The Nonlinear Output Regulation Problem

Local and Structurally Stable Regulation

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- Problem Formulation

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- The Construction of a Local Regulator

Problem formulation

Consider a nonlinear plant model of the form

$$\dot{x} = f(x, u, w, \mu)$$
$$e = h(x, w, \mu)$$

with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$, and error to be regulated $e \in \mathbb{R}^m$.

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The signal w is generated by a nonlinear exosystem of the form

$$\dot{w} = s(w)$$

with state $w \in \mathbb{R}^d$.

Standing Assumptions

The plant model is assumed to satisfy the following assumptions:

- The functions $f(x, u, w, \mu)$ and $h(x, w, \mu)$ are smooth.
- The nominal value of the parameter μ is $\mu = 0$.
- $f(0, 0, 0, \mu) = 0$ and $h(0, 0, \mu) = 0$ for all μ in an open neighborhood $\mathcal{P}$ of $\mu = 0$.
- The pair (A, B) is stabilizable and the pair (C, A) is detectable, where

$$A = \left[\frac{\partial f}{\partial x}\right]_0, \qquad B = \left[\frac{\partial f}{\partial u}\right]_0, \qquad C = \left[\frac{\partial h}{\partial x}\right]_0.$$

Standing Assumptions

The exosystem is assumed to be neutrally stable:

- The equilibrium $w = 0$ is stable in the sense of Lyapunov
- Each initial state $w_0 \in \mathcal{W}$ is stable in the sense of Poisson

Note that this implies that

$$S = \left[\frac{\partial s}{\partial w}\right]_0$$

has all eigenvalues on the imaginary axis.

Caveat: *This excludes interesting situations in which $\dot{w} = s(w)$ generates stable limit cycles. For such a case the theory is still incomplete, although results have started to appear (see Byrnes and Isidori, IEEE Tr-AC 48(10), 2003.)*

Problem Formulation

The problem of local and structurally stable regulation is to find a smooth controller of the form

$$\begin{aligned} \dot{\xi} &= \phi(\xi, e) \\ u &= \theta(\xi), \end{aligned}$$

with $\xi \in \mathbb{R}^\nu$, satisfying $\phi(0,0) = 0$, $\theta(0,0) = 0$, and

$$F = \left[\frac{\partial \phi}{\partial \xi}\right]_0, \qquad G = \left[\frac{\partial \phi}{\partial e}\right]_0, \qquad H = \left[\frac{\partial \theta}{\partial \xi}\right]_0,$$

such that

Problem Formulation

- The origin is a locally exponentially stable equilibrium of the unforced closed loop system

$$\begin{aligned}\dot{x} &= f(x, \theta(\xi), 0, \mu) \\ \dot{\xi} &= \phi(\xi, h(x, 0, \mu))\end{aligned}$$

for all μ in an *open neighborhood* $\mathcal{P} \subset \mathbb{R}^p$ of $\mu = 0$.

Problem Formulation

- The trajectories of the closed loop system

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{x} &= f(x, \theta(\xi), 0, \mu) \\ \dot{\xi} &= \phi(\xi, h(x, w, \mu)) \\ e &= h(x, w, \mu) \end{aligned}$$

originating within a neighborhood $\mathcal{W} \times \mathcal{X} \times \Xi \subset \mathbb{R}^{d+n+\nu}$ of the origin are bounded and satisfy

$$\lim_{t \to \infty} h(x(t), w(t), \mu) = 0$$

for all μ in an *open neighborhood* $\mathcal{P} \subset \mathbb{R}^p$ of $\mu = 0$.

Solvability of the Problem

Since μ satisfies

$$\dot{\mu} = 0$$

the role of w and that of μ need not be kept separate. As a result, we incorporate μ within the exosystem state w.

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the role of w and that of μ need not be kept separate. As a result, we incorporate μ within the exosystem state w.

The closed-loop system can be written as

$$\begin{aligned}
\dot{x} &= Ax + BH\xi + Pw + \varphi(x,\xi,w) \\
\dot{\xi} &= GCx + F\xi + GQw + \chi(x,\xi,w) \\
\dot{w} &= Sw + \psi(w)
\end{aligned}$$

for all $(x,\xi,w) \in \mathcal{X} \times \Xi \times \mathcal{W}$, where $\varphi(x,\xi,w)$, $\chi(x,\xi,w)$, and $\psi(w)$ vanish at the origin with their first derivatives.

Solvability of the Problem

Assume that $\{\phi, \theta\}$ locally exponentially stabilizes the origin of the unforced closed-loop system. Then

$$A_{cl} = \left(\begin{array}{cc|c} A & BH & P \\ GC & F & GQ \\ \hline 0 & 0 & S \end{array}\right) = \begin{pmatrix} J & \star \\ 0 & S \end{pmatrix}$$

with

$$\operatorname{spec}\{J\} \subset \mathbb{C}^{-}\,, \qquad \operatorname{spec}\{S\} \subset \mathbb{C}^{0}\,.$$

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with

$$\text{spec}\{J\} \subset \mathbb{C}^- \,, \qquad \text{spec}\{S\} \subset \mathbb{C}^0 \,.$$

As a result, the system has a center manifold at the origin, that is, a d-dimensional hypersurface

$$\mathcal{M} = \{(x, \xi, w) \in \mathbb{R}^{n+\nu+d} : \; x = \pi(w), \; \xi = \sigma(w) \,, \; w \in \mathcal{W}\}$$

