ML Sequence Demodulation [Ch. 18]:

- Goal: Show that "orthogonal modulation with memory" enables MLWD with $\mathcal{O}(K_b)$ complexity.
- Key Idea: Leverage trellis structure induced by the FSM.
- This problem is solved by the Viterbi algorithm, an instance of *dynamic programming*.
- For simplicity, we restrict our discussion to rate-1 coding (i.e., $R = 1$).

Recall MLWD:

$$\underline{\hat{I}} \;=\; \arg \min_{i \in \{0,\ldots,2^{K_b}-1\}} \sum_{k=1}^{N_f} \left| Q^{(k)} - \sqrt{E_b}\, \tilde{d}_i^{(k)} \right|^2$$

Observations:

- Since there are 2^{K_b} possible sequences, brute-force minimization would require calculating $\mathcal{O}(2^{K_b})$ metrics. This is impractical for typical K_b!
- The total number of trellis edges is only about $2N_sK_b$, suggesting that an $\mathcal{O}(K_b)$ scheme might exist.

Some definitions:

$$\Omega_{k_s}^{(l)} \triangleq \Big\{ i = \underline{I} \;\text{ s.t. }\; \sigma^{(l)} = k_s \Big\} \;\text{ for }\; k_s \in \{1, \dots, N_s\}$$

(i.e., the set of sequences in state k_s at time l)

$$\Delta_{k_s}^{(l)} \triangleq \min_{i \in \Omega_{k_s}^{(l)}} \sum_{k=1}^{l} \Big| Q^{(k)} - \sqrt{E_b}\, \tilde{d}_i^{(k)} \Big|^2$$

(i.e., the lowest time-l accumulated cost

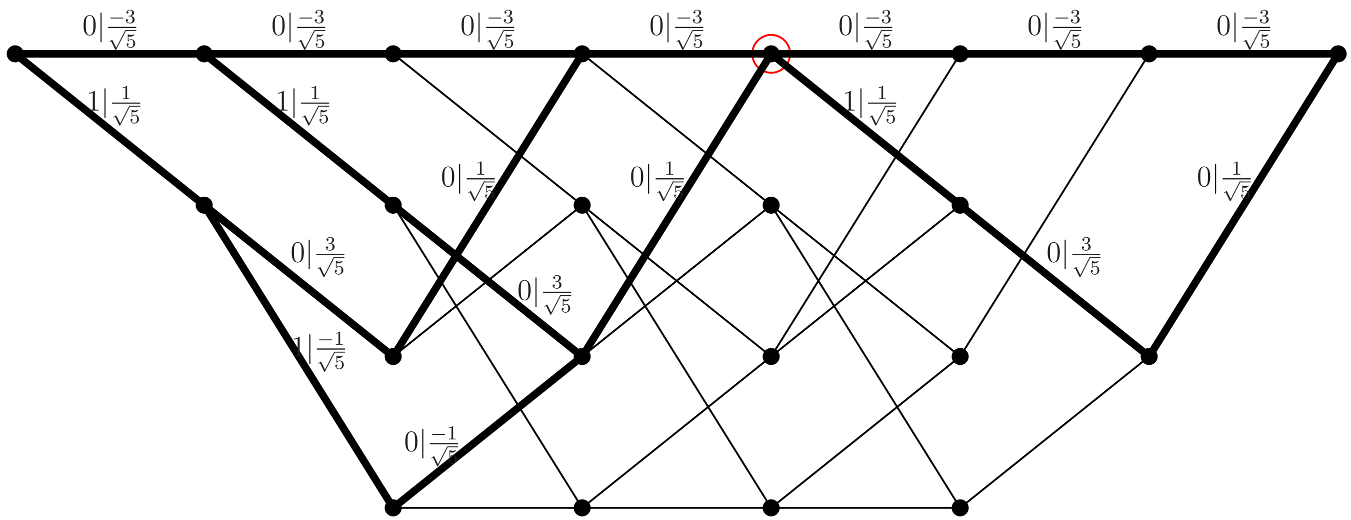
among sequences in state k_s at time l.)

$$\hat{\underline{I}}_{k_s}^{(l)} \triangleq \text{partial bit sequence } [m_1, \dots, m_l] \text{ minimizing } \Delta_{k_s}^{(l)}$$

