A Separation Principle for a Class of Non-UCO Systems

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***Abstract*—This paper introduces a new approach to output feedback stabilization of single-input–single-output systems which, unlike other techniques found in the literature, does not use quasilinear high-gain observers and control input saturation to achieve separation between the state feedback and observer designs. Rather, we show that by using nonlinear high-gain observers working in state coordinates, together with a dynamic projection algorithm, the same kind of separation principle is achieved for a larger class of systems which are not *uniformly completely observable*. By working in state coordinates, this approach avoids using knowledge of the inverse of the observability mapping to estimate the state of the plant, which is otherwise needed when using high-gain observers to estimate the output time derivatives.**

***Index Terms*—Nonlinear control, nonlinear observer, output feedback, separation principle.**

I. Introduction

The area of nonlinear output feedback control has received much attention after the publication of the work [3], in which the authors developed a systematic strategy for the output feedback control of input-output linearizable systems with full relative degree, which employed two basic tools: an high-gain observer to estimate the derivatives of the outputs (and hence the system states in transformed coordinates), and control input saturation to isolate the peaking phenomenon of the observer from the system states. Essentially the same approach has later been applied in a number of papers by various researchers (see, e.g., [9], [12], [7], [13], and [1]) to solve different problems in output feedback control. In most of the papers found in the literature, (see, e.g., [3], [9], [12], [13], [7], and [14]) the authors consider input-output feedback linearizable systems with either full relative degree or minimum phase zero dynamics. The work in [21] showed that for nonminimum phase systems the problem can be solved by extending the system dynamics with a chain of integrators at the input side. However, the results contained there are local. In [19], by putting together this idea with the approach found in [3], the authors were able to show how to solve the output feedback stabilization problem for smoothly stabilizable and uniformly completely observable (UCO) systems. The work in [1] unifies these approaches to prove a separation principle for a rather general class of nonlinear systems. The recent work in [16] relaxes the uniformity requirement of the UCO assumption by assuming the existence of one control input for which the system is observable. On the other hand, however, [16] requires the observability property to be complete, i.e., to hold on the entire state space. Another feature of that work is the relaxation of the smooth stabilizability assumption, replaced by the notion of asymptotic controllability (which allows for possibly nonsmooth stabilizers).

A common feature of the papers previously mentioned is their *input-output variable approach*, which entails using the vector $\mathrm{col}\,(y, \dot{y}, \ldots, y^{(n_y)}, u, \dot{u}, \ldots, u^{(n_u)})$ as feedback, for some integers n_y, n_u, where y and u denote the system output and input, respectively. This in particular implies that, when dealing with systems which are not input-output feedback linearizable, such approach requires the explicit knowledge of the inverse of the observability mapping, which in some cases may not be available.

This paper develops a different methodology for output feedback stabilization which is based on a *state-variable approach* and achieves a separation principle for a class of non-UCO systems, specifically systems that are observable on an open region of the state space and input space, rather than everywhere. We impose a restriction on the topology of such an "observability region" assuming, among other things, that it contains a sufficiently large simply connected neighborhood of the origin. The main contributions are the development of a nonlinear observer working in state coordinates (which is proved to be equivalent to the standard high-gain observer in output coordinates), and a dynamic projection operating on the observer dynamics which *eliminates* the peaking phenomenon in the observer states, thus avoiding the need to use control input saturation. One of the benefits of a *state-variable approach* is that the knowledge of the inverse of the observability mapping is not needed.

It is proved that, provided the observable region satisfies suitable topological properties, the proposed methodology yields closed-loop stability. In the particular case when the plant is globally stabilizable and UCO, this approach yields semiglobal stability, as in [19], provided a convexity requirement is satisfied. As in [21], [19], and [1], our results rely on adding integrators at the input side of the plant and designing a stabilizing control law for the resulting *augmented* system. Thus, a drawback of our approach (as well as the approaches in [21], [19], and [1]) is that separation is only achieved between the state feedback control design for the *augmented* system and the observer design and not between the state feedback control design for the *original* system and the observer design.

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II. Problem Formulation and Assumptions

Consider the following dynamical system:

$$\begin{aligned}\dot{x} &= f(x, u)\\ y &= h(x, u)\end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u, y \in \mathbb{R}$, f and h are known smooth functions, and $f(0, 0) = 0$. Our control objective is to construct a stabilizing controller for (1) without the availability of the system states x. In order to do so, we need an observability assumption. Define the observability mapping

$$\begin{aligned} y_e \triangleq \begin{bmatrix} y \\ \vdots \\ y^{(n-1)} \end{bmatrix} &= \mathcal{H}\left(x, u, \ldots, u^{(n_u-1)}\right) \\ &= \begin{bmatrix} h(x, u) \\ \varphi_1(x, u, u^{(1)}) \\ \vdots \\ \varphi_{n-1}\left(x, u, \ldots, u^{(n_u-1)}\right) \end{bmatrix} \end{aligned}$$

($y^{(n-1)}$ is the $n-1$th derivative) where

$$\begin{aligned} \varphi_1(x, u, u^{(1)}) &= \frac{\partial h}{\partial x} f(x, u) + \frac{\partial h}{\partial u} u^{(1)} \\ \varphi_2(x, u, u^{(1)}, u^{(2)}) &= \frac{\partial \varphi_1}{\partial x} f(x, u) + \frac{\partial \varphi_1}{\partial u} u^{(1)} + \frac{\partial \varphi_1}{\partial u^{(1)}} u^{(2)} \\ &\vdots \\ \varphi_{n-1}(x, \ldots) &= \frac{\partial \varphi_{n-2}}{\partial x} f(x, u) + \sum_{j=0}^{n_u-2} \frac{\partial \varphi_{n-2}}{\partial u^{(j)}} u^{(j+1)} \end{aligned}$$

where $0 \leq n_u \leq n$ ($n_u = 0$ indicates that there is no dependence on u). In the most general case, $\varphi_i = \varphi_i(x, u, \ldots, u^{(i)})$, $i = 1, 2, \ldots, n-1$. In some cases, however, we may have that $\varphi_i = \varphi_i(x, u)$ for all $i = 1, \ldots, r-1$ and some integer $r > 1$. This happens in particular when system (1) has a well-defined relative degree r. Here, we do not require the system to be input-output feedback linearizable, and hence to possess a well-defined relative degree. In the case of systems with well-defined relative degree, $n_u = 0$ corresponds to having $r \geq n$, while $n_u = n$ corresponds to having $r = 0$. Next, augment the system dynamics with n_u integrators on the input side, which corresponds to using a compensator of order n_u. System (1) can be rewritten as follows:

$$\dot{x} = f(x, z_1) \quad \dot{z}_1 = z_2, \ldots, \dot{z}_{n_u} = v. \tag{2}$$

Define the extended state variable $X = \operatorname{col}(x, z) \in \mathbb{R}^{n+n_u}$, and the associated *extended system*

$$\begin{aligned}\dot{X} &= f_e(X) + g_e v\\ y &= h_e(X)\end{aligned} \tag{3}$$

where $f_e(X) = \operatorname{col}(f(x, z_1), z_2, \ldots, z_{n_u}, 0)$, $g_e = \operatorname{col}(0, \ldots, 1)$, and $h_e(X) = h(x, z_1)$. Now, we are ready to state our first assumption.

Assumption A1 (Observability): System (1) is observable over an open set $\mathcal{O} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$ containing the origin, i.e., the mapping $\mathcal{F}\colon \mathcal{O} \to \mathcal{Y}$ [where $\mathcal{Y} = \mathcal{F}(\mathcal{O})$] defined by

$$Y = \begin{bmatrix} y_e \\ \cdots \\ z \end{bmatrix} = \mathcal{F}(X) = \begin{bmatrix} \mathcal{H}(x, z) \\ z \end{bmatrix} \tag{4}$$

has a smooth inverse $\mathcal{F}^{-1}\colon \mathcal{Y} \to \mathcal{O}$

$$\mathcal{F}^{-1}(Y) = \mathcal{F}^{-1}(y_e, z) = \begin{bmatrix} \mathcal{H}^{-1}(y_e, z) \\ z \end{bmatrix}. \tag{5}$$

Remark 1: In the existing literature, an assumption similar to A1 can be found in [21] and [19]. It is worth stressing, however, that in both works the authors assume the set $\mathcal{O}$ to be $\mathbb{R}^n \times \mathbb{R}^{n_u}$. When that is the case, the system is said to be [20], [19] UCO. In many practical applications, the system under consideration may not be UCO, but rather be observable in some subset of $\mathbb{R}^n \times \mathbb{R}^{n_u}$ only, thus preventing the use of most of the output feedback techniques found in the literature, including the ones found in [21], [19], and [1]. On the other hand in A1 the mapping $\mathcal{H}$, viewed as a mapping acting on x parameterized by z, is assumed to be square (i.e., it maps spaces of equal dimension), thus implying that x can be expressed as a function of y, its $n-1$ time derivatives and z, i.e., $x = \mathcal{H}^{-1}(\operatorname{col}(y, \dot{y}, \ldots, y^{(n-1)}), z)$. In the works [19], [16], x is allowed to be a function depending on a possibly higher number of derivatives of y, rather than just $n-1$. In our setting, this is equivalent to assuming that $\mathcal{F}$ in A1, rather than being invertible, is just left-invertible. We are currently working on relaxing A1 and replace it by the weaker left-invertibility of $\mathcal{F}$.

Assumption A2 (Stabilizability): The origin of (1) is locally stabilizable (or stabilizable) by a static function of x, i.e., there exists a smooth function $\overline{u}(x)$ such that the origin is an asymptotically stable (or globally asymptotically stable) equilibrium point of $\dot{x} = f(x, \overline{u}(x))$.

Remark 2: Assumption A2 implies that the origin of the extended system (3) is locally stabilizable (or stabilizable) by a function of X as well. A proof of the local stabilizability property for (3) may be found, e.g., in [17], while its global counterpart is a well known consequence of the integrator backstepping lemma (see, e.g., [5, Th. 9.2.3] or [10, Corollary 2.10]). Therefore, we conclude that for the extended system (3) there exists a smooth control $\overline{v}(X)$ such that its origin is asymptotically stable under closed-loop control. Let $\mathcal{D}$ be the domain of attraction of the origin of (3), and notice that, when A2 holds globally, $\mathcal{D} = \mathbb{R}^n \times \mathbb{R}^{n_u}$.

Remark 3: In [21], the authors consider affine systems and use a feedback linearizability assumption in place of our A2. Here, we consider the more general class of nonaffine systems for which the origin is locally stabilizable (stabilizable). In this respect, our assumption A2 relaxes also the stabilizability assumption found in [19], while it is equivalent to [1, Assumption 2].

