Robust Semiglobal Nonlinear Output Regulation

The case of systems in triangular form

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Problem formulation

We consider nonlinear systems of the form

$$\begin{aligned} \dot{x} &= f(x, u, w, \mu) \\ y &= h(x, w, \mu) \end{aligned}$$

with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}$, output $y \in \mathbb{R}$, unknown plant parameters $\mu \in \mathcal{P} \subset \mathbb{R}^p$.

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$$\begin{aligned} \dot{x} &= f(x, u, w, \mu) \\ y &= h(x, w, \mu) \end{aligned}$$

with state $x \in I\!R^n$, control input $u \in I\!R$, output $y \in I\!R$, unknown plant parameters $\mu \in \mathcal{P} \subset I\!R^p$.

The exogenous signal $w \in I\!R^d$ is generated by a linear, neutrally stable exosystem

$$\dot{w} = S(\sigma)w$$

with unknown parameters $\sigma \in \Sigma \subset I\!R^\nu$.

Problem formulation

We denote with

$$e_1 = y - q(w, \mu)$$

the regulated error, being $q(w, \mu)$ a smooth function.

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the regulated error, being $q(w, \mu)$ a smooth function.

The control input is to be provided by an *error-feedback controller* of the form

$$\begin{aligned} \dot{\xi} &= \Lambda(\xi, e_1) \\ u &= \Theta(\xi)\,, \end{aligned} \tag{1}$$

with state $\xi \in I\!R^m$, in which $\Lambda(\xi, e_1)$ and $\Theta(\xi)$ are smooth, and $\Lambda(0,0) = 0$, $\Theta(0) = 0$

Problem formulation

Given arbitrary compact sets $\mathcal{K}_x \subset \mathbb{R}^n$, $\mathcal{K}_w \subset \mathbb{R}^d$, find a controller (1) and a compact set $\mathcal{K}_\xi \subset \mathbb{R}^m$, such that

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- The equilibrium $(x, \xi) = (0, 0)$ of the unforced closed loop system

$$\begin{aligned} \dot{x} &= f(x, \Theta(\xi), 0, \mu) \\ \dot{\xi} &= \Lambda(\xi, h(x, 0, \mu)) \end{aligned}$$

  is asymptotically stable for every $\mu \in \mathcal{P}$, with domain of attraction containing the set $\mathcal{K}_x \times \mathcal{K}_\xi$

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- The trajectory $(x(t),\ \xi(t),\ w(t))$ of the closed loop system

$$\begin{aligned}\dot{w} &= S(\sigma)w \\ \dot{x} &= f(x, \Theta(\xi), w, \mu) \\ \dot{\xi} &= \Lambda(\xi, h(x, w, \mu) - q(w, \mu))\end{aligned}$$

  originating from $\mathcal{K}_x \times \mathcal{K}_\xi \times \mathcal{K}_w$ exists for all $t \geq 0$, is bounded for all $\mu \in \mathcal{P}$ and all $\sigma \in \Sigma$, and satisfies

$$\lim_{t \to \infty} e_1(t) = 0$$

Systems in lower-triangular form

We will consider systems in lower-triangular form

$$\begin{aligned}
\dot{z} &= f_0(z, x_1, w, \mu) \\
\dot{x}_1 &= a_2(\mu)x_2 + p_1(z, x_1, w, \mu) \\
\dot{x}_2 &= a_3(\mu)x_3 + p_2(z, x_1, x_2, w, \mu) \\
&\vdots \\
\dot{x}_r &= p_r(z, x_1, \ldots, x_r, w, \mu) + b(\mu)u \\
y &= x_1
\end{aligned}$$

with regulated error $e_1 = x_1 - q(w, \mu)$.

Systems in lower-triangular form

To ensure stabilizability by feedback from the partial state x, me make the following standard assumptions:

- $a_2(\mu) \neq 0\,, \ldots a_r(\mu) \neq 0$, for all $\mu \in \mathcal{P}$.

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- $b(\mu) \geq b_0 > 0$, for all $\mu \in \mathcal{P}$
- The equilibrium $z = 0$ of the unforced zero dynamics

$$\dot{z} = f_0(z, 0, 0, \mu)$$

  is globally asymptotically stable, uniformly in μ.

Outline of the talk

- Conditions for the solvability of the regulator equations

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- Intermediate case: partially known exosystem
- Illustrative example

Regulator equations

The existence of a globally-defined solution $\pi_\sigma(w,\mu)$, $c_\sigma(w,\mu)$ of the regulator equations

$$\begin{aligned} \frac{\partial \pi_\sigma(w,\mu)}{\partial w} S(\sigma) w &= f\left(\pi_\sigma(w,\mu), c_\sigma(w,\mu), w, \mu\right) \\ 0 &= h\left(\pi_\sigma(w,\mu), w, \mu\right) - q(w,\mu) \end{aligned}$$

for the considered class reposes of the following:

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for the considered class reposes of the following:

Assumption 1 *For every $\sigma \in \Sigma$, there exists a globally defined solution $\zeta_\sigma(w,\mu)$ to the equation*

$$\frac{\partial \zeta_\sigma(w,\mu)}{\partial w} S(\sigma) w = f_0(\zeta_\sigma(w,\mu), q(w,\mu), w, \mu).$$