With a terminated trellis, the ML solution $\hat{\underline{I}}$ becomes

$$\hat{\underline{I}} = \hat{\underline{I}}_1^{(N_f)}.$$

Example: HCV with 4-PAM, $K_b = 5$, and $E_b = 1$:



Say $\{Q^{(l)}\} = [-1.3, 0.5, 1.2, 0.6, -1.4, -1.3, -1.2]$. Then,

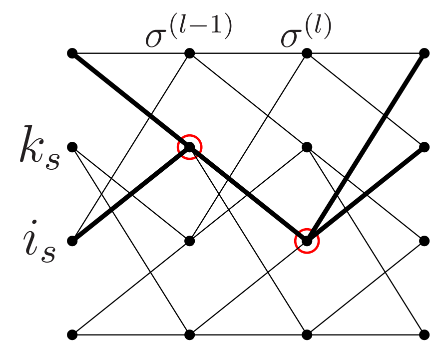
$$\Omega_1^{(4)} = \left\{\begin{matrix} [0,0,0,0,0] \\ [0,0,0,0,1] \\ [0,1,0,0,0] \\ [0,1,0,0,1] \\ [1,0,0,0,0] \\ [1,0,0,0,1] \\ [1,1,0,0,0] \\ [1,1,0,0,1] \end{matrix}\right\} \leftrightarrow \left\{\begin{matrix} [\frac{-3}{\sqrt5},\frac{-3}{\sqrt5},\frac{-3}{\sqrt5},\frac{-3}{\sqrt5},\frac{-3}{\sqrt5},\frac{-3}{\sqrt5},\frac{-3}{\sqrt5}] \\ [\frac{-3}{\sqrt5},\frac{-3}{\sqrt5},\frac{-3}{\sqrt5},\frac{-3}{\sqrt5},\frac{1}{\sqrt5},\frac{3}{\sqrt5},\frac{1}{\sqrt5}] \\ [\frac{-3}{\sqrt5},\frac{1}{\sqrt5},\frac{3}{\sqrt5},\frac{1}{\sqrt5},\frac{-3}{\sqrt5},\frac{-3}{\sqrt5},\frac{-3}{\sqrt5}] \\ [\frac{-3}{\sqrt5},\frac{1}{\sqrt5},\frac{3}{\sqrt5},\frac{1}{\sqrt5},\frac{1}{\sqrt5},\frac{3}{\sqrt5},\frac{1}{\sqrt5}] \\ [\frac{1}{\sqrt5},\frac{3}{\sqrt5},\frac{1}{\sqrt5},\frac{-3}{\sqrt5},\frac{-3}{\sqrt5},\frac{-3}{\sqrt5},\frac{-3}{\sqrt5}] \\ [\frac{1}{\sqrt5},\frac{3}{\sqrt5},\frac{1}{\sqrt5},\frac{-3}{\sqrt5},\frac{1}{\sqrt5},\frac{3}{\sqrt5},\frac{1}{\sqrt5}] \\ [\frac{1}{\sqrt5},\frac{-1}{\sqrt5},\frac{-1}{\sqrt5},\frac{1}{\sqrt5},\frac{-3}{\sqrt5},\frac{-3}{\sqrt5},\frac{-3}{\sqrt5}] \\ [\frac{1}{\sqrt5},\frac{-1}{\sqrt5},\frac{-1}{\sqrt5},\frac{1}{\sqrt5},\frac{1}{\sqrt5},\frac{3}{\sqrt5},\frac{1}{\sqrt5}] \end{matrix}\right\}, \quad \Delta_1^{(4)} = \min\left\{\begin{matrix} 13.6 \\ 13.6 \\ 0.05 \\ 0.05 \\ 8.10 \\ 8.10 \\ 6.69 \\ 6.69 \end{matrix}\right\},$$

$$\underline{\hat{I}}_1^{(4)} = [0,1,0,0].$$

Can partition the set $\Omega_{i_s}^{(l)}$ as follows:

$$\begin{aligned}\Omega_{i_s}^{(l)} &= \{ i = \underline{I} \;\text{ s.t. }\; \sigma^{(l)} = i_s \} \\ &= \bigcup_{k_s=1}^{N_s} \underbrace{\{ i = \underline{I} \;\text{ s.t. }\; \sigma^{(l-1)} = k_s \;\&\; \sigma^{(l)} = i_s \}}_{\triangleq\, \Omega_{k_s,i_s}^{(l-1,l)}}\end{aligned}$$

