Lagrange Stability and Boundedness of Discrete Event Systems*

Kevin M. Passino † and Kevin L. Burgess
Dept. of Electrical Engineering
The Ohio State University
2015 Neil Ave.
Columbus, OH 43210-1272

Anthony N. Michel ‡
Dept. of Electrical Engineering
University of Notre Dame
Notre Dame, IN 46556

Abstract

Recently it has been shown that the conventional notions of stability in the sense of Lyapunov and asymptotic stability can be used to characterize the stability properties of a class of "logical" discrete event systems (DES). Moreover, it has been shown that stability analysis via the choice of appropriate Lyapunov functions can be used for DES and can be applied to several DES applications including manufacturing systems and computer networks [1, 2]. In this paper we extend the conventional notions and analysis of uniform boundedness, uniform ultimate boundedness, practical stability, finite time stability, and Lagrange stability so that they apply to the class of logical DES that can be defined on a metric space. Within this stability-theoretic framework we show that the standard Petri net-theoretic notions of boundedness are special cases of Lagrange stability and uniform boundedness. In addition we show that the Petri net-theoretic approach to boundedness analysis is actually a Lyapunov approach in that the net-theoretic analysis actually produces an appropriate Lyapunov function. Moreover, via the Lyapunov approach we provide a sufficient condition for the uniform ultimate boundedness of General Petri nets. To illustrate the Petri net results, we study the boundedness properties of a rate synchronization network for manufacturing systems. In addition, we provide a detailed analysis of the Lagrange stability of a single-machine manufacturing system that uses a priority-based part servicing policy.

Keywords: Discrete Event Systems, Stability, Boundedness, Petri Nets, Manufacturing Systems

1 Introduction

Discrete event systems (DES) are dynamical systems which evolve in time by the occurrence of events at possibly irregular time intervals. "Logical" DES are a class of discrete time asynchronous

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DES with equations of motion that are most often non-linear and discontinuous with respect to the random occurrence of events. Recently there has been much interest in the characterization of the stability properties of logical DES and in [1, 3, 4] the authors show how to adapt the metric space approach to Lyapunov stability analysis in [5] so that a wide class of DES can be analyzed with this conventional approach. In fact in [1, 3, 4] the authors showed that the standard Lyapunov method via the choosing of appropriate Lyapunov functions can be applied to several particular classes of DES applications including manufacturing systems and computer networks. More recently, a detailed analysis of load balancing systems has been conducted in [2, 6] using the framework in [1, 3, 4]. Earlier work on the analysis of boundedness properties of DES is contained in [7].

In this paper we extend the conventional Lyapunov framework so that it applies to the study of uniform boundedness, uniform ultimate boundedness, practical stability, finite time stability, and Lagrange stability of the class of logical DES that can be defined on a metric space. As mentioned above, Lyapunov concepts have been already been studied on a metric space (see, e.g., [5] for an introductory treatment, [8] for more advanced studies of stability preserving mappings on metric spaces and their applications, and [9] for more recent work on the use of a metric space Lyapunov approach for interconnected systems). There have also been studies of stability for more general topological spaces (see, e.g., [10]). In addition, there have been studies of stability for automata (see, e.g., [11]) and in temporal logic systems (see, e.g., [12]); for an overview of other DES-theoretic work along these lines see [1]. This paper shows how to perform stability and boundedness analysis of logical DES that are defined on a metric space. Logical DES that can be defined on a metric space include Petri nets [13, 14], Vector DES [15, 16], and many applications (see, e.g., [2]). The DES to be studied, such as Petri nets, do not enjoy having a state space that is a vector space so that the general stability formulations in, for instance, [17, 18, 19, 20] for normed linear spaces do not directly apply. Moreover, the logical DES to be considered here are inherently asynchronous and at each state there may be up to an infinite number of events that can occur and hence there can possibly be an infinite number of next states. Hence, the approach in [11] (and similar automata-theoretic approahes) does not apply since it is for finite systems. In addition, the standard formulations in [8, 17, 18, 19, 20] do not directly apply since for them it is assumed that at each state there is a unique next state (for DES there is often non-deterministic behavior that results in uncertainty about what the next state is).

The results in Section 3 show that via relatively straightforward extensions, the Lyapunov analysis of uniform boundedness extends to the study of logical DES (Theorem 1). For uniform ultimate boundedness extensions must be made to ensure sufficient conditions that will be more generally applicable to logical DES (Theorem 2 and Corollary 1). Perhaps most importantly (due to its wide variety of possible applications), in Section 3 we show that the analysis of practical stability, finite time stability [21, 22], and Lagrange stability can be extended to include logical DES (Theorems 3 and 4 and Corollary 2).

We also investigate several applications of the theory of stability and boundedness for logical DES that is introduced in Section 3. For instance, in Section 4 we show how the standard notions of boundedness in Petri net theory are really special cases of the conventional notions of stability and boundedness in Section 3. In Theorem 5 we show that the standard approach to the analysis of "structural boundedness" for General Petri nets [13] is equivalent to a Lyapunov approach where an appropriate Lyapunov function is chosen. In addition, in Theorem 5 we introduce the notion of uniform ultimate boundedness for General Petri nets and using the Lyapunov approach provide sufficient conditions for uniform ultimate boundedness of General Petri nets. Finally, to illustrate the Petri net results we analyze a rate synchronization network for manufacturing systems.

In Section 5 we provide a detailed investgation into the Lagrange stability of a single-machine manufacturing system that uses a priority-based part servicing policy. The investigation was motivated by the work of Perkins and Kumar [23] and Lu and Kumar [24], but we conduct our studies in the stability framework established in Section 3 and investigate stability properties of a new scheduling policy. Although the priority-based policy that we study can be expected to be less efficient than, e.g., the Clear-a-Fraction or Clear-Largest-Buffer policies in [23], practical considerations in manufacturing systems (e.g., constraints due to the ordering of how parts must be processed) often dictate the use of the type of priority-based policy that we study. Hence, our manufacturing system appplication serves to illustrate the utility of the stability framework of Section 3 and provides a result that can be practically useful (See Theorem 6). Finally, we note that some concluding remarks are provided in Section 6.

2 A Discrete Event System Model

We study the stability of systems that can be accurately modeled with

$$G = (\mathcal{X},\ \mathcal{E},\ f_e,\ g,\ \boldsymbol{E}_v). \tag{1}$$

$\mathcal{X}$ is the set of states and $\mathcal{E}$ is the set of events. State transitions are defined by the operators, $f_e : \mathcal{X} \longrightarrow \mathcal{X}$ where $e \in \mathcal{E}$. An event, e, may only occur if it is in the set defined by the enable function, $g : \mathcal{X} \longrightarrow \mathcal{P}(\mathcal{E}) - \{\emptyset\}$, where $\mathcal{P}(\mathcal{E})$ denotes the power set of $\mathcal{E}$. We only require that f_e be defined when $e \in g(\boldsymbol{x})$. Notice that according to the definition of g, it can never be the case that no event is enabled. We can, however, model deadlock by defining a null event, e^0, so that

$f_{e^0}(\bar{\boldsymbol{x}}) = \bar{\boldsymbol{x}}$ where $\bar{\boldsymbol{x}} \in \mathcal{X}$ is the state that the system is deadlocked at.

We associate "logical time" indices with the states and events so that $\boldsymbol{x}_k \in \mathcal{X}$ represents the state at time $k \in \{0, 1, 2, \ldots\} = \mathbb{N}$ (the set of natural numbers) and $e_k \in g(\boldsymbol{x}_k)$ represents an *enabled* event at time $k \in \mathbb{N}$. Notice that there can be just one state at time k, but that many events may be enabled at time k. Should an enabled event e_k occur, then the next state, $\boldsymbol{x}_{k+1}$ is defined by $\boldsymbol{x}_{k+1} = f_{e_k}(\boldsymbol{x}_k)$.

We now define *state trajectories* and *event trajectories*. A state trajectory is any sequence $\{\boldsymbol{x}_k\} \in \mathcal{X}^{\mathbb{N}}$ such that $\boldsymbol{x}_{k+1} = f_{e_k}(\boldsymbol{x}_k)$ for some $e_k \in g(\boldsymbol{x}_k)$ for all $k \in \mathbb{N}$. An event trajectory is any sequence $\{e_k\} \in \mathcal{E}^{\mathbb{N}}$ such that there exists a state trajectory, $\{\boldsymbol{x}_k\} \in \mathcal{X}^{\mathbb{N}}$, where for every $k \in \mathbb{N}$, $e_k \in g(\boldsymbol{x}_k)$. The set of all such event trajectories is denoted by $\boldsymbol{E} \subset \mathcal{E}^{\mathbb{N}}$. Notice that corresponding to a given event trajectory, there can be only one state trajectory. In general, however, an event trajectory that produces a given state trajectory is not unique. Notice that all state and event trajectories must be infinite sequences.

