Tracking and Smoothing of Time-Varying Sparse Signals via Approximate Belief Propagation

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Outline

Fundamentals of the Dynamic CS Problem

The *dynamic CS* problem concerns recovering a temporal sequence of signals, $\{\mathbf{x}^{(t)}\}_{t=0}^{T}$, from an undersampled sequence of measurements, $\{\mathbf{y}^{(t)}\}_{t=0}^{T}$, where $\mathbf{y}^{(t)} = \mathbf{A}^{(t)}\mathbf{x}^{(t)} + \mathbf{e}^{(t)}$.

Assumed time-varying signal properties

1. The time-varying signal is sparse (or compressible) in an appropriately chosen basis.
2. The support set of the signal changes slowly over time.
3. The amplitudes of the non-zero coefficients evolve smoothly in time.

Example: Angiography Sequence

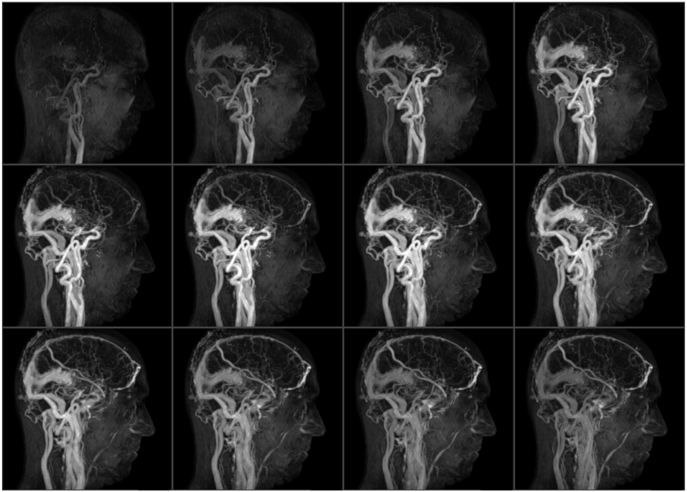


Image source: Koninklijke Philips Electronics N.V.

Related Work on Dynamic CS

- Related work
  - Video CS [Wakin et. al. '06]
  - Dynamic MRI [Gamper, Boesiger, Kozerke '08]
  - KF-CS [Vaswani, '08]
  - LS-CS [Vaswani '08]
  - Group-Fused Lasso [Angelosante, Giannakis, Grossi '09]
  - RLS Lasso [Angelosante, Giannakis '09]
  - Modified-CS [Vaswani, Lu '09]
  - Lasso-Kalman Smoother [Angelosante, Roumeliotis, Giannakis '09]
- Our goals
  - Unified approach to filtering and smoothing
  - Algorithm complexity that is linear in problem dimensions
  - Principled framework: Switching linear dynamical systems (SLDSs), Gaussian sum filtering/smoothing

A Model of Sparse Time-Evolving Signals

We write: $x_n^{(t)} = s_n^{(t)} \cdot \theta_n^{(t)}$ for $s_n^{(t)} \in \{0,1\}$ and $\theta_n^{(t)} \sim \mathcal{CN}(0,\sigma^2)$.

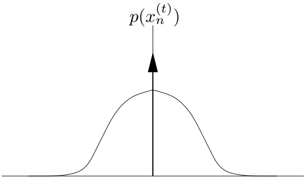
Support Set Evolution

Treat $\{s_n^{(t)}\}_{t=0}^T$ as a Markov chain characterized by two transition probabilities:
$p_{01} \triangleq Pr\{s_n^{(t)} = 0 | s_n^{(t-1)} = 1\}$ and
$p_{10} \triangleq Pr\{s_n^{(t)} = 1 | s_n^{(t-1)} = 0\}$.

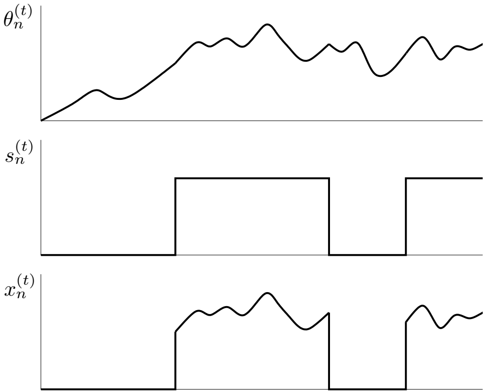
Amplitude Evolution

Treat $\{\theta_n^{(t)}\}_{t=0}^T$ as a Gauss-Markov process:
$\theta_n^{(t)} = (1-\alpha)\theta_n^{(t-1)} + \alpha w_n^{(t)}$, where $w_n^{(t)} \sim \mathcal{CN}(0,\rho)$, and α conrols the correlation in the random process.

