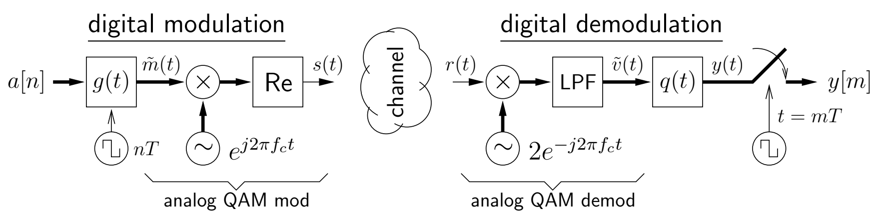
Digital Communication (Ch. 6,7,10):

Transmission consists of

1. pulse shaping: $\tilde{m}(t) = \sum_n a[n]g(t - nT)$,
2. modulation: $s(t) = \mathrm{Re}\{\tilde{m}(t)e^{j2\pi f_c t}\}$.

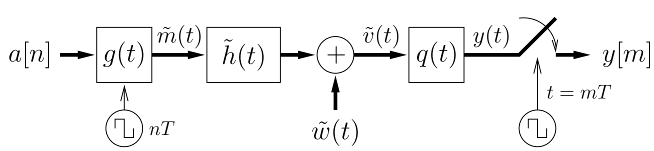
Reception consists of

1. demodulation: $\tilde{v}(t) = \mathsf{LPF}\{2r(t)e^{-j2\pi f_c t}\}$,
2. filtering: $y(t) = \tilde{v}(t) * q(t)$,
3. sampling: $y[m] = y(mT)$.



Building on analog QAM mod/demod components, digital mod adds pulse shaping & demod adds filtering/sampling.

Simplifying via the complex-baseband equivalent channel:



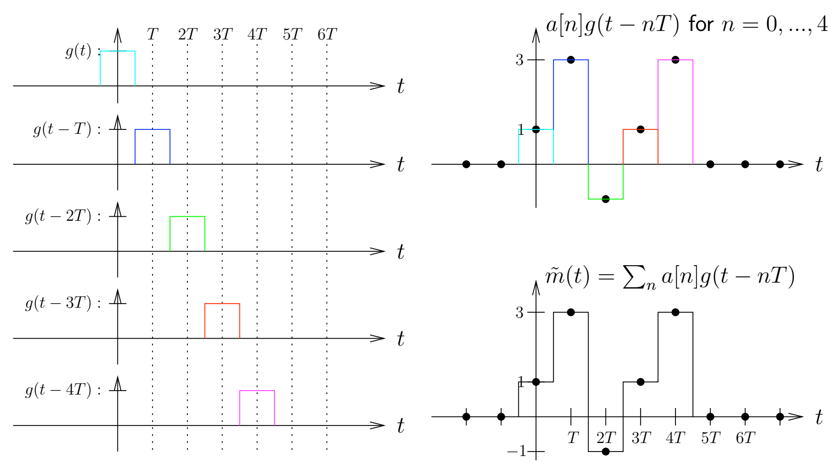
Transmitter pulse shaping is used to convert the symbol sequence $\{a[n]\}$ into the continuous message $\tilde{m}(t)$:

$$\tilde{m}(t) = \sum_n a[n]g(t - nT) \quad \text{“baseband message”}$$

$$T = \text{“symbol period”}$$

Thus, $\tilde{m}(t)$ can be seen to be a superposition of *scaled* and *time-shifted* copies of the pulse waveform $g(t)$.

Example, if the symbol sequence $[a[0], a[1], a[2], a[3], a[4]]$ equals $[1, 3, -1, 1, 3]$, then the square pulse $g(t)$ shown below left yields the message $\tilde{m}(t)$ shown below right.



Receiver filtering (via $q(t)$) has two goals:

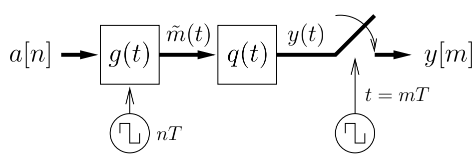
1. noise suppression (i.e., SNR improvement),
2. inter-symbol interference (ISI) prevention.

Noise suppression was briefly discussed on slide 13 and will soon be revisited in more detail. Next we describe ISI.