III. Nonlinear Observer: Its Need and Stability Analysis

Assumption A2 allows us to design a stabilizing state feedback control $v=\phi(x, z)$. In order to perform output feedback control x should be replaced by its estimate. Many researchers in the past adopted an input-output feedback linearizability assumption [3], [9], [13], [7], [14] and transformed the system into normal form

$$\begin{aligned}\dot{\pi}_i &= \pi_{i+1}, \qquad 1 \le i \le r-1\\ \dot{\pi}_r &= \overline{f}(\pi, \Pi) + \overline{g}(\pi, \Pi)\, u\\ \dot{\Pi} &= \Phi(\pi, \Pi), \qquad \Pi \in \mathbb{R}^{n-r}\\ y &= \pi_1. \end{aligned} \tag{6}$$

In this framework the problem of output feedback control finds a very natural formulation, as the first r derivatives of y are equal to the states of the π-subsystem (i.e., the linear subsystem). The works [3], [9], and [7] solve the output feedback control problem for systems with no zero dynamics (i.e., $r=n$), so that the first $n-1$ derivatives of y provide the entire state of the system. In the presence of zero dynamics (Π-subsystem), the use of input-output feedback linearizing controllers for (6) forces the use of a minimum phase assumption (e.g., [13]) since the states of the Π-subsystem are made unobservable by such controllers and, hence, cannot be controlled by output feedback. For this reason the output feedback control of nonminimum phase systems has been regarded in the past as a particularly challenging problem. Researchers who have addressed this problem (e.g., [21], [19]) rely on the explicit knowledge of $\mathcal{H}^{-1}$ in (5), $x=\mathcal{H}^{-1}(y_e, \mathrm{col}\,(z_1, \ldots, z_{n_u}))$, so that estimation of the first $n-1$ derivatives of y (the vector y_e) provides an estimate of x, $\hat{x}=\mathcal{H}^{-1}(\hat{y}_e, \mathrm{col}\,(z_1, \ldots, z_{n_u}))$, since the vector z, being the state of the controller, is known. Next, to estimate the derivatives of y, they employ an high-gain observer. Both the works [21] and [19] (the latter dealing with the larger class of stabilizable systems) rely on the knowledge of $\mathcal{H}^{-1}$ to prove closed-loop stability. In addition to this, the work in [1] proves that a separation principle holds for a quite general class of nonlinear systems which includes (1) provided that $\mathcal{H}^{-1}$ is explicitly known and that the system is uniformly completely observable. Sometimes however, even if it exists, $\mathcal{H}^{-1}$ cannot be explicitly calculated (see, e.g., the example in Section V) thus limiting the applicability of existing approaches. Hence, rather than estimating y_e and using $\mathcal{H}^{-1}(\cdot, \cdot)$ to get x, the approach adopted here is to estimate x directly using a nonlinear observer for (1) and using the fact that the z-states are known. The observer has the form[1]

$$\begin{aligned}\dot{\hat{x}} &= \hat{f}(\hat{x}, z, y)\\ &\triangleq f(\hat{x}, z_1) + \left[\frac{\partial \mathcal{H}(\hat{x}, z)}{\partial \hat{x}}\right]^{-1} \mathcal{E}^{-1} L\,[y(t)-\hat{y}(t)]\\ \hat{y}(t) &= h(\hat{x}, z_1)\end{aligned} \tag{7}$$

[1]Throughout this section, we assume A1 to hold globally, since we are interested in the ideal convergence properties of the state estimates. In the next section, we will show how to modify the observer equation in order to achieve the same convergence properties when A1 holds over the set $\mathcal{O} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$.

where L is a $n \times 1$ vector, $\mathcal{E}=\mathrm{diag}\,[\rho, \rho^2, \ldots, \rho^n]$, and $\rho \in (0, 1]$ is a fixed design constant.

Notice that (7) does not require any knowledge of $\mathcal{H}^{-1}$ and has the advantage of operating in x-coordinates. The observability assumption A1 implies that the Jacobian of the mapping $\mathcal{H}$ with respect to x is invertible, and hence the inverse of $\partial\mathcal{H}(\hat{x}, z)/\partial\hat{x}$ in (7) is well defined. In the work [2], the authors used an observer structurally identical to (7) for the more restrictive class of input-output feedback linearizable systems with full relative degree. Here, by modifying the definition of the mapping $\mathcal{H}$ and by introducing a dynamic projection, we considerably relax these conditions by just requiring the general observability assumption A1 to hold. Furthermore, we propose a different proof than the one found in [2] which clarifies the relationship among (7) and the high-gain observers commonly found in the literature.

Theorem 1: Consider system (2) and assume that A1 is satisfied for $\mathcal{O}=\mathbb{R}^{n+n_u}$, the state X belongs to a positively invariant, compact set Ω, $\sup_{t\ge 0} v(t) < \infty$, and that there exists a set $\hat{\Omega}$, $\hat{\Omega} \supset \Omega$, which is positively invariant for $(\hat{x}, z)$ and such that $\{y_e \in \mathbb{R}^n \,|\, (y_e, z) \in \mathcal{F}(\hat{\Omega})\}$ is a compact set. Choose $L=\mathrm{col}\,(l_1, \ldots, l_n)$ such that $s^n + l_1 s^{n-1} + \cdots + l_n$ is Hurwitz.

Under these conditions and using observer (7), the following two properties hold.

i) Asymptotic stability of the estimation error: There exists $\overline{\rho}$, $0<\overline{\rho}\le 1$, such that for all $\hat{x}(0) \in \{\hat{x} \in \mathbb{R}^n \,|\, (\hat{x}, z) \in \hat{\Omega}\}$, and all $\rho \in (0, \overline{\rho})$, $\hat{x}(t) \to x(t)$ as $t \to +\infty$.

ii) Arbitrarily fast rate of convergence: For each positive T, ϵ, there exists ρ^*, $0 < \rho^* \le 1$, such that for all $\rho \in (0, \rho^*]$, $\|\hat{x}(t)-x(t)\| \le \epsilon \;\forall\, t \ge T$.

In Section IV, we show that, by applying to the vector field $\hat{f}$ a suitable dynamic projection onto a fixed compact set, the existence of the compact positively invariant set $\hat{\Omega}$ is guaranteed.

Proof: Consider the smooth coordinate transformation

$$y_e = \mathcal{H}(x, z)$$

which maps (1) to

$$\dot{y}_e = A_c y_e + B_c[\alpha(y_e, z) + \beta(y_e, z)v] \tag{8}$$

where $\alpha(\cdot, \cdot)$ and $\beta(\cdot, \cdot)$ are smooth functions and the pair (A_c, B_c) is in Brunovsky canonical form. Similarly, it is not difficult to show that the coordinate transformation

$$\hat{y}_e = \mathcal{H}(\hat{x}, z)$$

maps the observer dynamics (7) to

$$\dot{\hat{y}}_e = A_c\hat{y}_e + B_c[\alpha(\hat{y}_e, z) + \beta(\hat{y}_e, z)v] + \mathcal{E}^{-1}L[y_{e,1} - \hat{y}_{e,1}]. \tag{9}$$

Define the observer error in the new coordinates, $\tilde{y}_e = \hat{y}_e - y_e$. Then, the observer error dynamics are given by

$$\begin{aligned}\dot{\tilde{y}}_e &= \left(A_c - \mathcal{E}^{-1}LC_c\right)\tilde{y}_e\\ &\quad + B_c\left[\alpha(\hat{y}_e, z) + \beta(\hat{y}_e, z)v - \alpha(y_e, z) - \beta(y_e, z)v\right]\end{aligned} \tag{10}$$

where $C_c=[1, 0, \ldots, 0]$. Note that, in y_e coordinates, the observer (7) is almost identical to the standard high-gain observer used in, e.g., [19] with an *important difference:* while α and β in [19] depend on $\mathrm{sat}(\hat{y}_e)$ [where $\mathrm{sat}(\cdot)$ denotes the saturation function] and, hence, the nonlinear portion of the observer error dynamics is globally bounded when $Y \in \mathcal{F}(\Omega)$, α and β in (10) do not contain saturation and, thus, one cannot guarantee global boundedness of the last term in (10). Since our observer operates in x coordinates, saturation *cannot be induced* in y_e coordinates and therefore, without any further assumption, we cannot prove stability of the origin of (10). The assumption concerning the existence of $\hat{\Omega}$ addresses precisely this issue since it guarantees that $\hat{y}_e(t) = \mathcal{H}(\hat{x}(t), z(t))$ is contained in a compact set for all $t \geq 0$.

Define the coordinate transformation

$$\tilde{\nu} = \mathcal{E}'\tilde{y}_e \quad \mathcal{E}' \triangleq \mathrm{diag}\left[\frac{1}{\rho^{n-1}}, \frac{1}{\rho^{n-2}}, \ldots, 1\right]. \qquad (11)$$

In the new coordinates the observer error dynamics are given by

$$\begin{aligned}\dot{\tilde{\nu}} = &\frac{1}{\rho}(A_c - LC_c)\tilde{\nu} \\ &+ B_c\left[\alpha(\hat{y}_e, z) + \beta(\hat{y}_e, z)v - \alpha(y_e, z) - \beta(y_e, z)v\right] \qquad (12)\end{aligned}$$

where, by assumption, $A_c - LC_c$ is Hurwitz. Let P be the solution to the Lyapunov equation

$$P(A_c - LC_c) + (A_c - LC_c)^\top P = -I \qquad (13)$$

and consider the Lyapunov function candidate $V_o(\tilde{\nu}) = \tilde{\nu}^\top P \tilde{\nu}$. Calculate the time derivative of V_o along the $\tilde{\nu}$ trajectories

$$\begin{aligned}\dot{V}_o = &-\frac{\tilde{\nu}^\top\tilde{\nu}}{\rho} + 2\tilde{\nu}^\top P B_c \\ &\cdot\left[\alpha(\hat{y}_e, z) + \beta(\hat{y}_e, z)v - \alpha(y_e, z) - \beta(y_e, z)v\right]. \qquad (14)\end{aligned}$$

Since $(\hat{x}(t), z(t)) \in \hat{\Omega}$ for all $t \geq 0$, we have that $(y_e(t), z(t)) \in \mathcal{F}(\hat{\Omega})$ and, hence, for all $t \geq 0$, $y_e(t) \in \{y_e \,|\, (y_e, z) \in \mathcal{F}(\hat{\Omega})\}$ which, by assumption, is a compact set. This, together with the fact that, for all $t \geq 0$, $X(t) \in \Omega$ and $\sup_{t\geq 0} v(t) < \infty$, implies that there exists a fixed scalar $\gamma > 0$, independent of ρ, such that the time derivative of V_0 can be bounded as

$$\dot{V}_o \leq -\frac{\|\tilde{\nu}\|^2}{\rho} + 2\|P\|\gamma\|\tilde{y}_e\|\,\|\tilde{\nu}\| \leq -\frac{\|\tilde{\nu}\|^2}{\rho} + 2\|P\|\gamma\|\tilde{\nu}\|^2. \qquad (15)$$

Defining $\overline{\rho} = \min\{1/(2\|P\|\gamma), 1\}$, we conclude that, for all $\rho \in (0, \overline{\rho})$, the origin of (12) [and, hence, also the origin of (10)] is asymptotically stable which, by the smoothness of $\mathcal{H}$, implies that $\hat{x}(t) \to x(t)$ as $t \to \infty$. This concludes the proof of part i).

