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A class of attractions/repulsion functions for stable swarm aggregations

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In this brief article, we consider an M-member 'individual-based' continuous time swarm model in an n-dimensional space and extend the results in Gazi and Passino (2003) by specifying a general class of attraction/repulsion functions that can be used to achieve swarm aggregation. These functions are odd functions that have terms for attraction and repulsion acting in opposite directions in compliance with real biological swarms. We present stability analysis for several cases of the functions considered to characterize swarm cohesiveness, size and ultimate motions while in a cohesive group. Moreover, we show how the model can be extended for achieving formation control. Furthermore, we discuss how the attraction repulsion functions can be modified to incorporate the finite body size of the swarm members. Numerical simulations are also presented for illustration purposes.

1. Introduction

In recent years the topic of distributed (or cooperative) coordination and control of multiple autonomous agents has gained a lot of attention in the engineering literature. This is mainly because of emergence of engineering applications such as formation control of multi-robot teams and autonomous air vehicles. For example, in Giulietti *et al*. (2000) the authors describe formation control strategies for autonomous air vehicles, whereas Balch and Arkin (1998), Desai *et al*. (2001), Leonard and Fiorelli (2001), Ögren *et al*. (2001) and Olfati-Saber and Murray (2002) describe different approaches for formation control of multi-agent (multi-robot) teams. Similarly, Reif and Wang (1999) consider a distributed control approach for groups of robots, called the *social potential fields* method.

Our results in Gazi and Passino (2003) were for an 'individual-based' continuous time model for swarm aggregations in n-dimensional space. There, we showed that for the given model the individuals will form a cohesive swarm in a finite time, and we obtained an explicit bound on the swarm size. In Gazi and Passino (2004) we used a similar model for a swarm moving in an environment with an attraction/repulsion profile (or nutrient/toxic substances environment) and showed stable convergence of the swarm to more favourable regions of the profile (i.e. the space). This article is an extension of the results in Gazi and Passino (2003) to a more general case. In Gazi and Passino (2003) our results were based on a particular attraction–repulsion function. Here, first we specify a class of attraction/repulsion functions which result in swarm aggregation. One of the characteristics of these functions is that they are odd functions which contain terms of attraction and repulsion between the individuals. As a difference from the attraction/repulsion function considered in Gazi and Passino (2003), the class presented here can have unbounded repulsion in order to prevent the individuals occupying the same space. We present stability analysis for several cases to characterize swarm cohesiveness, size and ultimate motion. We also show how with a simple modification the model can be extended/generalized to formation control, an issue not considered in Gazi and Passino (2003). Moreover, we discuss how the attraction/repulsion functions can be modified in order to incorporate a finite body size or 'safety' area for the swarm members, another issue not considered in Gazi and Passino (2003). This leads to swarm aggregation with more uniform density and size which scales with the number of individuals, similar to swarms in nature.

Our model is essentially a kinematic model for swarm aggregations. However, it can be viewed as an approximation for some swarms with point mass dynamics with negligible mass such as bacteria as was discussed in Gazi and Passino (2004) for foraging swarms. Moreover, given agents with predefined vehicle dynamics, it can serve as a virtual system generating reference trajectories. In other words, it can serve as a virtual system which the vehicles need to track and the control algorithms for that can be designed using variety of techniques including sliding mode control (Gazi 2003).

The inter-individual interactions in our model are based on *artificial potential functions*, a concept that has been used extensively for robot navigation and control (Khatib 1986, Rimon and Koditschek 1992) and were first used for multi-agent coordination in Reif and Wang (1999). However, we get our primary

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inspiration from the literature on swarming in mathematical biology (Warburton and Lazarus 1991, Grünbaum and Okubo 1994). (For a more complete list of articles on swarming, see Gazi and Passino (2003, 2004)). Note also that our results are similar in nature to the formation stability studies in the literature; however, the work here is also different in the sense that we consider not only formation control but also define cohesiveness independent from the relative positions of the agents as the main stability property and obtain results in this framework. The model and the set-up in § 2 is similar in nature to other models using artificial potential fields such as Reif and Wang (1999) and Leonard and Fiorelli (2001). However, note that in Reif and Wang (1999) no stability analysis was given. In Leonard and Fiorelli (2001), on the other hand, they have point mass dynamics for each agent, study group control using 'virtual leaders', and limit the analysis to two- or three-dimensional space, whereas here we consider a general n-dimensional space with different attraction/repulsion functions. Note also that our results were obtained independently from the ones in Leonard and Fiorelli (2001). Last, we would like to also mention that recently there have been important studies on swarming incorporating nearest-neighbours communications and obstacle avoidance including the work by Olfati-Saber and Murray (2003) and Tanner *et al.* (2003 a, b). An initial version of the current article can be found in Gazi and Passino (2002).

2. The class of attraction/repulsion functions

We use the same swarm model as in Gazi and Passino (2003), where we consider a swarm of M individuals in an n-dimensional Euclidean space. The position of individual i is described by $x^i \in \mathbb{R}^n$. We assume synchronous motion and no time delays, i.e. all the individuals move simultaneously and know the exact relative position of all the other individuals. The motion dynamics evolve in continuous time with the equation of motion of individual i given by

$$\dot{x}^i = \sum_{j=1, j\neq i}^{M} g(x^i - x^j), \quad i = 1, \ldots, M \tag{1}$$

where $g\colon \mathbb{R}^n \to \mathbb{R}^n$ represents the function of attraction and repulsion between the individuals. Note that in the above model we consider the individuals as points and ignore their dimensions. Later we will show how it can be modified in order to handle individuals with finite body size.

