Autonomous Helicopter Landing

A Nonlinear Output Regulation Perspective

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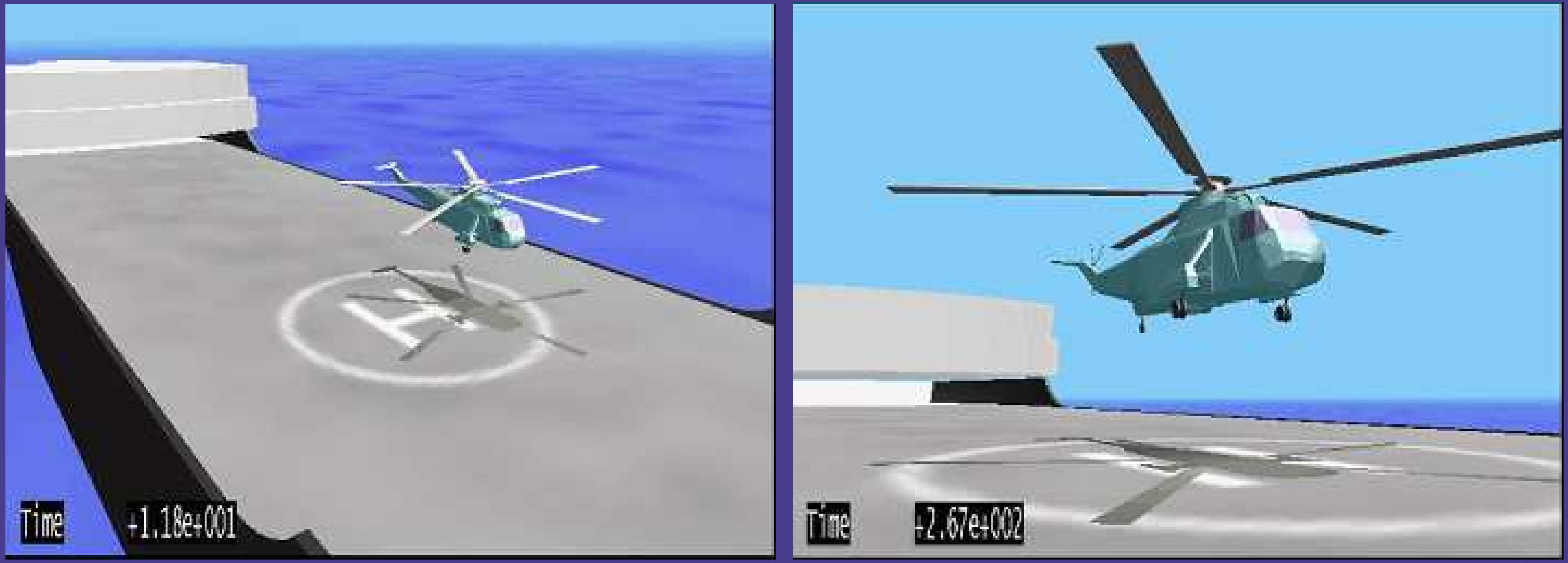
Introduction

Landing an autonomous helicopter on a vessel undergoing an unknown vertical motion



The vessel oscillates vertically, but the parameter of its motion (frequencies, amplitudes and phases) are unknown

Introduction

- The problem: tracking of unknown trajectories and/or rejection of unknown disturbances
- The technical approach: a feedback control that incorporates an “internal model" of the exogenous inputs
- Main features:
  - Continuous-time adaptation of the internal model
  - Stabilization based on combined low-amplitude / high-gain feedback
  - Guaranteed convergence and robustness

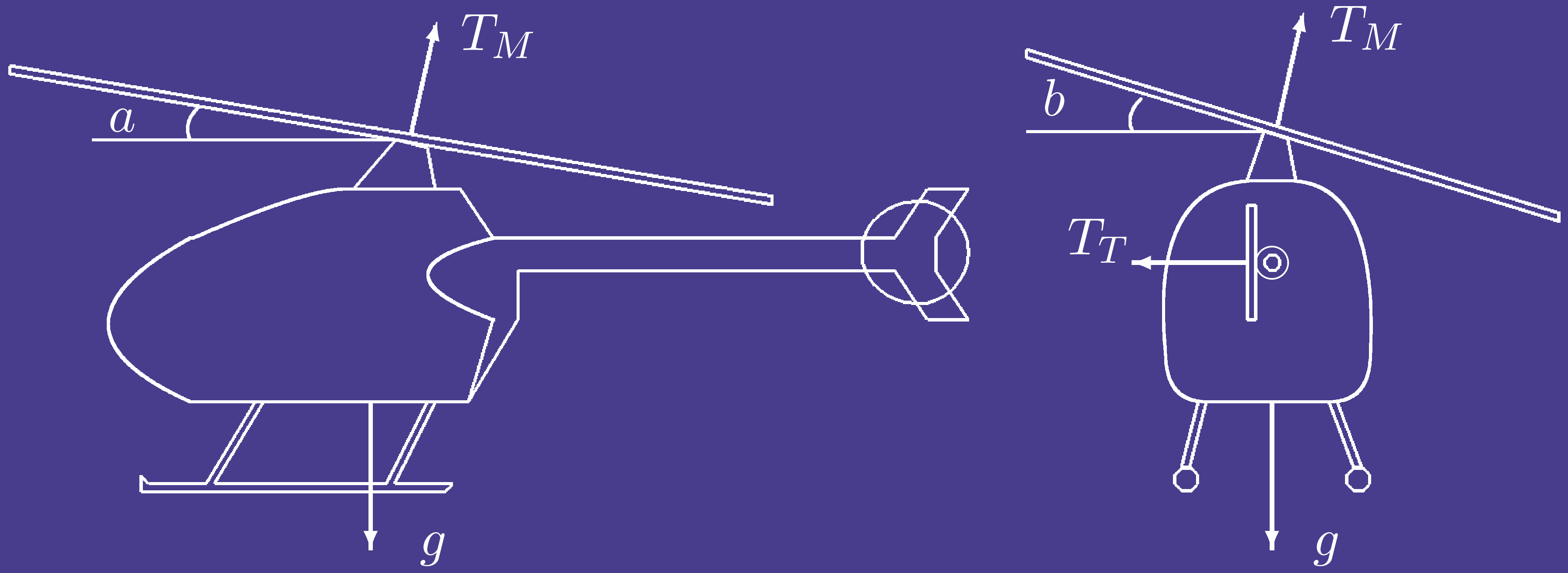
Helicopter model - rigid body

$$M\ddot{p}^i = Rf^b \qquad J\dot{\omega}^b = -S(\omega^b)J\omega^b + \tau^b$$

$$\dot{R} = RS(\omega^b) \quad \Rightarrow \quad \left\{ \begin{array}{rcl} \dot{q}_0 & = & -\frac{1}{2}q^{\mathrm{T}}\omega^b \\ \dot{q} & = & \frac{1}{2}[q_0 I + S(q)]\omega^b \end{array} \right.$$

p^i	position of the center of mass (inertial frame)
R	rotation matrix, (q_0, q) unit quaternions
ω^b	angular velocity (body frame)
f^b	external force (body frame)
τ^b	external torque (body frame)

Helicopter model - control inputs



T_M	Main thrust
T_T	Tail thrust
a	Longitudinal deflection of the rotor plane
b	Lateral deflection of the rotor plane

Full model

$$f^b = \begin{pmatrix} X_M \\ Y_M + Y_T \\ Z_M \end{pmatrix} + R^{\mathrm{T}} \begin{pmatrix} 0 \\ 0 \\ Mg \end{pmatrix}$$

