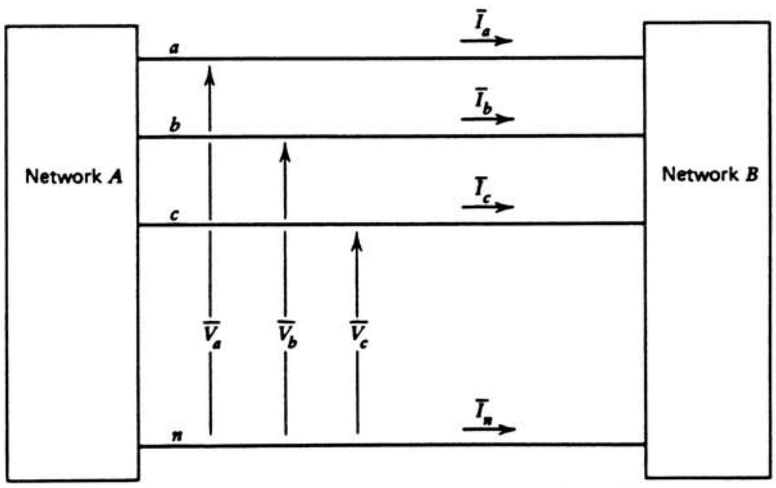
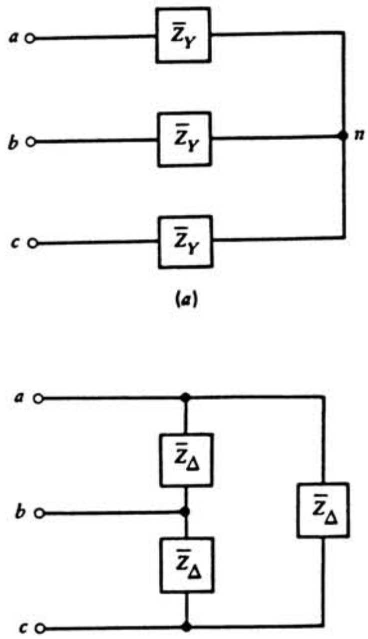


Figure 2.13. The general three-phase situation.

† The phase-to-phase voltage is frequently called the line voltage (V_L); thus, $V_L = \sqrt{3}V_p$.



(b)

Figure 2.14. Balanced three-phase loads. (*a*) The wye-connected load. (*b*) The delta-connected load.

The phase† currents in the balanced three-phase case are

$$\bar{I}_a = I_p\underline{/0^\circ - \psi} = I_L\underline{/0^\circ - \psi} \tag{2.45a}$$

$$\bar{I}_b = I_p\underline{/-120^\circ - \psi} = I_L\underline{/-120^\circ - \psi} \tag{2.45b}$$

$$\bar{I}_c = I_p\underline{/+120^\circ + \psi} = I_L\underline{/+120^\circ - \psi} \tag{2.45c}$$

$$\bar{I}_n = 0 \tag{2.45d}$$

The phase relationships between the various phase quantities are graphically summarized in the phasor diagram shown in Figure 2.15.

For the delta case, from equation (2.40c) the equivalent $\bar{Z}_\Delta$ is

$$\bar{Z}_\Delta = 2[1 - \cos(120^\circ)]\bar{Z}_Y \tag{2.46a}$$

$$= 3\bar{Z}_Y \tag{2.46b}$$

† The phase current is frequently called the line current (I_L); thus, $I_L = I_p$.

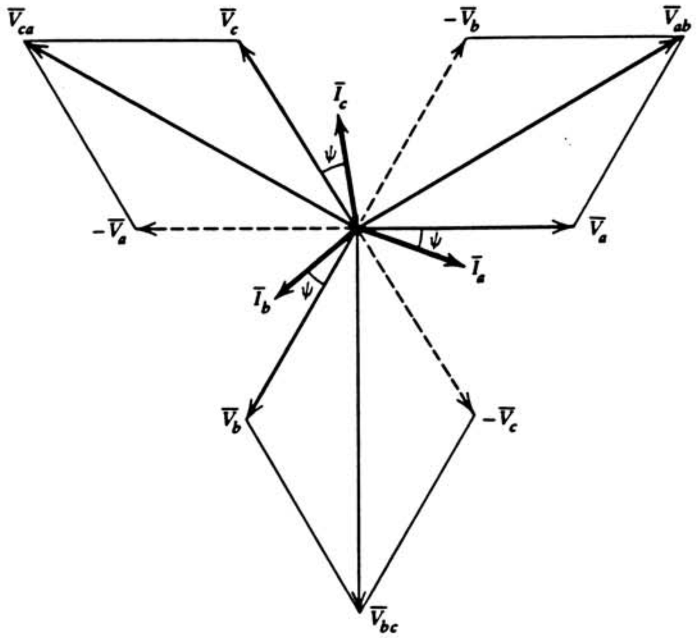


Figure 2.15. General phasor diagram for the balanced three-phase case: phase sequence *abc*.

If current flowing in each branch of the $\bar{Z}_\Delta$ is needed, the calculation is straightforward.

$$\bar{I}_{ab} = \frac{\bar{V}_{ab}}{\bar{Z}_\Delta} \tag{2.47a}$$

$$\bar{I}_{bc} = \frac{\bar{V}_{bc}}{\bar{Z}_\Delta} \tag{2.47b}$$

$$\bar{I}_{ca} = \frac{\bar{V}_{ca}}{\bar{Z}_\Delta} \tag{2.47c}$$

These currents are sometimes called the "phase currents," because they are the currents that flow in each "phase" ("arm," "leg," "branch") of the delta. However, we shall avoid that terminology because of the possibility of confusion with the currents $\bar{I}_a$, $\bar{I}_b$, and $\bar{I}_c$. By KCL,

$$\bar{I}_a = \bar{I}_{ab} - \bar{I}_{ca} \tag{2.48a}$$

$$\bar{I}_b = \bar{I}_{bc} - \bar{I}_{ab} \tag{2.48b}$$

$$\bar{I}_c = \bar{I}_{ca} - \bar{I}_{bc} \tag{2.48c}$$

There is a similar problem with voltage terminology in the delta. The voltages $\bar{V}_{ab}$, $\bar{V}_{bc}$, and $\bar{V}_{ca}$ are sometimes called the "phase voltages," since they appear across the "phases" of the delta. however, this terminology conflicts with our previous decision to define $\bar{V}_{an}$, $\bar{V}_{bn}$, and $\bar{V}_{cn}$ as the phase voltages.