Consider functions $g(\cdot)$ of type

$$g(y) = -y[g_a(\|y\|) - g_r(\|y\|)] \tag{2}$$

where $g_a\colon \mathbb{R}^+ \to \mathbb{R}^+$ represents (the magnitude of) the attraction term, whereas $g_r\colon \mathbb{R}^+ \to \mathbb{R}^+$ represents (the magnitude of) the repulsion term, and $\|y\| = \sqrt{y^\top y}$ is the Euclidean norm. We assume that on large distances attraction dominates, that on short distances repulsion dominates, and that there is a *unique distance* at which the attraction and the repulsion balance. In other words we assume that $g(\cdot)$ satisfies

(A1) There exist a *unique distance* δ at which we have $g_a(\delta) = g_r(\delta)$. Moreover, we have $g_a(\|y\|) > g_r(\|y\|)$ for $\|y\| > \delta$ and $g_r(\|y\|) > g_a(\|y\|)$ for $\|y\| < \delta$.

One issue to note here is that for the attraction/repulsion functions $g(\cdot)$ defined as above we have $g(y) = -g(-y)$. In other words, the above $g(\cdot)$ functions are *odd* (and therefore symmetric with respect to the origin). This is an important feature of the $g(\cdot)$ functions that leads to aggregation behaviour. Note also that the combined term $-yg_a(\|y\|)$ represents the actual attraction, whereas the combined term $yg_r(\|y\|)$ represents the actual repulsion, and they both act on the line connecting the two interacting individuals, but in opposite directions. The vector y determines the alignment (i.e. it guarantees that the interaction vector is along the line on which y is located), the terms $g_a(\|y\|)$ and $g_r(\|y\|)$ affect only the magnitude, whereas their difference determines the direction (along vector y).

It has been observed in nature that there are attraction and repulsion forces (with attraction having longer range than repulsion) between individuals that lead to the swarming behaviour (Warburton and Lazarus 1991, Grünbaum and Okubo 1994). For example, for fish attraction is generally based on vision and has a long range, whereas repulsion is based on the pressure on the side of the fish and has a short range (but is stronger than attraction). Moreover, it has been observed that both attraction and repulsion are always 'on' and the resulting behaviour is due to the *interplay* between these two forces, and there is a distance (called the 'equilibrium distance' in biology) at which attraction and repulsion between two individuals balance. Note that our model captures this by having attraction and repulsion terms in the motion equation acting in opposite directions, and the 'equilibrium distance' is the unique distance δ at which we have $g_a(\delta) = g_r(\delta)$.

The next assumption that we have about the attraction and repulsion functions is

(A2) There exist corresponding functions $J_a\colon \mathbb{R}^+ \to \mathbb{R}^+$ and $J_r\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\nabla_y J_a(\|y\|) = yg_a(\|y\|) \quad \text{and} \quad \nabla_y J_r(\|y\|) = yg_r(\|y\|).$$

In other words, we choose $g_a(\|y\|)$ and $g_r(\|y\|)$ such that these conditions are satisfied. Note that the functions

$J_a(\|y\|)$ and $J_r(\|y\|)$ can be viewed as potentials of attraction and repulsion, respectively, created around each individual. Then, the above assumption restricts the motion of the individuals toward each other along the gradient of these potentials (i.e. along the combined gradient field of these potentials). For simplicity (and easy reference) we will denote with $\mathcal{G}$ the set of attraction/repulsion functions $g(\cdot)$ satisfying Assumptions (A1) and (A2). An example of an attraction/repulsion function that satisfies the above conditions is the function that we used in Gazi and Passino (2003), which is

$$g(y) = -y\left[a - b\exp\left(-\frac{\|y\|^2}{c}\right)\right]. \quad (3)$$

With the above in mind, note that the motion of each individual is given by

$$\dot{x}^i = -\sum_{j=1, j\neq i}^{M}\left[\nabla_{x^i}J_a(\|x^i - x^j\|) - \nabla_{x^i}J_r(\|x^i - x^j\|)\right] \quad (4)$$

for all $i = 1, \ldots, M$. Note that $\dot{x}^i$ is along the negative gradient and leads to a motion towards a minimum. This together with the assumptions on $g_a(\cdot)$ and $g_r(\cdot)$ imply that we should choose the attraction and repulsion potentials such that the minimum of $J_a(\|x^i - x^j\|)$ occurs on or around $\|x^i - x^j\| = 0$, whereas the minimum of $-J_r(\|x^i - x^j\|)$ (or the maximum of $J_r(\|x^i - x^j\|)$) occurs on or around $\|x^i - x^j\| \to \infty$, and the minimum of the combined $J_a(\|x^i - x^j\|) - J_r(\|x^i - x^j\|)$ occurs at $\|x^i - x^j\| = \delta$. In other words, at $\|x^i - x^j\| = \delta$ the attraction/repulsion profile between two interacting individuals has a global minimum. Note, however, that when there are more than two individuals the minimum of the combined profile does not necessarily occur at $\|x^i - x^j\| = \delta$ for all $j \neq i$. Moreover, there exists a family of minima. If we view $J_a(\|x^i - x^j\|)$ and $-J_r(\|x^i - x^j\|)$ as potential energy profiles due to the relative positions of the individuals x^i and x^j, then their motions are towards a minimum energy configuration.

Swarming in nature normally occurs in a distributed fashion. In other words, there is no leader (or boss) and each individual decides independently its direction of motion. Our model captures this in its simplest form by having separate equations of motion of each individual that do not depend on an external variable (such as a command from a boss or another agent). In contrast, an individual's motion depends only on the position of the individual itself and its observation of the positions (or relative positions) of the other individuals. Note also that as was discussed in Gazi and Passino (2004) for foraging swarms our swarm model can be viewed as an approximation of a model with swarm members which have point mass dynamics for some organisms such as bacteria. In particular, for individuals which move based on the Newton's law $m_i a^i = F^i$ with negligible mass (i.e. $m_i \approx 0$) moving in an environment with high viscosity it can be shown that the model reduces to the model that we consider here with the appropriate choice of the force (control) input F^i. For more details see Gazi and Passino (2004).