Main / tail rotors

gravity

$$\tau^b = \begin{pmatrix} \tau_{f_1} \\ \tau_{f_2} \\ \tau_{f_3} \end{pmatrix} + \begin{pmatrix} R_M \\ M_M + M_T \\ N_M \end{pmatrix}$$

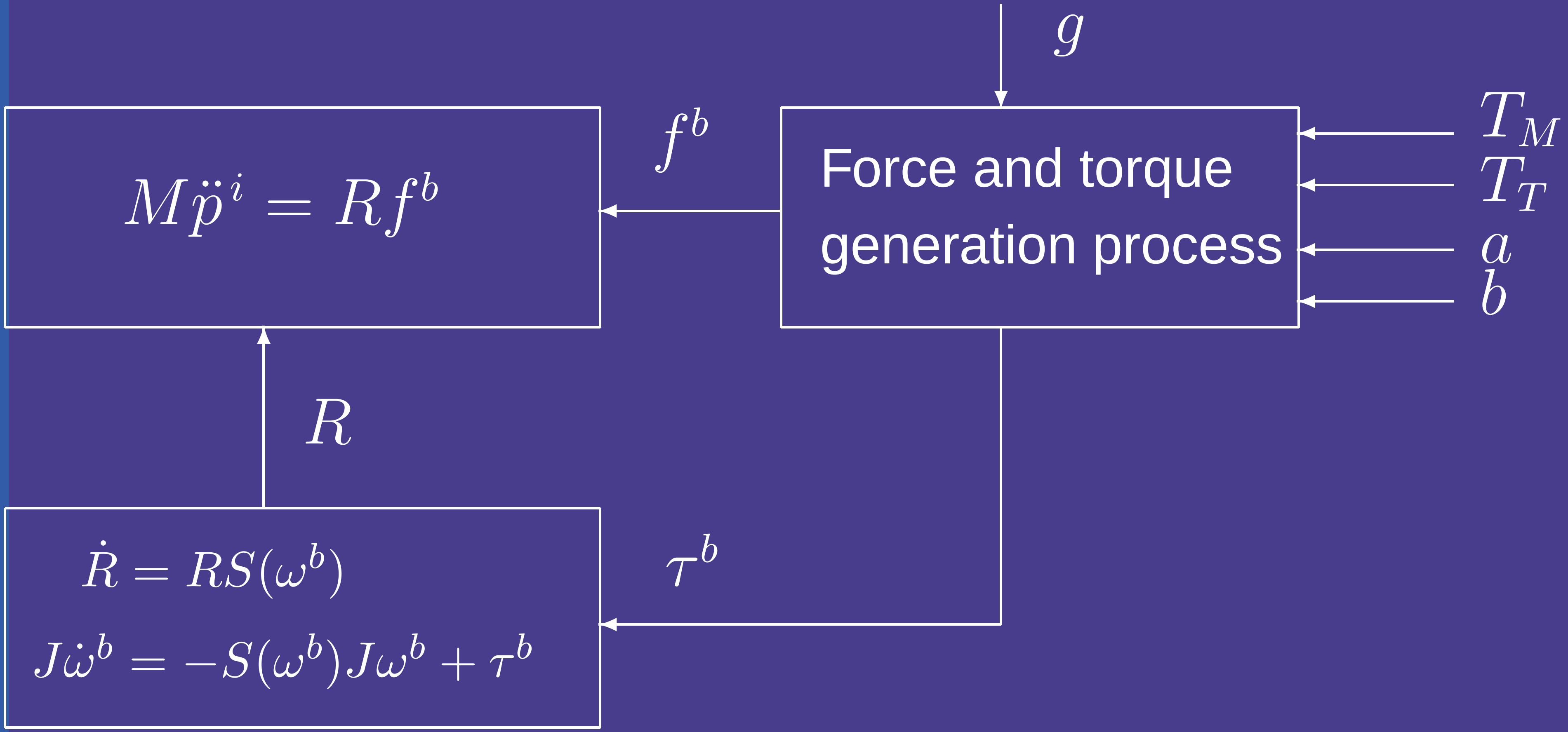
Aerodynamic forces

Full model

$$\begin{aligned} X_M &= -T_M \sin a \\ Z_M &= -T_M \cos a \cos b \\ Y_M &= T_M \sin b \\ Y_T &= -T_T \end{aligned} \qquad \begin{aligned} R_M &= c_b^M\, b - Q_M \sin a \\ M_M &= c_a^M\, a + Q_M \sin b \\ N_M &= -Q_M \cos a \cos b \\ M_T &= -c_T^Q\, T_T^{1.5} - D_T^Q \\ Q_M &= c_M^Q\, T_M^{1.5} + D_M^Q \end{aligned}$$

$$\begin{aligned} \tau_{f_1} &= Y_M h_M + Z_M y_M + Y_T h_T \\ \tau_{f_2} &= -X_M h_M + Z_M \ell_M \\ \tau_{f_3} &= -Y_M \ell_M - Y_T \ell_T \end{aligned}$$

Full model: structure



g

f^b

$$M\ddot{p}^i = Rf^b$$

Force and torque generation process

T_M

T_T

a

b

R

τ^b

$$\dot{R} = RS(\omega^b)$$

$$J\dot{\omega}^b = -S(\omega^b)J\omega^b + \tau^b$$

Simplified model

- We let

$$\sin(a) = a\,,\ \sin(b) = b\,,\ \cos(a) = \cos(b) = 1$$

- neglect the contribution of T_M and T_T along x^b, y^b

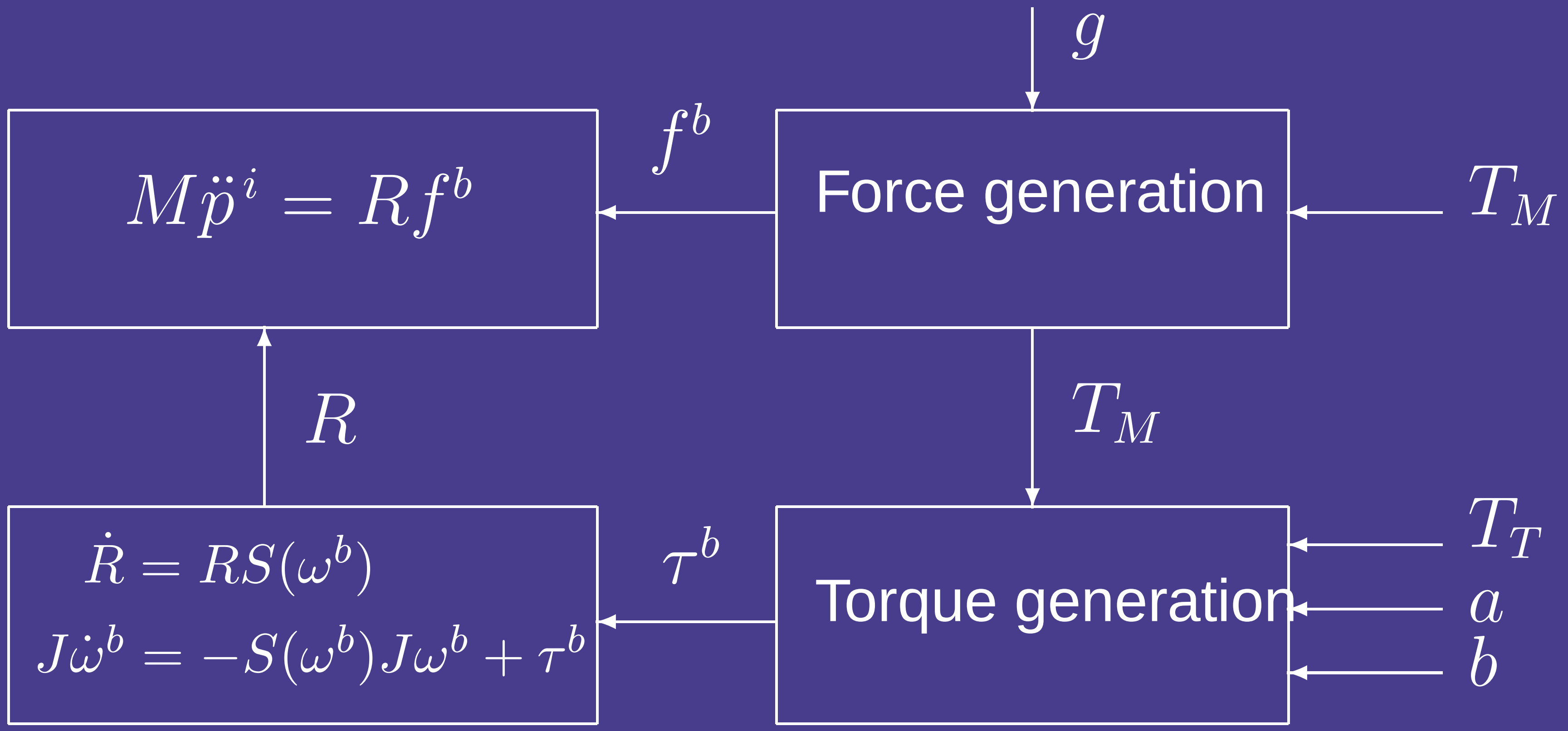
$$f^b = \begin{pmatrix} 0 \\ 0 \\ -T_M \end{pmatrix} + R^{\mathrm{T}} \begin{pmatrix} 0 \\ 0 \\ Mg \end{pmatrix}$$