Note that all sequences in $\Omega_{k_s,i_s}^{(l-1,l)}$ produce the same symbol at time l, i.e., $\forall i \in \Omega_{k_s,i_s}^{(l-1,l)},\; \tilde{d}_i^{(l)} = \tilde{d}_{k_s,i_s}$.



$$\tilde{d}_{k_s,i_s} \triangleq \begin{cases} a\big(g_2(k_s, m_s)\big) & \text{if } \exists\; m_s \text{ s.t. } i_s = g_1(k_s, m_s) \\ \infty & \text{else} \end{cases}$$

Main idea behind Viterbi algorithm:

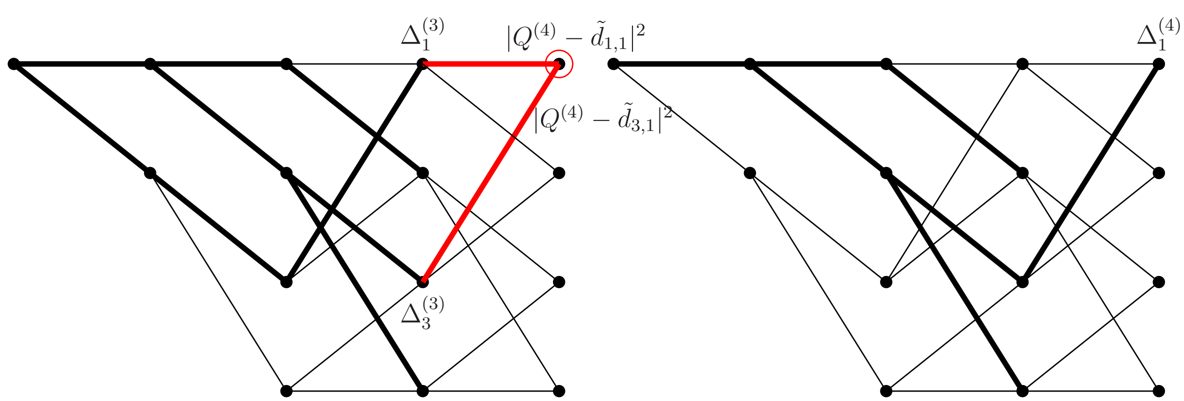
$$
\begin{aligned}
\Delta_{i_s}^{(l)} &= \min_{i \in \Omega_{i_s}^{(l)}} \sum_{k=1}^{l} \left| Q^{(k)} - \sqrt{E_b}\tilde{d}_i^{(k)} \right|^2 \\
&= \min_{i \in \Omega_{i_s}^{(l)}} \left\{ \sum_{k=1}^{l-1} \left| Q^{(k)} - \sqrt{E_b}\tilde{d}_i^{(k)} \right|^2 + \left| Q^{(l)} - \sqrt{E_b}\tilde{d}_i^{(l)} \right|^2 \right\} \\
&= \min_{k_s} \min_{j \in \Omega_{k_s, i_s}^{(l-1,l)}} \left\{ \sum_{k=1}^{l-1} \left| Q^{(k)} - \sqrt{E_b}\tilde{d}_j^{(k)} \right|^2 + \left| Q^{(l)} - \sqrt{E_b}\tilde{d}_j^{(l)} \right|^2 \right\} \\
&= \min_{k_s \in \{1,\ldots,N_s\}} \left\{ \Delta_{k_s}^{(l-1)} + \left| Q^{(l)} - \sqrt{E_b}\tilde{d}_{k_s,i_s} \right|^2 \right\}
\end{aligned}
$$

$$
\underline{\hat{I}}_{i_s}^{(l)} = \left[\underline{\hat{I}}_{k_s^*}^{(l-1)}, m_s^* \right] \text{ where } \begin{cases} k_s^* \triangleq \text{ the minimizing } k_s. \\ m_s^* \triangleq \text{ the bit value taking } k_s^* \text{ to } i_s. \end{cases}
$$

HCV Example:

- Say it's time $l = 4$. Assume we know the previous cumulative metrics $\{\Delta_{k_s}^{(3)}\}$ for all states k_s.
- For each state $i_s \in \{1, 2, 3, 4\}$, we compare the metrics for the two merging paths, calculate the next metric $\Delta_{i_s}^{(4)}$ using the better path, and discard the worse path.