Leads to Bernoulli-Gaussian distribution for $x_n^{(t)}$:
$p(x_n^{(t)}) = (1-\pi_n^{(t)})\delta(x_n^{(t)}) + \pi_n^{(t)}\mathcal{CN}(x_n^{(t)};0,\sigma^2)$



A Model of Sparse Time-Evolving Signals



Estimating the Time-Varying Signal

- Signal model is stochastic, thus our estimation procedure will be statistical in nature
- Belief propagation (BP) technique:
  - Suitable for performing inference when the posterior joint distribution factors into a product of marginal distributions that depend on small subsets of variables. Conveniently visualized via a *factor graph*.
  - Message passing algorithm in which messages traversing the factor graph are pdfs and pmfs
  - Messages described parametrically by just a few scalar variables, e.g. means and variances
  - Fast (if simplified), but approximate on loopy graphs

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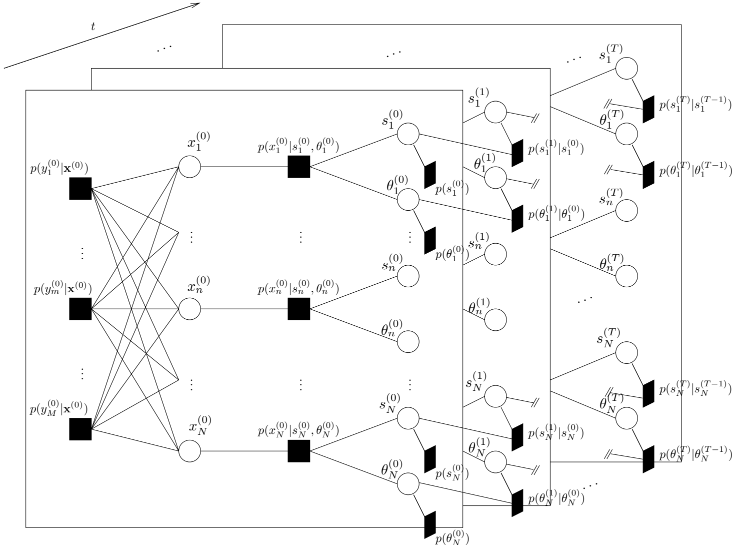
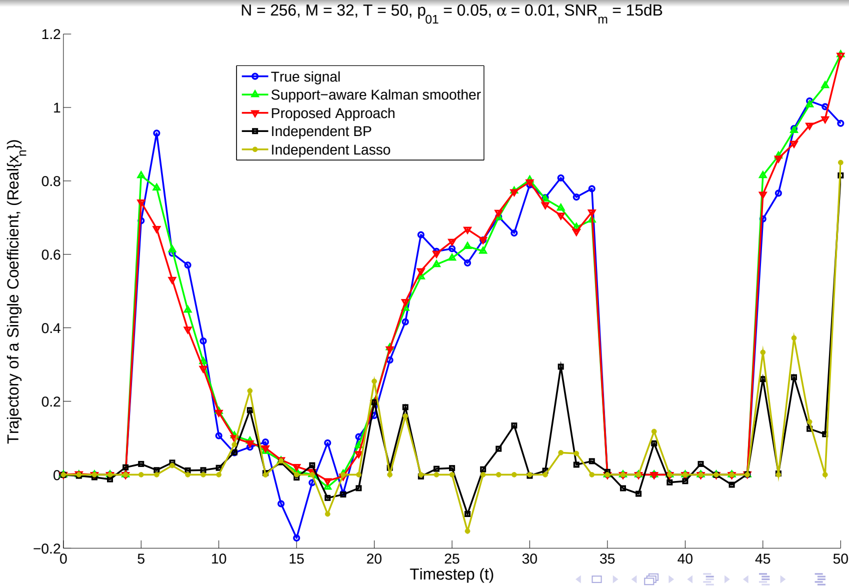
The Factor Graph Representation



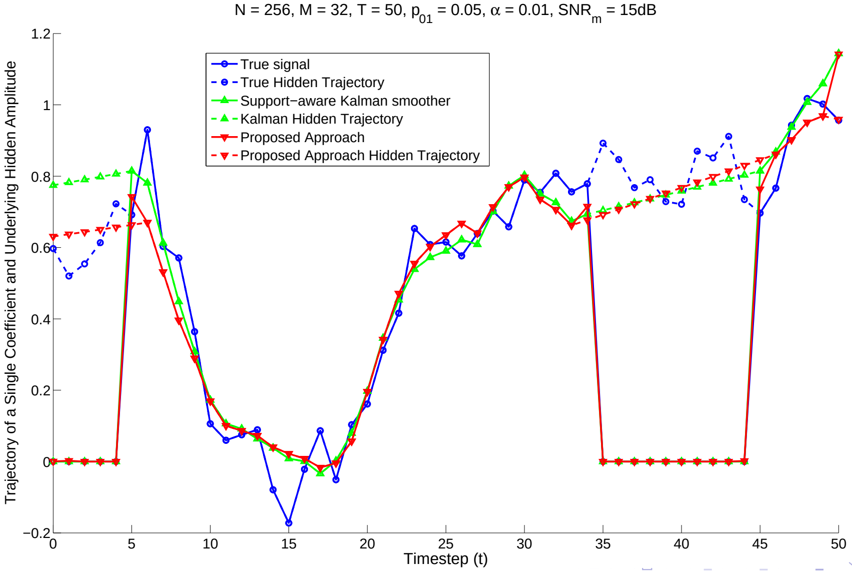
BP Implementation Highlights

- In general, CS measurement matrices, $\mathbf{A}^{(t)}$, are dense. Implementing standard BP would thus require evaluating multi-dimensional integrals that grow exponentially in number as problem dimensions grow
- Simplification: Approximate message passing (AMP) approach, [Donoho, Maleki, Montanari '09]
  - AMP has been shown to achieve asymptotic optimality, providing exact posteriors as $M, N \to \infty$ [Bayati, Montanari '10]
  - Complexity of iterative thresholding algorithms
    - Requires only $\mathcal{O}(MN)$ computations per timestep in the form of matrix-vector products