As for part ii), note that

$$\lambda_{\min}(\mathcal{E}'P\mathcal{E}') \geq \lambda_{\min}(\mathcal{E}')^2\lambda_{\min}(P) = \lambda_{\min}(P)$$

since $\lambda_{\min}(\mathcal{E}') = 1$. Next

$$\lambda_{\max}(\mathcal{E}'P\mathcal{E}') \leq 1/(\rho^{2(n-1)})\lambda_{\max}(P)$$

since $\lambda_{\max}(\mathcal{E}') = 1/\rho^{(n-1)}$. Therefore

$$\lambda_{\min}(P)\|\tilde{y}_e\|^2 \leq \tilde{y}_e^\top\mathcal{E}'P\mathcal{E}'\tilde{y}_e \leq \frac{1}{\rho^{2(n-1)}}\lambda_{\max}(P)\|\tilde{y}_e\|^2.$$

Define $\overline{\epsilon}$ so that $\|\tilde{y}_e\| \leq \overline{\epsilon}$ implies that $\|\hat{x} - x\| \leq \epsilon$ (the smoothness of $\mathcal{H}^{-1}$ guarantees that $\overline{\epsilon}$ is well-defined). By the aforementioned inequality, we have that $V_o \leq \overline{\epsilon}^2\lambda_{\min}(P)$ implies that $\|\tilde{y}_e\| \leq \overline{\epsilon}$, and $V_o(0) \triangleq V_o(\tilde{\nu}(0)) \leq (1/\rho^{2(n-1)})\lambda_{\max}(P)\|\tilde{y}_e(0)\|^2$. Moreover, from (15)

$$\begin{aligned}\dot{V}_o(t) &\leq -\left(\frac{1}{\rho} - 2\|P\|\gamma\right)\|\tilde{\nu}\|^2 \\ &\leq -\frac{1}{\lambda_{\max}(P)}\left(\frac{1}{\rho} - 2\|P\|\gamma\right)V_o(t).\end{aligned}$$

Therefore, by the comparison lemma (see, e.g., [22]), $V_o(t)$ satisfies the following inequality:

$$\begin{aligned}V_o(t) &\leq V_o(0)\exp\left\{-\frac{1}{\lambda_{\max}(P)}\left(\frac{1}{\rho} - 2\|P\|\gamma\right)t\right\} \\ &\leq \frac{\lambda_{\max}(P)}{\rho^{2(n-1)}}\|\tilde{y}_e(0)\|^2\exp\{-t/\lambda_{\max}(P)(\rho^{-1} - 2\|P\|\gamma)\}\end{aligned} \qquad (16)$$

which, for sufficiently small ρ, can be written as

$$V_o(t) \leq \frac{a_1}{\rho^{2n}}\exp\left\{-\frac{a_2}{\rho}t\right\}, \qquad a_1, a_2 > 0.$$

An upper estimate of the time T such that $\|\hat{y}_e(t) - y_e(t)\| \leq \overline{\epsilon}$ (and, thus, $\|\hat{x}(t) - x(t)\| \leq \epsilon$) for all $t \geq T$, is calculated as follows:

$$\frac{a_1}{\rho^{2n}}\exp\left\{-\frac{a_2}{\rho}t\right\} \leq \overline{\epsilon}^2\lambda_{\min}(P)$$

for all $t \geq T = \rho/a_2\ln(a_1/(\overline{\epsilon}^2\rho^{2n}\lambda_{\min}(P)))$. Noticing that $T \to 0$ as $\rho \to 0$, we conclude that T can be made arbitrarily small by choosing a sufficiently small ρ^*, thus concluding the proof of part ii). ■

Remark 4: Using inequality (16), we find the upper bound for the estimation error in y_e-coordinates

$$\begin{aligned}\|\tilde{y}_e\| \leq &\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\,\rho^{-(n-1)}\|\tilde{y}_e(0)\| \\ &\cdot\exp\{-t/(2\lambda_{\max}(P))(\rho^{-1} - 2\|P\|\gamma)\}. \qquad (17)\end{aligned}$$

Hence, during the initial transient, $\tilde{y}_e(t)$ may exhibit peaking, and the size of the peak grows larger as ρ decreases and the convergence rate is made faster. Refer to [18] for more details on the peaking phenomenon and to [3] for a study of its implications on output feedback control. The analysis in the latter paper shows that a way to *isolate* the peaking of the observer estimates from the system states is to saturate the control input outside of the compact set of interest. The same idea has then been adopted in several other works in the output feedback control literature (see, e.g., [3], [9], [19], [12]–[14], [7], and [1]). Rather than following this approach, in the next section we will present a new technique to *eliminate* (rather than just isolate) the peaking phenomenon which allows for the use of the weaker assumption A1.

IV. Output Feedback Stabilizing Control

Consider system (3), by using assumption A2 and Remark 2 we conclude that there exists a smooth stabilizing control $v = \phi(x, z) = \phi(X)$ which makes the origin of (3) an asymptotically stable equilibrium point with domain of attraction $\mathcal{D}$.

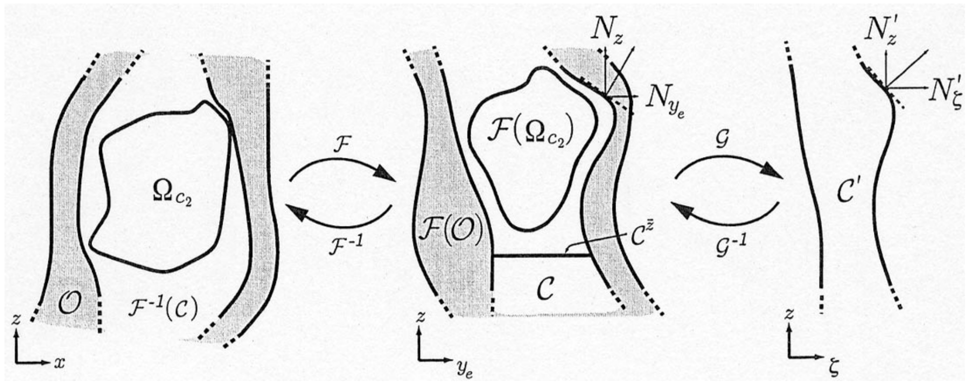

Fig. 1. Mechanism behind the observer estimates projection.

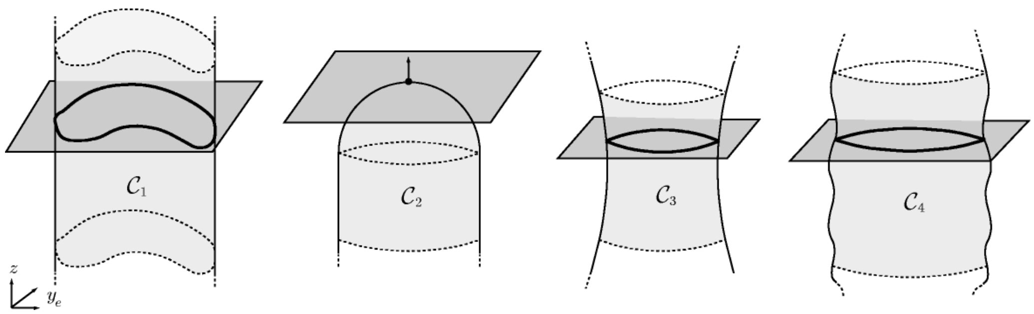

Fig. 2. Domains $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ violate some of the requirements i)–iv) in A3, while $\mathcal{C}_4$ does not.

By the converse Lyapunov theorem found in [11], there exists a continuously differentiable function V defined on $\mathcal{D}$ satisfying, for all $X \in \mathcal{D}$,

$$\alpha_1(\|X\|) \le V(X) \le \alpha_2(\|X\|) \tag{18}$$

$$\lim_{X\to\partial\mathcal{D}} \alpha_1(\|X\|) = \infty \tag{19}$$

$$\frac{\partial V}{\partial X}\,(f_e(X) + g_e v) \le -\alpha_3(\|X\|) \tag{20}$$

where α_i, $i = 1, 2, 3$ are class $\mathcal{K}$ functions (see [8] for a definition), and $\partial\mathcal{D}$ stands for the boundary of the set $\mathcal{D}$. Given any scalar $c > 0$, define

$$\Omega_c \triangleq \{X \in \mathbb{R}^{n+n_u} \mid V \le c\}.$$

Clearly, $\Omega_c \subset \mathcal{D}$ for all $c > 0$ and, from (19), Ω_c becomes arbitrarily close to $\mathcal{D}$ as $c \to \infty$. Next, the following assumption is needed.

Assumption A3 (Topology of $\mathcal{O}$): Assume that there exists a constant $\bar{c} > 0$ and a set $\mathcal{C}$ such that

$$\mathcal{F}(\Omega_{\bar c}) \subset \mathcal{C} \subset \mathcal{Y}\,(=\mathcal{F}(\mathcal{O})) \tag{21}$$

where $\mathcal{C}$ has the following properties.

i) The boundary of $\mathcal{C}$, $\partial\mathcal{C}$, is an $n-1$ dimensional, C^1 submanifold of $\mathbb{R}^n$, i.e., there exists a C^1 function g: $\mathcal{C} \to \mathbb{R}$ such that $\partial\mathcal{C} = \{Y \in \mathcal{C} \mid g(Y) = 0\}$, and $(\partial g/\partial Y)^\top \neq 0$ on $\partial\mathcal{C}$.

ii) $\mathcal{C}^{\bar z} = \{y_e \in \mathbb{R}^n \mid (y_e, \bar z) \in \mathcal{C}\}$ is convex for all $\bar z \in \mathbb{R}^{n_u}$.

iii) 0 is a regular value of $g(\cdot, \bar z)$ for each fixed $\bar z \in \mathbb{R}^{n_u}$, i.e., for all $y_e \in \mathcal{C}^{\bar z}$, $(\partial g/\partial y_e)(y_e, \bar z) \neq 0$.

iv) $\bigcup_{\bar z \in \mathbb{R}^{n_u}} \mathcal{C}^{\bar z}$ is compact.

Remark 5: See Fig. 1 for a pictorial representation of condition (21). This assumption requires *in primis* that there exists a set $\mathcal{C}$ in Y coordinates which contains the image under $\mathcal{F}$ of a level set $\Omega_{\bar c}$ of the Lyapunov function V and is contained in the image under $\mathcal{F}$ of the observable set $\mathcal{O}$. This guarantees in particular that $\Omega_{\bar c} \subset \mathcal{O}$, and thus, when the state feedback controller is employed, the phase curves leaving from $\Omega_{\bar c}$ never exit the observable region $\mathcal{O}$. Furthermore, it is required that $\mathcal{C}$ possesses some basic topological properties: It is asked that the boundary of $\mathcal{C}$ be continuously differentiable [part i)], every slice of $\mathcal{C}$ obtained by holding z constant at $\bar z$, $\mathcal{C}^{\bar z}$, is convex [part ii)], the normal vector to each slice $\mathcal{C}^{\bar z}$ (which is given by $[\partial g/\partial y_e(y_e, \bar z)]^\top$) does not vanish anywhere on the slice [part iii)] and, finally, it is asked that the set $\mathcal{C}$ is compact in the y_e direction [part iv)]. Part iv) can be replaced by the slightly weaker requirement that $\bigcup_{\bar z \in \Omega^z} \mathcal{C}^{\bar z}$ is compact (Ω^z is defined in the next section), with minor changes in the analysis to follow.