Thus, we see that the wye connection has some conceptual advantages over the delta connection, and we shall "think wye" each time there is a choice. Remember that we usually have the option of replacing delta connections with equivalent wyes. If this is not possible, we shall resort to standard double-subscript notation ($\bar{V}_{ab}$, $\bar{I}_{ab}$, etc.) to eliminate the possibility of confusion.

The equations for power flow in a balanced N-phase system were developed in section 2.4. They all apply to the three phase case. Thus, the power flow from network A to network B is

$$S_{3\phi} = 3V_pI_p \tag{2.49a}$$

$$P_{3\phi} = 3V_pI_p \cos\psi \tag{2.49b}$$

$$Q_{3\phi} = 3V_pI_p \sin\psi \tag{2.49c}$$

$$\psi = \psi_a = \psi_b = \psi_c \tag{2.50a}$$

$$\psi_a = \alpha_a - \beta_a \tag{2.50b}$$

$$\alpha_a = \text{Arg}(\bar{V}_a) \tag{2.50c}$$

$$\beta_a = \text{Arg}(\bar{I}_a) \tag{2.50d}$$

Recall the line quantities

$$I_L = I_p \tag{2.51a}$$

$$V_L = V_p\sqrt{3} \tag{2.51b}$$

Therefore, equivalent expressions for power are

$$S_{3\phi} = 3\left(\frac{V_L}{\sqrt{3}}\right)(I_L) = \sqrt{3}V_LI_L \tag{2.52a}$$

$$P_{3\phi} = \sqrt{3}V_LI_L \cos\psi \tag{2.52b}$$

$$Q_{3\phi} = \sqrt{3}V_LI_L \sin\psi \tag{2.52c}$$

The complex power is

$$\bar{S}_{3\phi} = P_{3\phi} + jQ_{3\phi} \tag{2.52d}$$

Example 2.5

A balanced 3ϕ load of 500 kVA, pf = 0.85 lagging is supplied from a balanced 3ϕ source such that $V_L = 4157$ V, phase sequence *abc*. Determine

(a) V_L, I_L, V_p, and I_p
(b) ψ
(c) $S_{3\phi}P_{3\phi}$, and$Q_{3\phi}$
(d) $\bar{Z}_Y$ and the equivalent $\bar{Z}_\Delta$ for the load

Solution

(a) $V_L = 4157$ V, which is given.

$$V_p = \frac{V_L}{\sqrt{3}} = 2400 \text{ V}$$

$$I_L = I_p = \frac{S_{3\phi}}{\sqrt{3}V_L} = \frac{500}{\sqrt{3}(4.157)} = 69.4 \text{ A}$$

(b) $\psi = \cos^{-1}(0.85) = \pm 31.8°$
Since the pf is lagging, $\psi = 31.8°$.

(c) $S_{3\phi} = 500$ kVA, which is given.
$P_{3\phi} = 500\,(0.85) = 425$ kW
$Q_{3\phi} = 500 \sin(31.8°) = 263$ kvar

(d) $Z_y = \dfrac{V_p}{I_p} = \dfrac{2400}{69.4} = 34.6\ \Omega$; $\bar{Z}_Y = 34.6\underline{/+31.8°}$
$\bar{Z}_\Delta = 3\bar{Z}_Y = 104\underline{/+31.8°}$

2.6 Symmetrical Components

The electrical power system normally operates in a balanced three-phase sinusoidal steady-state mode. However, there are certain situations that can cause unbalanced operation. The most severe of these is the so-called fault, or short circuit. An example would be a tree in contact with one phase of an overhead transmission line. To protect the system against such contingencies, we must size protective devices, such as fuses and circuit breakers. For these and other reasons, it is necessary to calculate currents and voltages in the system under such unbalanced operating conditions. Our first impulse is simply to extend our per-phase analysis approach to deal with all three phases separately, in effect, tripling our work. Unfortunately, it turns out that the work is significantly more than triple that of the balanced case.

In a classical paper, C. L. Fortescue† described how arbitrarily unbalanced N-phase voltages (or currents) could be transformed into N sets of balanced N-phase components. He called these components *symmetrical components*. The application to power system analysis is of fundamental importance. We can transform an arbitrarily unbalanced condition into symmetrical components, compute the system response by straightforward circuit analysis on simple circuit models, and transform the results back into the original phase variables. This approach proves to be simpler than the direct but much more complicated method of solving unbalanced problems in the original N-phase system.

† C. L. Fortescue, "Method of Symmetrical Coordinates Applied to the Solution of Polyphase Networks," *AIEE Transactions*, vol. 37, part 2, 1918.