- approximate τ^b with

$$\tau^b(\mathbf{v}) = A(T_M)\mathbf{v} + B(T_M)\,, \quad \mathbf{v} := \mathrm{col}(a, b, T_T)$$

Simplified model: structure

The simplified model neglects the weak couplings in the force/moment generation mechanism


$M\ddot{p}^i = R f^b$

f^b

Force generation

g

T_M

R

T_M

$\dot{R} = R S(\omega^b)$

$J\dot{\omega}^b = -S(\omega^b) J \omega^b + \tau^b$

τ^b

Torque generation

T_T

a

b

Simplified model

Since inertial and aerodynamic parameters are uncertain,

$$M = M_0 + M_\Delta\,, \qquad J = J_0 + J_\Delta$$

$$\begin{aligned} A(T_M) &= A_0(T_M) + A_\Delta(T_M) \\ B(T_M) &= B_0(T_M) + B_\Delta(T_M) \end{aligned}$$

All uncertain parameters are collected into a vector

$$\mu = \mu_0 + \mu_\Delta\,, \qquad \mu_\Delta \in \mathcal{P} \subset I\!R^p$$

where $\mathcal{P}$ is a compact set.

Problem statement

Control objective:

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- Synchronization of $z(t)$ with the vertical motion $z^*(t)$ of the ship, at a given safety distance H.
- Smooth landing, $H \to 0$, while keeping the center of mass on target and the orientation constant.

Reference trajectory:

$$\left(x^{\mathrm{ref}}(t), y^{\mathrm{ref}}(t), z^{\mathrm{ref}}(t)\right) = \left(0, 0, H + z^*(t)\right), \quad R^{\mathrm{ref}}(t) = I$$

Problem Statement

The motion $z^*(t)$ is modeled as the sum of a fixed number of sinusoidal signals

$$z^*(t) = \sum_{i=1}^{N} A_i \cos(\Omega_i t + \varphi_i)$$

of *unknown* amplitude, phase and frequency

$$\left(A_i, \varphi_i, \Omega_i\right), \qquad i = 1, \ldots, N$$

Problem Statement

The problem fits naturally in the framework of *nonlinear output regulation theory*, as $z^{\text{ref}}(t)$ is generated by

$$\text{exosystem} \left\{ \begin{array}{rcl} \dot{H} & = & 0 \\ \dot{w} & = & S(\varrho)w \\ z^{\text{ref}}(t) & = & H + r(w) \end{array} \right.$$

where

$$\varrho = \text{col}(\Omega_1, \ldots, \Omega_N), \quad r(w) = Qw$$

$$S(\varrho) = \text{diag}(S_1, \ldots, S_N)\,, \quad S_i = \begin{pmatrix} 0 & \Omega_i \\ -\Omega_i & 0 \end{pmatrix}$$

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$$\begin{aligned}\dot{\eta} &= \varphi(\eta, \mathbf{e}, \dot{\mathbf{e}}, q, \omega^b) \\ T_M &= \psi_{T_M}(\eta, \mathbf{e}, \dot{\mathbf{e}}, q, \omega^b) \\ \mathbf{v} &= \psi_{\mathbf{v}}(\eta, \mathbf{e}, \dot{\mathbf{e}}, q, \omega^b)\end{aligned}$$

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where $\mathbf{e} := (x,\ y,\ z - z^{\mathrm{ref}})$, such that

$$\lim_{t\to\infty} |z(t) - z^{\mathrm{ref}}(t)| = 0\,, \quad \|\mathbf{e}(t)\| \leq \delta\,,\ \|q(t)\| \leq \delta\,,\ \forall\, t \geq T$$

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with a semi-global domain of attraction, for all $\mu_\Delta \in \mathcal{P}$.

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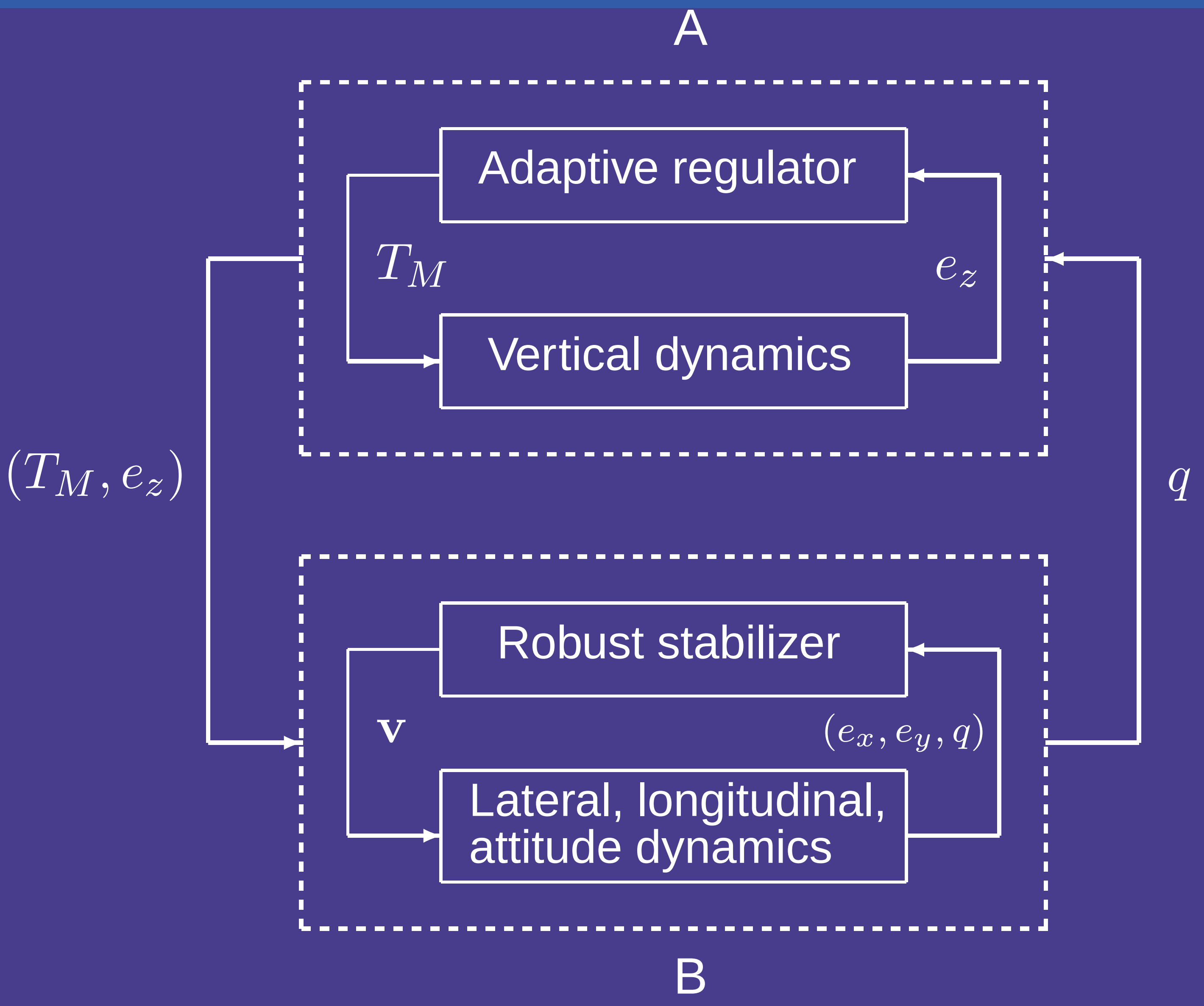
The two subsystems are not decoupled!