To further clarify the topology of the domains under consideration, consider the sets $\mathcal{C}_1$ to $\mathcal{C}_4$ in Fig. 2, corresponding to the case $y_e \in \mathbb{R}^2$, $z \in \mathbb{R}$. While they all satisfy part i), $\mathcal{C}_1$ does not satisfy part ii) since its slices along z are not convex. $\mathcal{C}_2$ satisfies part ii) but violates requirement iii) because the normal vector to one of the slices has no components in the y_e direction. $\mathcal{C}_3$ satisfies parts i)–iii) but violates iv) since the area of its slices grows unbounded as $z \to \infty$. $\mathcal{C}_4$ satisfies all the aforementioned requirements.

Note that, when the plant is UCO (and, thus, $\mathcal{O} = \mathbb{R}^n \times \mathbb{R}^{n_u}$) and $\mathcal{F}(\mathbb{R}^{n+n_u}) = \mathbb{R}^{n+n_u}$, A3 is always satisfied by a sufficiently large set $\mathcal{C}$ and any $\bar c > 0$. In order to see that, pick *any* $\bar c$ and choose $\mathcal{C}$ to be any cylinder $\{Y \in \mathbb{R}^{n+n_u} \mid \|y_e\| \le M\}$, where $M > 0$, containing $\mathcal{F}(\Omega_{\bar c})$. The existence of $\mathcal{C}$ is guaranteed by the fact that the set $\mathcal{F}(\Omega_{\bar c})$ is bounded. More generally, the same holds when $\mathcal{F}(\mathbb{R}^{n+n_u})$ is not all of $\mathbb{R}^{n+n_u}$ and $\mathcal{Y}^{\bar z} \triangleq \mathcal{F}(\mathbb{R}^n, \bar z)$ is convex for all $\bar z \in \mathbb{R}^{n_u}$.

Finally, notice that when the plant in *not* UCO but $\mathcal{O} = \mathcal{X} \times \mathbb{R}^{n_u}$, where $\mathcal{X}$ is an open set which is not all of $\mathbb{R}^n$, and the origin is globally stabilizable (i.e., A2 holds globally), one can choose $\mathcal{C} = D \times \mathbb{R}^{n_u}$, where $D \subset \mathbb{R}^n$ is *any* convex compact set with smooth boundary contained in $\mathcal{X}$ and containing the point $\mathcal{H}(0, 0)$ (i.e., the origin in y_e coordinates). The scalar $\overline{c}$ is then the largest number such that $\mathcal{F}(\Omega_{\overline{c}}) \subset \mathcal{C}$ (notice, however, that $\overline{c}$ does not need to be known for design purposes). Consequently, in the particular case when $n_u = 0$ (and hence the control input does not affect the mapping $\mathcal{H}$) and A2 holds globally, one can choose $\mathcal{C} = D$, where D is defined above.

A. Observer Estimates Projection

As we already pointed out in Remark 4, in order to isolate the peaking phenomenon from the system states, the approach generally adopted in several papers is to saturate the control input to prevent it from growing above a given threshold. This technique, however, does not eliminate the peak in the observer estimate and, hence, cannot be used to control general systems like the ones satisfying assumption A1, since even when the system state lies in the observable region $\mathcal{O} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$, the observer estimates may enter the unobservable domain where (7) is not well-defined. It appears that in order to deal with systems that are not completely observable, one has to eliminate the peaking from the observer by guaranteeing its estimates to be confined in a prespecified observable compact set.

A common procedure used in the adaptive control literature (see [4]) to confine vectors of parameter estimates within a desired convex set is gradient projection. This idea cannot be directly applied to our problem, mainly because $\hat{f}$ is not proportional to the gradient of the observer Lyapunov function and, thus, the projection cannot be guaranteed to preserve the convergence properties of the estimate. Inspired by this idea, however, we propose a way to modify the vector field $\hat{f}$ which confines $\hat{x}$ to within a prespecified compact set while preserving its convergence properties.

Recall the coordinate transformation defined in (4) and let

$$\begin{aligned} \hat{y}_e^P &= \mathcal{H}(\hat{x}^P, z) \quad \tilde{y}_e^P = \hat{y}_e^P - y_e \\ \hat{Y}^P &= \mathcal{F}(\hat{x}^P, z) = \mathrm{col}\,(\hat{y}_e^P, z), \end{aligned} \tag{22}$$

where $\hat{x}^P$ is the state of the *projected* observer defined as[2]

$$\dot{\hat{x}}^P = \begin{cases} \left[\dfrac{\partial \mathcal{H}}{\partial \hat{x}^P}\right]^{-1} \left\{ L_{\hat{F}}\mathcal{H} - \dfrac{\Gamma N_{y_e}(\hat{Y}^P) L_{\hat{G}} g}{N_{y_e}(\hat{Y}^P)^\top \Gamma N_{y_e}(\hat{Y}^P)} - \dfrac{\partial \mathcal{H}}{\partial z}\,\dot{z} \right\}, \\ \qquad\qquad \text{if } L_{\hat{G}} g \geq 0 \text{ and } \hat{Y}^P \in \partial\mathcal{C} \\ \hat{f}(\hat{x}^P, z, y), \qquad \text{otherwise} \end{cases} \tag{23}$$

where

$$N_{y_e}(\hat{Y}^P) = \left[\frac{\partial g}{\partial \hat{y}_e^P}(\hat{y}_e^P, z)\right]^\top \quad N_z(\hat{Y}^P) = \left[\frac{\partial g}{\partial z}(\hat{y}_e^P, z)\right]^\top$$

represent the y_e and z components of the normal vector $N(\hat{Y}^P)$ to the boundary of $\mathcal{C}$ at $\hat{Y}^P$, i.e., $N(\hat{Y}^P) = \mathrm{col}\,(N_{y_e}(\hat{Y}^P), N_z(\hat{Y}^P))$ (the function g is defined in A3). Further, $L_{\hat{F}}\mathcal{H}$ and $L_{\hat{G}}\,g$ are the Lie derivatives of $\hat{y}_e^P$ and $g(\hat{y}_e^P, z)$ along the vector fields $\hat{F} = \mathrm{col}\,(\hat{f}, z_2, \ldots, z_{n_u}, v)$ and $\hat{G} = L_{\hat{F}}\mathcal{F}$, respectively, i.e.,

$$\begin{aligned} L_{\hat{F}}\mathcal{H} &= \frac{\partial \mathcal{H}}{\partial \hat{x}^P}\,\hat{f}(\hat{x}^P, z, y) + \frac{\partial \mathcal{H}}{\partial z}\,\dot{z} \\ L_{\hat{G}} g &= \frac{\partial g}{\partial \hat{y}_e^P}\,L_{\hat{F}}\mathcal{H} + \frac{\partial g}{\partial z}\,\dot{z} = N_{y_e}(\hat{Y}^P)^\top L_{\hat{F}}\mathcal{H} + N_z(\hat{Y}^P)^\top \dot{z} \end{aligned}$$

and $\Gamma = (S\mathcal{E}')^{-1}(S\mathcal{E}')^{-\top}$, where $S = S^\top$ denotes the matrix square root of P [defined in (13)].

The dynamic projection (23) is well-defined since A3, part iii), guarantees that N_{y_e} does not vanish (see also Remark 5). The following lemma shows that (23) guarantees boundedness and preserves convergence of $\hat{x}^P$.

Lemma 1: If A3 holds and (23) is used:

i) Positive invariance of $\mathcal{F}^{-1}(\mathcal{C})$: if $(\hat{x}^P(0), z(0)) \in \mathcal{F}^{-1}(\mathcal{C})$, then $(\hat{x}^P(t), z(t)) \in \mathcal{F}^{-1}(\mathcal{C})$ for all $t \geq 0$.

If, in addition, $(x(t), z(t)) \in \Omega_{\overline{c}}$ for all $t \geq 0$ and the assumptions of Theorem 1 are satisfied, then the following property holds for the integral curves of the projected observer dynamics (7), (23).

ii) Preservation of original convergence characteristics: properties i) and ii) established by Theorem 1 remain valid for $\hat{x}^P$.

Proof: We begin by introducing another coordinate transformation, $\zeta = S\mathcal{E}' y_e$, (similarly, let $\hat{\zeta}^P = S\mathcal{E}'\hat{y}_e^P$, $\tilde{\zeta}^P = S\mathcal{E}'\tilde{y}_e^P$), letting $\mathcal{G} = \mathrm{diag}[S\mathcal{E}', I_{n_u \times n_u}]$, and letting $\mathcal{C}'$ be the image of the set $\mathcal{C}$ under the linear map $\mathcal{G}$, i.e., $\mathcal{C}' \triangleq \{(\zeta, z) \in \mathbb{R}^{n+n_u} \mid \mathcal{G}^{-1}\mathrm{col}\,(\zeta, z) \in \mathcal{C}\}$. Let $N'_\zeta(\zeta, z)$, $N'_z(\zeta, z)$ be the ζ and z components of the normal vector to the boundary of $\mathcal{C}'$. The reader may refer to Fig. 1 for a pictorial representation of the sets under consideration. In order to prove part i) of the Lemma, we have to show that the projection (23) renders $\mathcal{C}$ a positively invariant set for $\hat{Y}^P$. The coordinate transformation (22) maps (23) to

$$\dot{\hat{y}}_e^P = \frac{d}{dt}\{\mathcal{H}(\hat{x}^P, z)\} = \left[\frac{\partial \mathcal{H}}{\partial \hat{x}^P}\,\dot{\hat{x}}^P + \frac{\partial \mathcal{H}}{\partial z}\,\dot{z}\right] \tag{24}$$

$$= \begin{cases} L_{\hat{F}}\mathcal{H} - \Gamma\,\dfrac{N_{y_e}(\hat{Y}^P) L_{\hat{G}} g}{N_{y_e}(\hat{Y}^P)^\top \Gamma N_{y_e}(\hat{Y}^P)} \\ \qquad \text{if } L_{\hat{G}} g \geq 0 \text{ and } \hat{Y}^P \in \partial\mathcal{C} \\ L_{\hat{F}}\mathcal{H} \qquad \text{otherwise.} \end{cases} \tag{25}$$

In order to relate $N'_\zeta(\hat{\zeta}^P, z)$ and $N'_z(\hat{\zeta}^P, z)$ to $N_{y_e}(\hat{y}_e^P, z)$ and $N'_z(\hat{\zeta}^P, z)$, recall from A3 that the boundary of $\mathcal{C}$ is expressed as the set $\partial\mathcal{C} = \{Y \in \mathbb{R}^{n+n_u} \mid g(Y) = 0\}$ and hence the boundary of $\mathcal{C}'$ is the set $\partial\mathcal{C}' = \{\zeta \in \mathbb{R}^n \mid g((S\mathcal{E}')^{-1}\zeta, z) = 0\}$. From this definition we find the expression of N'_ζ and N'_z as

$$\begin{aligned} N'_\zeta(\hat{\zeta}^P, z) &= (S\mathcal{E}')^{-\top}[\partial g(\hat{Y}^P)/\partial y_e^P]^\top = (S\mathcal{E}')^{-\top} N_{y_e}(\hat{Y}^P) \\ N'_z(\hat{\zeta}^P, z) &= N_z(\hat{Y}^P). \end{aligned}$$

[2] The projection defined in (23) is discontinuous in the variable $\hat{y}_e$, therefore raising the issue of the existence and uniqueness of its solutions. We refer the reader to Remark 7, were this issue is addressed and a solution is proposed.