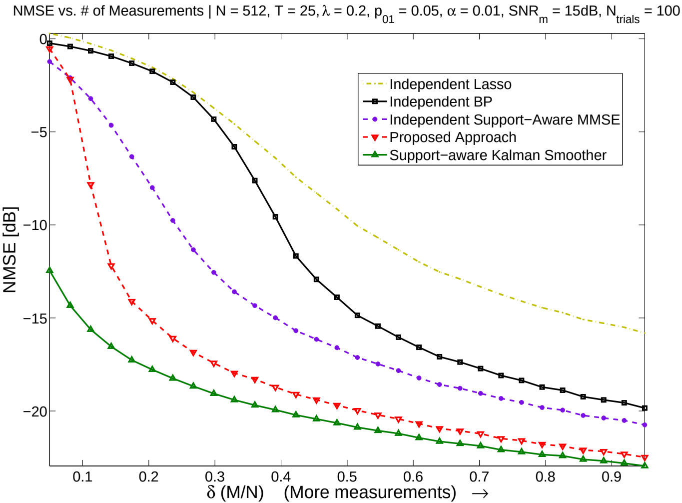
Sample Trajectory, Single Coefficient



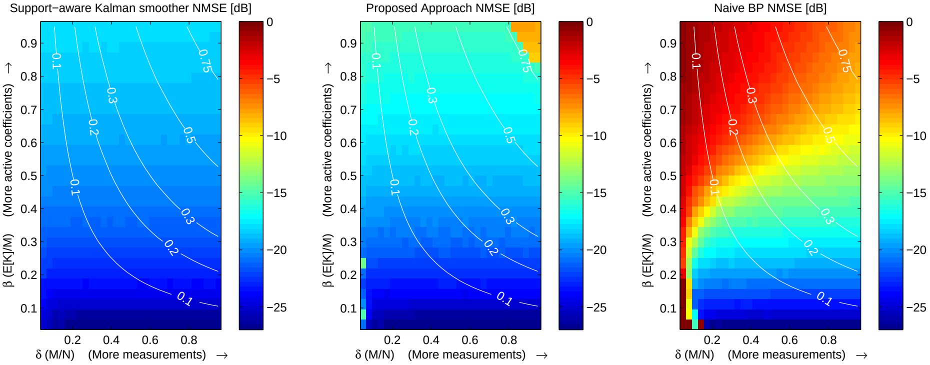
Sample Trajectory, Hidden Variables



MSE Performance vs. # of Measurements



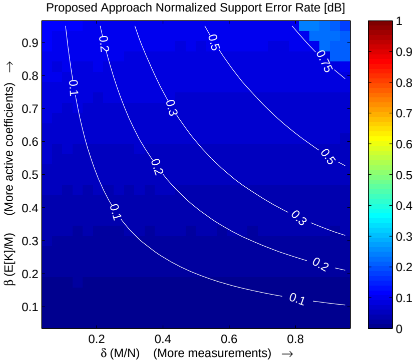
Sparsity-Undersampling Plane: MSE Performance

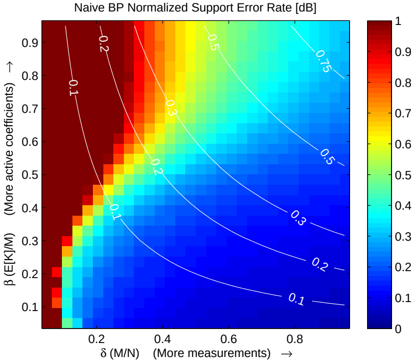


$$N = 512, p_{01} = 0.05, \alpha = 0.01, T = 25, \text{SNR} = 15 \text{ dB}$$

Sparsity-Undersampling Plane: Support Estimation Performance

$$\text{NSER} \triangleq \frac{\text{\# of Errors in Support Estimate}}{\text{\# of True Non-Zero Coefficients}}$$





Summary

- Presented a novel signal model for describing sparse, time-varying signals
- Described a belief propagation-based inference algorithm that implements both tracking and smoothing
    - AMP approach enables rapid estimation ($\mathcal{O}(MN)$ computations per timestep/pass)
- Compared performance against timestep-independent CS solvers and a support-aware Kalman smoother
    - Proposed approach drastically outperforms timestep-independent methods, and approaches the support-aware Kalman smoother in performance