Controller structure



A

Adaptive regulator

T_M

e_z

Vertical dynamics

(T_M, e_z)

q

Robust stabilizer

$\mathbf{v}$

(e_x, e_y, q)

Lateral, longitudinal, attitude dynamics

B

Vertical error dynamics

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$$T_M = \frac{gM_0 - \boxed{u}}{1 - \mathrm{sat}_c(2q_1^2 + 2q_2^2)}\,, \qquad 0 < c < 1$$

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$$T_M = \frac{gM_0 - \boxed{u}}{1 - \mathrm{sat}_c(2q_1^2 + 2q_2^2)}\,, \qquad 0 < c < 1 \qquad \text{external control}$$

we get

$$\begin{aligned} M\ddot{z} &= \phi_c^z(q)u + g[M - M_0\phi_c^z(q)] \\ &= u + gM_\Delta\,, \qquad \text{if } q \text{ is small!} \end{aligned}$$

Vertical error dynamics

If $q(t)$ is kept small so that $\phi_c^z(q(t)) \equiv 1$, the input u needed to keep $z(t) \equiv z^{\mathrm{ref}}(t)$ is

$$u_{\mathrm{ss}}(w, \mu) = M\ddot{r}(w) - gM_\Delta = MQS^2(\varrho)w - gM_\Delta$$

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The control $u_{\text{ss}}(w,\mu)$ is generated by the internal model

$$\boxed{\begin{array}{rcl} \frac{\partial\tau}{\partial w}S(\varrho)w & = & \Phi(\varrho)\tau(w,\mu) \\ u_{ss} & = & \Gamma(\varrho)\tau(w,\mu) \end{array}}$$

where

$$\tau(w,\mu) = \begin{pmatrix} -gM_\Delta \\ Mw \end{pmatrix}$$

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Then, there exists $H_2 \in I\!\!R^{1 \times 2N}$ such that the pair

$$F = \begin{pmatrix} 0 & H_2 \\ -G_2 & F_2 \end{pmatrix}, \qquad G = \begin{pmatrix} 0 \\ G_2 \end{pmatrix}$$

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is controllable, and F is Hurwitz.

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Then, for any $\varrho \in I\!R^N$, there exists $\Psi_{2,\varrho} \in I\!R^{1\times 2N}$ such that

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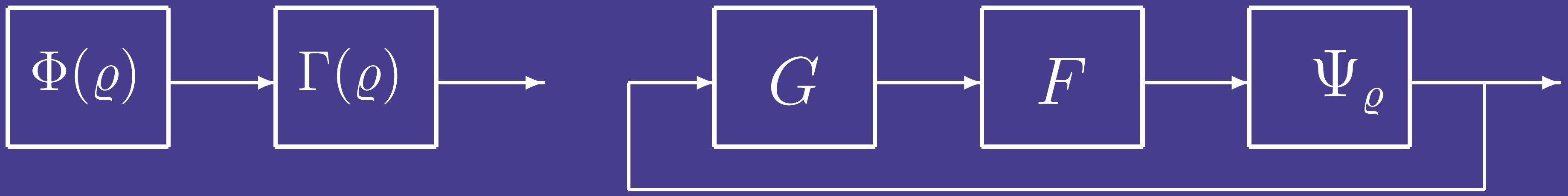
The advantage is that the uncertainties are now lumped in Ψ_ϱ.



Design of the regulator

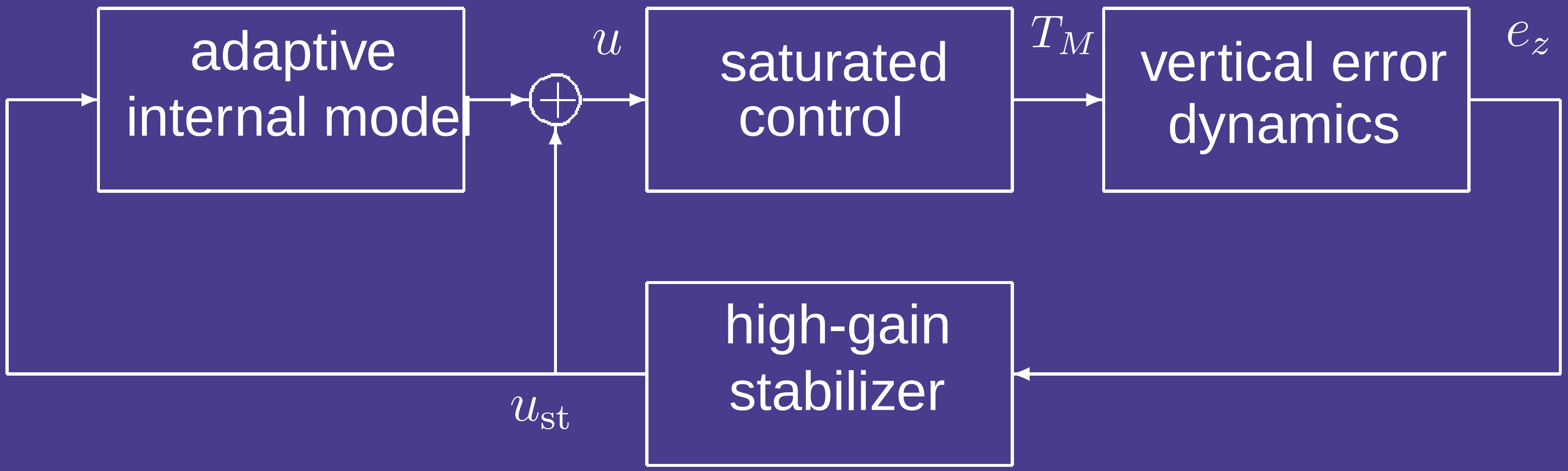
We replace Ψ_ϱ by an estimate $\hat{\Psi} = (1\ \hat{\Psi}_2)$, and implement the adaptive internal model-based regulator

$$\left\{ \begin{array}{rcl} \dot{\xi} & = & (F + G\hat{\Psi})\,\xi + G u_{\mathrm{st}} \\ \frac{\mathrm{d}}{\mathrm{d}t}\hat{\Psi}_2 & = & \gamma^{-1}\xi_2^{\mathrm{T}} u_{\mathrm{st}}\,, \qquad \gamma > 0 \\ u & = & \hat{\Psi}\xi + u_{\mathrm{st}} \end{array} \right.$$

with $\xi = \mathrm{col}(\xi_1,\, \xi_2) \in I\!R \times I\!R^{2N}$, where the stabilizing control u_{st} is selected as the high-gain feedback

$$u_{\mathrm{st}} = -k_2(\dot{e}_z + k_1 e_z)\,, \qquad k_1, k_2 > 0\,.$$

Regulator structure



The dynamic regulator yields boundedness of all internal variables.