Let $\boldsymbol{E}_v \subset \boldsymbol{E}$ denote a set of what we call "valid" event trajectories that we assume is specified as part of the modeling process. Let $\boldsymbol{E}_v(\boldsymbol{x}_0)$ be the set of valid event trajectories when the initial state is $\boldsymbol{x}_0 \in \mathcal{X}$. The framework provides another mechanism for further pruning $\boldsymbol{E}$. $\boldsymbol{E}_a \subset \boldsymbol{E}_v$ is the set of what we call "allowed" event trajectories. Including $\boldsymbol{E}_a$ in our model yields a great deal of modeling power. In particular, we will make use of $\boldsymbol{E}_a$ to model the decision-making policies which we impose on our systems.

If we fix $k \in \mathbb{N}$, then E_k denotes the sequence of events $e_0, e_1, \ldots, e_{k-1}$, and the $E_k E \in \boldsymbol{E}_v(\boldsymbol{x}_0)$ is used to denote the concatenation of E_k with a sequence of infinite length $E = e_k, e_{k+1}, \ldots$ such that $E_k E \in \boldsymbol{E}_v$ ($E_0 = \emptyset$, the string with no elements in it which we also use to denote the empty set). If E is a string then $|E|$ denotes the length of the string (i.e., the number of elements in the string). Let $\boldsymbol{E}_v^f = \{E' : E'E \in \boldsymbol{E}_v, |E'| < \infty\}$ (i.e., the set of all finite length valid event trajectories). Let $X : \mathcal{X} \times \boldsymbol{E}_v^f \times \mathbb{N} \to \mathcal{X}$. The value of the function $X(\boldsymbol{x}_0, E_k, k)$ will be used to denote the state reached at time k from $\boldsymbol{x}_0 \in \mathcal{X}$ by application of event sequence E_k such that $E_k E \in \boldsymbol{E}_v$. For fixed $\boldsymbol{x}_0$, the functions $X(\boldsymbol{x}_0, E_k, k)$, where $E_k E \in \boldsymbol{E}_v(\boldsymbol{x}_0)$, are called *motions*.

3 Sufficient Conditions for Stability and Boundedness of DES in a Metric Space

Let $\rho : \mathcal{X} \times \mathcal{X}$ denote a *metric* on $\mathcal{X}$, and $\{\mathcal{X}; \rho\}$ a metric space. Let $\mathcal{X}_z \subset \mathcal{X}$ and $\rho(\boldsymbol{x}, \mathcal{X}_z) = \inf\{\rho(\boldsymbol{x}, \boldsymbol{x}') : \boldsymbol{x}' \in \mathcal{X}_z\}$ denote the distance from point $\boldsymbol{x}$ to the set $\mathcal{X}_z$. The *r-neighborhood* of an arbitrary set $\mathcal{X}_z \subset \mathcal{X}$ is denoted by the set $S(\mathcal{X}_z; r) = \{\boldsymbol{x} : 0 < \rho(\boldsymbol{x}, \mathcal{X}_z) < r\}$ where $r > 0$.

Also, let $\bar{S}(\mathcal{X}_z; R) = \{\boldsymbol{x} \in \mathcal{X} : \ \rho(\boldsymbol{x}, \mathcal{X}_z) \geq R\}$. Let $\Re^+$ denote the nonnegative reals. A continuous function $\psi : \ [0, r_1] \to \Re^+$ (resp., $\psi : \ [0, \infty) \to \Re^+$) is said to belong to class K, i.e., $\psi \in K$, if $\psi(0) = 0$ and if ψ is strictly increasing on $[0, r_1]$ (resp., on $[0, \infty)$). If $\psi : \ \Re^+ \to \Re^+$, if $\psi \in K$, and if $\lim_{r\to\infty} \psi(r) = \infty$, then ψ is said to belong to class KR. Let $\boldsymbol{E}_a \subset \boldsymbol{E}_v$ be a set of allowed event trajectories and $\mathcal{X}_b \subset \mathcal{X}$ denote a bounded subset of $\mathcal{X}$ for the remainder of the paper.

<u>Defintion 1</u>: The motions $X(\boldsymbol{x}_0, E_k, k)$ of G which begin at $\boldsymbol{x}_0 \in \mathcal{X}$ are *bounded w.r.t* $\boldsymbol{E}_a$ and $\mathcal{X}_b$ if there exists a $\beta > 0$ such that $\rho(X(\boldsymbol{x}_0, E_k, k), \mathcal{X}_b) < \beta$ for all E_k such that $E_k E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$ and for all $k \in \mathbb{N}$. The DES G is said to possess *Lagrange Stability* w.r.t. $\boldsymbol{E}_a$ and $\mathcal{X}_b$ if for each $\boldsymbol{x}_0 \in \mathcal{X}$ the motions $X(\boldsymbol{x}_0, E_k, k)$ for all E_k such that $E_k E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$ and all $k \in \mathbb{N}$ are bounded w.r.t. $\boldsymbol{E}_a$ and $\mathcal{X}_b$.

<u>Definition 2</u>: The motions of G are *uniformly bounded w.r.t* $\boldsymbol{E}_a$ and $\mathcal{X}_b$ if for any $\alpha > 0$ there exists a $\beta > 0$ (that depends on α) such that if $\rho(\boldsymbol{x}_0, \mathcal{X}_b) < \alpha$ then $\rho(X(\boldsymbol{x}_0, E_k, k), \mathcal{X}_b) < \beta$ for all E_k such that $E_k E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$ and for all $k \in \mathbb{N}$.

<u>Definition 3</u>: The motions of G are *uniformly ultimately bounded with bound B w.r.t* $\boldsymbol{E}_a$ and $\mathcal{X}_b$ if there exists a $B > 0$ and if corresponding to any $\alpha > 0$ there exists $T(\alpha) > 0$ such that $\rho(\boldsymbol{x}_0, \mathcal{X}_b) < \alpha$ implies that $\rho(X(\boldsymbol{x}_0, E_{k'}, k'), \mathcal{X}_b) < B$ for all $E_{k'}$ such that $E_{k'} E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$ where $k' \geq T(\alpha)$.

<u>Definition 4</u>: Fix α and β such that $\beta \geq \alpha > 0$, let ρ be a specified metric on $\mathcal{X}$, and let $\mathcal{X}_b \subset \mathcal{X}$ and $\boldsymbol{E}_a \subset \boldsymbol{E}_v$. The DES G is said to be *practically stable w.r.t.* $(\alpha, \beta, \rho, \mathcal{X}_b, \boldsymbol{E}_a)$ if for all $\boldsymbol{x}_0 \in \mathcal{X}$ such that $\rho(\boldsymbol{x}_0, \mathcal{X}_b) < \alpha$, $\rho(X(\boldsymbol{x}_0, E_k, k), \mathcal{X}_b) < \beta$ for all E_k such that $E_k E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$ and all $k \in \mathbb{N}$.

<u>Definition 5</u>: Fix α and β such that $\beta \geq \alpha > 0$, let ρ be a specified metric on $\mathcal{X}$, and let $\mathcal{X}_b \subset \mathcal{X}$ and $\boldsymbol{E}_a \subset \boldsymbol{E}_v$. Furthermore, let T_f denote a fixed *final time*. The DES G is said to be *finite-time stable w.r.t.* $(\alpha, \beta, T_f, \rho, \mathcal{X}_b, \boldsymbol{E}_a)$ if for all $\boldsymbol{x}_0 \in \mathcal{X}$ such that $\rho(\boldsymbol{x}_0, \mathcal{X}_b) < \alpha$, $\rho(X(\boldsymbol{x}_0, E_{k'}, k'), \mathcal{X}_b) < \beta$ for all $E_{k'}$ such that $E_{k'} E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$ where $k' < T_f$.

Notice that if the above properties hold for some $\boldsymbol{E}_a$ then they also hold for all $\boldsymbol{E}'_a$ such that $\boldsymbol{E}'_a \subset \boldsymbol{E}_a$.