Another way to view the motion equations is as if the swarm members in the model presented here are virtual agents generating trajectories for real agents to follow as is sometimes done in formation control. Then, given agents with predefined vehicle dynamics, one can design the control inputs to these agents so that to follow the trajectories generated by our model (provided that aggregation is the desired behaviour). For example, given mobile robots with fully actuated dynamics $M_i(x^i)\ddot{x}^i + f_i(x^i, \dot{x}^i) = u^i$ under some boundedness conditions it is possible to design a sliding mode controller in order to force them to follow the gradient of the combined potential and therefore to obey the motion in (4). Some work in this direction was already done in Gazi (2003).

There are several reasons for considering the class of $g(\cdot)$ of form of (2). First of all, we show that the results in Gazi and Passino (2003) are not limited to the $g(\cdot)$ function in (3) and will hold for a class of functions satisfying certain properties. Moreover, we allow for unbounded repulsions in $g(\cdot)$ in (2) leading to avoidance of possible collisions between the individuals. Furthermore, we show that with appropriate choice of $g(\cdot)$ from within the class considered here, it is possible to incorporate a finite body size or safety area for each individual and therefore achieve a 'more uniform' swarm density or a swarm size scaling with the number of individuals as seen in real biological swarms.

As in Gazi and Passino (2003) define the *centre* of the swarm as $\bar{x} = 1/M\sum_{i=1}^{M} x^i$. Note that since the functions $g(\cdot)$ are odd, and therefore symmetric with respect to the origin, the centre $\bar{x}$ of the swarm is stationary for all t. This is stated formally in the following lemma.

Lemma 1: *The centre $\bar{x}$ of the swarm described by the model in* (1) *with an attraction/repulsion function $g(\cdot) \in \mathcal{G}$ is stationary for all t.*

Proof: The time derivative of centre is given by

$$\begin{aligned}\dot{\bar{x}} &= -\frac{1}{M}\sum_{i=1}^{M}\sum_{j=1, j\neq i}^{M}\left[g_a(\|x^i - x^j\|) - g_r(\|x^i - x^j\|)\right](x^i - x^j) \\ &= -\frac{1}{M}\sum_{i=1}^{M-1}\sum_{j=i+1}^{M}\Big[\left[g_a(\|x^i - x^j\|) - g_r(\|x^i - x^j\|)\right](x^i - x^j) \\ &\quad + \left[g_a(\|x^j - x^i\|) - g_r(\|x^j - x^i\|)\right](x^j - x^i)\Big] = 0\end{aligned}$$

□

Now, let us define the state x of the system as the vector of the positions of the swarm members $x = [x^{1\top}, \dots, x^{M\top}]^\top$. Note that $x \in \mathbb{R}^{nM}$. Let the invariant set of equilibrium (or stationary) points be $\Omega_e = \{x\colon \dot{x} = 0\}$. Note that $x \in \Omega_e$ implies that $\dot{x}^i = 0$ for all $i = 1, \dots, M$, implying that all individuals stop.

Theorem 1: *Consider the swarm described by the model in* (1) *with an attraction/repulsion function* $g(\cdot) \in \mathcal{G}$. *For any* $x(0) \in \mathbb{R}^{nM}$, *as* $t \to \infty$ *we have* $x(t) \to \Omega_e$.

Proof: We choose the (generalized) Lyapunov function $J\colon \mathbb{R}^{nM} \to \mathbb{R}$ defined as

$$J(x) = \sum_{i=1}^{M-1} \sum_{j=i+1}^{M} \left[J_a(\|x^i - x^j\|) - J_r(\|x^i - x^j\|) \right]. \quad (5)$$

Taking the gradient of $J(x)$ with respect to the position x^i of individual i we obtain

$$\nabla_{x^i} J(x) = \sum_{j=1, j\neq i}^{M} \left[\nabla_{x^i} J_a(\|x^i - x^j\|) - \nabla_{x^i} J_r(\|x^i - x^j\|) \right] = -\dot{x}^i \quad (6)$$

which follows from (4).

Now, taking the time derivative of the Lyapunov function along the motion of the system we obtain

$$\dot{J}(x) = [\nabla_x J(x)]^\top \dot{x} = \sum_{i=1}^{M} \left[\nabla_{x^i} J(x) \right]^\top \dot{x}^i = \sum_{i=1}^{M} \left[-\dot{x}^i \right]^\top \dot{x}^i = -\sum_{i=1}^{M} \|\dot{x}^i\|^2 \leq 0$$

for all t implying decrease in $J(x)$ unless $\dot{x}^i = 0$ for all $i = 1, \dots, M$. If the function $g(\cdot)$ is chosen such that the set defined as $\Omega_0 = \{x\colon J(x) \leq J(x(0))\}$ is compact, then using the LaSalle's invariance principle we can conclude that as $t \to \infty$ the state $x(t)$ converges to the largest invariant subset of the set defined as

$$\Omega_1 = \{x \in \Omega_0 : \dot{J}(x) = 0\} = \{x \in \Omega_0 : \dot{x} = 0\} \subset \Omega_e.$$