Regulator equations

The triangular structure allows the solution of the regulator equations to be computed recursively as

$$
\begin{aligned}
\vartheta_{\sigma 1}(w,\mu) &= q(w,\mu) \\
\vartheta_{\sigma 2}(w,\mu) &= \frac{1}{a_2(\mu)}[L_{S(\sigma)w}q - p_1(\zeta_\sigma, \vartheta_{\sigma 1}, w, \mu)] \\
&\dots \\
\vartheta_{\sigma r}(w,\mu) &= \frac{1}{a_r(\mu)}[L^{r-1}_{S(\sigma)w}q - p_{r-1}(\zeta_\sigma, \vartheta_{\sigma 1}, \dots, \vartheta_{\sigma r-2}, w, \mu)]
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\end{aligned}$$

$$\boxed{\begin{aligned}
\pi_\sigma(w,\mu) &= \operatorname{col}(\zeta_\sigma(w,\mu),\ \vartheta_\sigma(w,\mu)) \\
c_\sigma(w,\mu) &= \frac{1}{b(\mu)}[L^r_{S(\sigma)w}q - p_r(\zeta_\sigma, \vartheta_{\sigma 1}, \dots, \vartheta_{\sigma r}, w, \mu)]
\end{aligned}}$$

The error system

The global change of coordinates

$$\tilde{z} = z - \zeta_\sigma(w, \mu), \quad e = x - \vartheta_\sigma(w, \mu)$$

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puts the system in the *error system* form

$$\begin{aligned}
\dot{\tilde{z}} &= \tilde{f}_0(\tilde{z}, e_1, w, \rho) \\
\dot{e}_1 &= e_2 \\
&\vdots \\
\dot{e}_{r-1} &= e_r \\
\dot{e}_r &= \tilde{p}_r(\tilde{z}, e_1, \ldots, e_r, w, \rho) + b(\mu)[u - c_\sigma(w,\mu)]
\end{aligned}$$

where $\rho = \mathrm{col}(\mu, \sigma) \in \mathcal{R}$.

The error system

Setting $u = v + c_\sigma(w, \mu)$

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the resulting system has an equilibrium at $(\tilde{z}, e) = (0, 0)$, $v = 0$, which corresponds to the invariant error-zeroing manifold.

The error system

From the error system it is evident that the problem of robust nonlinear output regulation is solved if:

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- The feed-forward control $c_\sigma(w, \mu)$ can be reconstructed, at least asymptotically, by means of an internal model.
- The interconnection of the internal model and the error system can be robustly asymptotically stabilized by error feedback from the input v.

Issues in robust regulation

What makes the problem complicated?

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- The internal model adds zeros on the imaginary axis. The resulting system is critically minimum phase, and must be stabilized using output feedback.
- The exosystem and, consequently, the internal model depend on the unknown parameters σ.

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What are the available tools?

- Tools for semiglobal stabilization
- Nonlinear separation principle
- Passivity theory
- Adaptive control

Existence of an internal model

If the function $c_\sigma(w,\mu)$ satisfies the following

Assumption 2 *There exist $q \in \mathbb{N}$ and a set of real numbers $\alpha_0(\sigma), \alpha_1(\sigma), \ldots, \alpha_{q-1}(\sigma)$ such that the identity*

$$L^q_{S(\sigma)w} c_\sigma(w,\mu) = \sum_{i=0}^{q-1} \alpha_i(\sigma) L^i_{S(\sigma)w} c_\sigma(w,\mu)$$

holds for all $(w,\mu) \in \mathbb{R}^d \times \mathcal{P}$ all $\sigma \in \Sigma$

then there exists a linear observable internal model for $c_\sigma(w,\mu)$

Existence of an internal model

If assumption 2 holds, the mapping $\tau_\sigma(w,\mu)$ given by

$$\tau_\sigma(w,\mu) = \begin{pmatrix} c_\sigma(w,\mu) \\ L_{S(\sigma)w} c_\sigma(w,\mu) \\ \cdots \\ L^{q-1}_{S(\sigma)w} c_\sigma(w,\mu) \end{pmatrix}$$

defines an immersion between the systems

$$\left\{ \begin{array}{rcl} \dot{w} & = & S(\sigma)w \\ \dot{\mu} & = & 0 \\ u & = & c_\sigma(w,\mu) \end{array} \right. \longrightarrow \left\{ \begin{array}{rcl} \dot{\tau} & = & \Phi(\sigma)\tau \\ u & = & \Gamma\tau \end{array} \right.$$

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Existence of an internal model

where the pair $(\Phi(\sigma), \Gamma)$ is observable for any σ, as

$$\Phi(\sigma) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha_0(\sigma) & \alpha_1(\sigma) & \alpha_2(\sigma) & \cdots & \alpha_{q-1}(\sigma) \end{pmatrix},$$

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

The pair $(\Phi(\sigma), \Gamma)$ constitutes the candidate internal model.

The canonical internal model

In order to circumvent the obstruction given by σ, we look for a more manageable realization of $(\Phi(\sigma), \Gamma)$.

Lemma 1 (Nikiforov, 1998) *Given any Hurwitz matrix* $F \in \mathbb{R}^{q \times q}$ *and any vector* $G \in \mathbb{R}^{q}$ *such that the pair* (F, G) *is controllable, the Sylvester equation*

$$M_\sigma \Phi(\sigma) - F M_\sigma = G\Gamma$$

has a unique solution M_σ*, which is non singular.*

The canonical internal model

Then, the change of coordinates $\bar{\tau} = M_\sigma \tau$ yields

$$(\Phi(\sigma), \Gamma) \xrightarrow{M_\sigma} (F + G\Psi_\sigma, \Psi_\sigma)$$

where

$$\Psi_\sigma := \Gamma M_\sigma^{-1}\,.$$

Note that

$$\begin{aligned} c_\sigma(w,\mu) &= \Gamma \tau_\sigma(w,\mu) \\ &= \Psi_\sigma \bar{\tau}_\sigma(w,\mu)\,. \end{aligned}$$

The pair $(F + G\Psi_\sigma, \Psi_\sigma)$ is referred to as the canonical parameterization of the internal model.