The expression of the projection (23) in ζ coordinates is found by noting that

$$\dot{\hat{\zeta}}^P = S\mathcal{E}'\dot{\hat{y}}_e^P = \begin{cases} S\mathcal{E}'L_{\hat{F}}\mathcal{H} - (S\mathcal{E}')^{-\top}\dfrac{N_{y_e}\left(N_{y_e}^\top L_{\hat{F}}\mathcal{H} + N_z^\top \dot{z}\right)}{N_{y_e}^\top \Gamma N_{y_e}} & \text{if } L_{\hat{G}}g \geq 0 \text{ and } \hat{Y}^P \in \partial\mathcal{C} \\ S\mathcal{E}'L_{\hat{F}}\mathcal{H} & \text{otherwise} \end{cases} \tag{26}$$

and then substituting $N'_\zeta = (S\mathcal{E}')^{-\top}N_{y_e}$, $N'_z = N_z$, and $L_{\hat{F}}\mathcal{H} = (S\mathcal{E}')^{-1}(L_{\hat{F}}S\mathcal{E}'\mathcal{H})$, to find that

$$\dot{\hat{\zeta}}^P = \begin{cases} (L_{\hat{F}}S\mathcal{E}'\mathcal{H}) - \dfrac{N'_\zeta\left({N'_\zeta}^\top(L_{\hat{F}}S\mathcal{E}'\mathcal{H}) + {N'_z}^\top\dot{z}\right)}{{N'_\zeta}^\top N'_\zeta} & \text{if } {N'_\zeta}^\top(L_{\hat{F}}S\mathcal{E}'\mathcal{H}) + {N'_z}^\top\dot{z} \geq 0 \text{ and } (\hat{\zeta}^P, z) \in \partial\mathcal{C}' \\ (L_{\hat{F}}S\mathcal{E}'\mathcal{H}) & \text{otherwise.} \end{cases} \tag{27}$$

Next, we show that the boundary of the domain $\mathcal{C}'$ is positively invariant with respect to (27). In order to do that, consider the continuously differentiable function $V_{\mathcal{C}'} = g((S\mathcal{E}')^{-1}\zeta, z)$ and calculate its time derivative along the vector field of (27) when $(\hat{\zeta}^P, z) \in \partial\mathcal{C}'$

$$\begin{aligned}\dot{V}_{\mathcal{C}'} &= N'_\zeta(\hat{\zeta}^P, z)^\top\dot{\hat{\zeta}}^P + N'_z(\hat{\zeta}^P, z)\dot{z} \\ &= {N'_\zeta}^\top(L_{\hat{F}}S\mathcal{E}'\mathcal{H}) - \frac{{N'_\zeta}^\top N'_\zeta\left({N'_\zeta}^\top(L_{\hat{F}}S\mathcal{E}'\mathcal{H}) + {N'_z}^\top\dot{z}\right)}{{N'_\zeta}^\top N'_\zeta} \\ &\quad + N'_z\dot{z} = 0.\end{aligned}$$

Since $\mathcal{C}'$ is positively invariant, so is $\mathcal{C}$ and, therefore, $\mathcal{F}^{-1}(\mathcal{C})$.

The proof of part ii) is based on the knowledge of a Lyapunov function for the observer in $\tilde{\nu}$ coordinates [see (11)]. Letting $\tilde{\zeta} = S\tilde{\nu}$, we have that, in new coordinates, $V_o = \tilde{\nu}^\top P\tilde{\nu} = (\tilde{\nu}^\top S)(S\tilde{\nu}) = \tilde{\zeta}^\top\tilde{\zeta}$. Now let $V_o^P = \tilde{\zeta}^{P^\top}\tilde{\zeta}^P$ be a Lyapunov function candidate for the projected observer error dynamics in transformed coordinates and recall that, by assumption, $\mathcal{F}(\Omega_{\overline{c}}) \subset \mathcal{C}$ and, thus, $(x, z) \in \Omega_{\overline{c}}$. The latter fact implies that $(y_e, z) \in \mathcal{C}$ or, what is the same, $(\zeta, z) \in \mathcal{C}'$. From (27), when $(\hat{\zeta}^P, z)$ is in the interior of $\mathcal{C}'$, or $(\hat{\zeta}^P, z)$ is on the boundary of $\mathcal{C}'$ and ${N'_\zeta}^\top(L_{\hat{F}}S\mathcal{E}'\mathcal{H}) + {N'_z}^\top\dot{z} < 0$ (i.e., the *unprojected* update is pointed to the interior of $\mathcal{C}'$), we have that $\dot{V}_o^P = \dot{V}_o < 0$. Let us now consider all remaining cases, i.e., $(\hat{\zeta}^P, z) \in \partial\mathcal{C}$ and ${N'_\zeta}^\top(L_{\hat{F}}S\mathcal{E}'\mathcal{H}) + {N'_z}^\top\dot{z} \geq 0$

$$\begin{aligned}\dot{V}_o^P &= 2\tilde{\zeta}^P\dot{\tilde{\zeta}}^P = 2\tilde{\zeta}^{P^\top}\dot{\tilde{\zeta}}^P = 2\tilde{\zeta}^{P^\top} \\ &\quad\cdot\left[(L_{\hat{F}}S\mathcal{E}'\mathcal{H}) - \dot{\zeta} - p(\hat{\zeta}^P, (L_{\hat{F}}S\mathcal{E}'\mathcal{H}), z, \dot{z})N'_\zeta(\hat{\zeta}^P, z)\right] \\ &= \dot{V}_o(\tilde{\zeta}^P) - 2p\tilde{\zeta}^{P^\top}N'_\zeta\end{aligned} \tag{28}$$

where

$$p(\hat{\zeta}^P, (L_{\hat{F}}S\mathcal{E}'\mathcal{H}), z, \dot{z}) = \frac{{N'_\zeta}^\top(L_{\hat{F}}S\mathcal{E}'\mathcal{H}) + {N'_z}^\top\dot{z}}{{N'_\zeta}^\top N'_\zeta}$$

is nonnegative since, by assumption

$${N'_\zeta}^\top(L_{\hat{F}}S\mathcal{E}'\mathcal{H}) + {N'_z}^\top\dot{z} \geq 0.$$

Using the fact that $(\zeta, z) \in \mathcal{C}'$ and that $(\hat{\zeta}^P, z)$ lies on the boundary of $\mathcal{C}'$, we have that the difference vector $(\hat{\zeta}^P - \zeta, 0)$ points outside of $\mathcal{C}'$ or, equivalently, $\hat{\zeta}^P - \zeta$ points outside of the slice ${\mathcal{C}'}^z = \{\zeta \in \mathbb{R}^n \,|\, (\zeta, z) \in \mathcal{C}'\}$. Using the definition of $\mathcal{C}'$, we have that the set ${\mathcal{C}'}^z$ is the image of the convex compact set $\mathcal{C}^z$, defined in A3, under the linear map $S\mathcal{E}'$ and is, therefore, compact and convex as well. Combining these two facts we have that $\tilde{\zeta}^{P^\top}N'_\zeta \geq 0$, thus proving that $\dot{V}_o^P \leq \dot{V}_o < 0$, which concludes the proof of part ii). ■

Remark 6: Part i) of Lemma 1 implies that, with dynamic projection, the requirement in Theorem 1 that there exists a positively invariant set $\hat{\Omega}$ for $(\hat{x}, z)$ is satisfied with $\hat{\Omega} = \mathcal{F}^{-1}(\mathcal{C})$. To see that, note that

$$\mathcal{C} \subset \left(\bigcup_{z\in\mathbb{R}^{n_u}} \mathcal{C}^z\right) \times \mathbb{R}^{n_u}$$

and, thus,

$$\{y_e \in \mathbb{R}^n \,|\, (y_e, z) \in \mathcal{F}(\hat{\Omega})\} = \{y_e \in \mathbb{R}^n \,|\, (y_e, z) \in \mathcal{C}\} \subset \bigcup_{z\in\mathbb{R}^{n_u}} \mathcal{C}^z$$

which, by part iv) in A3, is a compact set.

Remark 7: In order to avoid the discontinuity in the right-hand side of (23) one can employ the smooth projection idea introduced in [15]. In this case, (21) in A3 should be replaced by the following:

$$\mathcal{F}(\Omega_{\overline{c}}) \subset \mathcal{C} \subset \overline{\mathcal{C}} \subset \mathcal{Y} \tag{29}$$

where $\mathcal{C}$ and $\overline{\mathcal{C}}$ satisfy properties i)–iv) in A3 and $\partial\mathcal{C} = \{Y \in \mathcal{C} \,|\, g(Y) = 0\}$, $\partial\overline{\mathcal{C}} = \{Y \in \mathcal{C} \,|\, g(Y) = 1\}$. The ratio $N_{y_e}L_{\hat{G}}g/N_{y_e}^\top\Gamma N_{y_e}$ in (23) should be multiplied by g, and a slight modification of Lemma 1 would show that, by applying the smooth projection to $\hat{f}$, the set $\mathcal{F}^{-1}(\overline{C})$ (rather than the set $\mathcal{C}$) is made positively invariant.

Remark 8: From the proof of Lemma 1, we conclude that (23) confines $(\hat{x}^P, z)$ within the set $\mathcal{F}^{-1}(\mathcal{C})$ which is, in general, unknown since we do not know $\mathcal{H}^{-1}$, and is generally *not* convex (see Fig. 1). It is interesting to note that applying a standard gradient projection for $\hat{x}$ over an arbitrary convex domain does not necessarily preserve the convergence result ii) in Theorem 1.

B. Closed-Loop Stability

To perform output feedback control we replace the state feedback law $v = \phi(x, z)$ with $\hat{v} = \phi(\hat{x}^P, z)$ which, by the smoothness of ϕ and the fact that $\hat{x}^P$ is guaranteed to belong to $\mathcal{H}^{-1}(\mathcal{C})$, is bounded provided that z is confined to within a compact set. In the following we will show that $\hat{v}$ makes the origin of (3) asymptotically stable and that $\Omega_{\underline{c}}$ is contained in its region of attraction, for all $0 < \underline{c} < \overline{c}$.[3] The proof is divided in three steps.

1) (Lemma 2). *Invariance of $\Omega_{\overline{c}}$ and uniform ultimate boundedness:* Using the arbitrarily fast rate of convergence of the observer [see part ii) in Theorem 1], we show that any phase curve leaving from $\Omega_{\underline{c}}$ cannot exit the set $\Omega_{\overline{c}}$ and converges in finite time to an arbitrarily small neighborhood of the origin. Here, Lemma 1 plays an important role, in that it guarantees that the peaking phenomenon is eliminated and thus it does not affect $\hat{v}$, allowing us to use the same idea found in [3] to prove stability.
2) (Lemma 3). *Asymptotic stability of the origin:* By using Lemma 2 and the exponential stability of the observer estimate, we prove that the origin of the closed-loop system is asymptotically stable.
3) (Theorem 2). *Closed-loop stability:* Finally, by putting together the results of Lemma 2 and 3, we conclude the closed-loop stability proof.