Realize that, in the *ideal* digital comm system, the n^{th} output $y[n]$ would simply equal the n^{th} input $a[n]$. But in practice, $y[n]$ can be corrupted by interference from the other symbols $\{a[m]\}_{m\neq n}$, known as "inter-symbol interference," and noise.

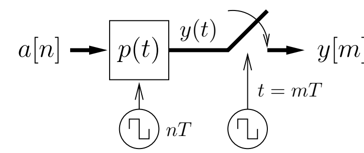
ISI-prevention for the noiseless trivial channel:

Consider the idealized system



$$\begin{aligned} y(t) &= \int q(\tau)\tilde{m}(t-\tau)d\tau \quad \text{for} \quad \tilde{m}(t) = \sum_n a[n]g(t-nT) \\ &= \sum_n a[n] \int q(\tau)g(t-nT-\tau)d\tau \\ &= \sum_n a[n]p(t-nT) \quad \text{for} \quad p(t) = g(t) * q(t). \end{aligned}$$

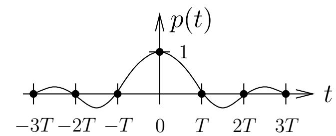
Thus, the idealized system can be re-drawn as



where

$$y[m] = y(mT) = \sum_n a[n]p(mT-nT) = \sum_n a[n]p\big((m-n)T\big).$$

To make $y[m] = a[m]$ (i.e., prevent ISI), we need



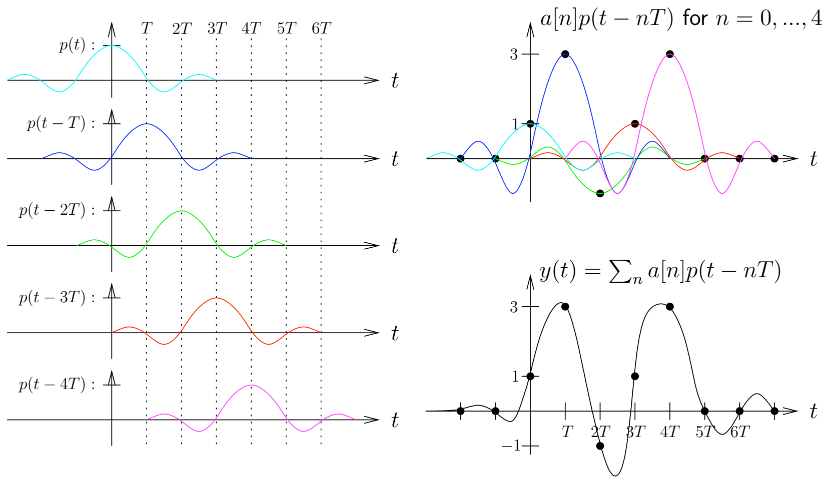
$$p(mT) = \begin{cases} 1 & m = 0 \\ 0 & m \neq 0 \end{cases}$$

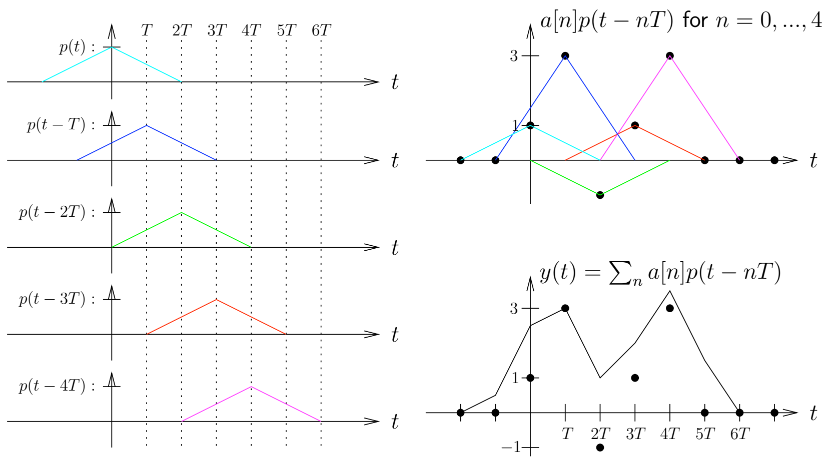
which is known as the "Nyquist Criterion." This criterion can be simply stated as $\boxed{p(mT) = \delta[m]}$ using

$$\delta[m] = \begin{cases} 1 & m = 0 \\ 0 & m \neq 0 \end{cases} \qquad \text{"discrete-time impulse," or "Kronecker delta."}$$

$$a[n] = \sum_{m=-\infty}^{\infty} a[m]\delta[n-m] \qquad \text{"sifting property."}$$