System augmentation

We augment the system with the q-dimensional internal model

$$\dot{\xi} = F\xi + Gu$$

which yields

$$\begin{aligned}
\dot{\xi} &= F\xi + Gu \\
\dot{\tilde{z}} &= \tilde{f}_0(\tilde{z}, e_1, w, \rho) \\
\dot{e}_1 &= e_2 \\
&\vdots \\
\dot{e}_{r-1} &= e_r \\
\dot{e}_r &= \tilde{p}_r(\tilde{z}, e_1, \ldots, e_r, w, \rho) + b(\mu)[u - \Psi_\sigma \bar{\tau}_\sigma(w, \mu)]
\end{aligned}$$

Augmented error system

For the augmented system, we make the following

Assumption 3 *There exists a smooth, positive definite function $V_0(\tilde{z})$ such that*

$$\underline{\alpha}_0(\|\tilde{z}\|) \le V_0(\tilde{z}) \le \overline{\alpha}_0(\|\tilde{z}\|)$$

$$\frac{\partial V_0(\tilde{z})}{\partial \tilde{z}} \tilde{f}_0(\tilde{z}, 0, w, \rho) \le -\alpha_0(\|\tilde{z}\|)\,,$$

for all $\tilde{z} \in \mathbb{R}^{n-r}$, all $w(0) \in \mathcal{K}_w$ and all $\rho \in \mathcal{R}$, where $\underline{\alpha}_0(\cdot)$, $\overline{\alpha}_0(\cdot)$ and $\alpha_0(\cdot)$ are class-$\mathcal{K}_\infty$ functions, locally quadratic near the origin.

Augmented error system

Assumption 3 states that the zero dynamics of the error system is GAS and LES, uniformly in (w, ρ).

Consider first the subsystem with virtual input e_r

$$
\begin{aligned}
\dot{\xi} &= F\xi + Gu \\
\dot{\tilde{z}} &= \tilde{f}_0(\tilde{z}, e_1, w, \rho) \\
\dot{e}_1 &= e_2 \\
&\vdots \\
\dot{e}_{r-1} &= e_r \\
\dot{e}_r &= \tilde{p}_r(\tilde{z}, e_1, \ldots, e_r, w, \rho) + b(\mu)[u - \Psi_\sigma \bar{\tau}_\sigma(w, \mu)]
\end{aligned}
$$

Augmented error system

The system

$$\begin{aligned} \dot{\tilde{z}} &= \tilde{f}_0(\tilde{z}, e_1, w, \rho) \\ \dot{e}_1 &= e_2 \\ &\vdots \\ \dot{e}_{r-1} &= v \end{aligned}$$

is robustly semi-globally stabilizable using

$$\boxed{v = -k^{r-1} b_0 e_1 - k^{r-2} b_1 e_2 - \cdots - k b_{r-2} e_{r-1}}$$

where $k > 0$ and $b_0, b_1, \ldots, b_{r-1}$ are the coefficients of a Hurwitz polynomial.

Augmented error system

Changing coordinates as

$$\theta = e_r + k^{r-1} b_0 e_1 + k^{r-2} b_1 e_2 + \cdots + k b_{r-2} e_{r-1}$$

and defining

$$\zeta := \mathrm{col}\left(\tilde{z}, e_1, e_2, \ldots, e_{r-1}\right) \in \mathbb{R}^{n-1}$$

we write the augmented system as

$$\begin{aligned} \dot{\xi} &= F\xi + Gu \\ \dot{\zeta} &= f_k(\zeta, w, \rho) + G_{\mathrm{a}}\theta \\ \dot{\theta} &= \phi_k(\zeta, \theta, w, \rho) + b(\mu)[u - \Psi_\sigma \bar{\tau}_\sigma(w, \mu)] \end{aligned}$$

Augmented error system

where

$$f_k(\zeta, w, \rho) = \begin{pmatrix} \tilde{f}_0(\tilde{z}, e_1, w, \rho) \\ e_2 \\ \vdots \\ -k^{r-1}b_0e_1 - k^{r-2}b_1e_2 - \cdots - kb_{r-2}e_{r-1} \end{pmatrix}$$

$$G_{\mathrm{a}}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & \cdots & 1 \end{pmatrix}^{\mathrm{T}}$$

$$\phi_k(\zeta, \theta, w, \rho) = \tilde{p}_r(\tilde{z}, e_1, \ldots, \theta - k^{r-1}b_0e_1 - \cdots - kb_{r-2}e_{r-1}, w, \rho)$$

Solution for known σ

Assume first that σ is known, and choose the control as

$$u = u_{\rm st} + u_{\rm im}$$

where $u_{\rm st}$ is a stabilizing control (yet to be defined) and

$$u_{\rm im} = \Psi_\sigma \xi \,.$$

The last equation of the augmented systems reads as

$$\dot\theta = \phi_k(\zeta, \theta, w, \rho) + b(\mu)\Psi_\sigma[\xi - \bar\tau_\sigma(w, \mu)] + b(\mu) u_{\rm st}$$

Solution for known σ

To quantify the error between the state of the internal model ξ and the immersion mapping $\bar{\tau}_\sigma(w,\mu)$ define

$$\chi := \xi - \bar{\tau}_\sigma(w,\mu) - \frac{1}{b(\mu)} G\theta.$$

The system in the (χ, ζ, θ)-coordinates reads as

$$\begin{aligned}
\dot{\chi} &= F\chi + \tfrac{1}{b(\mu)}[FG\theta - G\phi_k(\zeta,\theta,w,\rho)] \\
\dot{\zeta} &= f_k(\zeta,w,\rho) + G_{\mathrm{a}}\theta \\
\dot{\theta} &= \phi_k(\zeta,\theta,w,\rho) + \Psi_\sigma G\theta + b(\mu)\Psi_\sigma\chi + b(\mu)u_{\mathrm{st}}\,.
\end{aligned}$$

Solution for known σ

The zero dynamics with respect to the output θ

$$\begin{aligned} \dot{\chi} &= F\chi - \frac{1}{b(\mu)}[G\phi_k(\zeta, 0, w, \rho)] \\ \dot{\zeta} &= f_k(\zeta, w, \rho) \end{aligned} \tag{2}$$

is *LES and S-GAS in the parameter* k.