We shall first investigate symmetrical components applied to the three-phase case. In a general three-phase circuit, instead of a single voltage and currrent, we must deal with a minimum of three voltages and three currents, which we shall refrer to as the phase values. Their specific definitions are indicated in Figure 2.13. Each is to be considered as sinusoidal steady state and therefore is represented with phasor notation. Let us direct our attention to the voltages, realizing that similar statements apply to the currents.

$$\bar{V}_a = V_a \underline{/\alpha_a}$$

Note that the phase voltage $\bar{V}_a$ has two parts, magnitude and phase angle (two degrees of freedom). Thus, to specify three unbalanced voltages, we need six numbers:

$$V_a,\ V_b,\ V_c,\ \alpha_a,\ \alpha_b,\ \text{and } \alpha_c$$

Let us think of each phase voltage as having three components, such that each voltage is the sum of its components

$$\bar{V}_a = \bar{V}_{a0} + \bar{V}_{a1} + \bar{V}_{a2} \tag{2.53a}$$

$$\bar{V}_b = \bar{V}_{b0} + \bar{V}_{b1} + \bar{V}_{b2} \tag{2.53b}$$

$$\bar{V}_c = \bar{V}_{c0} + \bar{V}_{c1} + \bar{V}_{c2} \tag{2.53c}$$

Since the phase voltages have only a total of six degrees of freedom, these components cannot be completely independent. Suppose we force the components $\bar{V}_{a1}$, $\bar{V}_{b1}$, and $\bar{V}_{c1}$ to make up a balanced three-phase set with phase sequance *abc*. This will require two degrees of freedom (not six) and accomplish the desirable objective of preserving the balanced case. We shall refer to this set as the positive-sequence components and use the subscript 1 to indicate them.

Let us force the set $\bar{V}_{a2}$, $\bar{V}_{b2}$, and $\bar{V}_{c2}$ to be balanced with phase sequence *cba* and refer to these as negative-sequence components (use subscript 2 to indicate them). Again, two degrees of freedom are required, and we have a second balanced set. The remaining set $\bar{V}_{a0}$, $\bar{V}_{b0}$, and $\bar{V}_{c0}$ we shall call the zero-sequence set. Note that they cannot be balanced three phase, because if so, they could be combined with either the positive or negative set. Also, the set can have only two degrees of freedom. We simply force $\bar{V}_{a0}$, V_{b0}, and $\bar{V}_{c0}$ to be equal in magnitude and phase.

When the components are interrelated as just described, they become Fortescue's symmetrical components.

2.7 The *a* Operator As Used in Symmetrical Component Representation

Recall the operator *j*. In polar form, $j = 1\underline{/90^\circ}$. Note multiplication by *j* has the effect of rotating a phasor forward 90° without affecting the magnitude.

Example 2.6

Compute $j\bar{A}$ where $\bar{A} = 10\angle 60^\circ$. Refer to Figure 2.16(b).

Solution

$$\begin{aligned} j\bar{A} &= 1\angle 90^\circ(10\angle 60^\circ) \\ &= 10\angle 150^\circ \end{aligned}$$

Equivalently,

$$\begin{aligned} j\bar{A} &= j(10\angle 60^\circ) \\ &= j(5 + j8.66) \\ &= j5 - 8.66 \\ &= 10\angle 150^\circ \end{aligned}$$

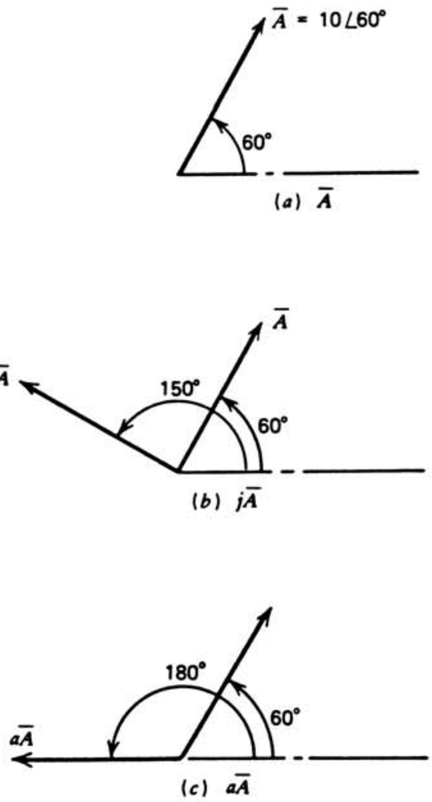


Figure 2.16. *j* and *a* effects.

In a similar manner, define $a = 1\angle 120^\circ$. Note that multiplication by a rotates a phasor forward 120° and does not affect the magnitude.

Example 2.7

Compute $a\bar{A}$ where $\bar{A} = 10\angle 60^\circ$. Refer to Figure 2.16(c).

Solution

$$a\bar{A} = (1\angle 120^\circ)(10\angle 60^\circ)$$
$$= 10\angle 180^\circ$$

Note that

$$a = 1\angle 120^\circ \tag{2.54a}$$
$$= 1\angle -240^\circ$$
$$a^2 = a \cdot a \tag{2.54b}$$
$$= (1\angle 120^\circ)(1\angle 120^\circ)$$
$$= 1\angle 240^\circ$$
$$= 1\angle -120^\circ$$
$$a^3 = 1\angle 360^\circ \tag{2.54c}$$
$$= 1\angle 0^\circ$$

Now recall the definition of symmetrical components. $\bar{V}_{b_1}$ *always lags* $\bar{V}_{a_1}$ by a fixed angle of 120° and always has the same magnitude as $\bar{V}_{a_1}$. Similarly, $\bar{V}_{c_1}$ *leads* $\bar{V}_{a_1}$ by 120°. It follows then that

$$\bar{V}_{b_1} = a^2 \bar{V}_{a_1} \tag{2.55a}$$
$$\bar{V}_{c_1} = a\bar{V}_{a_1} \tag{2.55b}$$