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- The restriction of the flow of the closed-loop system to $\mathcal{M}$ is diffeomorphic to that of the exosystem.
- $\mathcal{M}$ is tangent at the origin to the center subspace $\mathcal{V}^0$:

$$\pi(0) = 0, \ \sigma(0) = 0 \qquad \text{and} \qquad \frac{\partial \pi}{\partial w}(0) = 0 \, , \ \frac{\partial \sigma}{\partial w}(0) = 0 \, .$$

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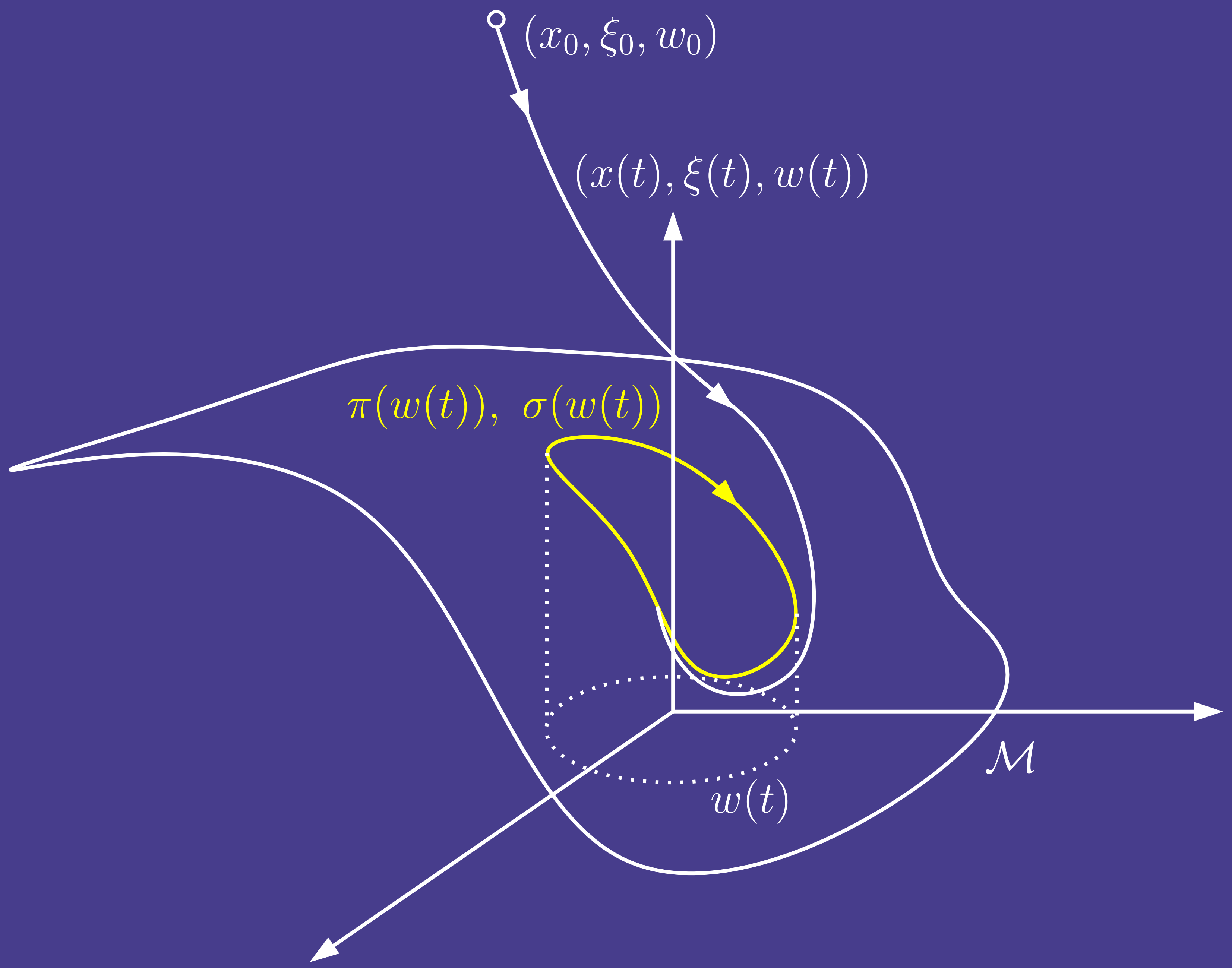
- $\mathcal{M}$ is locally exponentially attractive, i.e.,

$$\lim_{t\to\infty} \|x(t) - \pi(w(t))\| = 0\,, \qquad \lim_{t\to\infty} \|\xi(t) - \sigma(w(t))\| = 0$$

for all $(x(0), \xi(0), w(0)) \in \mathcal{X} \times \Xi, \times \mathcal{W}$.

The Center Manifold



Solvability of the Problem

The condition of invariance of $\mathcal{M}$ is expressed by the homology equations

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w), \theta(\sigma(w)), w)\,, \qquad \frac{\partial \sigma}{\partial w} s(w) = \phi(\sigma(w), h(\pi(w), w))$$

which hold for all $w \in \mathcal{W}$.