In particular, there exists a Lyapunov function $V(\chi, \zeta)$ with the following properties:

Solution for known σ

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- For any $r > 0$ there exists $c > 0$ such that the level set $\{V(\chi, \zeta) \leq c\}$ is compact and

$$\overline{\mathcal{B}}_r \subset \{V(\chi, \zeta) < c\}$$

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- For any $c > 0$ there exists $k^\star > 0$ such that, for any $k > k^\star$

$$\dot{V}(\chi, \zeta)_{(2)} < 0$$

for all $(\chi, \zeta) \in \{V(\chi, \zeta) \leq c\}$, and for all $w \in \mathcal{K}_w$, $\rho \in \mathcal{R}$.

Solution for known σ

As a result, the augmented system is semiglobally asymptotically stabilized by the high-gain feedback

$$u_{\mathrm{st}} = -K\theta\,, \qquad K > 0$$

with associated control-Lyapunov function

$$W(\chi, \zeta, \theta) = V(\chi, \zeta) + \frac{1}{2}\theta^2$$

for

$$\begin{aligned} \dot{\chi} &= F\chi + \frac{1}{b(\mu)}[FG\theta - G\phi_k(\zeta, \theta, w, \rho)] \\ \dot{\zeta} &= f_k(\zeta, w, \rho) + G_{\mathrm{a}}\theta \\ \dot{\theta} &= \phi_k(\zeta, \theta, w, \rho) + [\Psi_\sigma G - K]\theta + b(\mu)\Psi_\sigma\chi\,. \end{aligned} \qquad (3)$$

Solution for known σ

In particular, for any compact set $\mathcal{K} \subset I\!\!R^{n+q}$ there exist $d > 0$, $k^\star > 0$, $K^\star(k) > 0$ and a positive definite function $\alpha(\cdot)$ such that:

- $\{W(\chi, \zeta, \theta) \leq d\}$ is compact

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- $\{W(\chi, \zeta, \theta) \le d\}$ is compact
- $\mathcal{K} \subset \{W(\chi, \zeta, \theta) < d\}$
- Any choice $k > k^\star$, $K > K^\star(k)$ yields

$$\dot{W}(\chi, \zeta, \theta)_{(3)} \le -\alpha(\chi, \zeta, \theta)$$

for all $(\chi, \zeta, \theta) \in \{W(\chi, \zeta, \theta) \le d\}$ and all $w \in \mathcal{K}_w$, $\rho \in \mathcal{R}$.

Solution for known σ

The dynamic controller

$$\begin{aligned}\dot{\xi} &= (F + G\Psi_\sigma)\xi - KG\theta \\ u &= \Psi_\sigma \xi - K\theta \\ \theta &= e^{(r-1)} + k^{r-1} b_0 e + k^{r-2} b_1 e^{(1)} + \cdots + k b_{r-2} e^{(r-2)}\end{aligned}$$

solves the problem of robust semiglobal output regulation in case:

- The exosystem is known
- The partial state e is available for measurement

Solution for unknown σ

The σ-dependent term of the feedback law, $u_{\mathrm{im}} = \Psi_\sigma \xi$ is replaced by an estimate

$$u_{\mathrm{im}} = \hat{\Psi}\xi$$

where $\hat{\Psi}(t)$ is generated by an adaptation law of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\Psi} = \varphi(\xi, \theta)\,.$$

The update law is derived from the Lyapunov equation.

Solution for unknown σ

Change coordinate as $\tilde{\Psi} = \hat{\Psi} - \Psi_\sigma$, and define

$$\bar{W}(\chi, \zeta, \theta, \tilde{\Psi}) = W(\chi, \zeta, \theta) + b(\mu)\tilde{\Psi}\tilde{\Psi}^{\mathrm{T}}.$$

Letting $u_{\mathrm{im}} = \hat{\Psi}\xi$, we obtain

$$\dot{\theta} = \phi_k(\zeta, \theta, w, \rho) + [\Psi_\sigma G - K]\theta + b(\mu)\Psi_\sigma \chi + b(\mu)\tilde{\Psi}\xi.$$

Then, the obvious choice

$$\varphi(\xi, \theta) = -\theta\xi^{\mathrm{T}}$$

yields

$$\dot{\bar{W}}(\chi, \zeta, \theta, \tilde{\Psi})_{(3)} \leq -\alpha(\chi, \zeta, \theta).$$

Solution for unknown σ

The adaptive dynamic controller

$$\begin{aligned}
\dot{\xi} &= (F + G\hat{\Psi})\xi - KG\theta \\
\dot{\hat{\Psi}} &= -\theta\xi^{\mathrm{T}} \\
u &= \hat{\Psi}\xi - K\theta \\
\theta &= e^{(r-1)} + k^{r-1}b_0 e + k^{r-2}b_1 e^{(1)} + \cdots + kb_{r-2}e^{(r-2)}
\end{aligned}$$

yields

- Boundedness of all trajectories.
- Convergence of $(\chi(t), \zeta(t), \theta(t))$ to $(0, 0, 0)$, which implies $\lim_{t\to\infty} e_1(t) = 0$.

Error-feedback controller

In order to realize a device that uses information from the error signal e_1 only, the partial state $e_2, \ldots, e_r$ must be estimated.

- We use the high-gain observer of Khalil to generate “dirty derivatives” of the error.
- To prevent the occurrence of finite escape times, the estimates are saturated outside a compact set.
- If appropriate local conditions hold, the performance of the original partial-state feedback controller can be asymptotically recovered.