Note, however, that Ω_0 may not necessarily be compact for every $g(\cdot) \in \mathcal{G}$, which may happen if the corresponding $J(\cdot)$ is not radially unbounded. Therefore, the fact that $\dot{J}(x) \leq 0$ does not, in general, directly imply boundedness. Note, however, that in our swarm for every individual i we have $\left[\nabla_{x^i} J(x) \right]^\top \dot{x}^i = -\|\dot{x}^i\|^2 \leq 0$, which implies that every individual moves in a direction of decrease of $J(x)$. Therefore, the set defined as $\Omega_x = \{x(t) : t \geq 0\} \subset \Omega_0$ is compact and we still can apply LaSalle's invariance principle arriving at the conclusion that as $t \to \infty$ the state $x(t)$ converges to the largest invariant subset of the set defined as

$$\Omega_2 = \{x \in \Omega_x : \dot{J}(x) = 0\} = \{x \in \Omega_x : \dot{x} = 0\} \subset \Omega_e.$$

Since in both of the above cases both Ω_1 and Ω_2 are invariant themselves and satisfy $\Omega_1 \subset \Omega_e$ and $\Omega_2 \subset \Omega_e$, we have $x(t) \to \Omega_e$ as $t \to \infty$ and this concludes the proof. □

Note that in some engineering swarm applications such as *uninhabited air vehicles* (UAVs) individuals never stop. Therefore, the results here may seem not to be applicable. However, in some biological examples such as fruiting body formation by bacteria or engineering applications in which a group of agents are required to 'gather together' to be loaded on a vehicle and transferred to a new area it is possible (or desirable) to have the agents aggregate and stop. Moreover, note also that the results here are based on relative inter-individual interactions and describe only aggregation. It is possible to extend them to mobile swarms by having a motion (or drift) term in the equation of motion together with the aggregation term described here. As a result, if all the individuals share exactly the same motion term (e.g. a predefined speed profile or trajectory of motion), then we will achieve a cohesive swarm moving collectively since the aggregating term would decay as they would arrange in the minimum energy configuration (relative arrangement) as the above result suggests. In other words, the results here will guarantee cohesiveness during motion. For example, in Gazi and Passino (2004) it was shown how the model of a swarm moving in a 'plane' profile (i.e. environment) of nutrients of toxic substances can be reduced to the model in Gazi and Passino (2003) (and therefore the one here) with appropriate definition of the error variables.

Note that our approach is distributed in a sense that the individuals do not have to know the global Lyapunov or potential energy function $J(x)$ given in (5). Instead, it is sufficient if they know the local or their internal Lyapunov or potential energy function defined as

$$J_i(x) = \sum_{j=1, j\neq i}^{M} \left[J_a(\|x^i - x^j\|) - J_r(\|x^i - x^j\|) \right]$$

since

$$\dot{x}^i = -\nabla_{x^i} J_i(x) = -\nabla_{x^i} J(x)$$

where $J(x)$ can be written as $J(x) = \frac{1}{2}\sum_{i=1}^{M} J_i(x)$. Note also that for implementation each individual i may, instead of using actual position difference $(x^i - x^j)$ to the other individuals $j \neq i$, use some observation or estimate $\hat{e}^{ij} = (x^i - \hat{x}^j)$ of the position errors in determining its motion. However, the stability for this case needs to be investigated further. For initial results on this topic see Passino (2003) and Liu and Passino (2004).

The result in Theorem 1 is important. It proves that asymptotically the individuals will converge to a constant position and therefore to a constant relative arrangement. However, it does not say anything about where these positions will be. We conjecture that given the initial positions of the individuals $x^i(0)$, $i = 1, \ldots, M$, the final configuration (i.e. the relative arrangement) to which the individuals in the swarm will converge is unique. However, it is not easy to find a direct relation between $x(0)$ and the final position $x(\infty)$. This is an important problem, since it will solve the formation control problem for autonomous agents obeying our model. Then, given any desired formation, one would need to choose the initial conditions such that this formation is achieved. Unfortunately, it will still have a shortcoming since in formation control it is desirable that the formation is achieved independently of initial conditions. Later we will show how with a little modification the model can also be used for formation control.

One disadvatage of Theorem 1 is that it does not specify any bound on the resulting size of the swarm. Therefore, we need to investigate this issue further.

3. Swarm cohesion analysis

In this section, we will try to find bounds on the ultimate swarm 'size'. To this end, we define the distance between the position x^i of individual i and the centre $\bar{x}$ of the swarm as $e^i = x^i - \bar{x}$. The ultimate bound on the magnitude of e^i will quantify the size of the swarm. Taking the time derivative of e^i we have $\dot{e}^i = \dot{x}^i - \dot{\bar{x}} = \dot{x}^i$, since from Lemma 1 we have $\dot{\bar{x}} = 0$. Now, let the Lyapunov function for each individual be $V_i = \frac{1}{2}\|e^i\|^2 = \frac{1}{2}e^{i\top}e^i$. Taking the time derivative of V_i we obtain

$$\dot{V}_i = \dot{e}^{i\top}e^i = -\sum_{j=1, j\neq i}^{M}\left[g_a(\|x^i - x^j\|) - g_r(\|x^i - x^j\|)\right](x^i - x^j)^\top e^i. \quad (7)$$

Below, we analyse the case in which we have a linear attraction and a constant or bounded repulsion.