Similarly, we deduce

$$V_{b_2} = a\bar{V}_{a_2} \tag{2.55c}$$
$$V_{c_2} = a^2 \bar{V}_{a_2} \tag{2.55d}$$
$$V_{b_0} = \bar{V}_{a_0} \tag{2.55e}$$
$$V_{c_0} = \bar{V}_{a_0} \tag{2.55f}$$

In other words, it is possible to write all nine of our symmetrical components in terms of three, namely, those referred to the a phase. Rewriting equation (2.53) and substituting equation (2.55), we produce

$$\bar{V}_a = \bar{V}_{a0} + \bar{V}_{a1} + \bar{V}_{a2} \tag{2.56a}$$

$$\begin{aligned}\bar{V}_b &= \bar{V}_{b0} + \bar{V}_{b1} + \bar{V}_{b2} \\ &= \bar{V}_{a0} + a^2\bar{V}_{a1} + a\bar{V}_{a2}\end{aligned} \tag{2.56b}$$

$$\begin{aligned}\bar{V}_c &= \bar{V}_{c0} + \bar{V}_{c1} + \bar{V}_{c2} \\ &= \bar{V}_{a0} + a\bar{V}_{a1} + a^2\bar{V}_{a2}\end{aligned} \tag{2.56c}$$

We may simplify the notation as follows; define

$$\bar{V}_0 = \bar{V}_{a0} \tag{2.57a}$$

$$\bar{V}_1 = \bar{V}_{a1} \tag{2.57b}$$

$$\bar{V}_2 = \bar{V}_{a2} \tag{2.57c}$$

Remember that the symmetrical components $\bar{V}_0$, $\bar{V}_1$, and $\bar{V}_2$ are referred to the a phase. From equation (2.56), we obtain

$$\bar{V}_a = \bar{V}_0 + \bar{V}_1 + \bar{V}_2 \tag{2.58a}$$

$$\bar{V}_b = \bar{V}_0 + a^2\bar{V}_1 + a\bar{V}_2 \tag{2.58b}$$

$$\bar{V}_c = \bar{V}_0 + a\bar{V}_1 + a^2\bar{V}_2 \tag{2.58c}$$

These equations may be manipulated to solve for $\bar{V}_0$, $\bar{V}_1$, and $\bar{V}_2$ in terms of $\bar{V}_a$, $\bar{V}_b$, and $\bar{V}_c$. Doing this, we obtain

$$\bar{V}_0 = \tfrac{1}{3}(\bar{V}_a + \bar{V}_b + \bar{V}_c) \tag{2.59a}$$

$$\bar{V}_1 = \tfrac{1}{3}(\bar{V}_a + a\bar{V}_b + a^2\bar{V}_c) \tag{2.59b}$$

$$\bar{V}_2 = \tfrac{1}{3}(\bar{V}_a + a^2\bar{V}_b + a\bar{V}_c) \tag{2.59c}$$

Equation (2.59) may be used to convert phase voltages (or currents) to symmetrical component voltages (or currents) and vice versa [equation (2.58)].

Example 2.8

Given $\bar{I}_a = 1\underline{/60^\circ}$, $\bar{I}_b = 1\underline{/-60^\circ}$, and $\bar{I}_c = 0$, find the symmetrical components.

Solution

$$\bar{I}_1 = \tfrac{1}{3}(\bar{I}_a + a\bar{I}_b + a^2\bar{I}_c)$$
$$= \tfrac{1}{3}(1\angle 60^\circ + 1\angle 60^\circ + 0)$$
$$= \tfrac{1}{3}(0.5 + j0.866 + 0.5 + j0.866)$$
$$= \tfrac{1}{3}(2\angle 60^\circ)$$
$$= 0.667\angle 60^\circ$$
$$\bar{I}_2 = \tfrac{1}{3}(\bar{I}_a + a^2\bar{I}_b + a\bar{I}_c)$$
$$= \tfrac{1}{3}(1\angle 60^\circ + 1\angle 180^\circ)$$
$$= \tfrac{1}{3}(0.5 + j0.866 - 1)$$
$$= 0.333\angle 120^\circ$$
$$\bar{I}_0 = \tfrac{1}{3}(\bar{I}_a + \bar{I}_b + \bar{I}_c)$$
$$= \tfrac{1}{3}(0.5 + j0.866 + 0.5 - j0.866)$$
$$= 0.333\angle 0^\circ$$

Note

$$\bar{I}_{c_1} = a\bar{I}_1 = 0.667\angle 180^\circ$$
$$\bar{I}_{c_2} = a^2\bar{I}_2 = 0.333\angle 360^\circ$$
$$\bar{I}_{c0} = \bar{I}_0 = 0.333\angle 0^\circ$$

Check

$$\bar{I}_c = \bar{I}_{c_1} + \bar{I}_{c_2} + \bar{I}_{c0}$$
$$= -0.667 + 0.333 + 0.333 \cong 0$$

as given.

The transformation equations (2.58) and (2.59) may be written in matrix form as

$$\begin{bmatrix} \bar{V}_a \\ \bar{V}_b \\ \bar{V}_c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} \bar{V}_0 \\ \bar{V}_1 \\ \bar{V}_2 \end{bmatrix} \tag{2.60a}$$

$$\begin{bmatrix} \bar{V}_0 \\ \bar{V}_1 \\ \bar{V}_2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix} \bar{V}_a \\ \bar{V}_b \\ \bar{V}_c \end{bmatrix} \tag{2.61a}$$

or more compactly

$$\tilde{\mathbf{V}}_{abc} = [\mathbf{T}]\, \tilde{\mathbf{V}}_{012} \tag{2.60b}$$

$$\tilde{\mathbf{V}}_{012} = [\mathbf{T}]^{-1} \tilde{\mathbf{V}}_{abc} \tag{2.61b}$$

where the definitions of $\tilde{\mathbf{V}}_{abc}$, $\tilde{\mathbf{V}}_{012}$, [T], and $[\mathbf{T}]^{-1}$ are obvious, comparing equations (2.60a) and (2.61a) with (2.60b) and (2.61b).