Error-feedback controller

The high-gain observer is given by

$$\dot{\hat{x}} = M_g \hat{x} + L_g e$$

$$M_g = \begin{pmatrix} -gc_{r-1} & 1 & 0 & \cdots & 0 \\ -g^2 c_{r-2} & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -g^{r-1} c_1 & 0 & 0 & \cdots & 1 \\ -g^r c_0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad L_g = \begin{pmatrix} gc_{r-1} \\ g^2 c_{r-2} \\ \cdot \\ g^{r-1} c_1 \\ g^r c_0 \end{pmatrix}$$

where $g > 0$ and $c_{r-1}, \ldots, c_0$ are the coefficients of a Hurwitz polynomial.

Error-feedback controller

The resulting error-feedback controller is

$$
\begin{aligned}
\dot{\hat{x}} &= M_g \hat{x} + L_g e_1 \\
\dot{\xi} &= (F + G\hat{\Psi})\xi - KG\,\mathrm{sat}(l, \hat{\theta}) \\
\dot{\hat{\Psi}} &= \mathrm{sat}(l, \hat{\theta})\xi^{\mathrm{T}} \\
\hat{\theta} &= \hat{x}_r + k^{r-1} b_0 \hat{x}_1 + \cdots + k b_{r-2} \hat{x}_{r-1} \\
u &= \hat{\Psi}\xi - K\,\mathrm{sat}(l, \hat{\theta})
\end{aligned}
$$

where

$$
\mathrm{sat}(l, s) = \begin{cases} s\,, & \text{if } |s| \le l \\ \dfrac{s}{|s|}\,, & \text{if } |s| > l. \end{cases}
$$

Avoiding overparameterization

In several cases, some eigenvalues of the exosystem may be known in advance (for instance, $\lambda = 0$)

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To avoid overparameterization, we devise a method to incorporate the a priori knowledge of the exosystem on the canonical internal model.

Let

$$\operatorname{spec}\left(\Phi(\sigma)\right) = \underbrace{\{\lambda_{0_1}, \ldots, \lambda_{0_k}\}}_{\text{known}} \cup \underbrace{\{\lambda_{\sigma_1}, \ldots, \lambda_{\sigma_h}\}}_{\text{unknown}}$$

with corresponding modal decomposition

$$\mathbb{R}^q = \mathcal{V}_0 \oplus \mathcal{V}_\sigma$$

Avoiding overparameterization

Let

$$\Phi_0 = \Phi(\sigma)_{|\mathcal{V}_0}\,, \qquad F_1 + G_1\Psi_{\sigma_1} = \Phi(\sigma)_{|\mathcal{V}_\sigma}$$

and choose the internal model

$$\left\{ \begin{array}{rcl} \dot{\xi}_0 & = & \Phi_0\xi_0 + H\xi_1 \\ \dot{\xi}_1 & = & F_1\xi_1 + G_1 u \\ u_{\rm im} & = & \Gamma_0\xi_0 + \Psi_{\sigma_1}\xi_1\,, \end{array} \right. \qquad \longleftrightarrow \qquad \left\{ \begin{array}{rcl} \dot{\xi} & = & F\xi + Gu \\ u_{\rm im} & = & \Psi_\sigma\xi \end{array} \right.$$

Avoiding overparameterization

The internal model has the canonical parameterization (F, G, Ψ_σ), where

$$F = \begin{pmatrix} \Phi_0 & H \\ -G_1\Gamma_0 & F_1 \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ G_1 \end{pmatrix}, \quad \Psi_\sigma = \begin{pmatrix} \Gamma_0 & \Psi_{\sigma_1} \end{pmatrix}$$

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Only the estimate $\hat{\Psi}_1$ is needed, with update law

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\Psi}_1 = -\theta\xi_1^{\mathrm{T}}.$$

Avoiding overparameterization

We are only left to show that it is possible to choose H in such a way that the matrix F is Hurwitz.

Let P_0 and P_1 denote the positive definite solutions of the Lyapunov equations

$$P_0\Phi_0 + \Phi_0^{\mathrm{T}} P_0 \leq 0\,, \qquad P_1 F_1 + F_1^{\mathrm{T}} P_1 = -I$$

and choose

$$H = P_0^{-1}\Gamma_0^{\mathrm{T}} G_1^{\mathrm{T}} P_1$$

Avoiding overparameterization

The system

$$\dot{\xi} = F\xi$$

is GAS, being the negative loop interconnection of a strictly passive system and a passive observable system

$$\begin{aligned} \dot{\xi}_1 &= F_1\xi_1 + G_1u_1 \\ y_1 &= G_1^{\mathrm{T}}P_1\xi_1 \,, \end{aligned}$$

$$\begin{aligned} \dot{\xi}_0 &= \Phi_0\xi_0 + P_0^{-1}\Gamma_0^{\mathrm{T}}u_0 \\ y_0 &= \Gamma_0\xi_0 \,, \end{aligned}$$

Illustrative example

Consider the controlled Van der Pol equation

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \mu_1 x_2 - x_2^3 + \delta(x_1, x_2, \mu) + u \\ e &= x_1 - w_1\end{aligned}$$

- Model perturbation: $\delta(x_1, x_2, \mu) = -\mu_2\, x_1 x_2^2$
- Unknown parameters: $\mu \in \{|\mu_1| \leq 3\,,\ 0 \leq \mu_2 \leq 5\}$
- Initial conditions: $\mathcal{K} = \{x : |x_i| \leq 1\,,\ i = 1, 2\}$

Illustrative example

Exosystem

$$\begin{aligned} \dot{w}_1 &= \sigma\, w_2 \\ \dot{w}_2 &= -\sigma\, w_1\,, \end{aligned}$$

- Uncertain frequency (rad/s): $1 \leq \sigma \leq 4$
- Initial conditions: $\mathcal{K}_w = \{w_1^2 + w_2^2 \leq 4\}$

Illustrative example

Solution of the regulator equations:

$$\begin{aligned} \pi_{\sigma_1}(w,\mu) &= w_1 \\ \pi_{\sigma_2}(w,\mu) &= \sigma\, w_2 \\ c_\sigma(w,\mu) &= (1-\sigma^2)w_1 - \sigma\mu_1 w_2 + \sigma^2\mu_2 w_1 w_2^2 + \sigma^3 w_2^3\,. \end{aligned}$$

Internal model $(\Phi(\sigma),\Gamma)$:

$$\Phi(\sigma) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9\sigma^4 & 0 & -10\sigma^2 & 0 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}.$$

Illustrative example

Canonical internal model

$$\begin{aligned} \dot{\xi} &= F\xi + Gu \\ u_{\text{im}} &= \Psi_\sigma \xi \end{aligned}$$

- $\text{spec}(F) = \{-12, -10, -9, -8\}$
- $\text{spec}\left(F + G\Psi_{\sigma\,\text{nom}}\right) = \{j, -j, 3j, -3j\}$
- (F, G, Ψ_σ) in balanced realization.