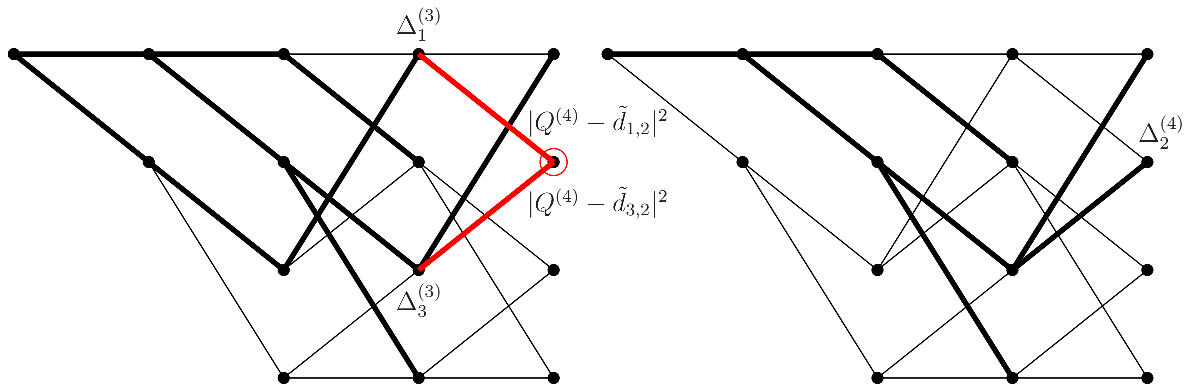
$\underline{i_s = 1}$:



$\Delta_1^{(3)} = 4.33, \qquad |Q^{(4)} - \tilde{d}_{1,1}|^2 = 3.77, \qquad \Delta_1^{(4)} = 0.04.$

$\Delta_3^{(3)} = 0.02, \qquad |Q^{(4)} - \tilde{d}_{3,1}|^2 = 0.02,$

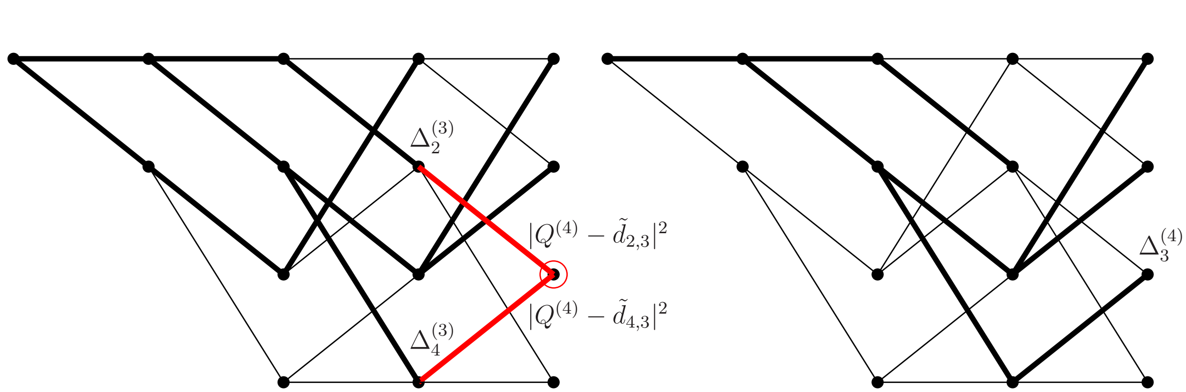
$\underline{i_s = 2}$:



$$\Delta_1^{(3)} = 4.33, \qquad |Q^{(4)} - \tilde{d}_{1,2}|^2 = 0.02, \qquad \Delta_2^{(4)} = 3.81.$$