Examples of Nyquist, and non-Nyquist, combined-pulses $p(t)$ for $\big[a[0], a[1], a[2], a[3], a[4]\big] = [1, 3, -1, 1, 3]$:





There is an interesting frequency-domain interpretation. Since

$$\frac{1}{T}\sum_{k=-\infty}^{\infty}\delta\Big(f-\frac{k}{T}\Big) \overset{\mathcal{F}}{\longleftrightarrow} \sum_{m=-\infty}^{\infty}\delta(t-mT),$$

we can see that

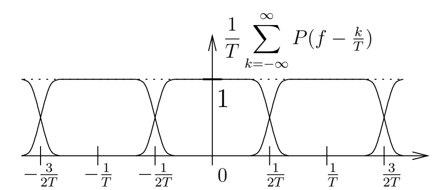
$$\underbrace{P(f) * \frac{1}{T}\sum_{k=-\infty}^{\infty}\delta\Big(f-\frac{k}{T}\Big)}_{\frac{1}{T}\sum_{k=-\infty}^{\infty}P\left(f-\frac{k}{T}\right)} \overset{\mathcal{F}}{\longleftrightarrow} \underbrace{p(t)\cdot\sum_{m=-\infty}^{\infty}\delta(t-mT)}_{\sum_{m=-\infty}^{\infty}p(mT)\delta(t-mT)}.$$

So, the time-domain Nyquist criterion $p(mT) = \delta[m]$ implies

$$\frac{1}{T}\sum_{k=-\infty}^{\infty}P\Big(f-\frac{k}{T}\Big) \overset{\mathcal{F}}{\longleftrightarrow} \delta(t),$$

which in turn implies

$$\frac{1}{T}\sum_{k=-\infty}^{\infty}P\Big(f-\frac{k}{T}\Big) = 1.$$



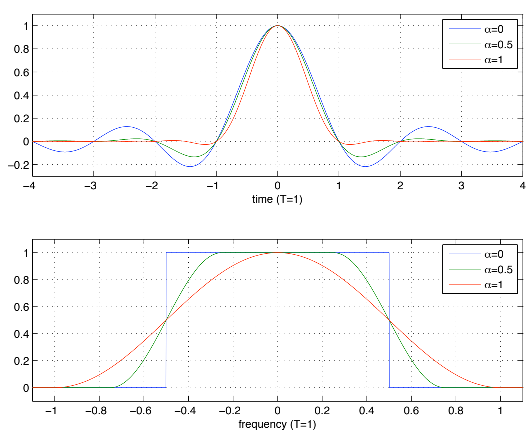
In other words, the superposition of $\big\{\frac{1}{T}P(f-\frac{k}{T})\big\}_{k\in\mathbb{Z}}$ must sum to one. This frequency-domain version of the Nyquist Criterion will soon come in handy...

A popular choice of combined pulse $p(t) = g(t) * q(t)$ is the "raised-cosine pulse" with rolloff parameter $\alpha \in [0, 1]$:

$$p_{\text{RC}}(t) = \frac{\cos(2\pi\alpha t/T)}{1-(2\alpha t/T)^2}\,\text{sinc}(t/T), \quad \text{sinc}(x) := \frac{\sin(\pi x)}{\pi x}$$

$$P_{\text{RC}}(f) = \begin{cases} T & |f| \le \frac{(1-\alpha)}{2T} \\ T\cos^2\left(\frac{\pi T}{2\alpha}\left(|f| - \frac{1-\alpha}{2T}\right)\right) & \frac{(1-\alpha)}{2T} \le |f| \le \frac{(1+\alpha)}{2T} \\ 0 & \frac{(1+\alpha)}{2T} \le |f| \end{cases}.$$

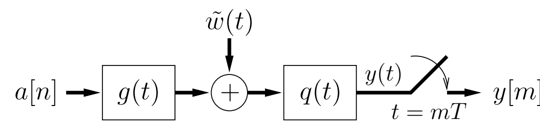
Tradeoff: larger $\alpha \Rightarrow$ less time-spread but more freq-spread:



So, we now know how to design the combined pulse $p(t)$. But what about the individual pulses $g(t)$ and $q(t)$?