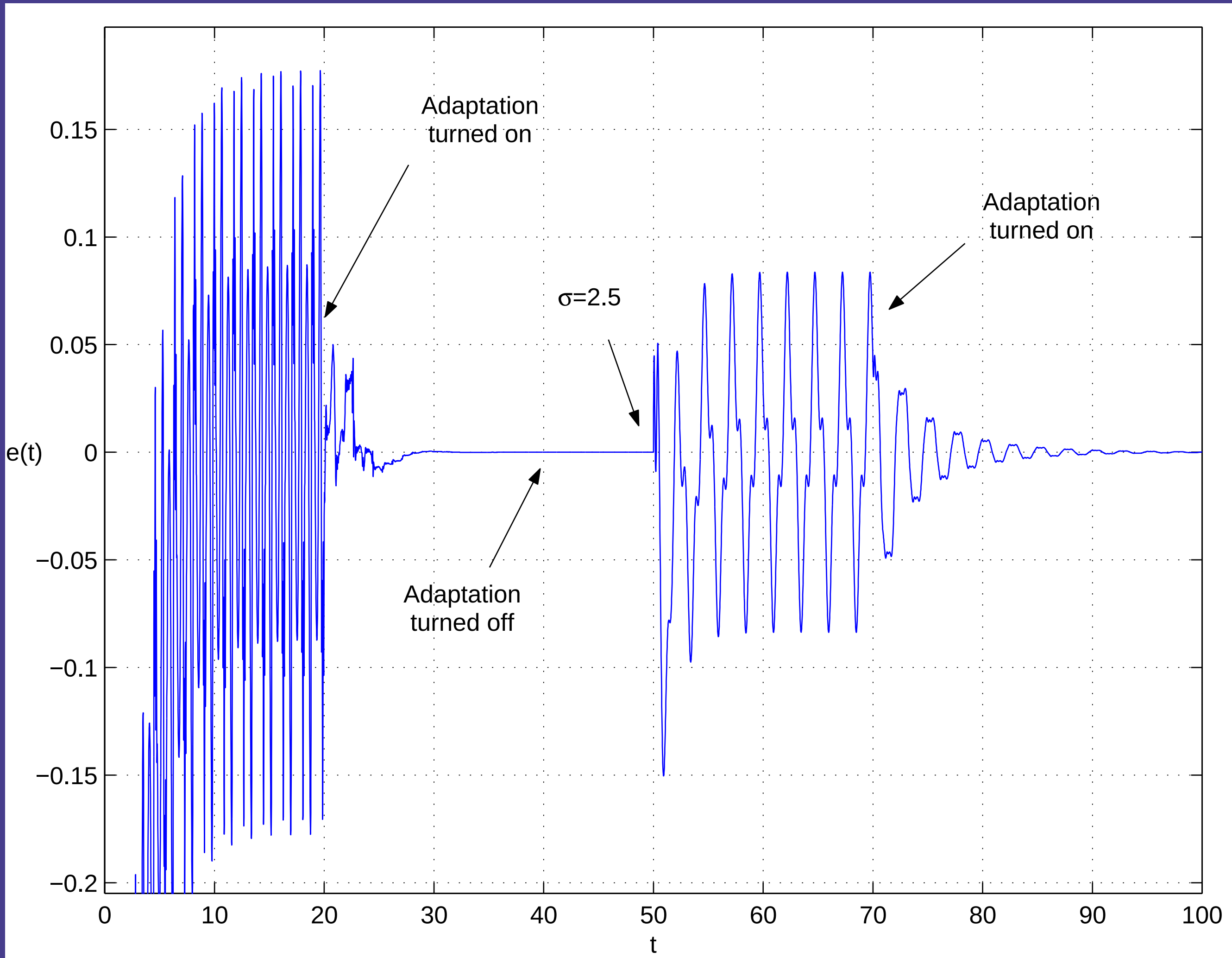
$k = 0.5$	$K = 75$	$\gamma = 1$	$g = 100$
$l = 30$	$b_0 = 1$	$c_0 = 2$	$c_1 = 3$

Simulation 1

- Exosystem frequency: $\sigma = 3.5$ rad/s.
- Internal model frequency: $\sigma_0 = 1$ rad/s.
- Adaptation turned on at time $t = 20$ s.
- Adaptation disconnected at time $t = 40$ s.
- Exosystem frequency changed to $\sigma = 2.5$ rad/s at time $t = 50$.
- Adaptation turned on again at time $t = 70$ s.