$$\Delta_3^{(3)} = 0.02, \qquad |Q^{(4)} - \tilde{d}_{3,2}|^2 = 3.79,$$

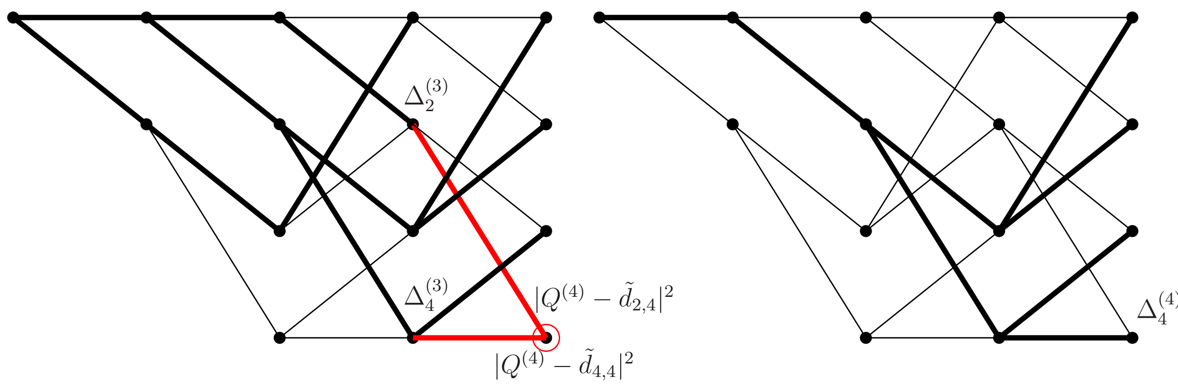
$\underline{i_s = 3}$:



$$\Delta_2^{(3)} = 3.96, \qquad |Q^{(4)} - \tilde{d}_{2,3}|^2 = 0.55, \qquad \Delta_3^{(4)} = 3.82.$$

$$\Delta_4^{(3)} = 2.72, \qquad |Q^{(4)} - \tilde{d}_{4,3}|^2 = 1.10,$$

$\underline{i_s = 4}$:



$\Delta_2^{(3)} = 3.96, \quad |Q^{(4)} - \tilde{d}_{2,4}|^2 = 1.10, \quad \Delta_4^{(4)} = 3.27.$

$\Delta_4^{(3)} = 2.72, \quad |Q^{(4)} - \tilde{d}_{4,4}|^2 = 0.55,$

At this point we have $\{\Delta_{k_s}^{(4)}\}$ for all states k_s, so we are ready to proceed onto the next time step (i.e., $l = 5$).

Viterbi Algorithm Summary:

$$\Delta_1^{(0)} = 0, \quad \Delta_{k_s}^{(0)}\Big|_{k_s>1} = \infty, \quad \underline{\hat{I}}^{(0)} = [\,]$$

for $l = 1, \ldots, N_f$,

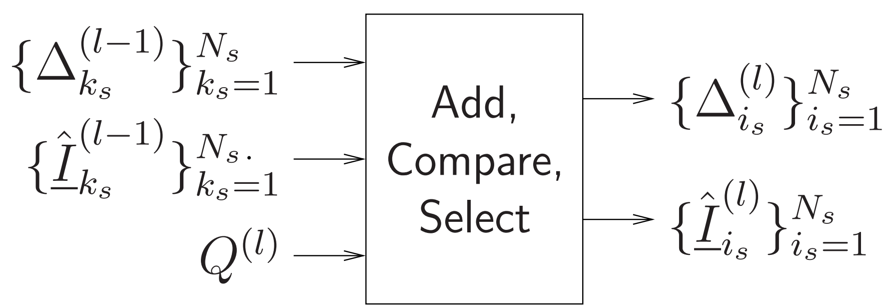
for $i_s = 1, \ldots, N_s$,

$$\Delta_{i_s}^{(l)} = \min_{k_s \in \{1,\ldots,N_s\}} \left\{ \Delta_{k_s}^{(l-1)} + \left| Q^{(l)} - \sqrt{E_b}\,\tilde{d}_{k_s,i_s} \right|^2 \right\}$$

$$\underline{\hat{I}}_{i_s}^{(l)} = \left[\underline{\hat{I}}_{k_s^*}^{(l-1)}, m_s^* \right] \text{ for } \begin{cases} k_s^* \triangleq \text{ the minimizing } k_s. \\ m_s^* \triangleq \text{ the bit taking } k_s^* \text{ to } i_s. \end{cases}$$

end

end



Viterbi complexity (for $R = 1$):

$$N_f \frac{\text{symbol intervals}}{\text{per block}} \times N_s \frac{\text{states}}{\text{symbol interval}} \times 2 \frac{\text{path comparisons}}{\text{per state}} \times$$

(2 real multiplies plus 3 real adds per comparison).

Thus, complexity of MLWD is $\mathcal{O}(N_f) = \mathcal{O}(K_b)$!