Maximizing SNR for ISI-free Pulses in White Noise:

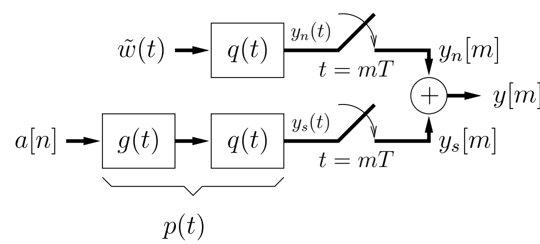
Now let's bring the noise back into consideration. Given



$$\text{E}\{\tilde{w}(t)\,\tilde{w}^*(t-\tau)\} = N_0\delta(\tau) \quad \text{(complex white noise)}$$

we want $\{g(t), q(t)\}$ pair that maximizes the SNR of $y[m]$.

Separating the noise and signal contributions to $y[m]$ via



the SNR can be written

$$\text{SNR} = \frac{\mathcal{E}_s}{\mathcal{E}_n} = \frac{\text{E}\{|y_s[m]|^2\}}{\text{E}\{|y_n[m]|^2\}},$$

where $\mathcal{E}_s$ and $\mathcal{E}_n$ are average signal and noise energies. Here, we treat both $\tilde{w}(t)$ and $a[n]$ as random, implying that $y_s[m]$ and $y_n[m]$ are both random.

Notice that, with an ISI-free combined pulse $p(t)$, we get

$$y_s[m] \;=\; \sum_n a[n]p\big((m-n)T\big) \;=\; a[m]p(0)$$
$$p(0) \;=\; \int_{-\infty}^{\infty} q(\tau)g(0-\tau)d\tau,$$

so that

$$\mathcal{E}_s \;=\; \mathrm{E}\{|y_s[m]|^2\} \;=\; \mathrm{E}\Big\{\Big|a[m]\int_{-\infty}^{\infty} q(\tau)g(-\tau)d\tau\Big|^2\Big\}$$
$$= \underbrace{\mathrm{E}\big\{|a[m]|^2\big\}}_{\sigma_a^2}\Big|\int_{-\infty}^{\infty} q(\tau)g(-\tau)d\tau\Big|^2,$$

where σ_a^2 denotes average symbol energy. Next, notice that

$$y_n[m] \;=\; y_n(mT) \;=\; \int_{-\infty}^{\infty} q(\tau)\,\tilde{w}(mT-\tau)d\tau,$$

so that

$$\mathcal{E}_n \;=\; \mathrm{E}\{|y_n[m]|^2\} \;=\; \mathrm{E}\Big\{\Big|\int_{-\infty}^{\infty} q(\tau)\tilde{w}(mT-\tau)d\tau\Big|^2\Big\}$$
$$= \mathrm{E}\Big\{\int_{-\infty}^{\infty} q(\tau)\tilde{w}(mT-\tau)d\tau\int_{-\infty}^{\infty} q^*(\tau')\tilde{w}^*(mT-\tau')d\tau'\Big\}$$
$$= \int_{-\infty}^{\infty} q(\tau)\int_{-\infty}^{\infty} q^*(\tau')\underbrace{\mathrm{E}\big\{\tilde{w}(mT-\tau)\tilde{w}^*(mT-\tau')\big\}}_{N_0\delta(\tau'-\tau)}d\tau' d\tau$$
$$= N_0\int_{-\infty}^{\infty} |q(\tau)|^2 d\tau.$$

Putting these together, we find

$$\mathsf{SNR} \;=\; \frac{\sigma_a^2}{N_0}\,\frac{\big|\int_{-\infty}^{\infty} q(\tau)g(-\tau)d\tau\big|^2}{\int_{-\infty}^{\infty}|q(\tau)|^2 d\tau}.$$

Cauchy-Schwarz says

$$\Big|\int_{-\infty}^{\infty} b(\tau)c(\tau)d\tau\Big|^2 \;\le\; \int_{-\infty}^{\infty}|b(\tau)|^2 d\tau\cdot\int_{-\infty}^{\infty}|c(\tau)|^2 d\tau$$

with equality iff $b(\tau) = Kc^*(\tau)$ for any K,

which implies

$$\mathsf{SNR} \;\le\; \frac{\sigma_a^2}{N_0}\int_{-\infty}^{\infty}|g(-\tau)|^2 d\tau$$

with equality iff $q(\tau) = Kg^*(-\tau)$ for any K.

