Derivation of Binary MAP Demodulator

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For the binary MAP demodulator, we know that

$$\begin{aligned}
\hat{i} &= \arg\max_i \Pr\{I = i | V_{\mathsf{I}}(T_p) = v_{\mathsf{I}}\} \\
&= \arg\max_i f_{V_{\mathsf{I}}(T_p)|I}(v_{\mathsf{I}})\pi_i \\
&= \arg\max_i \exp\left[-\frac{(v_{\mathsf{I}} - m_i(T_p))^2}{2\sigma_{N_{\mathsf{I}}}^2}\right]\pi_i
\end{aligned}$$

If we choose the front-end filter $H_{\max}(f)$ (which we justified in the notes for the equal-priors case, and which can be justified more generally using detection theory), then

$$\begin{aligned}
\sigma_{N_{\mathsf{I}}}^2 &= \frac{N_0}{2}\int_{-\infty}^{\infty} |H_{\max}(f)|^2 df \\
&= \frac{N_0}{2}\int_{-\infty}^{\infty} |h_{\max}(t)|^2 dt \\
&= \frac{N_0}{2}\int_{0}^{T_p} |x_1(t) - x_0(t)|^2 dt \\
&= \frac{N_0}{2}\Delta_E(1,0),
\end{aligned}$$

where

$$\Delta_E(1,0) = E_1 + E_0 - 2\,\mathrm{Re}\sqrt{E_0E_1}\rho_{10}.$$

Recall from the notes that

$$\begin{aligned}
m_1(T_p) &= E_1 - \mathrm{Re}\sqrt{E_0E_1}\rho_{10} \\
m_0(T_p) &= -E_0 + \mathrm{Re}\sqrt{E_0E_1}\rho_{10}.
\end{aligned}$$

Recall also that $V_{\mathsf{I}}(T_p) = \mathrm{Re}\,V_1(T_p) - \mathrm{Re}\,V_0(T_p)$. Using v_i, v_0, and v_1 to denote the observed realizations of the random variables $V_i(T_p)$, $V_0(T_p)$, and $V_1(T_p)$, we see that

$$v_{\mathsf{I}} = \mathrm{Re}\,v_1 - \mathrm{Re}\,v_0.$$

Putting these findings together yields the decision rule

$$\exp\left[-\frac{\left(\mathrm{Re}\,v_1 - \mathrm{Re}\,v_0 - E_1 + \sqrt{E_0E_1}\,\mathrm{Re}\,\rho_{10}\right)^2}{N_0\Delta_E(1,0)}\right]\pi_1 \ \underset{\hat{i}=0}{\overset{\hat{i}=1}{\gtrless}} \ \exp\left[-\frac{\left(\mathrm{Re}\,v_1 - \mathrm{Re}\,v_0 + E_0 - \sqrt{E_0E_1}\,\mathrm{Re}\,\rho_{10}\right)^2}{N_0\Delta_E(1,0)}\right]\pi_0,$$

which is equivalently restated as

$$\exp\left[-\frac{\left((\operatorname{Re}v_1-\frac{E_1}{2})-(\operatorname{Re}v_0-\frac{E_0}{2})-\frac{E_1}{2}-\frac{E_0}{2}+\sqrt{E_0E_1}\operatorname{Re}\rho_{10}\right)^2}{N_0\Delta_E(1,0)}\right]\pi_1$$

$$\underset{\hat{i}=0}{\overset{\hat{i}=1}{\gtrless}}\quad \exp\left[-\frac{\left((\operatorname{Re}v_1-\frac{E_1}{2})-(\operatorname{Re}v_0-\frac{E_0}{2})+\frac{E_0}{2}+\frac{E_1}{2}-\sqrt{E_0E_1}\operatorname{Re}\rho_{10}\right)^2}{N_0\Delta_E(1,0)}\right]\pi_0$$

or, with $T_1=\operatorname{Re}v_1-\frac{E_1}{2}$ and $T_0=\operatorname{Re}v_0-\frac{E_0}{2}$, as

$$\exp\left[-\frac{\left(T_1-T_0-\Delta_E(1,0)/2\right)^2}{N_0\Delta_E(1,0)}\right]\pi_1 \quad \underset{\hat{i}=0}{\overset{\hat{i}=1}{\gtrless}}\quad \exp\left[-\frac{\left(T_1-T_0+\Delta_E(1,0)/2\right)^2}{N_0\Delta_E(1,0)}\right]\pi_0.$$

Expanding the quadratic terms, we have

$$\exp\left[-\frac{\left(T_1-T_0\right)^2+\Delta_E(1,0)^2/4-(T_1-T_0)\Delta_E(1,0)}{N_0\Delta_E(1,0)}\right]\pi_1$$

$$\underset{\hat{i}=0}{\overset{\hat{i}=1}{\gtrless}}\quad \exp\left[-\frac{\left(T_1-T_0\right)^2+\Delta_E(1,0)^2/4+(T_1-T_0)\Delta_E(1,0)}{N_0\Delta_E(1,0)}\right]\pi_0,$$

and, after dividing both sides by common terms, we have

$$\exp\left[\frac{(T_1-T_0)\Delta_E(1,0)}{N_0\Delta_E(1,0)}\right]\pi_1 \quad \underset{\hat{i}=0}{\overset{\hat{i}=1}{\gtrless}}\quad \exp\left[\frac{-(T_1-T_0)\Delta_E(1,0)}{N_0\Delta_E(1,0)}\right]\pi_0$$

or

$$\exp\left[\frac{T_1-T_0}{N_0}\right]\pi_1 \quad \underset{\hat{i}=0}{\overset{\hat{i}=1}{\gtrless}}\quad \exp\left[\frac{-T_1+T_0}{N_0}\right]\pi_0$$

or

$$\exp\left[\frac{2T_1}{N_0}\right]\pi_1 \quad \underset{\hat{i}=0}{\overset{\hat{i}=1}{\gtrless}}\quad \exp\left[\frac{2T_0}{N_0}\right]\pi_0.$$

We can restate this as

$$\hat{i} \quad = \quad \arg\max_i \; \exp\left[\frac{2T_i}{N_0}\right]\pi_i.$$

Or, if we treat T_i as a random variable, i.e., $T_i=\operatorname{Re}V_i-\frac{E_i}{2}$, then the decision itself becomes random and we have

$$\hat{I} \quad = \quad \arg\max_i \; \exp\left[\frac{2T_i}{N_0}\right]\pi_i.$$