The arguments of the proofs are relatively standard (see, e.g., [1] and [19]) and so are sketched.

Lemma 2: For all $\underline{c} \in (0, \overline{c})$, $\epsilon > 0$ and $\mu > 1$ such that

$$d_\epsilon = \alpha_2 \circ \alpha_3^{-1}(\mu A \overline{\gamma} \epsilon) < \overline{c}$$

there exists a number $\overline{\rho} \in (0, 1)$ such that, for all $\rho \in (0, \overline{\rho}]$, for all $(X(0), \hat{x}^P(0)) \in \mathcal{A}$, where

$$\mathcal{A} = \left\{(X, \hat{x}^P) \in \mathbb{R}^{2n+n_u} \mid X \in \Omega_{\underline{c}}, (\hat{x}^P, z) \in \mathcal{F}^{-1}(\mathcal{C})\right\}$$

the phase curves $X(t)$ of the closed-loop system remain confined in $\Omega_{\overline{c}}$, the set $\Omega_{d_\epsilon} \subset \Omega_{\underline{c}}$ is positively invariant, and is reached in finite time.

Sketch of the Proof: Since $(\hat{x}^P(0), z(0)) \in \mathcal{F}^{-1}(\mathcal{C})$, by Lemma 1, part i), $(\hat{x}^P(t), z(t)) \in \mathcal{F}^{-1}(\mathcal{C})$ for all $t \geq 0$. Let $\Omega_{\overline{c}}^z = \{z \in \mathbb{R}^{n_u} \mid (x, z) \in \Omega_{\overline{c}}\}$ and notice that

$$X \in \Omega_{\overline{c}} \quad \text{and} \quad (\hat{x}^P, z) \in \mathcal{F}^{-1}(\mathcal{C}) \implies \hat{x}^P \in \left(\bigcup_{z \in \Omega_{\overline{c}}^z} \mathcal{C}^z\right) \times \Omega_{\overline{c}}^z$$

which is a compact set independent of ρ. Hence, there exists a bounded positive real number D independent of ρ such that, for all $X \in \Omega_{\overline{c}}$ and all $(\hat{x}^P, z) \in \mathcal{F}^{-1}(\mathcal{C})$,

$$\|f_e(X) + g_e \phi(\hat{x}^P, z)\| \leq D.$$

Therefore, for all t such that $X(t) \in \Omega_{\overline{c}}$, $\|X(t) - X(0)\| \leq Dt$, proving that the exit time from the set $\Omega_{\overline{c}}$ has a positive lower bound, $T_1^{X(0)}$, independent of ρ. Further, letting $d = \text{dist}(\Omega_{\overline{c}}, \Omega_{\underline{c}}) = \inf_{X_1 \in \Omega_{\overline{c}}, X_2 \in \Omega_{\underline{c}}} \|X_1 - X_2\|$, it is readily seen that $\inf_{X(0) \in \Omega_{\underline{c}}}(T_1^{X(0)}) = d/D \triangleq T_1$. We have, thus, shown that for any phase curve $X(t)$ leaving from the set $\Omega_{\underline{c}}$ there exists a uniform lower bound T_1 to the exit time from $\Omega_{\overline{c}}$ which is independent of ρ. Choose $T_0 \in (0, T_1)$, let $A = \sup_{X \in \Omega_{\overline{c}}} \|\partial V / \partial X\|$, and $\overline{\gamma}$ be the Lipschtiz constant of ϕ over the compact set

$$\mathcal{F}^{-1}\left(\bigcup_{z \in \Omega_{\overline{c}}^z} \mathcal{C}^z \times \Omega_{\overline{c}}^z\right).$$

By A3, $X(t) \in \Omega_{\underline{c}} \subset \Omega_{\overline{c}}$ implies $X(t) \in \mathcal{F}^{-1}(\mathcal{C})$ and thus from Theorem 1, part ii), and Lemma 1 there exists $\overline{\rho} \in (0, 1)$ such that, for all $\rho \in (0, \overline{\rho}]$, $\|x(t) - \hat{x}^P(t)\| \leq \epsilon$ for all $t \in [T_0, T_1)$. Now, using the function V defined in (18)–(20), a standard Lyapunov analysis shows that, for all $t \in [T_0, T_1)$

$$V \geq d_\epsilon \quad \Longrightarrow \quad \dot{V} \leq -(\mu - 1) A \overline{\gamma} \epsilon$$

which, since $d_\epsilon < \overline{c}$, implies that $\Omega_{\overline{c}}$ is positively invariant ($T_1 = \infty$), and that the phase curves $X(t)$ enter and stay in the set Ω_ϵ in finite time. ■

Remark 9: The use of the projection for the observer estimate plays a crucial role in the proof of Lemma 2. As ρ is made smaller, the observer peak may grow larger, thus generating a large control input, which in turn might drive the system states X outside of $\Omega_{\underline{c}}$ in shorter time. The dynamic projection makes sure that the exit time T_1 is independent of ρ, thus allowing one to choose ϵ independently of T_1.

Lemma 3: Under the assumptions of Lemma 2, there exists a positive scalar ϵ^* such that for all $\epsilon \in (0, \epsilon^*]$ the origin $(X, x - \hat{x}^P) = (0, 0)$ is asymptotically stable.

Sketch of the Proof: Let $\tilde{x}^P = x - \hat{x}^P$. By (17) and Lemma 1, the origin of the $y_e - \hat{y}_e^P$ dynamics is exponentially stable. Recalling that $\tilde{x}^P = \mathcal{H}^{-1}(y_e, z) - \mathcal{H}^{-1}(\hat{y}_e^P, z)$, from the smoothness of $\mathcal{H}$ we conclude that the origin of the $\tilde{x}^P$ dynamics is exponentially stable as well. By the converse Lyapunov theorem there exists a Lyapunov function $V_o'(\tilde{x}^P)$, a positive number r, and positive constants k_1, k_2, k_3 such that, for all $\tilde{x}^P \in B_r = \{\tilde{x}^P \in \mathbb{R}^n \mid \|\tilde{x}^P\| \leq r\}$

$$k_1 \|\tilde{x}^P\|^2 \leq V_o' \leq k_2 \|\tilde{x}^P\|^2$$
$$\dot{V}_o' \leq -k_3 \|\tilde{x}^P\|^2.$$

Choose $\epsilon^* \in (0, r)$ such that $d_{\epsilon^*} = \alpha_2 \circ \alpha_3^{-1}(\mu A \overline{\gamma} \epsilon^*) < \overline{c}$. For any $\epsilon \in (0, \epsilon^*]$, choose $\overline{\rho}$ as in Lemma 2 and pick any $\rho \in (0, \overline{\rho}]$. Then, the following Lyapunov function:

$$V_c(X, \tilde{x}^P) = V(X) + \lambda \sqrt{V_o'(\tilde{x}^P)} \quad \lambda > \frac{2\sqrt{\overline{c}_2} \overline{\gamma} A}{\overline{c}_3}$$

readily allows one to conclude that the origin $(X, \tilde{x}^P) = (0, 0)$ is asymptotically stable. ■

We are now ready to state the following closed-loop stability theorem.

Theorem 2: Suppose that assumptions A1, A2, and A3 are satisfied. Then, for the closed-loop system (3), (7), (23), with control law $\hat{v} = \phi(\hat{x}^P, z)$, there exists a scalar $\rho^* \in (0, 1)$ such

[3] Recall that $\overline{c}$ is a positive constant satisfying A3 and hence its size is constrained by the topology of the observable set $\mathcal{O}$.

that, for all $\rho \in (0, \rho^*]$, the set $\mathcal{A}$ is contained in the region of attraction of the origin $(X, \hat{x}^P) = (0, 0)$.

Sketch of the Proof: By Lemma 3, there exists $\epsilon^* > 0$ such that, for all $\epsilon \in (0, \epsilon^*]$, the origin $(X, \tilde{x}^P) = (0, 0)$ is asymptotically stable. Let Δ_ϵ be its domain of attraction. For any $(X(0), \hat{x}^P(0)) \in \mathcal{A}$, use Lemma 2 to find $\overline{\rho} \in (0, 1)$ so that for all $\rho \in (0, \overline{\rho}]$ the phase curves of the closed-loop system enter Δ_ϵ in finite time. ■

Remark 10: Theorem 2 proves regional stability of the closed-loop system, since given an observability domain $\mathcal{O}$, and provided A3 is satisfied, the control law $\hat{v}$, together with (7) and (23), make the compact set $\mathcal{A}$ an estimate of the domain of attraction for the origin of the closed-loop system. The difference between Theorem 2 and a local stability result lies in the fact that here the domain of attraction for x is at least as large as $\Omega_{\underline{c}}$ and not restricted to be a small *unknown* neighborhood of the origin. Further, the size of $\Omega_{\underline{c}}$ is independent of ρ and thus the domain of attraction *does not* shrink as the rate of convergence of the observer is made faster (that is, when $\rho \to 0$). The next corollary gives conditions for the recovery of the domain of attraction of the state feedback controller by output feedback.

Corollary 1: Assume that A1 is satisfied on $\mathcal{O} = \mathbb{R}^{n+n_u}$, $\mathcal{Y}^{\overline{z}} = \mathcal{F}(\mathbb{R}^n, \overline{z})$ is convex for all $\overline{z} \in \mathbb{R}^{n_u}$, and A2 holds. Then, given *any* set $\mathcal{D}' \subset \mathrm{int}(\mathcal{D})$, there exists a scalar $\rho^* \in (0, 1)$ such that, for all $\rho \in (0, \rho^*]$, the set $\mathcal{D}'$ is contained in the region of attraction of the origin $X = 0$. Moreover, if $\mathcal{D} = \mathbb{R}^{n+n_u}$, the origin $X = 0$ is semiglobally asymptotically stable.

Proof: By property (19) of the Lyapunov function $V(X)$, given any set $\mathcal{D}' \subset \mathrm{int}(\mathcal{D})$, there exists $\underline{c} > 0$ such that $\mathcal{D}' \subset \Omega_{\underline{c}}$. By Remark 5, if $\mathcal{O} = \mathbb{R}^{n+n_u}$, A3 is satisfied by *any* $\overline{c} \in (0, \infty)$ and a sufficiently large $\mathcal{C}$. Choose $\overline{c} > \underline{c}$. By Theorem 2 we conclude that there exists $\rho^* \in (0, 1)$ such that, for all $\rho \in (0, \rho^*]$, the origin of the closed-loop system is asymptotically stable and the set $\mathcal{A} = \{(X, \hat{x}^P) \in \mathbb{R}^{2n+n_u} \mid X \in \Omega_{\underline{c}}, (\hat{x}^P, z) \in \mathcal{F}^{-1}(\mathcal{C})\}$ is contained in its domain of attraction. Thus, in particular, $\mathcal{D}'$ is contained in the domain of attraction of $X = 0$. If $\mathcal{D} = \mathbb{R}^{n+n_u}$ then $\mathcal{D}'$ can be chosen to be any compact set and thus the origin $X = 0$ is semiglobally asymptotically stable. ■

Remark 11: A drawback of the result in Theorem 2, shared by the works in [21], [19], and [1], is that separation is achieved between the observer design and the state feedback control design for the *augmented* system (3). In order to avoid increased complexity of the controller, it would be more desirable to achieve separation between the observer design and the state feedback control design for the *original* system (1).