It steers asymptotically the vertical error to zero, only if the attitude error is kept sufficiently small.

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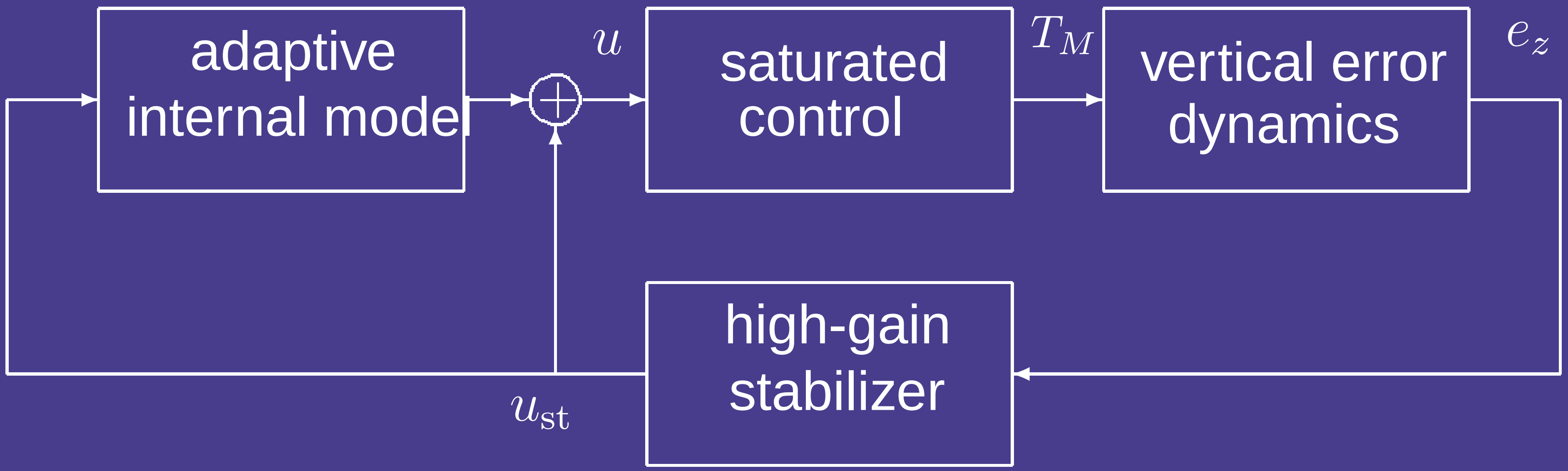

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$\Rightarrow$ We need finite-time convergence of $q(t)$ to a “small” ball

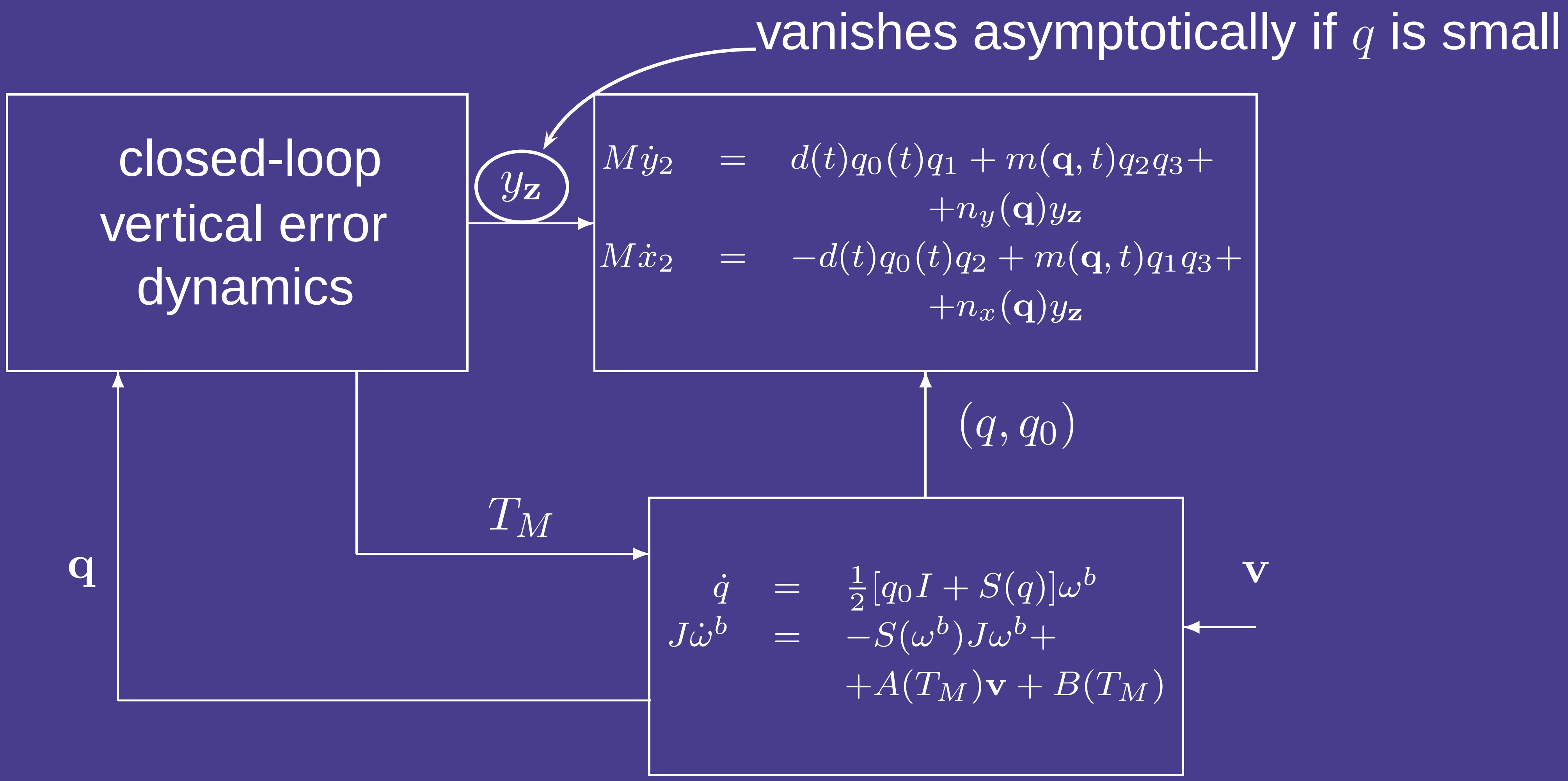
Lateral/ longitudinal dynamics

The choice of T_M to regulate the vertical error dynamics affects the lateral and longitudinal dynamics as well.

$$\begin{aligned} \dot{x} &= x_2 \\ M\dot{x}_2 &= -d(t)q_0q_2 + m(\mathbf{q},t)q_1q_3 + n_x(\mathbf{q})y_{\mathbf{z}}(e_z,w) \\ \dot{y} &= y_2 \\ M\dot{y}_2 &= d(t)q_0q_1 + m(\mathbf{q},t)q_2q_3 + n_y(\mathbf{q})y_{\mathbf{z}}(e_z,w) \end{aligned}$$

The dynamics are time-varying due to the exogenous system, and perturbed by $e_z(t)$ and $w(t)$.