Theorem 1 *In order for the motions of G to be uniformly bounded w.r.t. $\boldsymbol{E}_a$ and $\mathcal{X}_b$ it is sufficient that there exists a function V defined on $\bar{S}(\mathcal{X}_b; R)$ (where R may be large), and $\psi_1, \psi_2 \in KR$ such that*

(*i*) $\psi_1(\rho(\boldsymbol{x}, \mathcal{X}_b)) \leq V(\boldsymbol{x}) \leq \psi_2(\rho(\boldsymbol{x}, \mathcal{X}_b))$, $\ \boldsymbol{x} \in \bar{S}(\mathcal{X}_b; R)$, *and* *(ii)* $V(X(\boldsymbol{x}_0, E_k, k))$ *is a non-increasing function for* $\boldsymbol{x}_0 \in \bar{S}(\mathcal{X}_b; R)$, *for all* E_k *such that* $E_k E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$ *and all* $k \in \mathbb{N}$ *(i.e., V is non-increasing along all possible motions of the system).*

Proof:

Fix $r' > R$ and let $\boldsymbol{x}_0 \in S(\mathcal{X}_b; r')$ with $\rho(\boldsymbol{x}_0, \mathcal{X}_b) > R$. By conditions (i) and (ii), $V(X(\boldsymbol{x}_0, E_k, k)) \leq V(X(\boldsymbol{x}_0, \emptyset, 0)) \leq \psi_2(r')$ for all E_k such that $E_k E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$. By condition (ii) it is the case that $\psi_1(\rho(X(\boldsymbol{x}_0, E_k, k), \mathcal{X}_b)) \leq V(X(\boldsymbol{x}_0, E_k, k)) \leq \psi_2(r')$ for all E_k such that $E_k E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$ provided that $\rho(X(\boldsymbol{x}_0, E_k, k), \mathcal{X}_b) > R$. Since $\psi_1 \in KR$, its inverse exists, so $\rho(X(\boldsymbol{x}_0, E_k, k), \mathcal{X}_b)) \leq \psi_1^{-1}(\psi_2(r')) \doteq \beta$ for all E_k such that $E_k E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$ provided that $\rho(X(\boldsymbol{x}_0, E_k, k), \mathcal{X}_b) > R$. If $\boldsymbol{x}_0 \notin \bar{S}(\mathcal{X}_b; R)$ or if $\boldsymbol{x}_0 \in \bar{S}(\mathcal{X}_b; R)$ and there exists k', $E_{k'}$ such that $E_{k'} E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$ where $X(\boldsymbol{x}_0, E_{k'}, k') \notin \bar{S}(\mathcal{X}_b; R)$ then it could be that for all $k \geq k'$, $X(\boldsymbol{x}_0, E_k, k) \in S(\mathcal{X}_b; r')$ or it could be that for this $E_{k'}$ there exist $k_1 \geq k'$, $k_2 \geq k'$ such that $X(\boldsymbol{x}_0, E_{k''}, k'') \notin S(\mathcal{X}_b; r')$ for all k'', $k_1 < k'' < k_2 \leq \infty$. However, the above argument yeilds $\rho(X(\boldsymbol{x}_0, E_{k''}, k''), \mathcal{X}_b)) \leq \beta$ for all such k'' so that $\rho(X(\boldsymbol{x}_0, E_k, k), \mathcal{X}_b)) \leq \max\{R, \beta\}$ for all E_k such that $E_k E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$.

Theorem 2 *In order for the motions of G to be uniformly ultimately bounded with bound B w.r.t.* $\boldsymbol{E}_a$ *and* $\mathcal{X}_b$ *it is sufficient that there exists a function V defined on* $\bar{S}(\mathcal{X}_b; R)$ *(where R may be large),* ψ_1, $\psi_2 \in KR$, *and* $\psi_3 \in K$ *such that*

(i) $\psi_1(\rho(\boldsymbol{x}, \mathcal{X}_b)) \leq V(\boldsymbol{x}) \leq \psi_2(\rho(\boldsymbol{x}, \mathcal{X}_b))$, $\boldsymbol{x} \in \bar{S}(\mathcal{X}_b; R)$, *and*

(ii) $V(X(\boldsymbol{x}_0, E_{k+1}, k+1)) - V(X(\boldsymbol{x}_0, E_k, k)) \leq -\psi_3(\rho(X(\boldsymbol{x}_0, E_k, k), \mathcal{X}_b))$ *for all* $\boldsymbol{x}_0 \in \bar{S}(\mathcal{X}_b; R)$, *and for all* E_k *such that* $E_{k+1} = E_k e$ $(e \in \mathcal{E})$ *and* $E_{k+1} E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$ *and all* $k \in \mathbb{N}$.

Proof:

Fix $r_1 > R$, choose $B > r_1$ such that $\psi_2(r_1) < \psi_1(B)$ (which is always possible), choose $r_2 > B$, and let $T' = (\psi_2(r_2)/\psi_3(r_1)) + 1$. With $B < \rho(\boldsymbol{x}_0, \mathcal{X}_b) \leq r_2$, assume that $\rho(X(\boldsymbol{x}_0, E_k, k), \mathcal{X}_b) > r_1$ for all E_k such that $E_k E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$. By condition (ii), $V(X(\boldsymbol{x}_0, E_k, k)) \leq V(X(\boldsymbol{x}_0, \emptyset, 0)) - \sum_k \psi_3(\rho(X(\boldsymbol{x}_0, E_k, k), \mathcal{X}_b))$ for all E_k such that $E_k E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$. But $V(X(\boldsymbol{x}_0, \emptyset, 0)) \leq \psi_2(\rho(\boldsymbol{x}_0, \mathcal{X}_b)) \leq \psi_2(r_2)$ and $\psi_3(\rho(X(\boldsymbol{x}_0, E_k, k), \mathcal{X}_b)) > \psi_3(r_1)$ so that we get $V(X(\boldsymbol{x}_0, E_{k'}, k')) \leq \psi_2(r_2) - k\psi_3(r_1)$ for all E_k such that $E_k E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$. Let $k = T' = (\psi_2(r_2)/\psi_3(r_1)) + 1$ as above so that, $V(X(\boldsymbol{x}_0, E_k, k)) \leq -\psi_3(r_1)$ for all E_k such that $E_k E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$ which is a contradiction. Then there exists k^* such that $\rho(X(\boldsymbol{x}_0, E_{k^*}, k^*), \mathcal{X}_b)) \leq r_1$ where $E_{k^*} E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$. Suppose now that $\rho(X(\boldsymbol{x}_0, E_{k^*}, k^*), \mathcal{X}_b)) \leq r_1$ and $\rho(X(\boldsymbol{x}_0, E_k, k), \mathcal{X}_b)) > r_1$ for k such that $k^* < k \leq k' \leq \infty$ and

$E_k E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$ then

$$\psi_1(\rho(X(\boldsymbol{x}_0, E_k, k), \mathcal{X}_b)) \leq V(X(\boldsymbol{x}_0, E_k, k)) \leq V(X(\boldsymbol{x}_0, E_{k^*}, k^*)) \leq \psi_2(\rho(X(\boldsymbol{x}_0, E_{k^*}, k^*), \mathcal{X}_b))$$

and $\psi_2(\rho(X(\boldsymbol{x}_0, E_{k^*}, k^*), \mathcal{X}_b)) \leq \psi_2(r_1) < \psi_1(B)$ so that $\rho(X(\boldsymbol{x}_0, E_k, k), \mathcal{X}_b) < \psi_1^{-1}(\psi_1(B)) = B$ for all $k \geq k^*$, and E_k such that $E_k E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$.

Corollary 1: In order for the motions of G to be uniformly ultimately bounded with bound B w.r.t. $\boldsymbol{E}_a$ and $\mathcal{X}_b$ it is sufficient that there exists a function V defined on $\bar{S}(\mathcal{X}_b; R)$ (where R may be large), $D \in \mathbb{N}$, and $\psi_1, \psi_2 \in KR$, $\psi_3 \in K$ such that

- (i) Conditions (i) and (ii) of Theorem 1 hold, and
- (ii) $V(X(\boldsymbol{x}_0, E_{k+1}, k+1)) - V(X(\boldsymbol{x}_0, E_k, k)) \leq -\psi_3(\rho(X(\boldsymbol{x}_0, E_k, k), \mathcal{X}_b))$ for all $\boldsymbol{x}_0 \in \bar{S}(\mathcal{X}_b; R)$, and for all E_k such that $E_{k+1} = E_k e$ $(e \in \mathcal{E})$, $E_{k+1}E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$, $k \in [0, D)$ and if this inequality holds for $k' \in \mathbb{N}$ then it holds for each E_k such that $E_k E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$ for some $k \in (k', k'+D]$ (i.e., for each $E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$ the inequality holds at least once every D steps).

Proof:

Choose r_1, r_2, and B as above and T' as above and by condition (ii) for $k' \geq kD$, $k \in \mathbb{N}$, $V(X(\boldsymbol{x}_0, E_{k'}, k')) \leq V(X(\boldsymbol{x}_0, \emptyset, 0)) - k\psi_3(\rho(\boldsymbol{x}_0, \mathcal{X}_b))$ for all $E_{k'}$ such that $E_{k'}E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$. As above, we find that for $k' \geq kD$, $k \in \mathbb{N}$, $V(X(\boldsymbol{x}_0, E_{k'}, k')) \leq \psi_2(r_2) - k\psi_3(r_1)$ for all $E_{k'}$ such that $E_{k'}E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$. Choosing $k = T'$ we get a contradiction for $k' \geq DT'$. The remainder of the proof is the same as for Theorem 2.