Remark 12: As mentioned in Remark 5, if the plant is UCO (and hence $\mathcal{O} = \mathbb{R}^n \times \mathbb{R}^{n_u}$) and $\mathcal{Y}^{\overline{z}} = \mathcal{F}(\mathbb{R}^n, \overline{z})$ is a convex set for all $\overline{z} \in \mathbb{R}^{n_u}$, assumption A3 is automatically satisfied by a sufficiently large cylindrical set $\mathcal{C}$. Even in this case, if $\mathcal{H}^{-1}$ is not explicitly known and one wants to directly estimate the state of the plant, one should employ the dynamic projection (23) since the standard saturation used, e.g., in [3] and [19] can only be applied to a high-gain observer in y_e coordinates. Clearly, the only instance when dynamic projection can be replaced by saturation of the observer estimates is when the observability mapping is the identity.

V. Example

Consider the following input–output linearizable system:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= (1 + x_1)\exp(x_1^2) + u - 1\\ y &= (x_2 - 1)^2.\end{aligned} \tag{30}$$

The control input appears in the first derivative of the output

$$\dot{y} = 2(x_2 - 1)(1 + x_1)\exp(x_1^2) + 2(x_2 - 1)(u - 1).$$

Notice, however, that the coefficient multiplying u vanishes when $x_2 = 1$, and hence system (30) does not have a well-defined relative degree everywhere. Since u appears in $\dot{y}$, we have that $n_u = 1$, therefore, we add one integrator at the input side

$$\dot{z}_1 = v \quad u = z_1. \tag{31}$$

The mapping $\mathcal{F}$ is given by

$$\begin{aligned}Y = \begin{bmatrix} y\\ \dot{y}\\ z_1\end{bmatrix} &= \mathcal{F}(x, z_1) = \begin{bmatrix}\mathcal{H}(x, z_1)\\ z_1\end{bmatrix}\\ &= \begin{bmatrix}(x_2 - 1)^2\\ 2(x_2 - 1)[(1 + x_1)\exp(x_1^2) + (z_1 - 1)]\\ z_1\end{bmatrix}.\end{aligned} \tag{32}$$

The first equation in (32) is invertible for all $x_2 < 1$, and its inverse is given by $x_2 = 1 - \sqrt{y}$. Substituting x_2 into the second equation in (32) and isolating the term in x_1, we get

$$(1 + x_1)\exp(x_1^2) = \frac{\dot{y} + 2\sqrt{y}(z_1 - 1)}{-2\sqrt{y}}. \tag{33}$$

Since $(1 + x_1)\exp(x_1^2)$ is a strictly increasing function, it follows that (33) is invertible for all $x_1 \in \mathbb{R}$, however, an analytical solution to this equation cannot be found. In conclusion, Assumption A1 is satisfied on the domain $\mathcal{O} = \{x \in \mathbb{R}^2 \mid x_2 < 1\} \times \mathbb{R}$, but an explicit inverse $(x, z_1) = \mathcal{F}^{-1}(y_e, z_1)$ is not known. The fact that system (30) is not UCO, together with the nonexistence of an explicit inverse to (32), prevents the application of the output feedback control approaches in [3], [21], [9], [19], [12], [13], [7], [14], and [1].

To find a stabilizing state feedback controller, note that the extended system (30), (31) can be feedback linearized by letting $x_3 = (1 + x_1)\exp(x_1^2) + z_1 - 1$ and rewriting the system in new coordinates $x_e \triangleq \mathrm{col}\,(x_1, x_2, x_3)$:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= x_2\exp(x_1^2)(2x_1^2 + 2x_1 + 1) + v.\end{aligned} \tag{34}$$

Choose $v = -x_2\exp(x_1^2)(2x_1^2 + 2x_1 + 1) - Kx_e$, where $K = [1, 3, 3]$, so that the closed-loop system becomes $\dot{x}_e = (A_c - B_cK)\,x_e$ with poles placed at -1. Then, the origin $x_e = 0$ is a globally asymptotically equilibrium point of (34), and Assumption A2 is satisfied with $\mathcal{D} = \mathbb{R}^3$. Let $\overline{P}$ be the solution of the Lyapunov equation associated to $A_c - B_cK$, so that a

Lyapunov function for system (34) is $V = x_e^\top \overline{P} x_e$, and any set $\Omega_c \triangleq \{x_e \in \mathbb{R}^3 \,|\, V(x_e) \leq c\}$, with $c > 0$, is contained in the region of attraction for the origin. Equation (35) is shown at the bottom of the page.

Next, we seek to find a set $\mathcal{C}$ satisfying Assumption A3. To this end, notice that

$$\begin{aligned}\mathcal{Y} &= \mathcal{F}(\mathcal{O}) = \mathcal{F}\left(\{x \in \mathbb{R}^2,\, z_1 \in \mathbb{R} \,|\, x_2 < 1\}\right)\\ &= \{\mathbb{R}^+ - 0\} \times \mathbb{R} \times \mathbb{R}.\end{aligned}$$

Next recall from Remark 5 that, since A2 holds globally, $\mathcal{C}$ can be chosen to be the cylinder $D \times \mathbb{R}$, where D is any compact convex set in the upper half plane $\{\mathbb{R}^+ - 0\} \times \mathbb{R}$ containing the point $\mathcal{H}(0, 0) = (1, 0)$. For the sake of simplicity choose D to be the disk of radius $\omega < 1$ centered at $(1, 0)$, so that $\mathcal{C} = \{Y \in \mathbb{R}^3 \,|\, (Y_1 - 1)^2 + Y_2^2 < \omega^2\}$, and $C \subset \mathcal{Y}$, as depicted in Fig. 3. We stress that our choice is quite conservative and is made exclusively for the sake of illustration. Once $\mathcal{C}$ has been chosen, the control design is complete and the output feedback controller is given by

$$\begin{aligned}\dot{z}_1 &= -\hat{x}_2^P \exp\left\{(\hat{x}_1^P)^2\right\}\left[2(\hat{x}_1^P)^2 + 2\hat{x}_1^P + 1\right] - \hat{x}_1^P - 3\hat{x}_2^P\\ &\quad -3\left[(1 + \hat{x}_1^P)\exp\left\{(\hat{x}_1^P)^2\right\} + z_1 - 1\right] \qquad (36)\end{aligned}$$

where $\hat{x}^P(t)$ is the solution of (23) with $\hat{f}(\hat{x}, z, y)$ defined in (35) and

$$\frac{\partial \mathcal{H}}{\partial z} = \begin{bmatrix} 0 \\ 2(\hat{x}_2^P - 1)\end{bmatrix} \quad N_{y_e}(\hat{y}_e^P, z) = \hat{y}_e^P - \begin{bmatrix}1\\0\end{bmatrix}$$

$$N_z(\hat{y}_e^P, z) = 0$$

$$\hat{y}_e^P = \begin{bmatrix} (\hat{x}_2^P - 1)^2 \\ 2(\hat{x}_2^P - 1)[(1 + \hat{x}_1^P)\exp((\hat{x}_1^P)^2) + (z_1 - 1)]\end{bmatrix}$$

$$L_{\hat{F}}\mathcal{H} = \frac{\partial \mathcal{H}}{\partial \hat{x}^P}\hat{f}(\hat{x}^P, z, y) + \frac{\partial \mathcal{H}}{\partial z}\dot{z}. \qquad (37)$$

Using controller (37), Theorem 2 guarantees that the origin $x_e = 0$ of the closed-loop system is asymptotically stable and it provides an estimate of its domain of attraction. Specifically, given any positive scalar $\underline{c} < \overline{c}$, there exists $\rho^* > 0$ such that $\Omega_{\underline{c}}$ is contained in the domain of attraction for all $0 < \rho < \rho^*$. In what follows we will find the set $\Omega_{\overline{c}}$ satisfying A3. Recalling that, in x_e-coordinates, $\Omega_{\overline{c}}$ is expressed as $\{x_e \in \mathbb{R}^2 \,|\, x_e^\top \overline{P} x_e\}$, we have that $x_e \in \Omega_{\overline{c}}$ implies $|x_i| \leq (\overline{c}/\lambda_{\min}(\overline{P}))$, $i = 1, 2, 3$, and hence $\Omega_{\overline{c}} \subset \Xi \triangleq \{x_e \in \mathbb{R}^3 \,|\, |x_i| \leq (\overline{c}/\lambda_{\min}(\overline{P})),\, i = 1, 2, 3\}$. Now

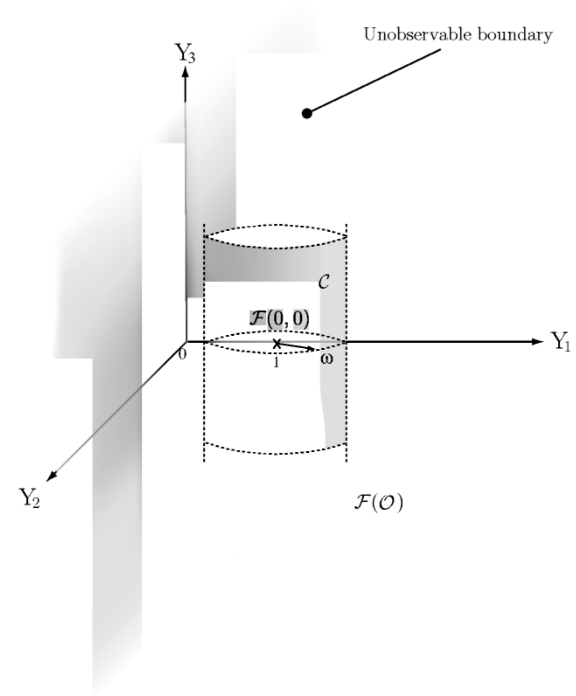

Fig. 3. Projection domain $\mathcal{C}$.

let $\overline{c} < \lambda_{\min}(\overline{P})$ and note that, for all $x_e \in y_e$,

$$\left(1 - \frac{\overline{c}}{\lambda_{\min}(\overline{P})}\right)^2 \leq |x_2 - 1|^2 \leq \left(1 + \frac{\overline{c}}{\lambda_{\min}(\overline{P})}\right)^2$$

$$|2(x_2 - 1)x_3| \leq 2\left(1 + \frac{\overline{c}}{\lambda_{\min}(\overline{P})}\right)\frac{\overline{c}}{\lambda_{\min}(\overline{P})}.$$

Next, we seek to find $\overline{c}$ such that $\mathcal{F}(\Omega_{\overline{c}}) \subset \mathcal{C}$. Using the inequalities above we have that if $\overline{c} = \min\{c^*, \lambda_{\min}(\overline{P})\}$, where c^* is the largest scalar satisfying

$$\left[\left(1 + \frac{c^*}{\lambda_{\min}(\overline{P})}\right)^2 - 1\right]^2 + 4\left(1 + \frac{c^*}{\lambda_{\min}(\overline{P})}\right)^2\left(\frac{\overline{c}_2}{\lambda_{\min}(\overline{P})}\right)^2 < \omega^2 \qquad (38)$$

(note that $c^* > 0$ satisfying the inequality above always exists) then $\mathcal{F}(\Omega_{\overline{c}}) \subset \mathcal{F}(\Xi) \subset \mathcal{C}$ and, hence, $\overline{c}$ satisfies A3.