The same transformation holds for currents

$$\tilde{\mathbf{I}}_{abc} = [\mathbf{T}]\, \tilde{\mathbf{I}}_{012} \tag{2.62}$$

$$\tilde{\mathbf{I}}_{012} = [\mathbf{T}]^{-1}\, \tilde{\mathbf{I}}_{abc} \tag{2.63}$$

2.8 The Effect on Impedance

The effect on impedance must be derived. Suppose we start with

$$\tilde{\mathbf{V}}_{abc} = [\mathbf{Z}_{abc}]\, \tilde{\mathbf{I}}_{abc} \tag{2.64a}$$

where $[\mathbf{Z}_{abc}]$ is a 3×3 matrix giving the self- and mutual impedance in and between phases. Substitute equations (2.60b) and (2.62) into (2.64a)

$$[\mathbf{T}]\, \tilde{\mathbf{V}}_{012} = [\mathbf{Z}_{abc}][\mathbf{T}]\, \tilde{\mathbf{I}}_{012} \tag{2.64b}$$

or

$$\tilde{\mathbf{V}}_{012} = [\mathbf{T}]^{-1}[\mathbf{Z}_{abc}][\mathbf{T}]\, \tilde{\mathbf{I}}_{012} \tag{2.64c}$$

Define

$$[\mathbf{Z}_{012}] = [\mathbf{T}]^{-1}[\mathbf{Z}_{abc}][\mathbf{T}] \tag{2.65}$$

so that

$$\tilde{\mathbf{V}}_{012} = [\mathbf{Z}_{012}]\, \tilde{\mathbf{I}}_{012} \tag{2.64d}$$

The key to understanding the importance of symmetrical components lies in equation (2.65). For typical power system components, the matrix $[\mathbf{Z}_{abc}]$ is not diagonal, but does possess certain symmetries. These symmetries are such that $[\mathbf{Z}_{012}]$ is diagonal, either exactly or approximately. When $[\mathbf{Z}_{abc}]$ possesses certain symmetries, analysis is greatly simplified. Example 2.9 illustrates the point.

Example 2.9

Evaluate $[\mathbf{Z}_{012}]$ for the line shown in Figure 2.17.

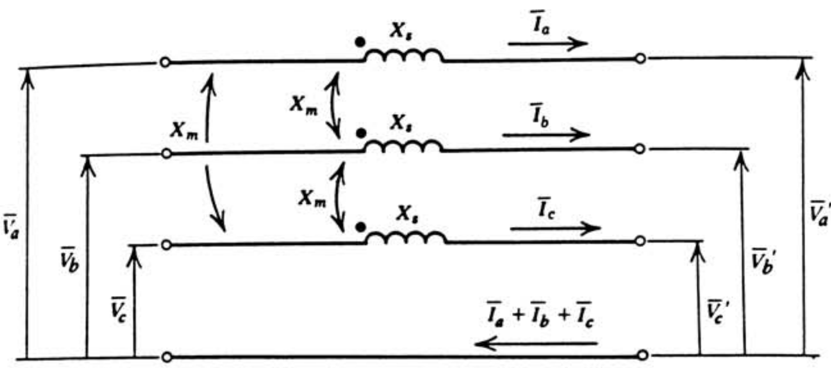


Figure 2.17. Simplified transmission line circuit diagram.

Solution

Kirchhoff's voltage law produces

$$\bar{V}_a - \bar{V}'_a = jX_s\bar{I}_a + jX_m\bar{I}_b + jX_m\bar{I}_c \tag{2.66a}$$

$$\bar{V}_b - \bar{V}'_b = jX_m\bar{I}_a + jX_s\bar{I}_b + jX_m\bar{I}_c \tag{2.66b}$$

$$\bar{V}_c - \bar{V}'_c = jX_m\bar{I}_a + jX_m\bar{I}_b + jX_s\bar{I}_c \tag{2.66c}$$

or in matrix notation

$$\begin{bmatrix}\bar{V}_a\\ \bar{V}_b\\ \bar{V}_c\end{bmatrix} - \begin{bmatrix}\bar{V}'_a\\ \bar{V}'_b\\ \bar{V}'_c\end{bmatrix} = \mathbf{j}\begin{bmatrix}X_s & X_m & X_m\\ X_m & X_s & X_m\\ X_m & X_m & X_s\end{bmatrix}\begin{bmatrix}\bar{I}_a\\ \bar{I}_b\\ \bar{I}_c\end{bmatrix} \tag{2.66d}$$

Even more compactly

$$\bar{\mathbf{V}}_{abc} - \bar{\mathbf{V}}'_{abc} = [\mathbf{Z}_{abc}]\,\bar{\mathbf{I}}_{abc} \tag{2.66e}$$

Transforming into sequence values,

$$\bar{\mathbf{V}}_{012} - \bar{\mathbf{V}}'_{012} = [\mathbf{Z}_{012}]\,\mathbf{I}_{012} \tag{2.66f}$$

where

$$[\mathbf{Z}_{012}] = [\mathbf{T}]^{-1}[\mathbf{Z}_{abc}][\mathbf{T}] \tag{2.65}$$

$$= \tfrac{1}{3}\begin{bmatrix}1 & 1 & 1\\ 1 & a & a^2\\ 1 & a^2 & a\end{bmatrix} j \begin{bmatrix}X_s & X_m & X_m\\ X_m & X_s & X_m\\ X_m & X_m & X_s\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 1 & a^2 & a\\ 1 & a & a^2\end{bmatrix}$$

$$= \tfrac{1}{3}j\begin{bmatrix}(X_s + 2X_m) & (X_s + 2X_m) & (X_s + 2X_m)\\ (X_s - X_m) & [aX_s + (1 + a^2)X_m] & [a^2X_s + (1 + a)X_m]\\ (X_s - X_m) & [a^2X_s + (1 + a)X_m] & [aX_s + (1 + a^2)X_m]\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 1 & a^2 & a\\ 1 & a & a^2\end{bmatrix}$$

$$= j\begin{bmatrix}X_s + 2X_m & 0 & 0\\ 0 & X_s - X_m & 0\\ 0 & 0 & X_s - X_m\end{bmatrix} \tag{2.67}$$