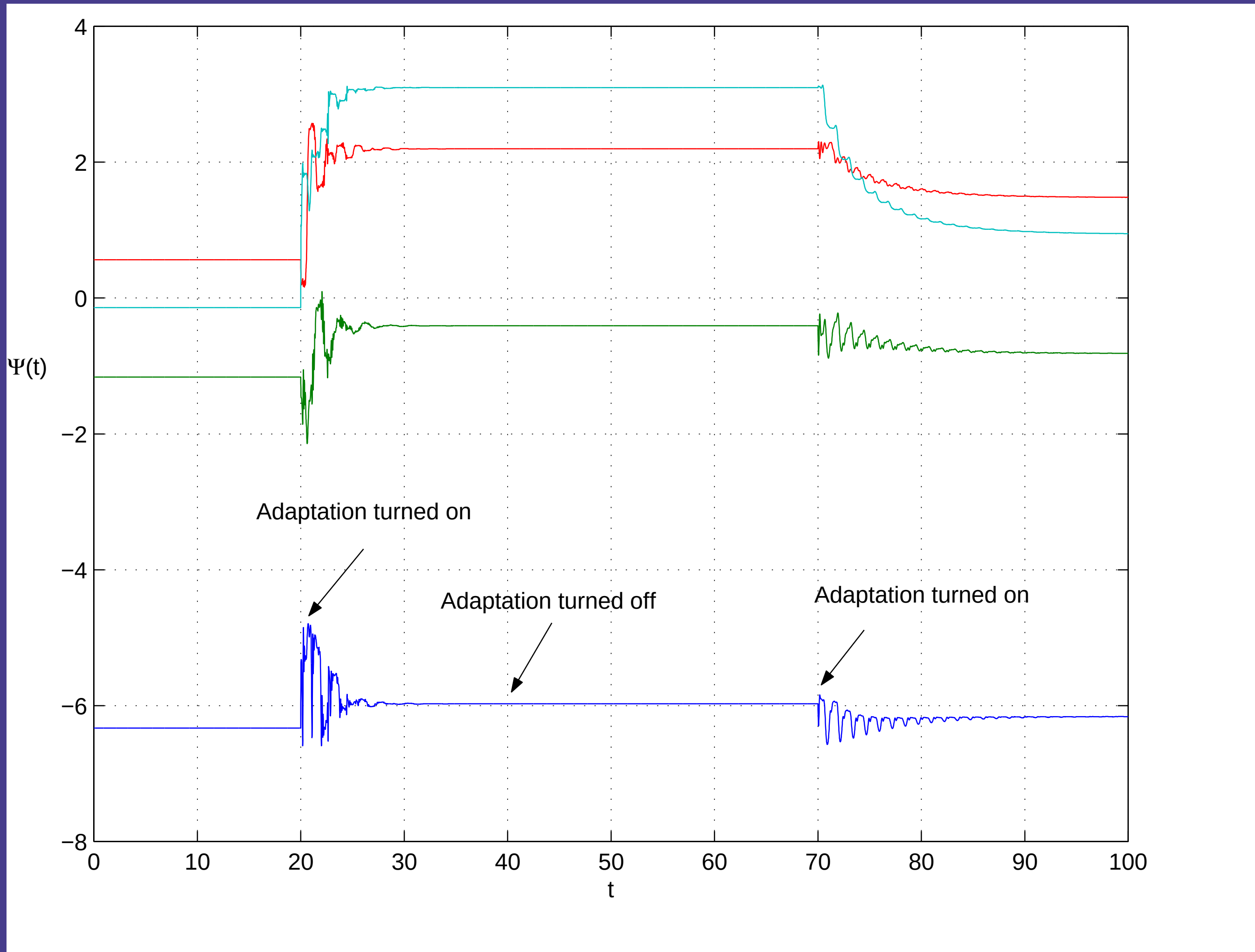
Results of simulation 1

Regulation error $e(t)$



Results of simulation 1

Parameter estimates $\hat{\Psi}(t)$

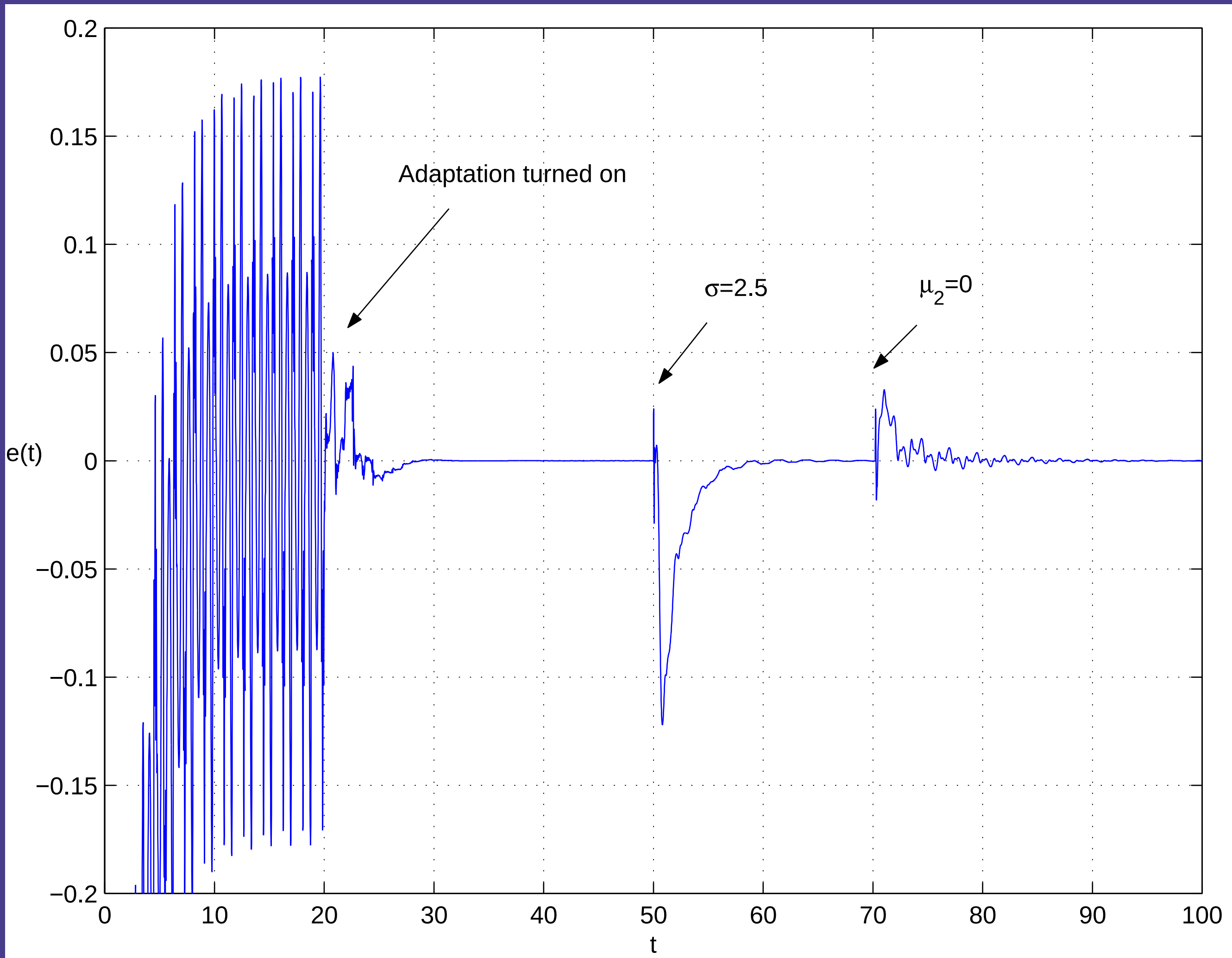


Simulation 2

- Same experiment, with the adaptation always active.
- Exosystem frequency at $t = 0$: $\sigma = 3.5$ rad/s.
- Exosystem frequency changed to $\sigma = 2.5$ rad/s at time $t = 50$.
- Parameter variation: μ_2 set to zero at $t = 70$ s.