Noting that SNR doesn't depend on K, we choose $K=1$.
Thus, given pulse $g(t)$, the SNR-maximizing receiver filter is

$$q(\tau) = g^*(-\tau) \quad \text{known as a "matched filter".}$$

We can write this in the frequency domain as

$$Q(f) \;=\; \int_{-\infty}^{\infty} q(\tau)e^{-j2\pi f\tau}d\tau \;=\; \int_{-\infty}^{\infty} g^*(-\tau)e^{-j2\pi f\tau}d\tau$$
$$= \int_{-\infty}^{\infty} g^*(t)e^{j2\pi ft}dt \;=\; \Big[\int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt\Big]^*$$
$$= G^*(f).$$

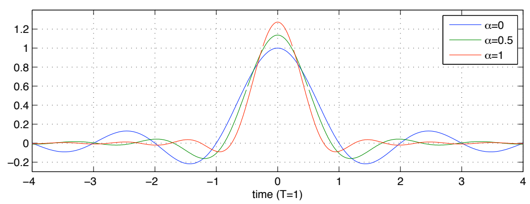
Summary: For SNR-maximizing ISI-free pulses, we need

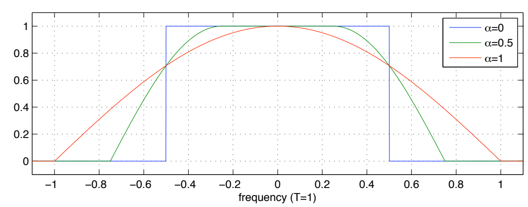
1. $G(f)Q(f) = P(f)$ satisfying the Nyquist crit,
2. $Q(f) = G^*(f)$,

which together imply $|G(f)|^2$ must satisfy the Nyquist crit.

One option is $G(f) = \sqrt{P_{\text{RC}}(f)}$, since $P_{\text{RC}}(f)$ was Nyquist. We call this the "square-root raised cosine" (SRRC) pulse. Working out the details of $\mathcal{F}^{-1}\{G_{\text{SRRC}}(f)\}$, we find

$$g_{\text{SRRC}}(t) = \frac{(1-\alpha)\,\text{sinc}(\frac{t}{T}(1-\alpha))}{1-(4\alpha\frac{t}{T})^2} + \frac{4\alpha\cos(\pi\frac{t}{T}(1+\alpha))}{\pi(1-(4\alpha\frac{t}{T})^2)}.$$

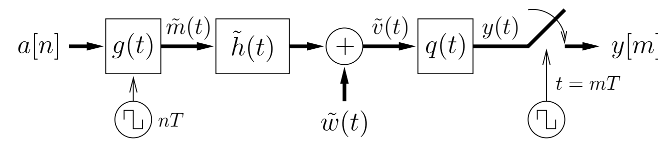




At the receiver, we would use $q_{\text{SRRC}}(t) = g^*_{\text{SRRC}}(-t) = g_{\text{SRRC}}(t)$; the latter equality is due to $g_{\text{SRRC}}(t)$ being real and symmetric.

DSP Implementation of Digital Comm:

Digital implementation of transmitter pulse-shaping and receiver filtering is much more practical than analog.



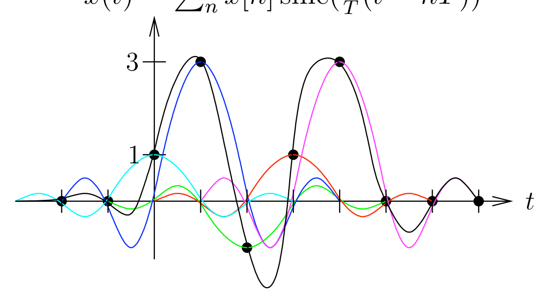
To proceed further, we need an important DSP concept called "sinc reconstruction":

If waveform $x(t)$ is bandlimited to $\frac{1}{2T_s}$ Hz, then

$$x(t) = \sum_{n=-\infty}^{\infty} x[n]\,\text{sinc}\left(\tfrac{1}{T_s}(t-nT_s)\right) \text{ for } x[n] = x(nT_s).$$