Lateral/ longitudinal dynamics



vanishes asymptotically if q is small

closed-loop vertical error dynamics

$$
\begin{aligned}
M\dot{y}_2 &= d(t)q_0(t)q_1 + m(\mathbf{q},t)q_2q_3 + \\
&\quad + n_y(\mathbf{q})y_{\mathbf{z}} \\
M\dot{x}_2 &= -d(t)q_0(t)q_2 + m(\mathbf{q},t)q_1q_3 + \\
&\quad + n_x(\mathbf{q})y_{\mathbf{z}}
\end{aligned}
$$

$$
\begin{aligned}
\dot{q} &= \tfrac{1}{2}[q_0 I + S(q)]\omega^b \\
J\dot{\omega}^b &= -S(\omega^b)J\omega^b + \\
&\quad + A(T_M)\mathbf{v} + B(T_M)
\end{aligned}
$$

Multi-objective control

The only DOF left is the choice of $\mathbf{v}$, which must accomplish the following tasks:

- Robustly stabilize the attitude dynamics, sending $q(t)$ in a neighborhood of the origin *in finite time*

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- Robustly stabilize the attitude dynamics, sending $q(t)$ in a neighborhood of the origin *in finite time*
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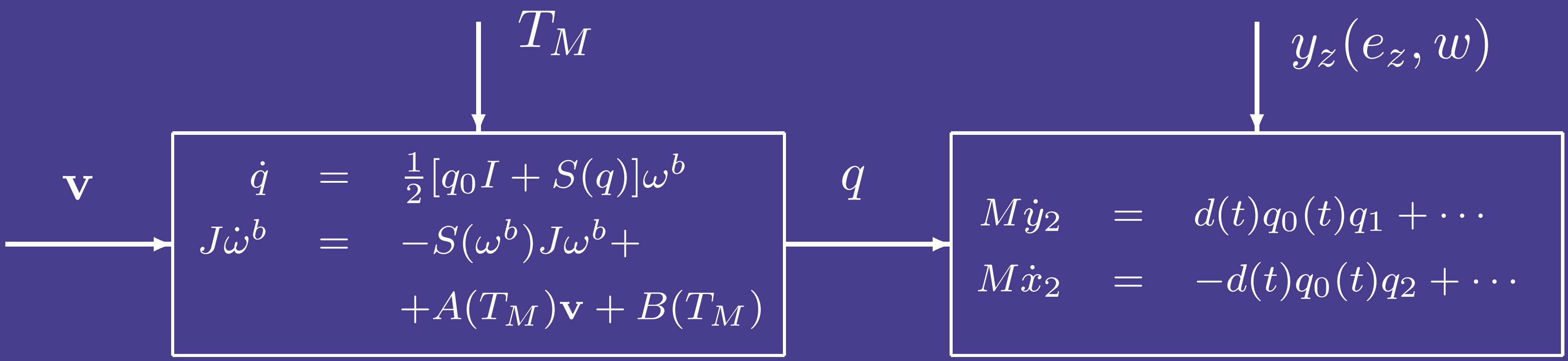
- Stabilize the interconnected subsystem (lat./long./attitude)

Multi-objective control



T_M

$y_z(e_z, w)$

$\mathbf{v}$

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q

$$\begin{aligned} M\dot{y}_2 &= d(t)q_0(t)q_1 + \cdots \\ M\dot{x}_2 &= -d(t)q_0(t)q_2 + \cdots \end{aligned}$$

In principle, we could use q as a virtual control for the lateral/long. dynamics, but the system is not in *feedback form*.

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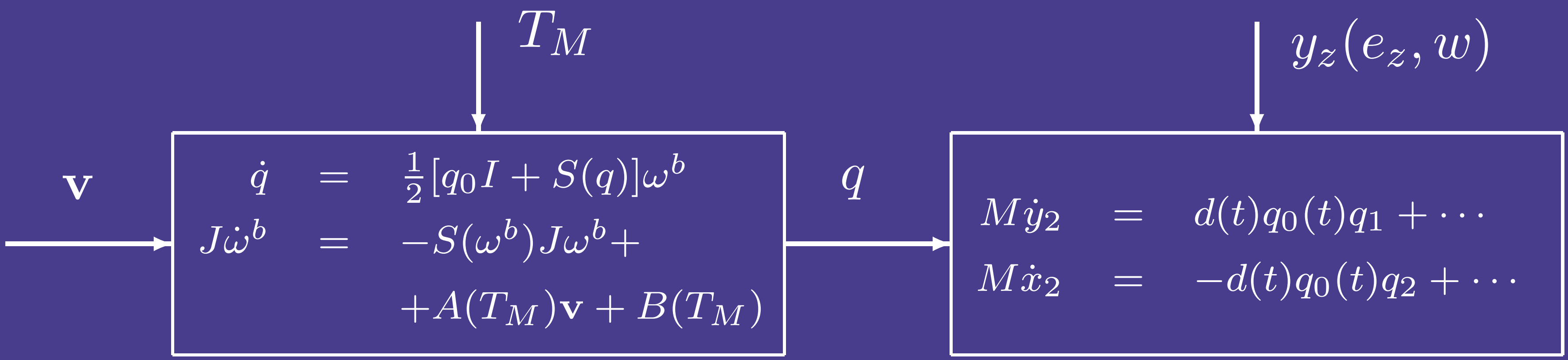

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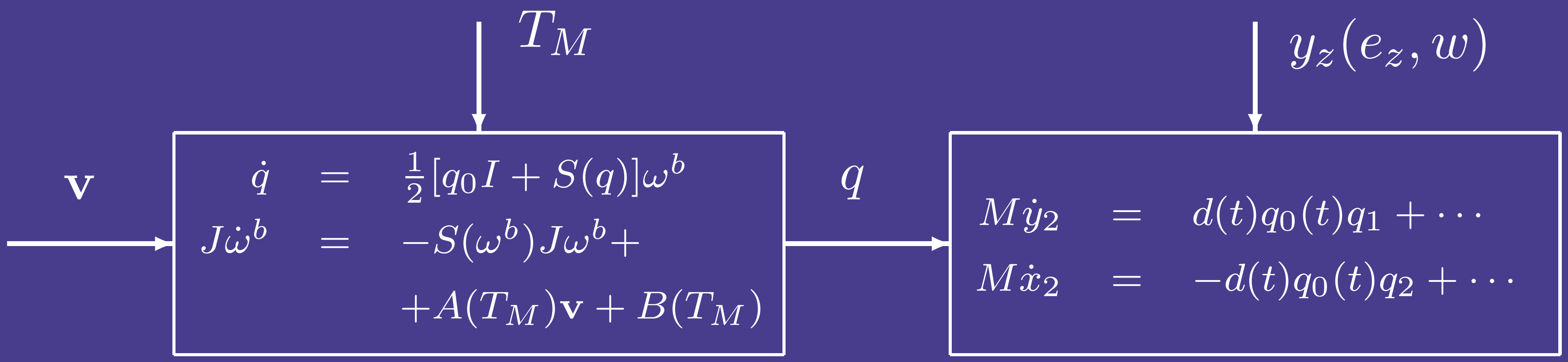

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- Use a combined high/gain - low/amplitude control to induce a time-scale separation between the two subsystems.