We define for this line

$\bar{Z}_0$ = zero sequence impedance $= \bar{Z}_{00} = j(X_s + 2X_m)$.
$\bar{Z}_1$ = positive sequence impedance $= \bar{Z}_{11} = j(X_s - X_m)$.
$\bar{Z}_2$ = negative sequence impedance $= \bar{Z}_{22} = j(X_s - X_m)$.

The sequence networks are shown in Figure 2.18. Be sure to recognize that mutual coupling has been eliminated.

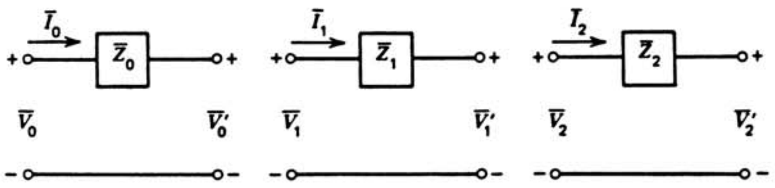


Figure 2.18. Sequence networks.

2.9 Power Considerations

We observe that the total complex power flowing from left to right in the generalized situation shown in Figure 2.13 is

$$\bar{S}_{3\phi} = \bar{V}_a\bar{I}_a^* + \bar{V}_b\bar{I}_b^* + \bar{V}_c\bar{I}_c^* \tag{2.27}$$

In matrix notation,

$$\bar{S}_{3\phi} = \bar{\mathbf{V}}_{abc_t}\bar{\mathbf{I}}_{abc}^* \tag{2.68a}$$

$$= \{[\mathbf{T}]\bar{\mathbf{V}}_{012}\}_t\{[\mathbf{T}]\bar{\mathbf{I}}_{012}\}^* \tag{2.68b}$$

$$= \bar{\mathbf{V}}_{012_t}[\mathbf{T}]_t[\mathbf{T}]^*\bar{\mathbf{I}}_{012}^* \tag{2.68c}$$

Now

$$[\mathbf{T}]_t[\mathbf{T}]^* = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore,

$$\bar{S}_{3\phi} = 3\bar{\mathbf{V}}_{012_t}\bar{\mathbf{I}}_{012}^* \tag{2.68d}$$

or

$$\bar{S}_{3\phi} = 3[\bar{V}_0\bar{I}_0^* + \bar{V}_1\bar{I}_1^* + \bar{V}_2\bar{I}_2^*] \tag{2.68e}$$

Think about this result. Notice that there are no "cross" terms (such as $\bar{V}_1\bar{I}_0{}^*$). This is an essential property of the transformation if we are to construct equivalent circuits and analyze them with conventional circuit analysis techniques. The factor of three is reasonable when we think in terms of nine components (b and c as well as a phase components). It is possible to remove the three by defining [T] with a $1/\sqrt{3}$ coefficient, which some workers prefer.

Example 2.10

Evaluate $S_{3\phi}$ two ways [equation (2.68a) and (2.68d)] given

$$\bar{\mathbf{V}}_{abc} = \begin{bmatrix} 100 \\ -100 \\ 0 \end{bmatrix} \quad \text{and} \quad \bar{\mathbf{I}}_{abc} = \begin{bmatrix} j10 \\ -10 \\ -10 \end{bmatrix}$$

Solution

$$\bar{\mathbf{S}}_{3\phi} = \bar{\mathbf{V}}_{abc_t}\mathbf{I}^*_{abc}$$

$$= [100 \quad -100 \quad 0]\begin{bmatrix} -j10 \\ -10 \\ -10 \end{bmatrix} = 1000 - j1000$$

Now,

$$\bar{\mathbf{V}}_{012} = [\mathbf{T}]^{-1}\bar{\mathbf{V}}_{abc}$$

$$= \tfrac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}\begin{bmatrix} 100 \\ -100 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 0 \\ 100\underline{/-30^\circ} \\ 100\underline{/+30^\circ} \end{bmatrix}$$

$$\bar{\mathbf{I}}_{012} = [\mathbf{T}]^{-1}\bar{\mathbf{I}}_{abc}$$

$$= \tfrac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}\begin{bmatrix} j10 \\ -10 \\ -10 \end{bmatrix} = \tfrac{1}{3}\begin{bmatrix} j10 - 20 \\ j10 + 10 \\ j10 + 10 \end{bmatrix}$$

$$\bar{\mathbf{S}}_{3\phi} = 3\bar{\mathbf{V}}_{012_t}\mathbf{I}^*_{012}$$

$$= \begin{bmatrix} 0 & \dfrac{100}{\sqrt{3}}\underline{/-30^\circ} & \dfrac{100}{\sqrt{3}}\underline{/+30^\circ} \end{bmatrix}\begin{bmatrix} -j10 - 20 \\ 10\sqrt{2}\,\underline{/-45^\circ} \\ 10\sqrt{2}\,\underline{/-45^\circ} \end{bmatrix}$$

$$= \frac{1000\sqrt{2}}{\sqrt{3}}[1\underline{/-75^\circ} + 1\underline{/-15^\circ}] = 1000 - j1000$$