In other words, a bandlimited waveform can be reconstructed from its samples via sinc pulse shaping.

$$x(t) = \textstyle\sum_n x[n]\,\text{sinc}(\frac{1}{T}(t-nT))$$

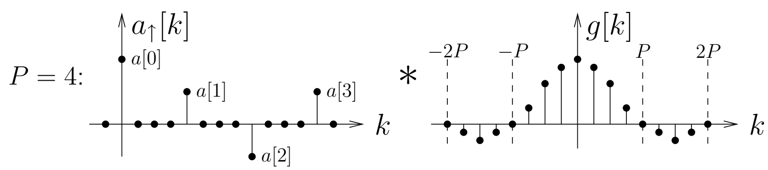


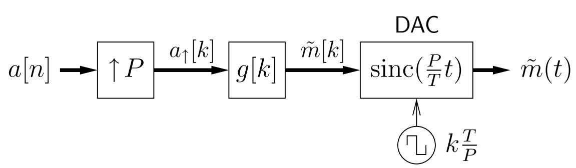
Discrete-Time Pulse-Shaping:

Applying $\frac{T}{P}$-sampling and reconstruction to $g(t)$ (where the SRRC pulse bandwidth $\frac{1+\alpha}{2T}$ requires the use of $P \geq 2$),

$$\begin{aligned} g(\tau) &= \textstyle\sum_l g[l]\,\mathrm{sinc}(\frac{P}{T}(\tau - l\frac{T}{P})) \quad \text{for } g[l] = g(l\tfrac{T}{P}) \\ \tilde{m}(t) &= \textstyle\sum_n a[n] g(t - nT) \\ &= \textstyle\sum_n a[n] \sum_l g[l]\,\mathrm{sinc}(\frac{P}{T}(t - nT - l\frac{T}{P})) \\ &= \textstyle\sum_n a[n] \sum_k g[k-nP]\,\mathrm{sinc}(\frac{P}{T}(t - k\frac{T}{P})) \text{ via } k = nP - l \\ &= \textstyle\sum_k \underbrace{\sum_n a[n] g[k-nP]}_{:= \tilde{m}[k]}\,\mathrm{sinc}(\frac{P}{T}(t - k\frac{T}{P})) \end{aligned}$$

The sequence $\tilde{m}[k]$, a weighted sum of P-shifted pulses $g[k]$, can be generated by P-upsampling $a[n]$ (i.e., inserting $P-1$ zeros between every pair of samples) and filtering with $g[k]$:





Note: sinc-pulse shaping = digital-to-analog conversion (DAC).

Discrete-Time Receiver Filtering:

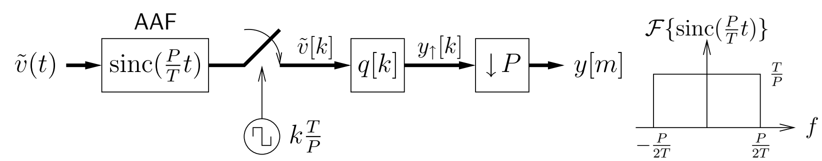
Applying $\frac{T}{P}$-sampling and reconstruction to bandlimited $q(\tau)$:

$$q(\tau) = \sum_{l=-\infty}^{\infty} q[l]\,\mathrm{sinc}\big(\tfrac{P}{T}(\tau - l\tfrac{T}{P})\big) \quad \text{for } q[l] = q(l\tfrac{T}{P})$$

where again we need $P \geq 2$, yields

$$\begin{aligned} y_\uparrow[k] = y(k\tfrac{T}{P}) &= \int_{-\infty}^{\infty} q(\tau)\,\tilde{v}(k\tfrac{T}{P} - \tau)d\tau \\ &= \int_{-\infty}^{\infty} \sum_{l=-\infty}^{\infty} q[l]\,\mathrm{sinc}\big(\tfrac{P}{T}(\tau - l\tfrac{T}{P})\big)\,\tilde{v}\big(k\tfrac{T}{P} - \tau\big)d\tau \\ &= \sum_{l=-\infty}^{\infty} q[l] \underbrace{\int_{-\infty}^{\infty} \mathrm{sinc}\big(\tfrac{P}{T}\tau'\big)\,\tilde{v}\big((k-l)\tfrac{T}{P} - \tau'\big)d\tau'}_{\{\mathrm{sinc}(\frac{P}{T}t) * \tilde{v}(t)\}_{t=(k-l)\frac{T}{P}} = \tilde{v}[k-l]} \\ &= \sum_{l=-\infty}^{\infty} q[l]\tilde{v}[k-l] = q[k] * \tilde{v}[k], \end{aligned}$$