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$$\dot{w} = s(w)\,, \qquad w(0) \in \mathcal{W}$$

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$$\dot{w} = s(w)\,, \qquad w(0) \in \mathcal{W}$$

and on $\mathcal{M}$ the error reads as $e(t) = h(\pi(w(t)), w(t))$. Since the exosystem is Poisson stable

$$\lim_{t \to \infty} e(t) = 0 \iff h(\pi(w), w) = 0 \quad \forall\, w \in \mathcal{W}$$

Solvability of the Problem

Theorem 1 (Isidori and Byrnes, 1990) *A controller which locally exponentially stabilizes the plant achieves regulation if only if there exist mappings* $\pi : \mathcal{W} \to \mathbb{R}^n$ *and* $\sigma : \mathcal{W} \to \mathbb{R}^\nu$*, with* $\pi(0) = 0$ *and* $\sigma(0) = 0$ *such that*

$$
\begin{aligned}
\frac{\partial \pi}{\partial w} s(w) &= f(\pi(w), \theta(\sigma(w)), w) \\
\frac{\partial \sigma}{\partial w} s(w) &= \phi(\sigma(w), 0) \\
0 &= h(\pi(w), w)
\end{aligned}
$$

for all $w \in \mathcal{W}$*.*

The Regulator Equations

The previous equations can be split into two sets of equations as follows:

$$\begin{aligned} \frac{\partial \pi}{\partial w} s(w) &= f(\pi(w), c(w), w) \\ 0 &= h(\pi(w), w) \end{aligned}$$

$$\begin{aligned} \frac{\partial \sigma}{\partial w} s(w) &= \phi(\sigma(w), 0, w) \\ c(w) &= \theta(\sigma(w)) \end{aligned}$$

where the mapping $c : \mathcal{W} \to \mathbb{R}^m$ satisfies $c(0) = 0$.

The Regulator Equations

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- Regulator Equations: analogous to

$$\begin{aligned} \Pi S &= A\Pi + BR + P \\ 0 &= C\Pi + Q\,, \end{aligned}$$

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\frac{\partial \sigma}{\partial w} s(w) &= \phi(\sigma(w), 0) \\
c(w) &= \theta(\sigma(w))
\end{aligned}
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- Internal Model Principle: analogous to

$$
\begin{aligned}
\Sigma S &= F \Sigma \\
R &= H \Sigma
\end{aligned}
$$

Necessary Condition

The first equation yields a necessary condition for regulation

Theorem 2 (Isidori and Byrnes, 1990) *The local ouptut regulation problem is solvable only if there exist mappings* $\pi : \mathcal{W} \to \mathbb{R}^n$ *and* $c : \mathcal{W} \to \mathbb{R}^m$*, with* $\pi(0) = 0$ *and* $c(0) = 0$ *such that*

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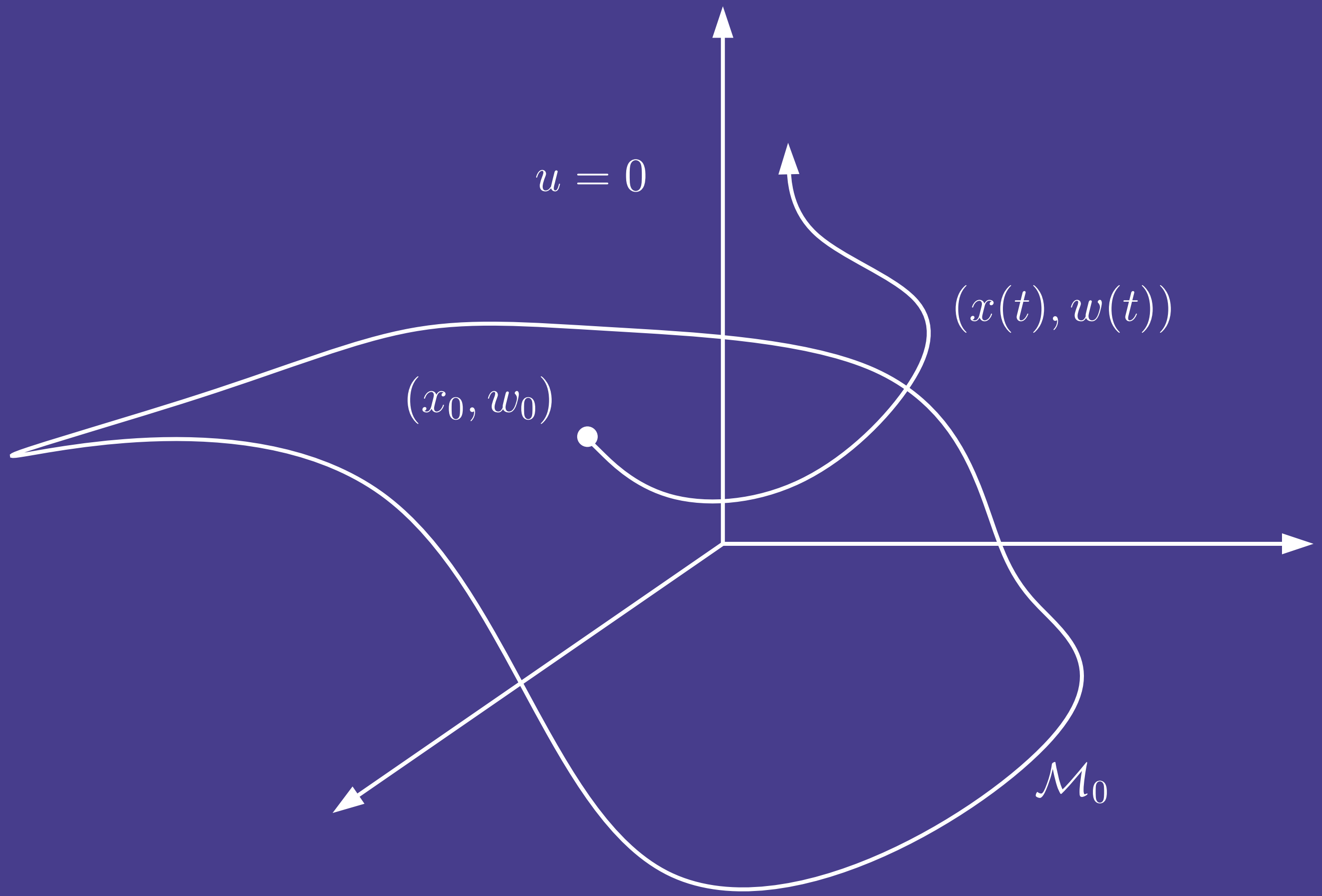
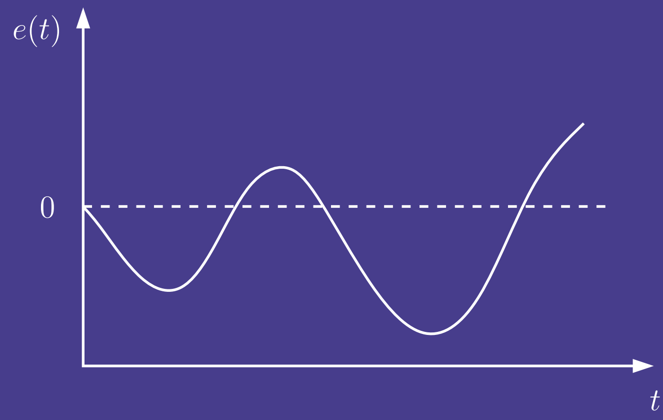
$$\begin{aligned} \frac{\partial \pi}{\partial w} s(w) &= f(\pi(w), c(w), w) \\ 0 &= h(\pi(w), w) \end{aligned}$$