Theorem 3 *For the DES G to be practically stable w.r.t. $(\alpha, \beta, \rho, \mathcal{X}_b, \boldsymbol{E}_a)$ it is sufficient that there exists a function V defined on $\mathcal{X}$ and a real valued function $\phi(k)$ such that*

- (i) $V(X(\boldsymbol{x}_0, E_{k+1}, k+1)) - V(X(\boldsymbol{x}_0, E_k, k)) \leq \phi(k)$ *for all* $E_{k+1} = E_k e$ $(e \in \mathcal{E})$, $E_{k+1}E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$ *and all* $k \in \mathbb{N}$, *and*
- (ii) $\sum_{i=0}^{k} \phi(i) < \inf\{V(\boldsymbol{x}) : \ \rho(\boldsymbol{x}, \mathcal{X}_b) \geq \beta\} - \sup\{V(\boldsymbol{x}) : \ \rho(\boldsymbol{x}, \mathcal{X}_b) < \alpha\}$ *for all* $k \in \mathbb{N}$.

Proof:

The result is shown via contradiction. Let $X(\boldsymbol{x}_0, E_k, k)$ be any motion with $\boldsymbol{x}_0 \in \mathcal{X}$ such that $\rho(\boldsymbol{x}_0, \mathcal{X}_b) < \alpha$ and with E_k such that $E_k E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$. Assume that there exists a $k' \geq 0$ which is the earliest time such that $\rho(X(\boldsymbol{x}_0, E_{k'}, k'), \mathcal{X}_b) \geq \beta$ for any $E_{k'}$ such that $E_{k'}E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$. From (ii), $\sum_{i=0}^{k'-1} \phi(i) < \inf\{V(\boldsymbol{x}) : \ \rho(\boldsymbol{x}, \mathcal{X}_b) \geq \beta\} - \sup\{V(\boldsymbol{x}) : \ \rho(\boldsymbol{x}, \mathcal{X}_b) < \alpha\}$ and substituting in (i) it is the case that

$$V(X(\boldsymbol{x}_0, E_{k'}, k')) - V(X(\boldsymbol{x}_0, \emptyset, 0)) < \inf\{V(\boldsymbol{x}) : \rho(\boldsymbol{x}, \mathcal{X}_b) \geq \beta\} - \sup\{V(\boldsymbol{x}) : \rho(\boldsymbol{x}, \mathcal{X}_b) < \alpha\}$$

for all $E_{k'}$ such that $E_{k'}E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$. Using the fact that

$$V(X(\boldsymbol{x}_0, \emptyset, 0)) - \sup\{V(\boldsymbol{x}) : \rho(\boldsymbol{x}, \mathcal{X}_b) < \alpha\} \leq 0$$

for $\boldsymbol{x}_0 \in \mathcal{X}$ such that $\rho(\boldsymbol{x}_0, \mathcal{X}_b) < \alpha$ it follows that $V(X(\boldsymbol{x}_0, E_{k'}, k')) < \inf\{V(\boldsymbol{x}) : \ \rho(\boldsymbol{x}, \mathcal{X}_b) \geq \beta\}$ for all $E_{k'}$ such that $E_{k'}E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$. This implies that $\rho(X(\boldsymbol{x}_0, E_{k'}, k'), \mathcal{X}_b) < \beta$ which is a contradiction. Therefore there does not exist k' such that $\rho(X(\boldsymbol{x}_0, E_{k'}, k'), \mathcal{X}_b) \geq \beta$ for any $E_{k'}$ such that $E_{k'}E \in \boldsymbol{E}_a(\boldsymbol{x}_0)$ and all $k' \in \mathbb{N}$ so $\rho(X(\boldsymbol{x}_0, E_k, k), \mathcal{X}_b) < \beta$ for all E_k such that $E_kE \in \boldsymbol{E}_a(\boldsymbol{x}_0)$ and all $k \in \mathbb{N}$.

Corollary 2: Replace in Theorem 3 the infinite time interval $[0, \infty)$ with the finite time interval $[0, T_f)$ and sufficient conditions for finite time stability w.r.t. $(\alpha, \beta, T_f, \rho, \mathcal{X}_b, \boldsymbol{E}_a)$ are obtained.

Using ideas from the proof for uniform boundedness (Theorem 1) and practical stability (Theorem 3) we state and prove the following result on Lagrange stability.

Theorem 4 *For a DES G to possess Lagrange stability w.r.t.* $\boldsymbol{E}_a$ *and* $\mathcal{X}_b$ *it is sufficient that there exists a function* V *defined on* $\mathcal{X}$ *and* $\psi_1, \psi_2 \in KR$ *such that*

- (i) $\psi_1(\rho(\boldsymbol{x}, \mathcal{X}_b)) \leq V(\boldsymbol{x}) \leq \psi_2(\rho(\boldsymbol{x}, \mathcal{X}_b))$*, for all* $\boldsymbol{x} \in \mathcal{X}$*,and*
- (ii) $V(X(\boldsymbol{x}_0, E_k, k)) - V(\boldsymbol{x}_0) < \beta(\boldsymbol{x}_0)$ *for each* $\boldsymbol{x}_0 \in \mathcal{X}$*,and all* E_k *such that* $E_kE \in \boldsymbol{E}_a(\boldsymbol{x}_0)$ *for all* $k \in \mathbb{N}$ *and some* $\beta(\boldsymbol{x}_0) > 0$*.*

Proof:

Fix $r' > 0$ and let $\boldsymbol{x}_0 \in S(\mathcal{X}_b; r')$ so that $V(\boldsymbol{x}_0) \leq \psi_2(r')$. For all E_k such that $E_kE \in \boldsymbol{E}_a(\boldsymbol{x}_0)$ and all $k \in \mathbb{N}$,

$$\psi_1(\rho(X(\boldsymbol{x}_0, E_k, k), \mathcal{X}_b)) \leq V(X(\boldsymbol{x}_0, E_k, k)) \leq V(\boldsymbol{x}_0) + \beta(\boldsymbol{x}_0) \leq \psi_2(r') + \beta(\boldsymbol{x}_0)$$

Since $\psi_1 \in KR$,

$$\rho(X(\boldsymbol{x}_0, E_k, k), \mathcal{X}_b) \leq \psi_1^{-1}(\psi_2(r') + \beta(\boldsymbol{x}_0)) \doteq \beta'(\boldsymbol{x}_0)$$

which shows that G possesses Lagrange stability.

4 A Lyapunov Stability-Theoretic Approach to the Analysis of Boundedness Properties of Petri Nets

4.1 Petri Net Model

For our discussions on Petri nets we will adhere (to the greatest extent possible) to the somewhat standard notation in [13] where a Petri net $PN = (P, T, F, W, M_0)$ where

(*i*) $P = \{p_1, p_2, \ldots, p_m\}$ is a finite set of places (represented with circles),

(*ii*) $T = \{t_1, t_2, \ldots, p_n\}$ is a finite set of transitions (represented with line segments),

(*iii*) $F \subset (P \times T) \bigcup (T \times P)$ is a set of arcs (represented with arrows),

(*iv*) $W : F \rightarrow \{1, 2, 3, \ldots\}$ is an *arc weight function* (represented with numbers labeling arcs and assume for convenience that if $(p, t) \notin F$ or if $(t, p) \notin F$ we will extend the arc weight function so that $W(t', p') = W(p, t) = 0$ for these cases and the arrow will be omitted), and

(*v*) $M_0 : P \rightarrow \mathbb{N}$ is a (initial) *marking* (represented with dark dots, i.e., *tokens*, in places).

It is the case that $P \bigcap T = \emptyset$ and $P \bigcup T \neq \emptyset$. The Petri net structure is $\mathcal{N} = (P, T, F, W)$ so $PN = (\mathcal{N}, M_0)$. The Petri net PN is normally referred to as the "General Petri net" while if "inhibitor arcs" are added it is called an "Extended Petri net" [13, 14] (also recall that "finite capacity nets" can be reduced to General Petri nets and that Marked Graphs and State Machines [13] are special cases of General Petri nets). If the initial marking is pre-specified then we will refer to the Petri net as $(\mathcal{N}, M_0)$ or simply PN, whereas, if the initial marking is not specified we will refer to the net as $\mathcal{N}$. Also note that if $W(p, t) = \alpha$ (or $W(t, p) = \alpha'$) then this is often represented graphically by $\alpha(\alpha')$ arcs from p to t (t to p) each with no numeric label.