3.1. *Linear attraction and bounded repulsion case*

In this section we consider the special case in which

$$g_a(\|y\|) = a$$

for some finite positive constant $a > 0$ and for all y (as is the one in (3)), which corresponds to linear attraction since the actual attraction is given by $yg_a(\|y\|) = ay$. Incorporating the value of $g_a(\|x^i - x^j\|)$ in (7) we obtain

$$\dot{V}_i = -a\sum_{j=1, j\neq i}^{M}(x^i - x^j)^\top e^i + \sum_{j=1, j\neq i}^{M} g_r(\|x^i - x^j\|)(x^i - x^j)^\top e^i.$$

Now, note that

$$\sum_{j=1, j\neq i}^{M}(x^i - x^j) = \sum_{j=1}^{M}(x^i - x^j) = Mx^i - \sum_{j=1}^{M}x^j = Mx^i - M\bar{x} = Me^i. \quad (8)$$

Substituting this in the $\dot{V}_i$ equation we obtain

$$\dot{V}_i = -aM\|e^i\|^2 + \sum_{j=1, j\neq i}^{M} g_r(\|x^i - x^j\|)(x^i - x^j)^\top e^i \leq -aM\|e^i\|\left[\|e^i\| - \frac{1}{aM}\sum_{j=1, j\neq i}^{M} g_r(\|x^i - x^j\|)\|x^i - x^j\|\right]$$

which implies that $\dot{V}_i < 0$ as long as $\|e^i\| > 1/aM\sum_{j=1, j\neq i}^{M} g_r(\|x^i - x^j\|)\|x^i - x^j\|$. This, on the other hand, implies that as $t \to \infty$ asymptotically we have

$$\|e^i\| \leq \frac{1}{aM}\sum_{j=1, j\neq i}^{M} g_r(\|x^i - x^j\|)\|x^i - x^j\|.$$

Note that this equation holds for any type of repulsion, provided that the attraction is linear. Now, assume that the repulsion is constant or bounded (as is the one in (3)), i.e. assume that

$$g_r(\|x^i - x^j\|)\|x^i - x^j\| \leq b$$

for some finite positive constant b. Then, we conclude that asymptotically for this case we will have

$$\|e^i\| \leq \frac{b(M-1)}{aM} < \frac{b}{a} \triangleq \varepsilon$$

which provides a bound on the maximum ultimate swarm size. Moreover, noting that for $\|e^i\| \geq \varepsilon$ we have $\dot{V}_i \leq -a\|e^i\|^2 = -2aV_i$, one can show that we will have $\|e^i\| < \varepsilon$ for all i in a finite time bounded by

$$\bar{t} \triangleq \max_{i\in S}\left\{-\frac{1}{2a}\ln\left(\frac{\varepsilon^2}{2V_i(0)}\right)\right\}$$

where S is the set of all individuals $S = \{1, \ldots, M\}$.

Remark: Note that if instead of having $g_a(\|y\|) = a$, we had $g(\cdot)$ such that

$$\sum_{j=1, j\neq i}^{M} g_a(\|x^i - x^j\|)(x^i - x^j)^\top e^i \geq \eta\|e^i\|^2$$

for some $\eta > 0$ and for all $i = 1, \ldots, M$, then, with a similar analysis to above, we would be able to conclude that asymptotically we have

$$\|e^i\| \leq \frac{1}{\eta}\sum_{j=1, j\neq i}^{M} g_r(\|x^i - x^j\|)\|x^i - x^j\| \leq \frac{b(M-1)}{\eta}.$$

Above we could establish an explicit bound on the swarm size for the case when the attraction is linear and the repulsion is bounded. This is a direct generalization of the results in Gazi and Passino (2003). Next, we will analyse the case for which we have an unbounded repulsion.

3.2. *Linearly bounded from below attraction and unbounded repulsion*

By linearly bounded from below attraction we mean the case in which we have

$$g_a(\|x^i - x^j\|) \geq a \tag{9}$$

for some finite positive constant a and for all $\|x^i - x^j\|$. For the repulsion functions, on the other hand, we will consider the unbounded functions satisfying

$$g_r(\|x^i - x^j\|) \leq \frac{b}{\|x^i - x^j\|^2}. \tag{10}$$

An example of attraction/repulsion function $g(\cdot)$ satisfying the above assumptions is shown in figure 1(a).

First, we define the cumulative (or overall) Lyapunov function as $V = \sum_{i=1}^{M} V_i$ and note that since at equilibrium $\dot{e}^i = \dot{x}^i = 0$, we have also $\dot{V}_i = 0$ for all i and therefore $\dot{V} = 0$. In other words, we have

$$\begin{aligned}
\dot{V} &= -\sum_{i=1}^{M}\sum_{j=1,j\neq i}^{M}\left[g_a(\|x^i - x^j\|) - g_r(\|x^i - x^j\|)\right](x^i - x^j)^\top e^i \\
&= -\sum_{i=1}^{M-1}\sum_{j=i+1}^{M}\left[\left[g_a(\|x^i - x^j\|) - g_r(\|x^i - x^j\|)\right](x^i - x^j)^\top e^i\right. \\
&\quad \left. + \left[g_a(\|x^j - x^i\|) - g_r(\|x^j - x^i\|)\right](x^j - x^i)^\top e^j\right] \\
&= -\sum_{i=1}^{M-1}\sum_{j=i+1}^{M}\left[g_a(\|x^i - x^j\|) - g_r(\|x^i - x^j\|)\right]\|x^i - x^j\|^2 \\
&= -\frac{1}{2}\sum_{i=1}^{M}\sum_{j=1,j\neq i}^{M}\left[g_a(\|x^i - x^j\|) - g_r(\|x^i - x^j\|)\right]\|x^i - x^j\|^2 \\
&= 0
\end{aligned}$$

where to obtain the third equality we used the fact that for any $\alpha \in \mathbb{R}$ we have

$$\alpha(x^i - x^j)^\top e^i + \alpha(x^j - x^i)^\top e^j = \alpha\|x^i - x^j\|^2 \tag{11}$$

which is true since $x^i - x^j = e^i - e^j$. From the above equation we obtain

$$\sum_{i=1}^{M}\sum_{j=1,j\neq i}^{M} g_a(\|x^i - x^j\|)\|x^i - x^j\|^2 = \sum_{i=1}^{M}\sum_{j=1,j\neq i}^{M} g_r(\|x^i - x^j\|)\|x^i - x^j\|^2. \tag{12}$$

This equation, in a sense, says that at equilibrium the attraction and repulsion will balance.