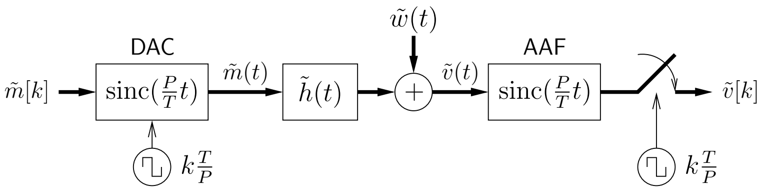
from which $y[m]$ is obtained by keeping only every P^{th} sample:



i.e., downsampling. Here $\mathrm{sinc}(\frac{P}{T}t)$ does anti-alias filtering.

Discrete-Time Complex-Baseband Channel:

Finally, we derive a dicrete-time representation of the channel between $\tilde{m}[k]$ and $\tilde{v}[k]$:



Using $\tilde{w}[k]$ to refer to the noise component of $\tilde{v}[k]$, it can be seen from the block diagram that

$$\tilde{w}[k] \;=\; \int_{-\infty}^{\infty} \tilde{w}(\tau)\,\mathrm{sinc}(\tfrac{P}{T}(k\tfrac{T}{P}-\tau))d\tau$$

To model the signal component of $\tilde{v}[k]$, realize that $\tilde{m}[k]$ is effectively pulse-shaped by $\mathrm{sinc}(\frac{P}{T}t) * \tilde{h}(t) * \mathrm{sinc}(\frac{P}{T}t)$. But since the frequency response of $\mathrm{sinc}(\frac{P}{T}t)$ has a flat gain of $\frac{T}{P}$ over the signal bandwidth, and thus the bandwidth of $\tilde{h}(t)$,

$$\mathrm{sinc}(\tfrac{P}{T}t) * \tilde{h}(t) * \mathrm{sinc}(\tfrac{P}{T}t) \;=\; \tfrac{T^2}{P^2}\,\tilde{h}(t).$$

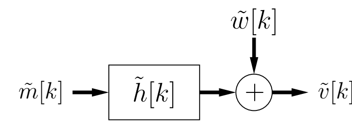
So, with $\frac{T}{P}$-sampling and reconstruction of $\frac{T^2}{P^2}\,\tilde{h}(t)$, i.e.,

$$\tfrac{T^2}{P^2}\,\tilde{h}(t) \;=\; \sum_{i=-\infty}^{\infty} h[i]\,\mathrm{sinc}\big(\tfrac{P}{T}(t-i\tfrac{T}{P})\big) \quad \text{for } h[i] = \tfrac{T^2}{P^2}\,h(i\tfrac{T}{P})$$

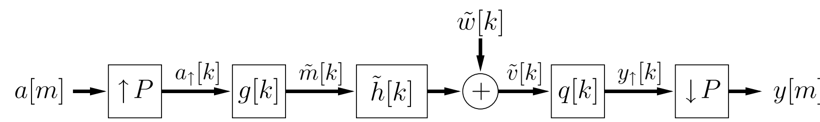
we can write $\tilde{v}[k]$ as

$$\begin{aligned}\tilde{v}[k] &= \tilde{w}[k] + \textstyle\sum_l \tilde{m}[l]\,\tfrac{T^2}{P^2}\,\tilde{h}(k\tfrac{T}{P} - l\tfrac{T}{P}) \\ &= \tilde{w}[k] + \textstyle\sum_l \tilde{m}[l] \sum_i \tilde{h}[i]\,\underbrace{\mathrm{sinc}(k-l-i)}_{\delta[k-l-i]} \\ &= \tilde{w}[k] + \textstyle\sum_l \tilde{m}[l]\tilde{h}[k-l]\end{aligned}$$

yielding the discrete-time channel



Merging the discrete-time channel with the discrete-time modulator and demodulator yields



known as the “fractionally sampled” system model. This model is very convenient for MATLAB simulation and acts as a foundation for further analysis.