Let $M_k(p_i)$ denote the marking (*i.e.*, the number of tokens) at place $p_i \in P$ at time k and let the marking (state) of PN at time k (the "k" will be dropped when it is not needed) be denoted by $M_k = [M_k(p_1) \cdots M_k(p_m)]^t$. A transition $t_j \in T$ is said to be *enabled* at time k if $M_k(p_i) \geq W(p_i, t_j)$ for all $p_i \in P$ such that $(p_i, t_j) \in F$. It is assumed that at each time k there exists at least one transition to fire. If a transition is enabled, then it can fire. If an enabled transition $t_j \in T$ fires at time k, then the next marking for place $p_i \in P$ is given by

$$M_{k+1}(p_i) = M_k(p_i) + W(t_j, p_i) - W(p_i, t_j)$$

where $(t_j, p_i) \in F$ and $(p_i, t_j) \in F$. Let $R(M_0)$ denote the set of makings of PN (states) that can be reached from M_0. Let $R_1(M)$ denote the set of all markings that are reachable from M in one transition firing. Let $A = [a_{ij}]$ denote an $n \times m$ matrix of integers (the incidence matrix) where $a_{ij} = a_{ij}^+ - a_{ij}^-$ with $a_{ij}^+ = W(t_i, p_j)$ and $a_{ij}^- = W(p_j, t_i)$. Let $u_k \in \{0,1\}^n$ denote a firing vector where if $t_j \in T$ is fired then its corresponding firing vector is $u_k = [0 \cdots 0\ 1\ 0 \cdots 0]^t$ with the "1" in the jth position in the vector and zeros are everywhere else. The matrix equations (nonlinear difference equations defined on $\mathbb{N}^m$ with non-unique solutions) describing the dynamical behavior represented by a Petri net are given by [13, 14]

$$M_{k+1} = M_k + A^t u_k \tag{2}$$

where if at step k, $a_{ij}^- \leq M_k(p_j)$ for all $p_j \in P$, then $t_i \in T$ is enabled and if this $t_i \in T$ fires then its corresponding firing vector u_k is utilized in equation 2 to generate the next state. Notice that if $M_d \in R(M_0)$, and we fire some sequence of d transitions with corresponding firing vectors $u_0,\ u_1,\ u_2,\ \ldots,\ u_{d-1}$ we will get $M_d = M_0 + A^t u$ with $u = \sum_{k=0}^{d-1} u_k$ where u is called the firing count vector.

An Extended Petri net is obtained from a General Petri net by adding inhibitor arcs (sometimes called "not arcs"). Let $F_n \subset (P \times T)$ denote the set of inhibitor arcs for the extended Petri net $EPN = (P, T, F, F_n, W, M_0)$ $(F \bigcap F_n = \emptyset)$. We use a line with a small circle on the end to graphically represent the inhibitor arc. The inhibitor arc does not change in any way what happens when a transition $t \in T$ fires (*i.e.*, equation 2 remains unchanged for the Extended Petri net). The inhibitor arc does, however, change which transitions are enabled at each step. The set of transitions in EPN enabled at time k is given by $\{t_j :\ M_k(p_i) \geq W(p_i, t_j) \text{ for all } p_i \in P \text{ s.t. } (p_i, t_j) \in F\} - \{t_j :\ (p_i, t_j) \in F_n \text{ and } M(p_i) = 0\}$. Hence, the inhibitor arc tests if a place has a zero marking. It is important to study properties of Extended Petri nets due to fact that the addition of the inhibitor arc greatly enhances the "modeling power" of the Petri net [14]. The characterization and analysis of the qualitative properties of systems represented via Petri nets is based on the the fact that Petri net models are a special case of the general DES model in equation 1 [3, 4].

4.2 Boundedness Properties of Petri Nets: A Lyapunov Approach

The fact that systems represented by Petri nets are amenable to Lyapunov stability analysis was first pointed out in [3, 4]. Below we show that the Petri net theoretic boundedness properties and analysis [14, 13] are actually special cases of the boundedness definitions in Section 3 and the Lyapunov approach to boundedness analysis. Let $\xi = [\xi_1 \xi_2 \ldots \xi_m]^t$ such that $\xi \in \Re^m$ and $\xi_i > 0$,

$i = 1, 2, ..., m$. Throughout this Section we will use the metric $\rho : \mathbb{N}^m \times \mathbb{N}^m \to \Re$ where

$$\rho(M, M') = \sum_{i=1}^{m} \xi_i |M(p_i) - M'(p_i)| \tag{3}$$

and we will use $D \subset \mathbb{N}^m$ to denote a bounded set. Next we state the standard definitions of boundedness for Petri nets [13, 14].

<u>Definition 6</u>: A Petri net $(\mathcal{N}, M_0)$ is said to be *γ-bounded* or simply *bounded* if for a given γ, $M(p_i) \leq \gamma$ for all $p_i \in P$ and $M \in R(M_0)$.

<u>Definition 7</u>: A Petri net $\mathcal{N}$ is said to be *structurally bounded* if it is bounded for any finite initial marking M_0.

For a Petri net $(\mathcal{N}, M_0)$: (i) $(\mathcal{N}, M_0)$ is γ-bounded for some $\gamma \geq 0$ iff the motions of $(\mathcal{N}, M_0)$ which begin at M_0 are bounded, (ii) $\mathcal{N}$ is structurally bounded iff $\mathcal{N}$ possesses Lagrange stability, and (iii) $\mathcal{N}$ is structurally bounded iff the motions of $\mathcal{N}$ are uniformly bounded. Next, we show how the Petri net-theoretic approach to the analysis of structural boundedness is actually a Lyapunov stability-theoretic approach. Moreover, we introduce the characterization and analysis of uniform ultimate boundedness for Petri nets.

Theorem 5 *For the Petri net $\mathcal{N}$ with $D = \{0\}$:*

- (i) *$\mathcal{N}$ is uniformly bounded if there exists an m-vector $\phi > 0$ such that $A\phi \leq 0$ and*
- (ii) *$\mathcal{N}$ is uniformly ultimately bounded if there exists an m-vector $\phi > 0$ and n-vector $\pi > 0$ such that $A\phi \leq -\pi$.*

Proof:

For (i) the proof follows by extending the one for structural boundedness in [13]. Let $\xi = \phi$ and choose

$$V(M) = \inf \left\{ \sum_{i=1}^{m} \phi_i |M(p_i) - M''(p_i)| : \ M'' \in D \right\} = M^t \phi \tag{4}$$

so that due to the choice of ρ in equation 3 the appropriate ψ_1 and ψ_2 exist so that $\psi_1(\rho(M, D)) \leq V(M) \leq \psi_2(\rho(M, D))$. Notice that V must only be defined and satisfy the appropriate properties on $\{M : \rho(M, D) \geq R\}$ where R may be large. Choose

$$R = \inf\{r' : 0 < \rho(M, D) < r' \ \text{ and all } t \in T \text{ are enabled at } M\}$$

(R is finite since $W(p_i, t_j)$ is finite.) For (i), it suffices to show that for all M and $M' \in R_1(M)$ such that $M \in \{M : \rho(M, D) \geq R\}$, $M'^t\phi \leq M^t\phi$. We know that for all M and $M' \in R_1(M)$, $M' = M + A^t u$ for some $u \geq 0$ (we know that $u \geq 0$ exists since $M \in \{M : \rho(M, D) \geq R\}$) and $M'^t = M^t + u^t A$ so $M'^t\phi = M^t\phi + u^t A\phi$. Since $u \geq 0$, $A\phi \leq 0$ implies that for all M and $M' \in R_1(M)$, $M'^t\phi \leq M^t\phi$ whenever $M \in \{M : \rho(M, D) \geq R\}$.

For (ii), it suffices to show that for all M and $M' \in R_1(M)$ such that $M \in \{M : \rho(M, D) \geq R\}$, $M'^t\phi \leq M^t\phi - \gamma$ for some $\gamma > 0$. From equation 2, if $M' \in R_1(M)$, then $M'^t\phi = M^t\phi + u^t A\phi$ so that $M'^t\phi \leq M^t\phi - u^t\pi$. Since $u \geq 0$ exists (as long as $M \in \{M : \rho(M, D) \geq R\}$) and $\pi_i > 0$,

$$M'^t\phi - M^t\phi \leq -\min\{\pi_i\}$$

for all M and $M' \in R_1(M)$. Hence, if we choose

$$\psi_3(\rho(M, D)) = \min\{\pi_i\}\left(\frac{\rho(M, D)}{1 + \rho(M, D)}\right)$$

(ii) holds.