For our simulations we choose $\omega = 0.9$ and, from (38), we get $\overline{c} = 0.06$. The initial condition of the extended system is set to $x_1(0) = 0.05$, $x_2(0) = 0.07$, $x_3(0) = 0.1$ [or $z_1(0) = 0.0474$], which is contained inside $\Omega_{\overline{c}}$ so that Theorem 2 can be applied. Finally, we choose the observer gain L to be $\mathrm{col}\,(4, 4)$,

$$\begin{aligned}\hat{f}(\hat{x}, z, y) &= \begin{bmatrix}\hat{x}_2 \\ (1 + \hat{x}_1)\exp\left\{(\hat{x}_1)^2\right\} + z_1 - 1\end{bmatrix}\\ &+ \begin{bmatrix} 0 & 2(\hat{x}_2 - 1) \\ 2(\hat{x}_2 - 1)\exp\left\{(\hat{x}_1)^2\right\}\left[2(\hat{x}_1)^2 + 2\hat{x}_1 + 1\right] & 2\left[(1 + \hat{x}_1)\exp\left\{(\hat{x}_1)^2\right\} + z_1 - 1\right]\end{bmatrix}^{-1}\mathcal{E}^{-1}L[y - (\hat{x}_2 - 1)^2]. \qquad (35)\end{aligned}$$

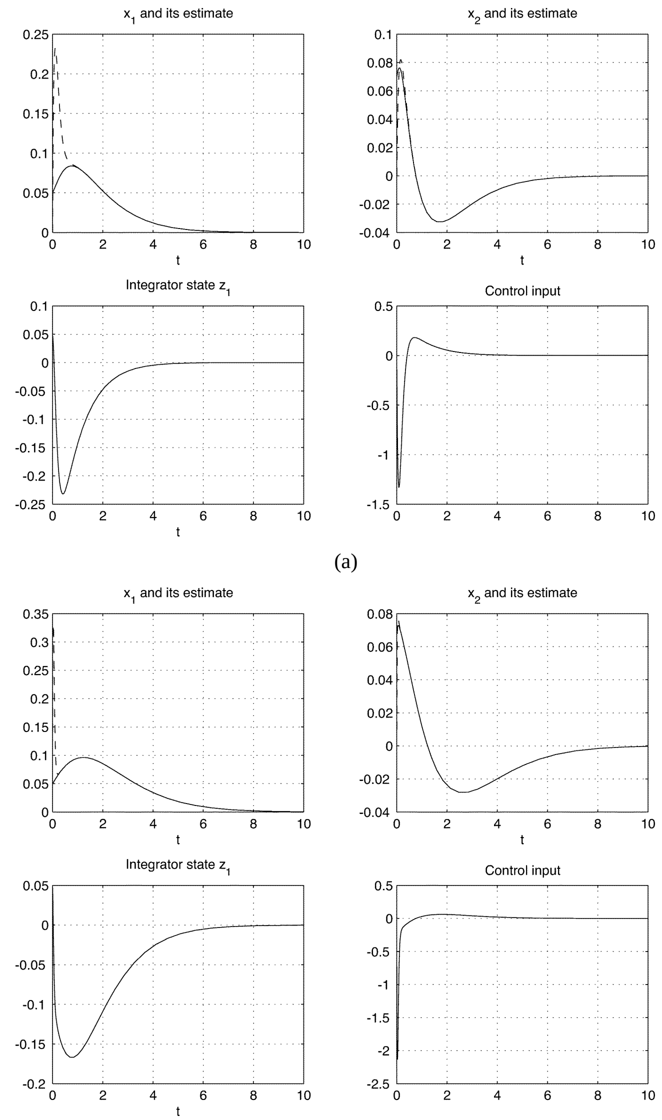

Fig. 4. Integral curve of the closed-loop system under output feedback. (a) $\rho = 0.2$. (b) $\rho = 0.05$.

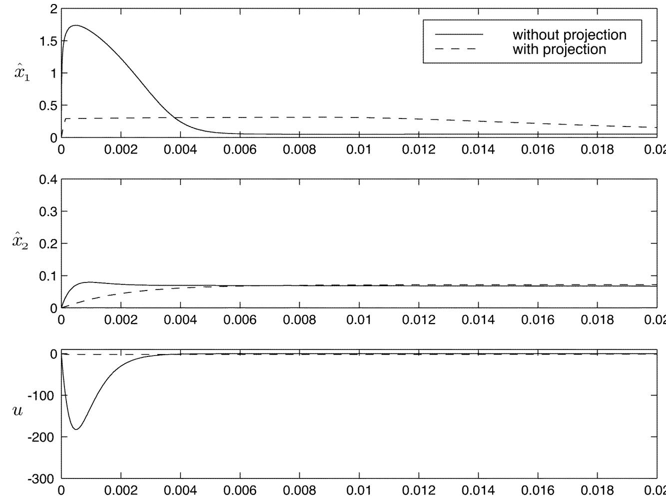

Fig. 5. Observer states during the initial peaking phase with and without projection, $\rho = 10^{-3}$.

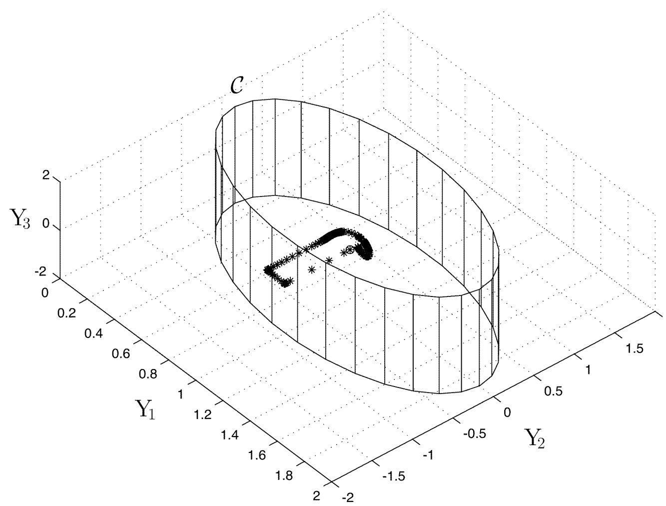

Fig. 6. Phase curve of the observer with dynamic projection in the transformed domain $Y = \mathcal{F}(X)$.

so that its associated polynomial is Hurwitz with both poles placed at −2. We present four different situations to illustrate four features of our output feedback controller.

1) **Arbitrary fast rate of convergence of the observer**. Fig. 4 shows the evolution of the integral curve $X(t)$, as well as the control input v, for $\rho = 0.2$ and $\rho = 0.05$. The convergence in the latter case is faster, as predicted by Theorem 1 (see Remark 4).
2) **Observer estimate projection**. Fig. 5 shows the evolution of $\hat{x}$ and v for $\rho = 10^{-3}$ with and without projection. The dynamic projection successfully eliminates the peak in the observer states, thus yielding a bounded control input, as predicted by the result of Lemma 1. Fig. 6 shows that the phase curve $Y(t) = \mathcal{F}(X(t))$ is contained within the set $\mathcal{C}$ for all $t \geq 0$, confirming the result of Lemma 1. In particular, Fig. 6 shows the operation of the projection when the phase curve of the observer (in Y coordinates) hits the boundary of $\mathcal{C}$: it forces $\hat{Y}^P(t)$ to "slide" along the boundary of $\mathcal{C}$ and preserves its convergence characteristics. This is equivalent, in the x domain, to saying that the phase curve $(\hat{x}^P(t), z_1(t))$ slides along the boundary of $\mathcal{F}^{-1}(\mathcal{C})$ and $\hat{x}^P(t)$ converges to $x(t)$.
3) **Observer estimate projection and closed-loop stability**. In Fig. 7 a phase plane plot for $x(t)$ is shown with and without observer projection when $\rho = 10^{-4}$. The small value of ρ generates a significant peak which, if projection is not employed, drives the phase curve of the output feedback system away form that of the state feedback system and, in general, may drive the system to instability (see Remark 9). On the other hand, using dynamic projection, the phase curve of the output feedback system is almost indistinguishable from the phase curve of the state feedback system.
4) **Trajectory recovery**. The evolution of the phase curve $X(t)$ for decreasing values of ρ, in Fig. 8, shows that the phase curve of the output feedback system approaches the phase curve of the state feedback one as $\rho \to 0$ (see Remark 11).

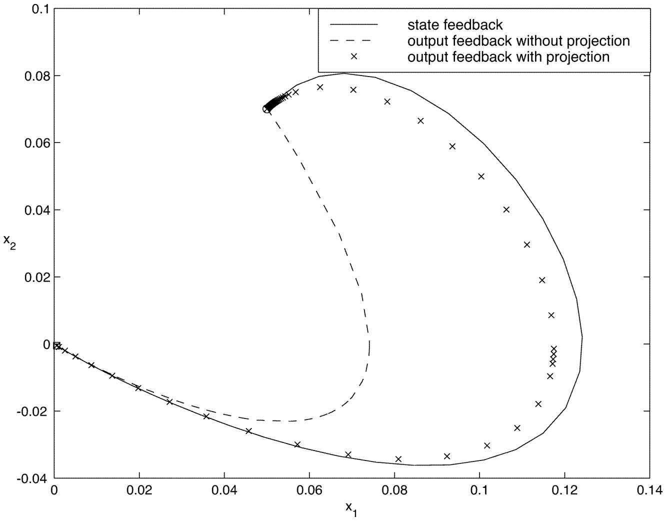

Fig. 7. Phase curve $x(t)$ with and without projection, $\rho = 10^{-3}$.

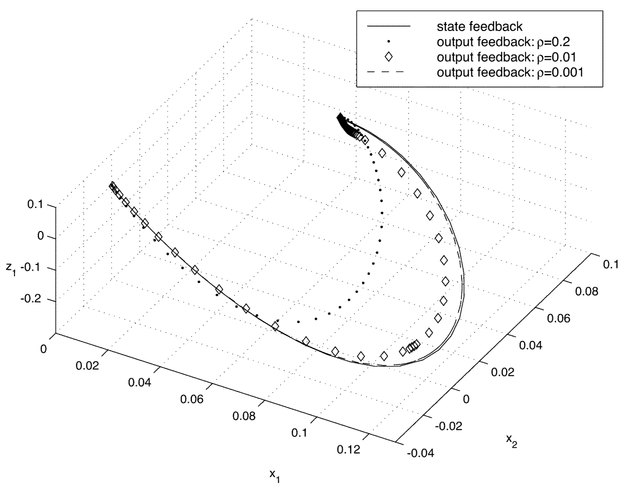

Fig. 8. Phase curve $X(t)$ of the closed-loop system for decreasing values of ρ.

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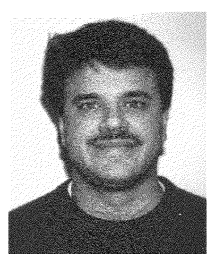

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