Remark: Note that the cumulative Lyapunov function V is only one way to quantify the cohesion/dispersion of the swarm. In other words, instead of V, we could equally well choose

$$\bar{V} = \frac{1}{2}\sum_{i=1}^{M-1}\sum_{j=i+1}^{M}\|x^i - x^j\|^2$$

which would quantify the interindividual distances instead of the distances to the centre. In some applications, where the centre is moving or the relative motion or positions to each other of the individuals is more important than their relative motion to a predefined point such as their centre, it may be better to use a function like $\bar{V}$. In fact, we arrive at the same conclusion using $\bar{V}$ since it can be shown that

$$\dot{\bar{V}} = -\frac{M}{2}\sum_{i=1}^{M}\sum_{j=1,j\neq i}^{M}\left[g_a(\|x^i - x^j\|) - g_r(\|x^i - x^j\|)\right]\|x^i - x^j\|^2 = M\dot{V}.$$

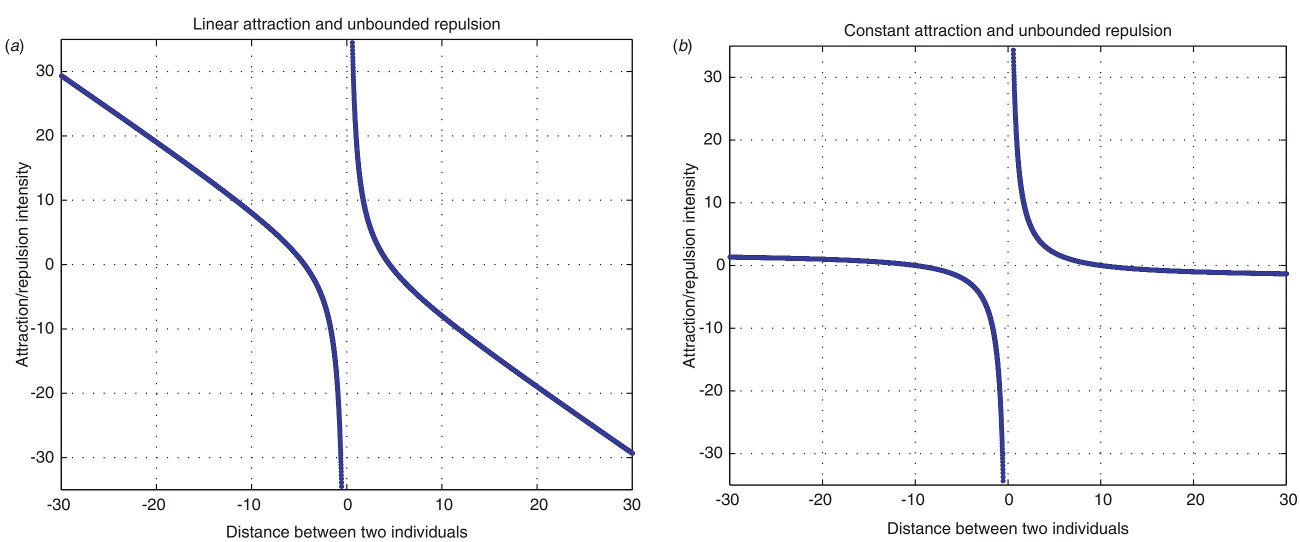

Figure 1. Example $g(\cdot)$ functions. (a) Linear attraction and unbounded repulsion. (b) Constant attraction and unbounded repulsion.

One issue to note here is that, if we had only attraction (i.e. if we had $g_r(\|x^i - x^j\|) \equiv 0$ for all i and $j, j \neq i$), then the above equation would imply that the swarm shrinks to a single point, which is the centre $\bar{x}$. In contrast, if we had only repulsion (i.e. if we had $g_a(\|x^i - x^j\|) \equiv 0$ for all i and $j, j \neq i$), then the swarm would disperse in all directions away from the centre $\bar{x}$ towards infinity. Having the attraction dominating at large distances prevents the swarm from dispersing, whereas having the repulsion dominating on short distances prevents it from collapsing to a single point, and the equilibrium is established in between.

Note that since the actual attraction term is $yg_a(\|y\|)$, we have $g_a(\|x^i - x^j\|)\|x^i - x^j\| \geq a\|x^i - x^j\|$ for this case (and hence the name linearly bounded from below attraction). Using this fact, we have

$$a\sum_{i=1}^{M}\sum_{j=1,j\neq i}^{M}\|x^i - x^j\|^2 \leq \sum_{i=1}^{M}\sum_{j=1,j\neq i}^{M} g_r(\|x^i - x^j\|)\|x^i - x^j\|^2.$$

Similarly, from the bound on $g_r(\|x^i - x^j\|)$ we know that $g_r(\|x^i - x^j\|)\|x^i - x^j\|^2 \leq b$ and obtain

$$\sum_{i=1}^{M}\sum_{j=1,j\neq i}^{M} g_r(\|x^i - x^j\|)\|x^i - x^j\|^2 \leq bM(M-1).$$

Now, note that using the fact that $e^i = 1/M\sum_{j=1}^{M}(x^i - x^j)$ (see (8)) and the equality in (11) for the sum of the squares of the error one can show that

$$\sum_{i=1}^{M}\|e^i\|^2 = \frac{1}{2M}\sum_{i=1}^{M}\sum_{j=1,j\neq i}^{M}\|x^i - x^j\|^2$$

holds.