Corollary 3: For the Petri net $\mathcal{N}$ with $D = \{0\}$ if for each $t_j \in T$,

$$\sum_p W(p, t_j) \geq \sum_p W(p, t_j) \quad \left(\sum_p W(p, t_j) > \sum_p W(p, t_j)\right)$$

then $\mathcal{N}$ is uniformly bounded (resp., uniformly ultimately bounded).

Proof:

Choose $V(M) = M^t\phi$ where $\phi = [1\,1 \ldots 1]^t$ and use Theorem 5.

Due to the fact that stability in the sense of Lyapunov and asymptotic stability are *local* properties, they hold trivially for any invariant set for a Petri net [1, 3, 4]. An analogous result to Theorem 5 part (ii) exists for asymptotic stability in the large. Note that the addition of inhibitor arcs to the General Petri net to obtain the Extended Petri net simply reduces the number of possible motions that can be generated by the system. Therefore, if a general Petri net is uniformly bounded or uniformly ultimately bounded, no matter what inhibitor arcs are added to obtain an Extended Petri net the Extended Petri net will maintain the corresponding properties. Theorem 5 shows that the standard approach to boundedness analysis for General Petri nets is actually a special case of a Lyapunov approach to boundedness analysis. Really what is shown is that in the Petri net-theoretic approach to the analysis of structural boundedness [13], in picking ϕ one is actually picking a Lyapunov function $V(M) = M^t\phi$. Once this is recognized it will perhaps be easier to study boundedness properties due to the wealth of experience there is with regard to the choice of Lyapunov functions. Part (*ii*) of Theorem 5 provides what seems to be the first characterization and analysis of uniform ultimate boundedness for Petri nets. It is important to note that the Lyapunov approach also applies to the many subclasses of Petri nets (e.g., Marked Graphs and State Machines) or for Extended Petri nets.

4.3 Petri Net Application: Manufacturing System

In [7] the authors provide simple computer network and production network applications that illustrate the use of the Lyapunov approach for analysis of boundedness properties. In this paper we introduce the study of boundedness properties of a special class of manufacturing lines with rate synchronization shown in Figure 1.

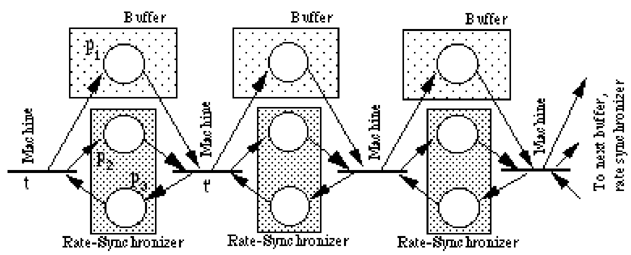

Figure 1: Manufacturing Line with Rate-Synchronization

Suppose that we are given the manufacturing system line shown in Figure 1 where transitions represent machines (a transition firing represents the completion of processing a part), and the places are used as shown to represent buffers where parts are passed through the system for processing (e.g., $M(p_1)$ represents the number of parts that have already been processed by the first machine and that are waiting to be processed by the second machine). The "rate-synchronizers" are used to ensure that the rates of processing of parts in the manufacturing system line are synchronized (to allow maximum flexibility in processing, we only seek to maintain a loosely coupled form of rate synchronization). Let $\mathcal{N} = (P, T, F, W)$ represent a manufacturing system with N such machines connected in series (similar analysis applies for other topologies). With this, $m = 3(N-1)$ and $n = N$.

For the analysis of boundedness properties choose $V(M) = M^t\phi$ where $\phi = [1\ 1\ 2\ 1\ 1\ 2 \ldots 2]^t$. Notice that if either t or t' fires $V(M_{k+1}) \leq V(M)$ so that the manufacturing line with rate-synchronization is uniformly bounded. The choice of the "2" in the ϕ vector weights the adding and subtacting of tokens to, *e.g.*, place p_3, so that the weighted sum of tokens for the network will not increase. Checking that $A\phi \leq 0$ per part (*i*) of Theorem 5 also verifies the uniform boundedness of the manufacturing line.

5 Lagrange Stability of DES: A Manufacturing Application

We consider machines as shown in Figure 2 which are capable of servicing parts of type i such that $i \in P$ where $P = \{1, 2, \ldots, N\}$. We fix the rate of arrival of parts to the machine. The machine can only service one part at a time and must be configured differently to service parts of different types. There is a set-up time when reconfiguring the machine for processing different part types. Parts that have arrived at the machine and have not yet been processed are accumulated in buffers. We will show that some such machines can be implemented with buffers of finite size.

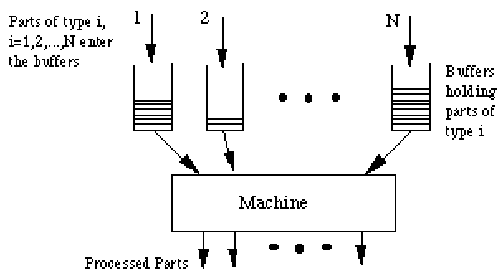

Figure 2: Machine with Buffers

Because we are concerned with arrival rates and because the processing of any part takes a finite amount of *real time*, we require that our DES model of the machine be synchronous. The events, e_k, will be required to occur with a fixed real time period. All references to real time will be given in terms of the event period, which we will call a *cycle*. Accordingly, we define the relevant rate and delay constants. There must be $b_i > 0$ cycles between arrivals of parts of type $i \in P$ at buffer i, the machine requires $m_i > 0$ cycles to process one part of type $i \in P$ (when the machine is producing parts of type $i \in P$), and $s_i > 0$ cycles are required to configure the machine to produce parts of type $i \in P$.

We further define

$$\begin{aligned} w_i &= \frac{m_i}{b_i} \\ w &= \sum_{i \in P} w_i. \end{aligned}$$

From the definitions of m_i and b_i, w_i is the number of parts that can arrive at buffer i per every part of type i that enters the machine to be processed (when parts of type i are being processed). In other words, the frequency of arrivals of part type i is 1 part per b_i cycles and the frequency of processing of part type i is 1 part processed per m_i cycles (assuming the machine is currently processing parts of type i); w_i is the ratio of the frequency of arrivals to the frequency of departures.

Clearly, it is not possible to bound the buffer levels in general if $w > 1$. Hence, we require that $w < 1$ so that $w_i < 1$ for all $i \in P$.

The machine described above is similar to the machine in [23]. The authors of [23] assume that the parts are fluid and *continuously* arrive at and depart from the machine. We assume that the parts arrive singularly at *discrete* points in time. In addition in [23] they study several different scheduling policies for this machine. Here, we shall study the Lagrange stability of a priority-based part servicing policy.

5.1 Single Machine Priority-Based Part Servicing Problem

Let $\mathcal{X} = \Re^{\mathrm{N}}$. The number of parts in buffer $i \in P$ at time $k \in \mathbb{N}$ is x_i, and $\boldsymbol{x}_k = [x_1 x_2 \ldots x_N]^t$. Let e_i^b represent that one part of type $i \in P$ arrives at buffer i, let e_i^m represent that one part of type $i \in P$ enters the machine for processing, and let e^0 be the null event. Let $B = \{e_i^b : \ i \in P\}$ and $M = \{e_i^m : \ i \in P\}$, so that the set of events is

$$\mathcal{E} = \mathcal{P}\left(B \bigcup M \bigcup \{e^0\}\right) - \{\emptyset\}.$$

Notice that each event $e_k \in \mathcal{E}$ is defined as a set of "sub-events".

We now specify g and f_e for $e_k \in g(\boldsymbol{x}_k)$. For $e_k \in g(\boldsymbol{x}_k)$, it is necessary that $e_k \in \mathcal{E}$ and that e_k satisfy the following conditions:

- If $e_i^m \in e_k$ then $x_i > 0$.
- If $e^0 \in e_k$ then $e_k = \{e^0\}$.