for all $w \in \mathcal{W}$*, that is, only if there exists a controlled-invariant submanifold* $\mathcal{M}_0 \subset \mathbb{R}^{n+d}$ *satisfying*

$$\mathcal{M}_0 \subset \{(x, w) : \ h(x, w) = 0\} .$$

Geometric Picture





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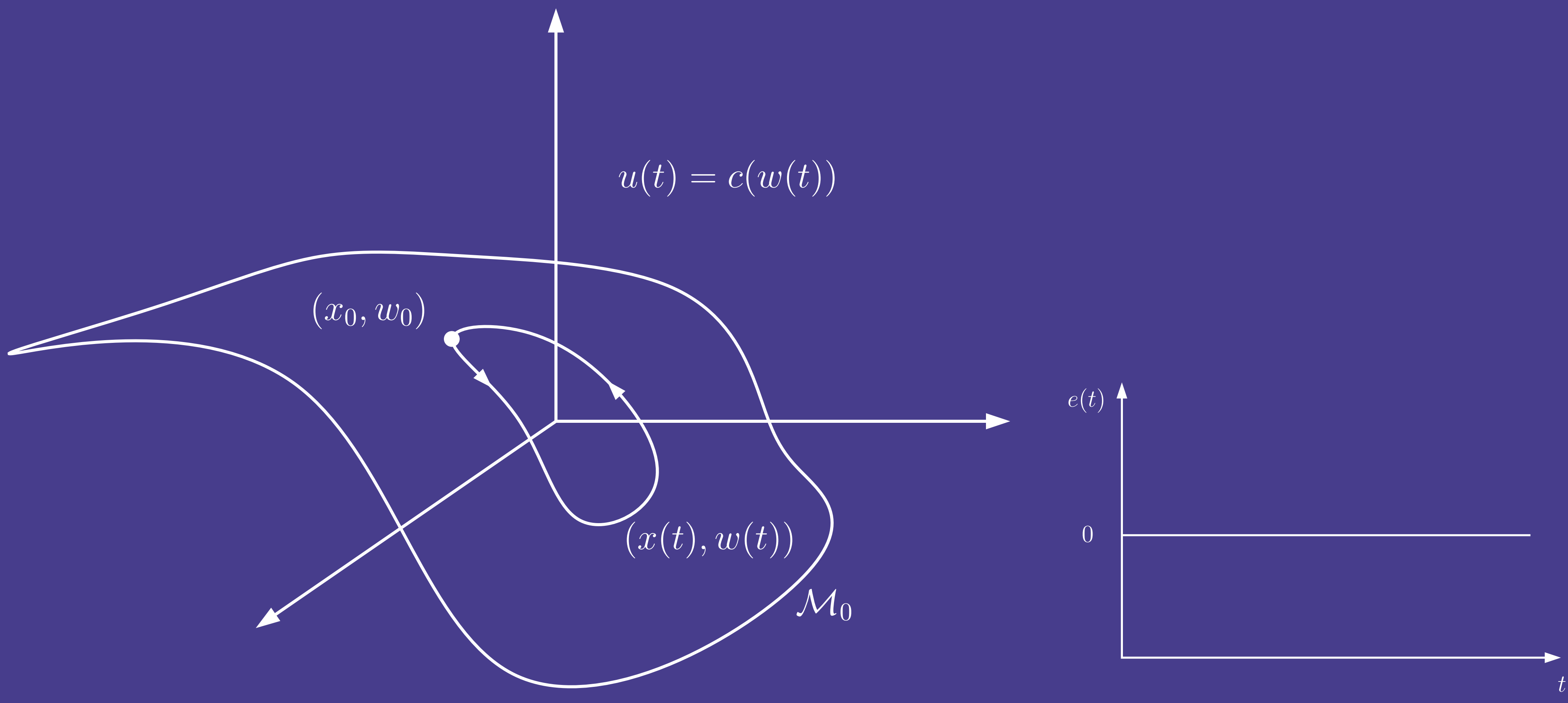