Combining these equations with (12) we obtain

$$2aM\sum_{i=1}^{M}\|e^i\|^2 \leq bM(M-1)$$

which implies that at equilibrium we have

$$\frac{1}{M-1}\sum_{i=1}^{M}\|e^i\|^2 \leq \frac{b}{2a}.$$

Then, for the *root mean square* of the error we have

$$\mathrm{e}_{\mathrm{rms}} \triangleq \sqrt{\frac{1}{M}\sum_{i=1}^{M}\|e^i\|^2} \leq \sqrt{\frac{b}{2a}} \triangleq \varepsilon_{\mathrm{rms}} \qquad (13)$$

which establishes a bound on the swarm size and implies that the swarm will be cohesive. Note also that this inequality shows that $\|e^i\|$ will be ultimately bounded by

$$\|e^i\| \leq \sqrt{\frac{bM}{2a}} = \varepsilon_{\mathrm{rms}}\sqrt{M}$$

for all t. In other words, no swarm member can diverge to infinity.

3.3. *Almost constant attraction and unbounded repulsion*

In this section we consider the attraction functions that satisfy $g_a(\|x^i - x^j\|) \to 0$ as $\|x^i - x^j\| \to \infty$. However, we assume also that

$$g_a(\|x^i - x^j\|) \geq \frac{a}{\|x^i - x^j\|}. \qquad (14)$$

For the repulsion function we use the same type of functions as in the previous section, i.e. functions satisfying (10). An example of attraction/repulsion function $g(\cdot)$ satisfying the above assumptions (with a constant attraction) is shown in figure 1(*b*).

For this case we have

$$a\sum_{i=1}^{M}\sum_{j=1,j\neq i}^{M}\|x^i - x^j\| \leq \sum_{i=1}^{M}\sum_{j=1,j\neq i}^{M} g_r(\|x^i - x^j\|)\|x^i - x^j\|^2.$$

Also, since we have

$$\|e^i\| = \frac{1}{M}\left\|\sum_{i=1}^{M}(x^i - x^j)\right\| \leq \frac{1}{M}\sum_{i=1}^{M}\|x^i - x^j\|$$

(which is obtained using the equality in (8)) in a similar manner to earlier we obtain

$$aM\sum_{i=1}^{M}\|e^i\| \leq bM(M-1)$$

which, on the other hand, implies that

$$\frac{1}{M-1}\sum_{i=1}^{M}\|e^i\| \leq \frac{b}{a}.$$

In other words, the *average* of the errors satisfies

$$e_{\mathrm{avg}} \triangleq \frac{1}{M}\sum_{i=1}^{M}\|e^i\| \leq \frac{b}{a} \triangleq \varepsilon_{\mathrm{avg}},$$

at equilibrium, implying cohesiveness of the swarm. Also, we can deduce from this inequality that for this case also $\|e^i\|$ will be ultimately bounded by the bound

$$\|e^i\| \leq \frac{Mb}{a} = M\varepsilon_{avg}.$$

One shortcoming of the results obtained so far is that the bounds obtained (i.e. ε, $\varepsilon_{\mathrm{rms}}$, and $\varepsilon_{\mathrm{avg}}$) are independent of the number of individuals M. Therefore, it could be the case that as the number of individuals increase the density of the swarm may also increase. This might happen even for the case with unbounded repulsion, which guarantees that the individuals will not occupy the same point, but does not necessarily guarantee uniform swarm density. Such a behaviour will not be consistent with real biological swarms. By creating a private or safety area for each individual, it is possible to account for the finite body sizes of the individuals and also to achieve swarm size which increases with the number of individuals implying, in a sense, more uniform swarm density. This will be discussed in the next section.

4. Individuals with finite body size

In this section, we will discuss how the model can be modified in order to handle finite body size or some private or safety area for the swarm members. In other words, the swarm members will be viewed as entities with finite body size instead of points without dimensions. In particular, we will consider individuals which are hyperspheres in the n-dimensional space.

Assume that all the individuals have the same size and let η be the radius of the hypersphere representing the body size or private (safety) area of each agent. Let x^i be the centre of the hypersphere for individual i. Then, in order for two individuals i and j not to collide we need $\|x^i - x^j\| > 2\eta$. Note that this is not guaranteed to be the case given the attraction/repulsion functions considered in the preceding sections. The main reason for that is the fact that for the repulsion function $g_r(\cdot)$ we have

$$\lim_{\|x^i - x^j\| \to 0^+} g_r(\|x^i - x^j\|)\|x^i - x^j\| = \infty$$

which, since as $\|x^i - x^j\| \to 0^+$ the repulsion becomes unbounded, prevents the individuals from occupying the same space, i.e. prevent collisions for point-individuals. However, it is not suitable for individuals with finite body size. Therefore, in order to incorporate for the body size or to create a safety area of the individuals, the repulsion function could be modified to be of 'hard-limiting' type satisfying

$$\lim_{\|x^i - x^j\| \to 2\eta^+} g_r(\|x^i - x^j\|)\|x^i - x^j\| = \infty$$

where η is the parameter representing the radius of the safety area or the body size of the individuals as mentioned above.

Note that with the assumption that initially all the swarm members are sufficiently far apart from each other, i.e. that we have $\|x^i(0) - x^j(0)\| > 2\eta$ for all (i, j), $j \neq i$, this type of repulsion function will guarantee that $\|x^i(t) - x^j(t)\| > 2\eta$ is satisfied for all t and all (i, j), $j \neq i$.

One issue we would like to emphasize is that for this case the results (i.e. the derived bounds on the swarm size) in the preceding sections will not hold. In fact, the hard-limiting repulsion functions will guarantee that the swarm size will scale with the number of individuals and therefore will result in a swarm density which is, in a sense, more uniform. To see this first consider the two dimensional space $\mathbb{R}^2$. The private area of an individual is a disk with centre x^i and radius η with occupation area equal to $V_{ai} = \pi\eta^2$. Given the fact that $\|x^i(t) - x^j(t)\| > 2\eta$ for all t and all (i, j), $j \neq i$ and the safety areas of the swarm members are disjoint, the total area occupied by the swarm will be $V_{\text{tas}} = M\pi\eta^2$.