2.10 Symmetrical Components Extended to the *N*-Phase Case

The transformation may be generalized to the N-phase case. Consider revised definitions for

$$\tilde{\mathbf{V}}_{abc} = [\mathbf{T}]\,\tilde{\mathbf{V}}_{012} \tag{2.69a}$$

where

$$\underset{(N\times 1)}{\tilde{\mathbf{V}}_{abc}} = \begin{bmatrix} \bar{V}_a \\ \bar{V}_b \\ \vdots \\ \bar{V}_z \end{bmatrix}; \qquad \underset{(N\times 1)}{\tilde{\mathbf{V}}_{012}} = \begin{bmatrix} \bar{V}_0 \\ \bar{V}_1 \\ \vdots \\ \bar{V}_{N-1} \end{bmatrix}$$

$$\underset{(N\times N)}{[\mathbf{T}]} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & a^{N-1} & a^{N-2} & \dots & a \\ 1 & a^{N-2} & a^{N-4} & \dots & a^2 \\ 1 & a^{N-3} & a^{N-6} & \dots & a^3 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & a & a^2 & \dots & a^{N-1} \end{bmatrix} \tag{2.69b}$$

with general entry

$$t_{ij} = a^{-(i-1)(j-1)} \tag{2.69c}$$

$$a = 1\underline{/360^\circ/N} \tag{2.69d}$$

Likewise,

$$\tilde{\mathbf{V}}_{012} = [\mathbf{T}]^{-1}\tilde{\mathbf{V}}_{abc} \tag{2.70a}$$

where

$$\underset{(N\times N)}{[\mathbf{T}]^{-1}} = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & a & a^2 & \dots & a^{N-1} \\ 1 & a^2 & a^4 & \dots & a^{2(N-1)} \\ 1 & a^3 & a^6 & \dots & a^{3(N-1)} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & a^{N-1} & a^{2(N-1)} & \dots & a^{(N-1)(N-1)} \end{bmatrix} \tag{2.70b}$$

with general entry

$$t_{ij}^{-1} = a^{(i-1)(j-1)} \tag{2.70c}$$

Likewise,

$$\tilde{\mathbf{I}}_{abc} = [\mathbf{T}]\tilde{\mathbf{I}}_{012} \tag{2.71a}$$

and

$$\mathbf{\bar{I}}_{012} = [\mathbf{T}]^{-1}\,\mathbf{\bar{I}}_{abc} \tag{2.71b}$$

Fortescue maintained that N-phase variables could be decomposed into N *sets* of symmetrical components, N components to a set. Thus, N^2 components were needed to completely represent an N-phase system. For example, in a six-phase system, there are *six* Fortescue zero-sequence voltages: $\bar{V}_{a0}$, $\bar{V}_{b0}$, $\bar{V}_{c0}$, $\bar{V}_{d0}$, $\bar{V}_{e0}$, and $\bar{V}_{f0}$.

In this work, only one sequence variable per sequence is used ($\bar{V}_0$, $\bar{V}_1$, etc.), and it is defined in equation (2.70a). The view taken is that of a transformation through

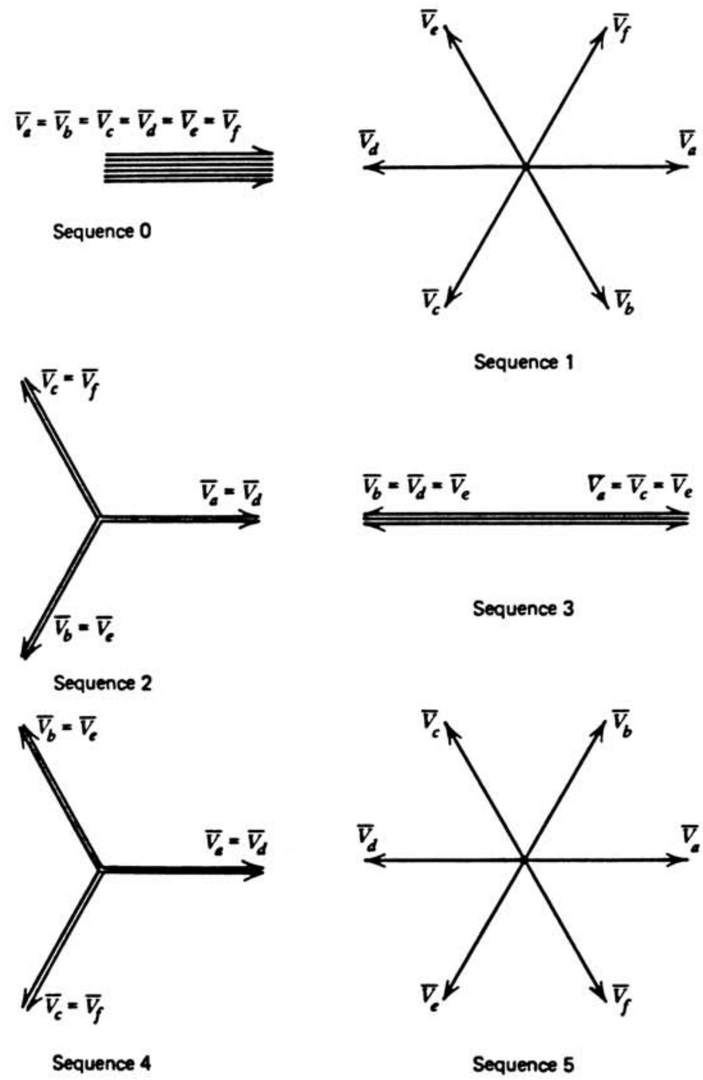


Figure 2.19. Phase voltages for isolated sequence excitation.

which N-phase variables are replaced by N-sequence variables. The two viewpoints are reconciled if we realise that

$$\begin{aligned}\bar{V}_0 &= \bar{V}_{a0}\\ \bar{V}_1 &= \bar{V}_{a1}\\ &\vdots\\ \bar{V}_{N-1} &= \bar{V}_{aN-1}\end{aligned}$$

and that b, c, and so forth, sequence variables can be ignored with no loss of generality and computed from $\bar{V}_0, \ldots, \bar{V}_{N-1}$ if necessary.