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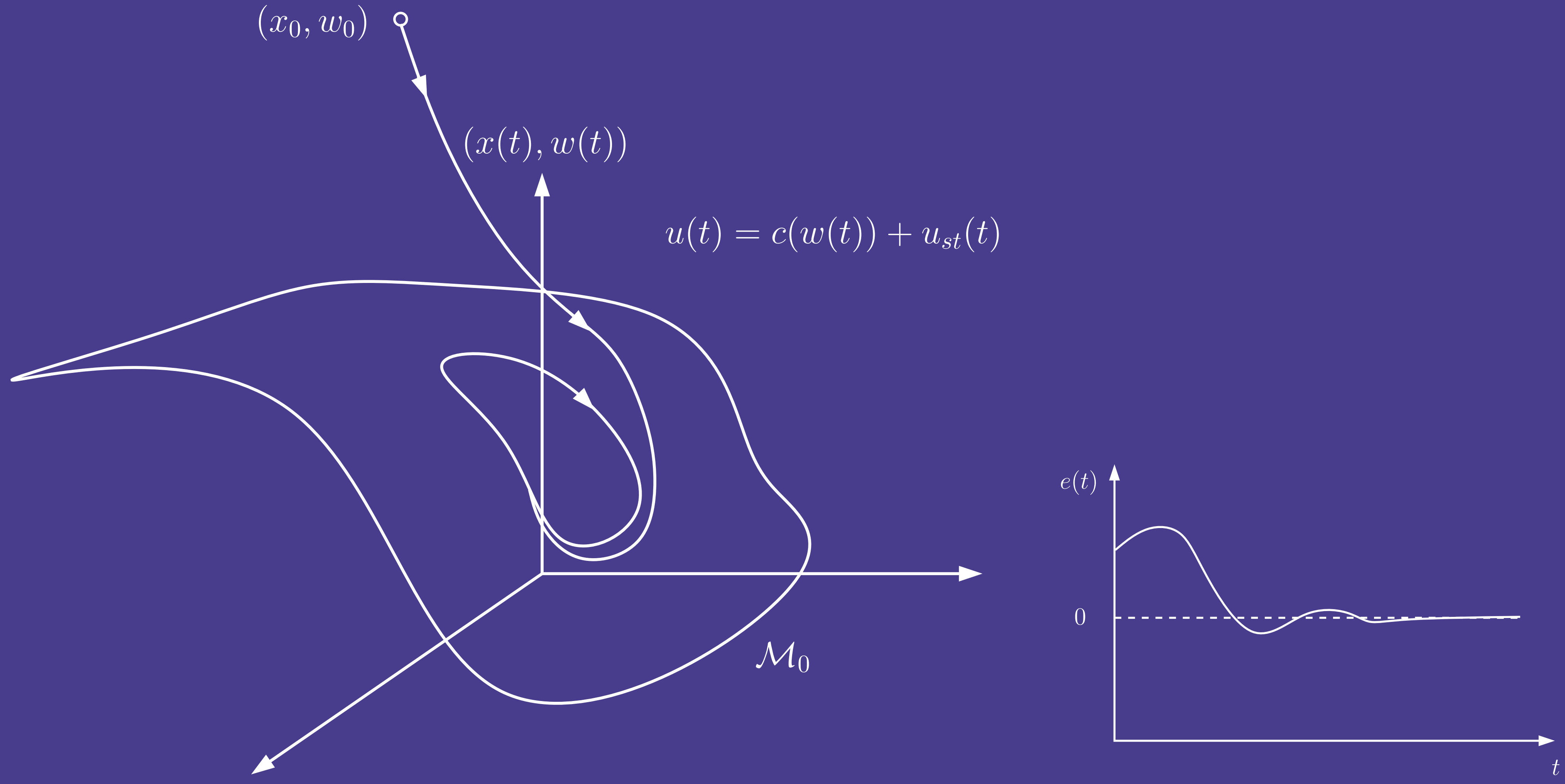

Any controller must render $\mathcal{M}_0$ invariant and attractive.

Sufficient Conditions

How far is the condition of Theorem 2 from being sufficient?