If $\boldsymbol{x}_{k+1} = f_{e_k}(\boldsymbol{x}_k)$ then

$$x_i(k+1) = \begin{cases} x_i, & e_i^b \in e_k, \ e_i^m \in e_k \\ x_i + 1, & e_i^b \in e_k, \ e_i^m \notin e_k \\ x_i - 1, & e_i^b \notin e_k, \ e_i^m \in e_k \\ x_i, & e_i^b \notin e_k, \ e_i^m \notin e_k. \end{cases}$$

If $\boldsymbol{x}_{k+1} = f_{e^0}(\boldsymbol{x}_k)$, then

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k.$$

Let $\boldsymbol{E}_v = \boldsymbol{E}$ be the set of valid event trajectories. We further specify the set of allowed event trajectories, $\boldsymbol{E}_a$, in order to specify the manner in which the machine chooses which parts to produce and to guarantee the synchronicity of the machine. We specify that the machine observe a *priority-based part servicing policy.* This policy mandates that once a production run is begun on a particular part type, the run must continue until the buffer of the chosen part type is emptied. Additionally, the parts in buffer $i \in P$ will be servicied before the parts in buffer $i+1$. Once all buffers have been serviced, the first buffer will be serviced again. For $\boldsymbol{E}_a \subset \boldsymbol{E}_v$, every $E \in \boldsymbol{E}_a$ must satisfy the following conditions:

- $e_i^b \in e_{k'}$ for some $0 \leq k' \leq b_i$ for all $i \in P$.
- If $e_i^b \in e_k$, then $e_i^b \in e_{k+b_i}$, and $e_i^b \notin e_{k'}$ for all k', $k < k' < k + b_i$.
- Let $k^* = \min\{k \geq 0 : \ x_i(k) > 0 \ \text{ for some } \ i \in P\}$ and $i^* = \min\{i \in P : \ x_i(k^*) > 0\}$. $e_{i^*}^m \in e_{k^*+s_{i^*}}$ where (i) $e_i^m \notin e_{k^*+s_{i^*}}$ for all $i \neq i^*$ and (ii) $e_i^m \notin e_{k'}$ for all $i \in P$ and all k', $0 \leq k' < k^* + s_{i^*}$.
- If $e_i^m \in e_k$, then
  - (i) if $x_i(k+1) > 0$, then
    - (a) $e_i^m \in e_{k+m_i}$ and $e_i^m \notin e_{k'}$ for all k', $k < k' < k + m_i$.
    - (b) for all $j \in P$, $j \neq i$, $e_j^m \notin e_{k'}$ for all k', $k < k' \leq k + m_i$.
  - (ii) if $x_i(k+1) = 0$, then
    - (a) $e_j^m \in e_{k+s_j}$ and $e_j^m \notin e_{k'}$ for all k', $k < k' < k + s_j$ where $j = i+1$ if $(i+1) \in P$ or $j = 1$ if $(i+1) \notin P$.
    - (b) for all k', $k < k' \leq k + s_j$, $e_i^m \notin e_{k'}$ for all $i \in P$, $i \neq j$.
- For any $k' \geq 0$, if $e_i^b \notin e_{k'}$ for all $i \in P$ and $e_i^m \notin e_{k'}$ for all $i \in P$, then $e_{k'} = \{e^0\}$.
- The *real time* between events e_k and e_{k+1} is fixed for all $k \geq 0$.

Notice that with this definition, for start-up the priority-based policy sets up for and processes the first part to arrive that has the highest priority. Following this, it cycles through the processing of part types according to their fixed priority ordering (where after the lowest priority part is processed, the highest priority part is serviced again).

5.2 Lagrange Stability Analysis

We will show that the machine whose operation is described above can be implemented with finite buffers by showing that the machine with a priority-based part processing policy possesses Lagrange stability. Choose

$$\mathcal{X}_b = \{[00\ldots0]^t\}$$

and

$$\rho(\boldsymbol{x}_k, \mathcal{X}_b) = \sum_{i \in P} m_i x_i. \tag{5}$$

For any variable a_i which is defined for all $i \in P$, we let $\underline{a} = \min_i\{a_i\}$ and $\bar{a} = \max_i\{a_i\}$.

Theorem 6 *The machine with priority-based part servicing policy possesses Lagrange stability w.r.t.* $\boldsymbol{E}_a$ *and* $\mathcal{X}_b$.

Proof:

Choose

$$\boldsymbol{V}(\boldsymbol{x}_k) = \rho(\boldsymbol{x}_k, \mathcal{X}_b) \tag{6}$$

so that condition (i) of Theorem 4 is satisfied. We define the set of times $R = \{k_0, k_1, k_2, \ldots\}$, $k_p < k_q$ if $p < q$, to include every time k' such that $e_i^m \in e_{k'-1}$ and $x_i(k') = 0$ for some $i \in P$ (these times define the ends of production runs). Additionally, let $k_0 = 0$. Let $j^*(k_p) \in P$, $k_p \in R$, denote the part type that is being processed by the machine between times k_p and k_{p+1}. We define $k_{p+1} - k_p \triangleq \Delta_p$. In order to bound Δ_p, we consider:

- $x_{j^*(k_p)}(k_p)$, the number of parts of type $j^*(k_p)$ that are in the buffer at time k_p and
- $\frac{\Delta_p}{b_{j^*(k_p)}} + 1$, the maximum number of parts of type $j^*(k_p)$ that can arrive during time Δ_p.

The sum of the above two classifications of parts of type $j^*(k_p)$ is the maximum number of parts that must be processed between times k_p and k_{p+1}. The maximum number of cycles, Δ'_p, that the machine may require to accomplish the necessary processing is simply $m_{j^*(k_p)}$ times the sum of parts:

$$\Delta'_p = m_{j^*(k_p)} \left(x_{j^*(k_p)}(k_p) + \frac{\Delta_p}{b_{j^*(k_p)}} + 1 \right).$$

Δ_p can be no larger than Δ'_p plus the number of cycles required to configure the machine to process parts of type $j^*(k_p)$. Hence, we find that

$$\Delta_p \quad \leq \quad \Delta'_p + s_{j^*(k_p)} \ ,$$

so that

$$\Delta_p \ \leq \ \frac{\left(x_{j^*(k_p)}(k_p) + 1 \right) m_{j^*(k_p)} + s_{j^*(k_p)}}{1 - w_{j^*(k_p)}}. \tag{7}$$

We now bound $\boldsymbol{V}(\boldsymbol{x}_{k_{p+1}})$ in terms of $\boldsymbol{V}(\boldsymbol{x}_{k_p})$. In order to do this, we consider the following relations, which are easily derived from the specification of $\boldsymbol{E}_a$:

$$x_{j^*(k_p)}(k_{p+1}) = 0 \tag{8}$$

$$x_i(k_{p+1}) - x_i(k_p) \leq \frac{\Delta_p}{b_i} + 1 \text{ for all } i \in P,\, i \neq j^*(k_p) \tag{9}$$

From equations 8, 9, and the definition of $\boldsymbol{E}_a$, it follows that

$$\sum_{i\in P} m_i x_i(k_{p+1}) \leq \sum_{i\in P} m_i x_i(k_p) - m_{j^*(k_p)} x_{j^*(k_p)}(k_p) + \Delta_p \sum_{i\in P, i\neq j^*(k_p)} \frac{m_i}{b_i} + \sum_{i\in P, i\neq j^*(k_p)} m_i \tag{10}$$

Applying equations 5, 6 , 7, and 10, we see that

$$\begin{aligned}
\boldsymbol{V}(\boldsymbol{x}_{k_{p+1}}) &\leq \boldsymbol{V}(\boldsymbol{x}_{k_p}) - x_{j^*(k_p)}(k_p) m_{j^*(k_p)} + \Delta_p \sum_{i\in P, i\neq j^*(k_p)} \frac{m_i}{b_i} + \sum_{i\in P, i\neq j^*(k_p)} m_i \\
&= \boldsymbol{V}(\boldsymbol{x}_{k_p}) - x_{j^*(k_p)}(k_p) m_{j^*(k_p)} + \Delta_p (w - w_{j^*(k_p)}) + \sum_{i\in P, i\neq j^*(k_p)} m_i \\
&\leq \boldsymbol{V}(\boldsymbol{x}_{k_p}) - x_{j^*(k_p)}(k_p) m_{j^*(k_p)} + \Big[\Big(x_{j^*(k_p)}(k_p) + 1\Big) m_{j^*(k_p)} + \\
&\quad s_{j^*(k_p)}\Big] \frac{w - w_{j^*(k_p)}}{1 - w_{j^*(k_p)}} + \sum_{i\in P, i\neq j^*(k_p)} m_i \\
&= \boldsymbol{V}(\boldsymbol{x}_{k_p}) - x_{j^*(k_p)}(k_p) m_{j^*(k_p)} \left(\frac{1-w}{1 - w_{j^*(k_p)}}\right) + \\
&\quad \Big(m_{j^*(k_p)} + s_{j^*(k_p)}\Big) \left(\frac{w - w_{j^*(k_p)}}{1 - w_{j^*(k_p)}}\right) + \sum_{i\in P, i\neq j^*(k_p)} m_i \,.
\end{aligned} \tag{11}$$

While up to this point the proof has been similar to the proof of stability for the CAF policy in [23], next we must account for the the fact that we are using the priority-based part servicing policy.