Next step: attitude dynamics

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high-gain feedback

low-amplitude, $\|u_2(t)\| \leq \lambda_2$

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high-gain feedback

low-amplitude, $\|u_2(t)\| \leq \lambda_2$

The control u_2 will be designed to stabilize the lateral dynamics

Attitude dynamics: main result

It can be shown that, for any compact set of initial conditions for $(q(t), \omega(t))$, and for any $T^* > 0$ there exists a choice of $K_3 > 0$, $K_4 > 0$ and $\lambda_2 > 0$ such that:

- The trajectory $(q(t), \omega(t))$ is bounded for all $t \geq 0$, and $q_0(t)$ does not change sign

Attitude dynamics: main result

It can be shown that, for any compact set of initial conditions for $(q(t), \omega(t))$, and for any $T^* > 0$ there exists a choice of $K_3 > 0$, $K_4 > 0$ and $\lambda_2 > 0$ such that:

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Hence, $q(t)$ is brought in finite time in a neighborhood of the origin. This is already enough to conclude that

$$\lim_{t \to \infty} |z(t) - z^{\text{ref}}(t)| = 0 \ .$$

Putting everything together

Now it's time to design the bounded control u_2 to stabilize the interconnection of the attitude and the lateral/long. dynamics.

We will use q_1 and q_2 as “virtual controls" for y and x respectively.

To remove drifts, we augment the dynamics with the bank of integrators

$$\dot{\eta}_x = x \, , \quad \dot{\eta}_y = y \, , \quad \dot{\eta}_q = q_3$$

and introduce *smooth vector saturation functions* $\sigma(s)$:

$$|\sigma'(s)| := |d\sigma(s)/ds| \leq 2 \; \forall \, s, \quad s\sigma(s) > 0 \; \forall \, s \neq 0, \, \sigma(0) = 0.$$

$$\sigma(s) = \text{sgn}(s) \text{ for } |s| \geq 1. \qquad |s| < |\sigma(s)| < 1 \text{ for } |s| < 1.$$

Putting everything together

Define new state variables as

$$\zeta_0 := \begin{pmatrix} \eta_y \\ \eta_x \end{pmatrix} , \quad \zeta_1 := \begin{pmatrix} y \\ x \end{pmatrix} + \lambda_0 \sigma(\frac{K_0}{\lambda_0} \zeta_0)$$

$$\zeta_2 := \begin{pmatrix} y_2 \\ x_2 \\ \eta_q \end{pmatrix} + \lambda_1 \sigma(\frac{K_1}{\lambda_1} \zeta_1)$$

and choose the “nested saturation" control

$$u_2 = -\lambda_2 \sigma(\frac{K_2}{\lambda_2} \zeta_2)$$

Main result

It can be shown that there exists a choice of the gains K_0, K_1, K_2 and the saturation levels λ_0, λ_1, λ_2 such that:

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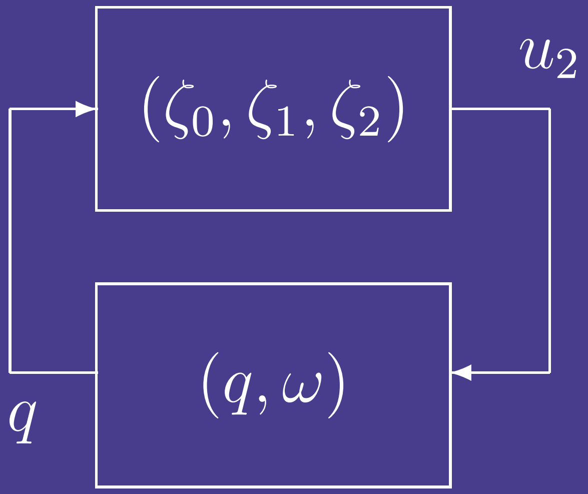
It can be shown that there exists a choice of the gains K_0, K_1, K_2 and the saturation levels λ_0, λ_1, λ_2 such that:

- The interconnected system satisfies an asymptotic I/O bound with respect to the external disturbance $\Rightarrow$ all trajectories are bounded.

Main result

It can be shown that there exists a choice of the gains K_0, K_1, K_2 and the saturation levels λ_0, λ_1, λ_2 such that:

- The system is indeed a small-gain theorem interconnection of two (weak) ISS-systems $\Rightarrow$ overall system is (weakly) ISS, and the gain can be assigned through K_4.



A case study: a small AUV

For the simulations, we use the full nonlinear model, with parameter uncertainties up to 20$\%$ of the nominal values.

$J_x = 0.142413$	$J_y = 0.271256$	$J_z = 0.271492$
$\ell_M = -0.015$	$y_M = 0$	$h_M = 0.2943$
$\ell_T = 0.8715$	$h_T = 0.1154$	$M = 4.9$
$C_M^Q = 0.004452$	$D_M^Q = 0.6304$	$c_M^Q = 25.23$
$C_T^Q = 0.005066$	$D_T^Q = 0.008488$	$c_T^Q = 25.23$