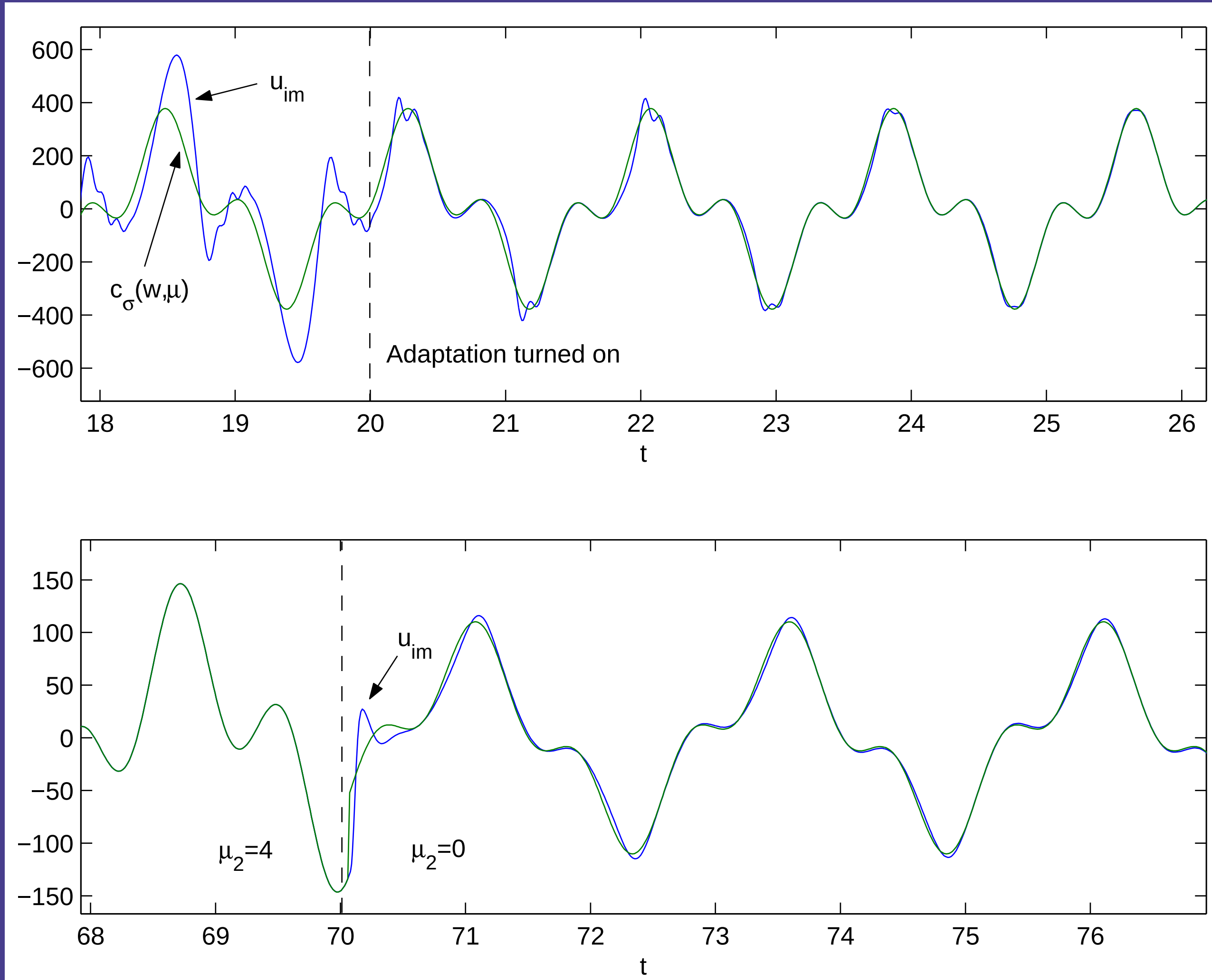
Results of simulation 2

Regulation error $e(t)$



Results of simulation 2

Internal model output $u_{\mathrm{im}}(t)$ vs. $c_\sigma(w, \mu)$



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- Parameter uncertainties on the exosystem model can be dealt with using a self-tuning internal model.
- Once again, robust stabilizability in the large is a major issues.