From the definition of $\boldsymbol{E}_a$ (which characterizes the priority-based part processing policy), it is evident that

$$\max_i \{x_i(k_p)\} < \frac{\Big(x_{j^*(k_p)}(k_p) + 1\Big) b_{j^*(k_p)}}{\underline{b}} + 1 \tag{12}$$

where $\Big(x_{j^*(k_p)}(k_p) + 1\Big) b_{j^*(k_p)}$ is an upper bound on the number of cycles that have transpired since the last time k_q, $q < p$, such that $x_{j^*(k_q)} = 0$. From equation 12, we see that

$$\sum_{i\in P} x_i(k_p) < N \left(\frac{\Big(x_{j^*(k_p)}(k_p) + 1\Big) b_{j^*(k_p)}}{\underline{b}} + 1\right) \,. \tag{13}$$

Manipulating equation 13 yields

$$x_{j^*(k_p)}(k_p) \quad > \quad \frac{\Big(\frac{1}{N}\sum_{i\in P} x_i(k_p) - 1\Big)\underline{b}}{b_{j^*(k_p)}} - 1 \tag{14}$$

$$
\begin{aligned}
&= \frac{\underline{b}}{N b_{j^*(k_p)}} \sum_{i \in P} x_i(k_p) - \frac{\underline{b}}{b_{j^*(k_p)}} - 1 \\
&> \frac{\underline{b}}{N \bar{b} \bar{m}} \sum_{i \in P} m_i x_i(k_p) - 2 \\
&= \frac{\underline{b}}{N \bar{b} \bar{m}} \boldsymbol{V}(\boldsymbol{x}_{k_p}) - 2 \\
&\triangleq \epsilon \boldsymbol{V}(\boldsymbol{x}_{k_p}) - 2
\end{aligned} \tag{15}
$$

where $\epsilon = \frac{\underline{b}}{N \bar{m} \bar{b}}$ and $0 < \epsilon < 1$. If we define

$$
\gamma = \max_i \left\{ 1 - \epsilon m_i \left(\frac{1 - w}{1 - w_i} \right) \right\}
$$

and

$$
\zeta = \max_i \left\{ (m_i + s_i) \left(\frac{w - w_i}{1 - w_i} \right) + \sum_{j \in P, j \neq i} m_j + 2 m_i \left(\frac{1 - w}{1 - w_i} \right) \right\},
$$

we see from equations 11 and 16 that

$$
\boldsymbol{V}(\boldsymbol{x}_{k_{p+1}}) < \gamma \boldsymbol{V}(\boldsymbol{x}_{k_p}) + \zeta. \tag{16}
$$

Notice that by definition, $0 < \gamma < 1$. We now show via induction that

$$
\boldsymbol{V}(\boldsymbol{x}_{k_{p+q}}) < \gamma^q \boldsymbol{V}(\boldsymbol{x}_{k_p}) + \sum_{n=0}^{q-1} \gamma^n \zeta. \tag{17}
$$

As the induction hypothesis, we assume that equation 17 is true for some general q. Given the induction hypothesis and equation 16, we have

$$
\begin{aligned}
\boldsymbol{V}(\boldsymbol{x}_{k_{p+q+1}}) &< \gamma \boldsymbol{V}(\boldsymbol{x}_{k_{p+q}}) + \zeta \\
&< \gamma \left(\gamma^q \boldsymbol{V}(\boldsymbol{x}_{k_p}) + \sum_{n=0}^{q-1} \gamma^n \zeta \right) + \zeta \\
&< \gamma^{q+1} \boldsymbol{V}(\boldsymbol{x}_{k_p}) + \sum_{n=0}^{q} \gamma^n \zeta.
\end{aligned}
$$

Hence equation 17 is true for $q + 1$. Because equation 16 is precisely equation 17 with $q = 1$, equation 17 must be true for all $q \geq 1$. If we let $p = 0$ in equation 17, we see that

$$
\begin{aligned}
\boldsymbol{V}(\boldsymbol{x}_{k_q}) &< \gamma^q \boldsymbol{V}(\boldsymbol{x}_0) + \sum_{n=0}^{q-1} \gamma^n \zeta \\
&< \boldsymbol{V}(\boldsymbol{x}_0) + \frac{\zeta}{1 - \gamma}.
\end{aligned} \tag{18}
$$

Thus, we have bounded $\boldsymbol{V}(\boldsymbol{x}_{k_q})$ for all $k_q \in R$.

Consider now the set of times S_q such that if $k \in Z$ and $k \in (k_q, k_{q+1})$, then $k \in S_q$. In equation 18, we have found a bound for $V(\boldsymbol{x}_k)$, for $k = k_q$ and $k = k_q + 1$. We now wish to bound $V(\boldsymbol{x}_k)$ for all $k \in S_q \bigcup \{k_q, k_{q+1}\}$. Clearly, the maximum of V over $S_q \bigcup \{k_q, k_{q+1}\}$ must occur at one of the following times: k_q, k_{q+1}, or at k_q^m, where $k_q^m \triangleq \max\{k \in S_q : \; e_i^m \notin e_k, \; i \in P\}$ (*i.e.* k_q^m is the time in S_q immediately before the beginning of part production). We can bound the increase in V that occurs between times k_q and k_q^m as

$$\begin{aligned} V(\boldsymbol{x}_{k_q^m}) - V(\boldsymbol{x}_q) &\leq \sum_{i \in P} m_i \left(\frac{\bar{s}}{b_i} + 1 \right) \\ &= \bar{s}w + \sum_{i \in P} m_i \; . \end{aligned}$$

Hence, for all $k > 0$,

$$\boldsymbol{V}(\boldsymbol{x}_k) \leq \boldsymbol{V}(\boldsymbol{x}_0) + \frac{\zeta}{1-\gamma} + \bar{s}w + \sum_{i \in P} m_i \; , \tag{19}$$

so that by Theorem 4, the machine posses Lagrange stability.

Remark 1: Utilizing equation 19 it is easy to see that the buffer levels will for all $k \geq 0$ be constrained by

$$\begin{aligned} \sum_{i \in P} x_i &\leq \frac{1}{\underline{m}} \left(\bar{s}w + \sum_{i \in P} m_i \left(x_i(0) + 1 \right) + \frac{\zeta}{1-\gamma} \right) \\ &= \frac{1}{\underline{m}} \left(\bar{s}w + \frac{N \bar{b} \bar{m} \max_i \left\{ (m_i + s_i) \left(\frac{w - w_i}{1 - w_i} \right) + \sum_{j \in P, j \neq i} m_j + 2m_i \left(\frac{1-w}{1-w_i} \right) \right\}}{\underline{b} \min_i \left\{ m_i \left(\frac{1-w}{1-w_i} \right) \right\}} \right. \\ &\quad \left. + \sum_{i \in P} m_i \left(x_i(0) + 1 \right) \right) . \end{aligned}$$

Remark 2: Notice that the results verify our intuition that increasing N, w, or $\sum_{i \in P} m_i x_i(0)$ for the priority-based part servicing policy will create the possibility that buffer levels can rise even higher.

Remark 3: If the CAF (clear a fraction) policy is used the authors in [23] show that the buffer levels will be constrained by

$$\sum_{i \in P} x_i = \frac{1}{\underline{m}} \left(\bar{s}w + \frac{\max_i \left\{ s_i \left(\frac{w - w_i}{1 - w_i} \right) \right\}}{\min_i \left\{ \frac{\epsilon m_i}{\bar{m}} \left(\frac{1-w}{1-w_i} \right) \right\}} + \sum_{i \in P} m_i x_i(0) \right) ,$$

where $0 < \epsilon < 1$. If ϵ is chosen to be $\frac{1}{N}$ and we have a machine such that $m_i = m_j$, $b_i = b_j$ for all $i, j \in P$, and we have all the same set-up times, then the bound given in Remark 1 is the same as the CAF bound (except for the differences induced by the discrete nature of our part flow). Intuitively, this is not surprising since it indicates that if all the arrival rates are the same and all the processing times are the same then in this special case the priority-based policy is a special case of the CAF policy so we get the same bounds as in [23]. It is easy, however, to pick the machine parameters so that the priority-based policy will perform much differently than the CAF policy (in fact this is most often the case), and for these cases our results provide bounds for this new policy.

6 Concluding Remarks

We have shown that it is straightforward to extend the conventional notions and analysis of uniform boundedness, uniform ultimate boundedness, practical stability, finite time stability, and Lagrange stability to the class of DES that can be defined on a metric space (e.g., Petri nets and Vector DES). We show that the Petri net-theoretic notions and analysis of boundedness are really a special case of conventional notions and analysis of boundedness. Also, we introduce the notion of uniform ultimate boundedness to Petri net theory and provide a sufficient condition for this property. We use a rate-synchronization network in manufacturing systems to illustrate some of the Petri net results. Moreover, we show that a machine with a priority-based part servicing policy possesses Lagrange stability. In fact we provide explicit bounds on the maximum number of parts that will be in the buffers at any one time in terms of the machine parameters and initial buffer levels.

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