- Attractivity of $\mathcal{M}_0$ is guaranteed by the properties of the center manifold (by local exponential stability of the origin)

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- Attractivity of $\mathcal{M}_0$ is guaranteed by the properties of the center manifold (by local exponential stability of the origin)
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- The capability of the controller to "reconstruct" $c(w)$ is the real issue.
- Constructing a controller that satisfies the internal model property is not easy. Further conditions are needed.
- A crucial role is played by the notion of system immersion.

System Immersion

Definition 1 *Given two systems with same output space*

$$\left\{ \begin{array}{rcll} \dot{x} & = & f(x), & x \in \mathcal{X} \\ y & = & h(x), & y \in \mathbb{R}^m \end{array} \right. \qquad \left\{ \begin{array}{rcll} \dot{X} & = & F(X), & X \in \mathbf{X} \\ Y & = & H(X), & Y \in \mathbb{R}^m \end{array} \right.$$

we say that $\{\mathcal{X}, f, h\}$ is immersed into $\{\mathbf{X}, F, H\}$ if there exists a smooth mapping $\tau : \mathcal{X} \to \mathbf{X}$ satisfying $\tau(0) = 0$ and

$$\begin{array}{rcl} \dfrac{\partial \tau}{\partial x} f(x) & = & F(\tau(x)) \\ h(x) & = & H(\tau(x)) \end{array}$$

for all $x \in \mathcal{X}$.

System Immersion

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$$\begin{cases} \dot{x} = f(x), & x \in \mathcal{X} \\ y = h(x), & y \in \mathbb{R}^m \end{cases} \qquad \begin{cases} \dot{X} = F(X), & X \in \mathbf{X} \\ Y = H(X), & Y \in \mathbb{R}^m \end{cases}$$

we say that $\{\mathcal{X}, f, h\}$ is immersed into $\{\mathbf{X}, F, H\}$ if there exists a smooth mapping $\tau : \mathcal{X} \to \mathbf{X}$ satisfying $\tau(0) = 0$ and

$$\begin{aligned} \frac{\partial \tau}{\partial x} f(x) &= F(\tau(x)) \\ h(x) &= H(\tau(x)) \end{aligned}$$

for all $x \in \mathcal{X}$.

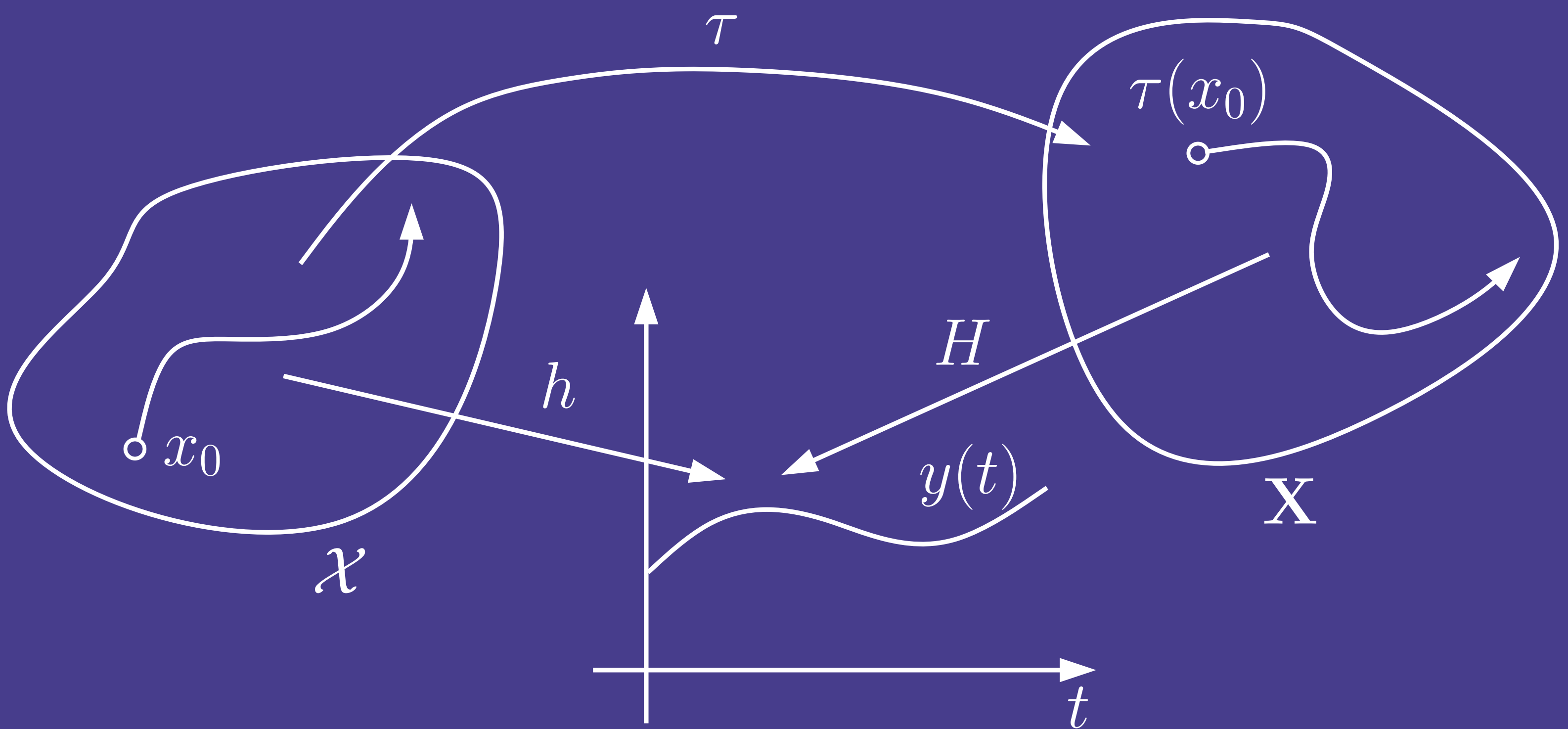
NOTE: τ need not be a diffeomorphism, as $\dim \mathcal{X} \leq \dim \mathbf{X}$.