Three of these N-sequence quantities have properties that are familiar from three-phase work. If an N-phase system is excited with isolated sequence 0 voltage, the phase voltages will be equal in magnitude and phase. Sequence 1 voltage produces equal phase voltages, equal in magnitude, 360°/N separated in phase. with phase sequence *abc*. Sequence $N - 1$ voltage produces equal phase voltages, equal in magnitude, 360°/N separated in phase, with phase sequence *cba*. Thus, sequence 0, sequence 1, and sequence $N - 1$ values in the N-phase case correlate to zero, positive, and negative sequence values in the three-phase case. Phase voltages for isolated sequence excitation are presented in diagram form for a six-phase system in Figure 2.19.

2.11 Balanced N-Phase Operation Using Symmetrical Components

Consider the general three-phase system shown in Figure 2.13 operating in an arbitrary but balanced condition, with phase sequence *abc* such that

$$\begin{aligned}\bar{V}_a &= \bar{V}_p\underline{/\alpha} = (1)\bar{V}_a\\ \bar{V}_b &= \bar{V}_p\underline{/\alpha - 120} = (a^2)\bar{V}_a\\ \bar{V}_c &= \bar{V}_p\underline{/\alpha + 120} = (a)\bar{V}_a\end{aligned}$$

Let us calculate the symmetrical components

$$\begin{aligned}\tilde{\mathbf{V}}_{012} &= [\mathbf{T}]^{-1}\,\tilde{\mathbf{V}}_{abc}\\ &= \frac{\bar{V}_a}{3}\begin{bmatrix}1 & 1 & 1\\ 1 & a & a^2\\ 1 & a^2 & a\end{bmatrix}\begin{bmatrix}1\\ a^2\\ a\end{bmatrix}\\ &= \begin{bmatrix}0\\ \bar{V}_a\\ 0\end{bmatrix}\end{aligned}$$

That is, the zero- and negative-sequence voltages $\bar{V}_0$ and $\bar{V}_2$ are zero, whereas the positive-sequence voltage $\bar{V}_1$ is $\bar{V}_a$!

This result is extremely important. It means that for balanced three-phase circuits, the problem reduces to a single-phase situation. We need only determine the positive-sequence equivalent network for the power system (a single-phase network), solve for currents and voltages throughout, and recognize that $\bar{V}_1$ is $\bar{V}_a$ and $\bar{I}_1$ is $\bar{I}_a$. everywhere. The zero- and negative-sequence quantities are everywhere trivially zero and thus may be ignored. This aproach is sometimes called per-phase analysis; it should be recognized that what is called the per-phase equivalent circuit is identical to the positive-sequence equivalent circuit.

This result is also valid for the balanced N-phase operation, phase sequence *abc*. That is,

$$\bar{\mathbf{V}}_{012} = [0 \quad \bar{V}_a \quad 0 \;\ldots\; 0]_t \qquad (N \times 1)$$

$$\bar{\mathbf{I}}_{012} = [0 \quad \bar{I}_a \quad 0 \;\ldots\; 0]_t \qquad (N \times 1)$$

Since many power system problems involve investigating system behavior in the balanced mode, the reduction of the problem to single-phase order is of major practical importance. Note that *all* phase values can easily be computed on demand from

$$\bar{\mathbf{V}}_{abc} = [\mathbf{T}]\,\bar{\mathbf{V}}_{012}$$

$$\bar{\mathbf{I}}_{abc} = [\mathbf{T}]\,\bar{\mathbf{I}}_{012}$$

2.12 Summary

Power system electrical performance is basically described using sinusoidal steady-state (ac) circuit concepts. The phasor method of describing voltages and current is invaluable and used extensively throughout the rest of this book. Power and energy calculations are of major importance and emphasized accordingly. It is important to understand the three kinds of power (S, P, and Q).

Since the power system is typically three phase, special attention is given to this topic. Perhaps you were curious why the balanced case was emphasized. The reasons are that many problems in power system analysis are concerned with the system operating in its normal balanced three-phase mode. Though it is a special case from a mathematical viewpoint, it is of considerable practical importance. Unbalanced systems are changed through the symmetrical component transformation into three balanced systems. Unexpectedly, balanced concepts are again of prime importance.

We have developed the method of symmetrical components as essentially a mathematical transformation or change of coordinates. We have asserted that circuit models for power system components are quite simple for each set of sequence components. The next step is to discuss such circuit models. We should recognize that when the system is operating in its normal balanced mode, the phase values become the positive-sequence values, the negative- and zero-sequence quantities evaluating to zero. In that sense, we think of the balanced case as a special case of a more general situation that involves all three sequence networks.

As in all of electrical engineering, we must be careful not to confuse the physical device with its mathematical model. Thus, when we discuss the ideal three-phase source, we are tempted to equate it to an actual three-phase generator. Actually, these two are dramatically different in many respects; we shall spend considerable time later developing a more satisfactory model. Even then, the model is different in some important respects from the real thing. There is *always* a trade off between accuracy and complexity in modeling, with the ideal of perfect accuracy unattainable. Where do we stop this process? The answer requires an understanding of what our calculated results are to be used for. This understanding comes only from experience. The models and methods we investigate represent the combined experience of many engineers and scientists. As our investigation of the power system develops, the student will acquire a growing appreciation of why a thorough understanding of the basics of sinusoidal steady-state circuit is necessary.

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