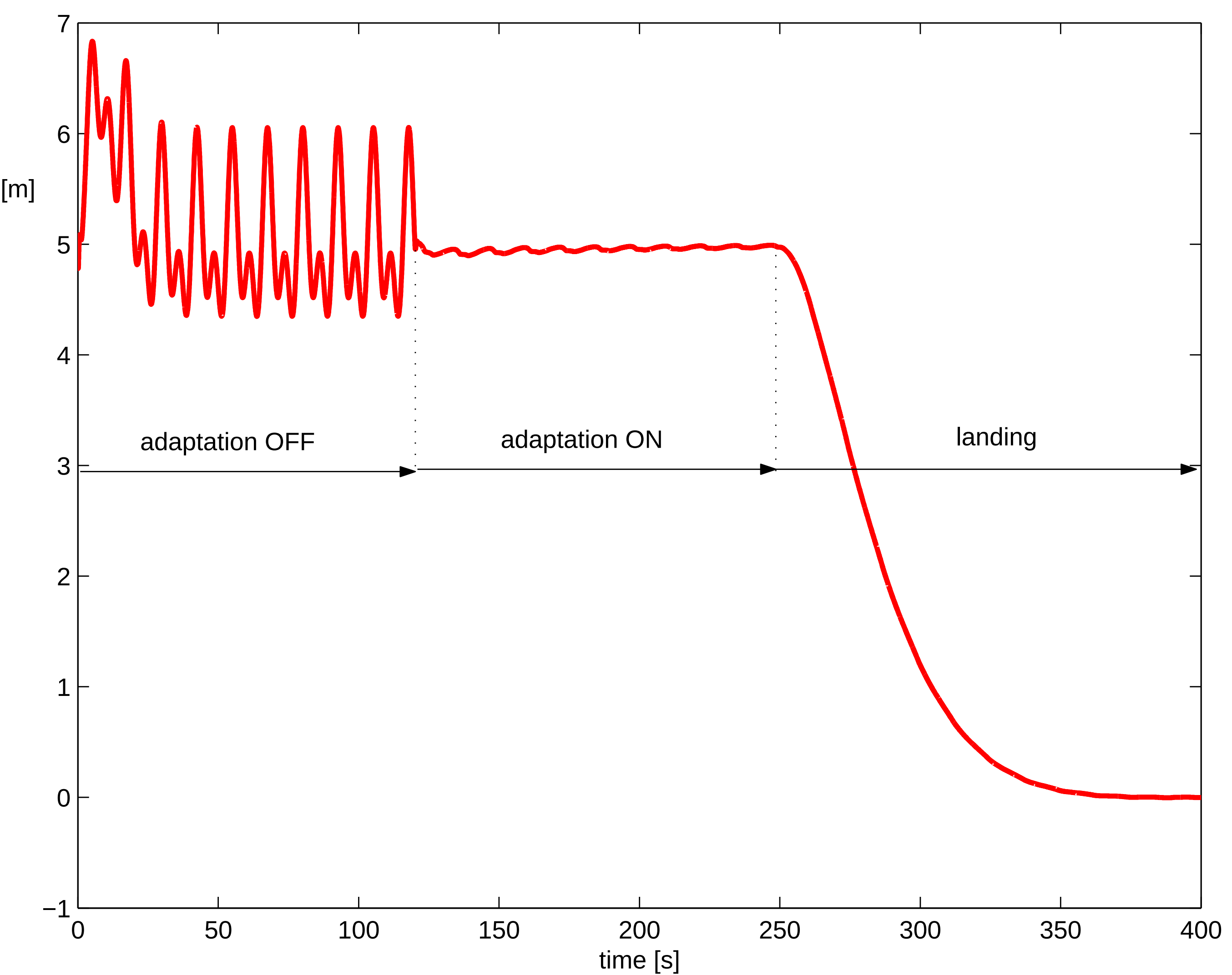
Nominal parameters of the plant

Vertical dynamics	$k_1 = 0.1$	$k_2 = 45$	$\gamma = 1$
Lateral/longit. dynamics	$K_0 = 0.09$	$K_1 = 0.081$	$K_2 = 0.75$
Attitude dynamics	$K_3 = 0.8$	$K_4 = 30$	$\varepsilon = 0.1$
Saturation levels	$\lambda_0 = 2000$	$\lambda_1 = 8.1$	$\lambda_2 = 0.2952$

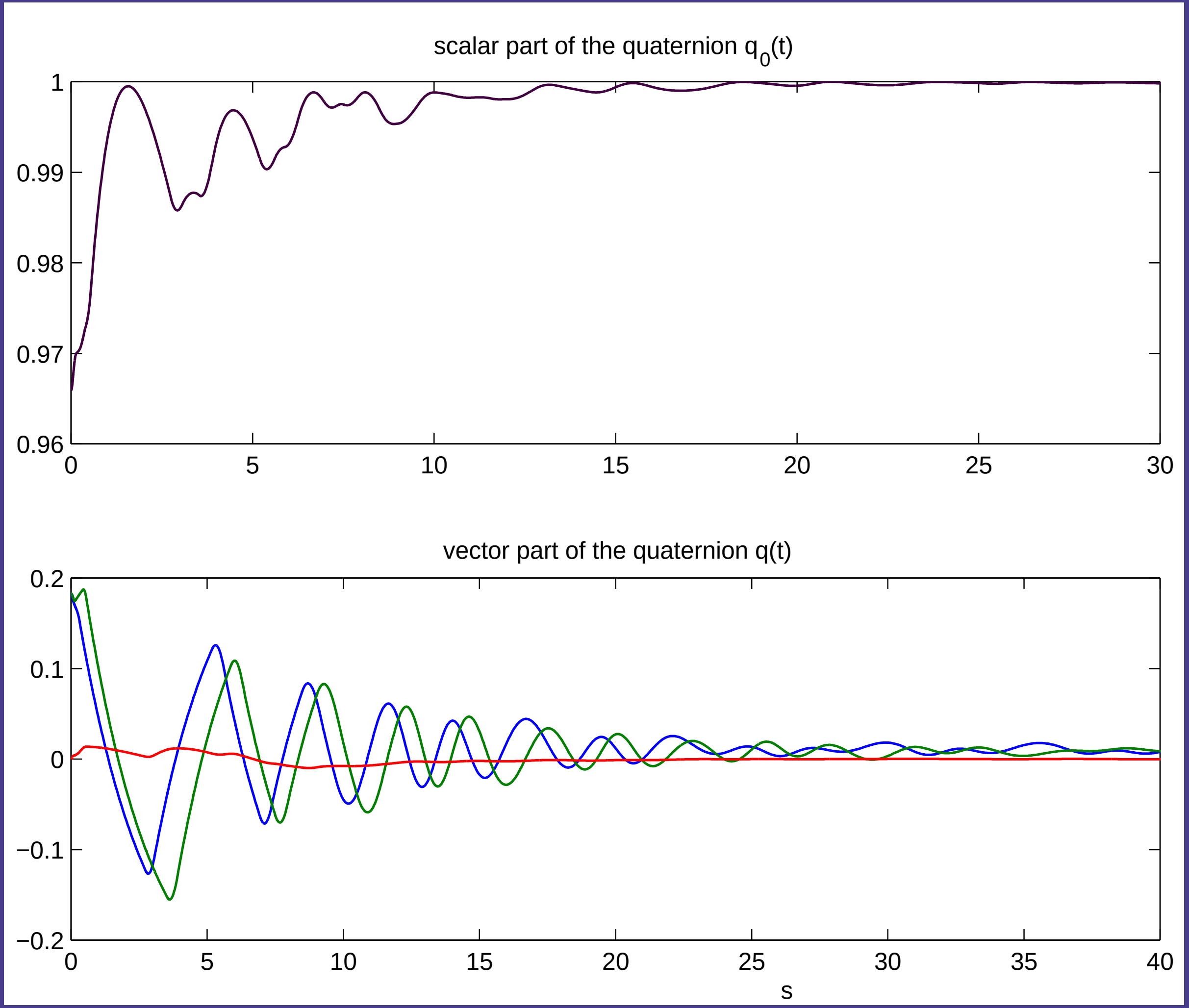
Controller parameters

Simulation results - vertical error

Regulation error $z(t) - z^*(t)$

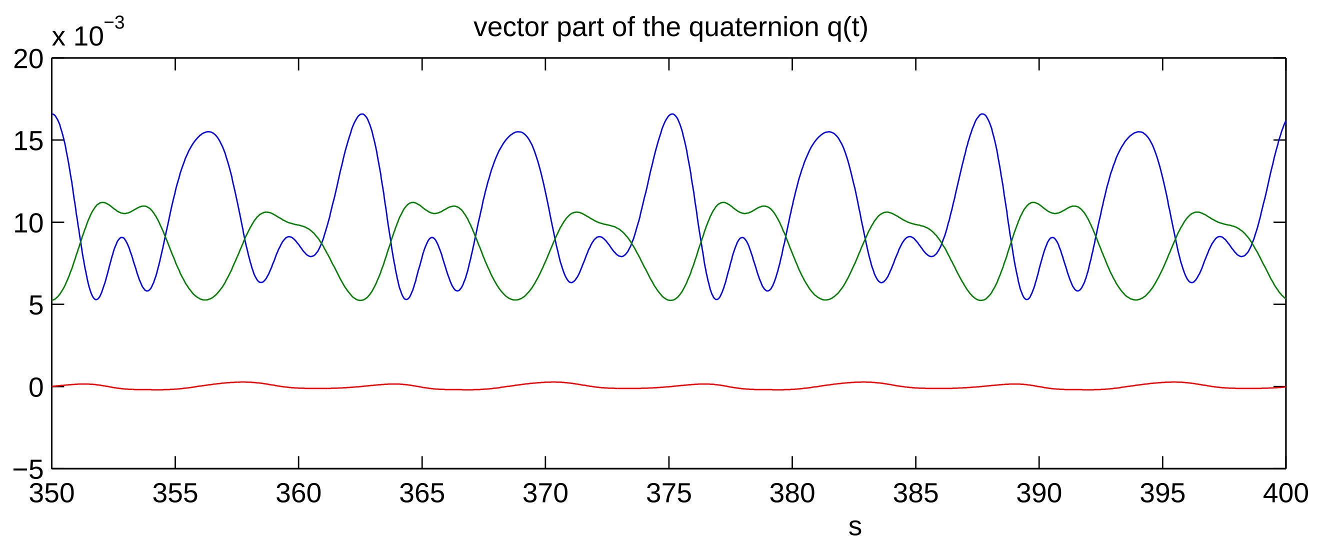


Simulation results - attitude



Simulation results - attitude

Steady-state behavior of the attitude



Simulation results - lateral/ long.