System Immersion

This means that the flows of the systems are τ-related and

$$h \circ \Phi_t^f(x) = H \circ \tau \circ \Phi_t^f(x) = H \circ \Phi_t^F(\tau(x)) \,.$$

Any output trajectory of $\{\mathcal{X}, f, h\}$ is an output trajectory of $\{\mathbf{X}, F, H\}$.



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Consider the exosystem with output map $y \in \mathbb{R}$

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If there exists $q \in \mathbb{N}$ and a smooth function $\alpha : \mathbb{R}^q \to \mathbb{R}$ s.t.

$$L_s^q c(w) = \alpha\left(c(w),\, L_s c(w), \ldots,\, L_s^{q-1} c(w)\right)$$

for all $w \in \mathcal{W}$, then the exosystem is immersed into $\{\varphi, \gamma\}$

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\vdots \\ \dot{\xi}_q &= \alpha\left(\xi_1, \xi_2, \ldots, \xi_{q-1}\right)\,, \qquad y = \xi_1 \,. \end{aligned}$$

System Immersion

Consider the exosystem with output map $y \in \mathbb{R}$

$$\begin{aligned} \dot{w} &= s(w) \\ y &= c(w)\,. \end{aligned}$$

If there exists $q \in \mathbb{N}$ and $a_i \in \mathbb{R}$, $i = 0, \ldots, q-1$ such that

$$L_s^q c(w) + a_{q-1} L_s^{q-1} c(w) + \cdots + a_1 L_s c(w) + a_0 c(w) = 0$$

for all $w \in \mathcal{W}$, then the exosystem is immersed into $\{\Phi, \Gamma\}$

$$\Phi = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{q-1} \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$$

Example

An observable LTI immersion always exists if

- The exosystem is linear, $\dot{w} = Sw$.

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Since the set $\mathbb{P}$ of polynomials is a linear vector space over $\mathbb{R}$, and the mapping $D_s : \mathbb{P} \to \mathbb{P}$ given by

$$c(w) \to L_s c(w) = \frac{\partial c}{\partial w} s(w)$$

is linear, there exist an integer q and real numbers $a_i \in \mathbb{R}$, $i = 0, \ldots, q-1$ such that

$$D_s^q + a_{q-1} D_s^{q-1} + \cdots + a_1 D_s + a_0 I = 0$$

Necessary and Sufficient Condition for Regulation

Theorem 3 *The Error Feedback Output Regulation Problem is solvable if and only if*

- *There exist mappings* $x = \pi(w)$ *and* $u = c(w)$, *with* $\tau(0) = 0$ *and* $c(0) = 0$, *satisfying*

$$\begin{aligned} \frac{\partial \pi}{\partial w} s(w) &= f(\pi(w), c(w), w) \\ 0 &= h(\pi(w), w) \end{aligned}$$

for all $w \in \mathcal{W}$.

Necessary and Sufficient Condition for Regulation

Theorem 3 *The Error Feedback Output Regulation Problem is solvable if and only if*

- *The autonomous system* $\{\mathcal{W}, s, c\}$ *is immersed into a system*

$$\begin{aligned} \dot{\xi} &= \varphi(\xi)\,, \quad \xi \in \Xi \subset \mathbb{R}^{\nu} \\ u &= \gamma(\xi) \end{aligned}$$

in which $\varphi(0) = 0$ *and* $\gamma(0) = 0$*, such that the linear approximation*

$$\Phi = \left[\frac{\partial \varphi}{\partial \xi}\right]_0 , \qquad \Gamma = \left[\frac{\partial \gamma}{\partial \xi}\right]_0 ,$$

satisfies the following property:

Necessary and Sufficient Condition for Regulation

Theorem 3 *The Error Feedback Output Regulation Problem is solvable if and only if*

- *The pair*

$$\begin{pmatrix} A & 0 \\ \Theta C & \Phi \end{pmatrix}, \qquad \begin{pmatrix} B \\ 0 \end{pmatrix}$$

  is stabilizable for some choice of the matrix $\Theta \in \mathbb{R}^{\nu \times m}$*, and the pair*

$$\begin{pmatrix} C & 0 \end{pmatrix}, \qquad \begin{pmatrix} A & B\Gamma \\ 0 & \Phi \end{pmatrix}$$

  is detectable.

Proof

Necessity. Given a regulator $\{\phi, \theta\}$, there exist mappings $x = \pi(w)$ and $\xi = \sigma(w)$ solving the regulator equations.

Proof

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Set

$$c(w) = \theta(\sigma(w)) \,, \quad \gamma(\xi) = \theta(\xi) \,, \quad \varphi(\xi) = \phi(\xi, 0)$$

and note that the system $\{\mathcal{W}, s, c\}$ is immersed into $\{\Xi, \varphi, \gamma\}$, with immersion mapping $\tau(w) = \sigma(w)$.

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and note that the system $\{\mathcal{W}, s, c\}$ is immersed into $\{\Xi, \varphi, \gamma\}$, with immersion mapping $\tau(w) = \sigma(w)$.

Since

$$\begin{pmatrix} A & BH \\ GC & F \end{pmatrix} = \begin{pmatrix} A & B\Gamma \\ \Theta C & \Phi \end{pmatrix}, \quad \Theta = G$$

is Hurwitz, the given pairs are stabilizable and detectable.

Proof

Sufficiency. Since

$$\begin{pmatrix} A & B\Gamma \\ \Theta C & \Phi \end{pmatrix}, \qquad \begin{pmatrix} B \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} C & 0 \end{pmatrix}$$

is stabilizable and detectable, there exist L, M, N such that

$$\begin{pmatrix} A & B\Gamma & BN \\ \Theta C & \Phi & 0 \\ MC & 0 & L \end{pmatrix} \quad \text{is Hurwitz.}$$

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Define the controller

$$\begin{cases} \dot{\xi}_0 &= \varphi(\xi_0) + \Theta e \\ \dot{\xi}_1 &= L\xi_1 + Me \\ u &= \gamma(\xi_0) + N\xi_1 \end{cases}$$

Proof

The controller solves the local output regulation problem:

- The Jacobian matrix of the unforced closed-loop system

$$f_{cl}(x,\xi,0) = \begin{pmatrix} f(x,\gamma(\xi_0)+N\xi_1,0) \\ \varphi(\xi_0)+\Theta h(x,0) \\ L\xi_1 + Mh(x,0) \end{pmatrix}$$

  is precisely $\begin{pmatrix} A & B\Gamma & BN \\ \Theta C & \Phi & 0 \\ MC & 0 & L \end{pmatrix}$.

Proof

The controller solves the local output regulation problem:

- The mappings

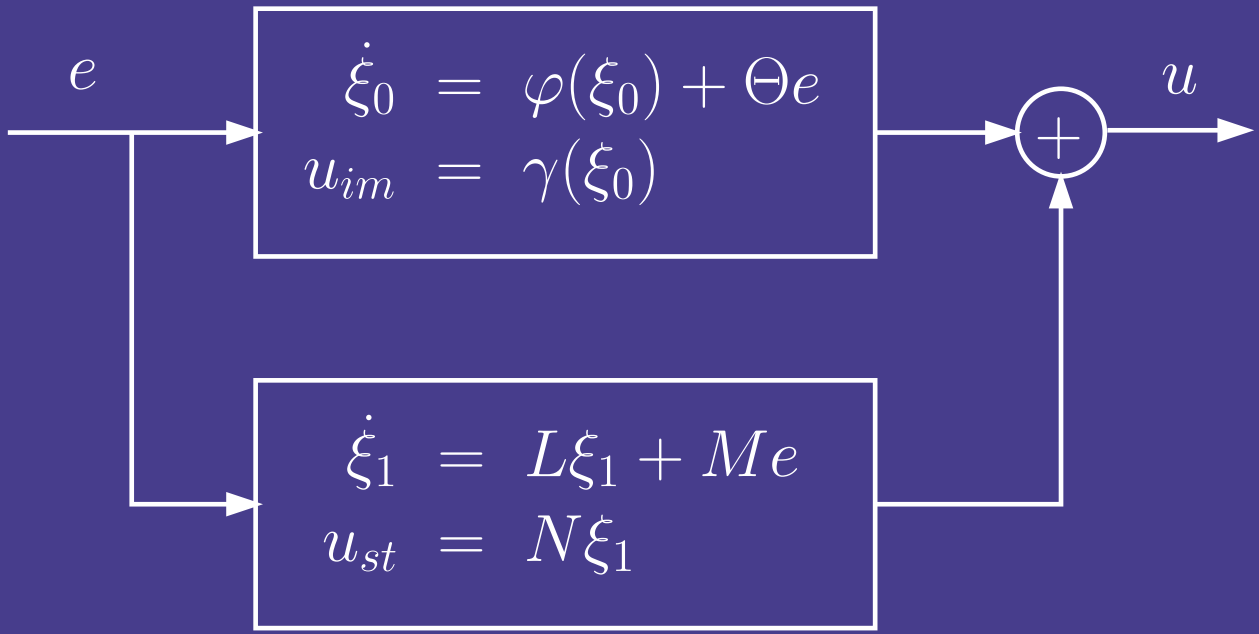
$$x = \pi(w)\,, \quad u = c(w) \qquad \text{(given)}$$

and

$$\begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix} = \sigma(w) = \begin{pmatrix} \tau(x) \\ 0 \end{pmatrix}$$

solve the regulator equations.

Regulator Structure



The regulator is given as the parallel interconnection of an internal model and a stabilizer.

- The internal model provides $u = c(w)$ on the set $\mathcal{M}_0$.
- The stabilizer locally exponentially stabilizes the origin of the closed-loop system, and induces local exponential attractivity of $\mathcal{M}_0$.

Conclusions

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- What is required to extend these results beyond local validity?