Assume that all the swarm members are 'squeezed' cohesively as close as possible in an area (a disk) of radius r around the swarm centre $\bar{x}$. Then, we have

$$\pi r^2 \geq M\pi\eta^2$$

from which we obtain that a lower bound on the radius of the smallest circle which can enclose all the individuals is given by

$$r_{\min} = \eta\sqrt{M}.$$

This, on the other hand, implies that in $\mathbb{R}^2$ the swarm will have a size which is always greater than $\eta\sqrt{M}$. This is important because it shows that the lower bound on the swarm size depends on M, implying that the swarm size will scale with the number of individuals. In particular, even the size of the smallest possible swarm will be greater than $r_{\min}$ because of the unoccupied area 'lost' between the swarm members. The most compact swarm is achieved when the individuals are located on a regular grid with the grid points as the edges of equilateral triangles with edge size equal to 2η and the total of $(M-2)$ triangles. Defining ρ as the density (the number of individuals per unit area/volume) of the swarm the above inequalities imply that

$$\rho \leq \frac{1}{\pi\eta^2}.$$

In other words, the density of the swarm is upper bounded implying that the swarm cannot become arbitrarily dense.

With similar analysis to the above, we can show that on $\mathbb{R}^n$ for any n the lower bound on the swarm size is given by

$$r_{\min} = \eta\sqrt[n]{M}.$$

This bound implies that as the dimension n of the state space gets larger, the relative effect of M on the swarm size gets smaller, which is an intuitively expected result. As in the two-dimensional case, the smallest swarm occurs when the individuals are placed on a regular grid where each individual is located at the vertex of an equilateral shape (triangle in $\mathbb{R}^2$, tetrahedron in $\mathbb{R}^3$, etc.). Similar to the $n=2$ case it can be shown that the density of the swarm is upper bounded by

$$\rho \leq \frac{\beta(n)}{\eta^n}$$

where $\beta(n)$ is a constant for a given n. In other words, it depends only on the dimension n of the state space.

Having relatively uniform swarm density is an important characteristic of real biological swarms and therefore, desired characteristic of mathematical models of swarms. The discussion in this section shows that incorporation of hard-limiting type of repulsion functions will lead to a behaviour which is more consistent with biology. Moreover, in engineering applications creating a safety area around each individual might be more effective in avoiding collisions.

5. Extension to formation control

The formation control problem is an important problem in the multi-agent and robotics literature and recently there has been substantial research in that area. As we mentioned in an earlier section, with a little modification our model can be extended (generalized) to handle the formation control (stabilization) problem. To see this consider the case in which the attraction/repulsion functions $g(\cdot)$ are pair dependent. In other words, consider the case in which we have

$$\dot{x}^i = \sum_{j=1, j\neq i}^{M} g_{ij}(x^i - x^j), \quad i = 1, \ldots, M \tag{15}$$

where $g_{ij}(\cdot) \in \mathcal{G}$ for all pairs (i, j) and $g_{ij}(x^i - x^j) = -g_{ji}(x^j - x^i)$. Note, however, that $(x^i - x^j) = (x^k - x^l)$ does not necessarily imply $g_{ij}(x^i - x^j) = g_{kl}(x^k - x^l)$. In other words, in this new model the attraction and repulsion functions and therefore the equilibrium distance δ_{ij} for different pairs of individuals can be different.

Now, consider the problem of formation stabilization, where the desired formation is *uniquely* specified with respect to rotation and translation (i.e. uniquely in terms of relative inter-individual arrangement) by the *formation constraints*

$$\|x^i - x^j\| = d_{ij}$$

for all (i, j), $j \neq i$. Note that with these constraints there are a family of possible arrangements since these conditions continue to be satisfied when the formation is rotated or translated in an arbitrary way. With the above requirements in mind, the idea is to choose each of the attraction/repulsion functions $g_{ij}(\cdot)$ such that $\delta_{ij} = d_{ij}$ for every pair of individuals (i, j). This, in turn, results in the fact that the generalized Lyapunov function $J(x)$ defined in (5) has a minimum achieved at the desired formation and that once the formation is achieved we have $\dot{x}^i = 0$ for all i. Note that Theorem 1 still holds implying that all the individuals will eventually stop. If the attraction/repulsion functions have been chosen such that $J(x)$ has a unique minimum (again with respect to rotation and translation implying a family of minima) at the desired formation, then Theorem 1 will imply global asymptotic stability of the desired formation. However, usually there exist local minima in $J(x)$ (i.e. the minimum corresponding to the desired formation is not necessarily unique). Therefore, it is not always possible to globally guarantee convergence to the desired formation. Nevertheless, from Theorem 1 still it is possible to deduce local asymptotic stability for the equilibrium at the desired formation. This could be stated formally as follows.

Corollary 1: *Consider the generalized swarm model in* (15) *with pair dependent attraction/repulsion functions* $g_{ij}(\cdot) \in \mathcal{G}$. *Assume that* $g_{ij}(\cdot)$ *are chosen such that* $\delta_{ij} = d_{ij}$, *where* d_{ij} *are the desired formation distances. Then, the equilibrium at the desired formation is locally asymptotically stable. Moreover, if* $g_{ij}(\cdot)$ *are such that* $J(x)$ *has unique minimum at the desired formation, then asymptotic stability holds globally.*

We would like also to emphasize that for this case, i.e. for the case with generalized pair dependent attraction/repulsion functions and its application to formation control, Lemma 1 also still holds, i.e. the centre $\bar{x}$ of the swarm is stationary. This implies that the generalized model is for stabilizing stationary formations around their centre $\bar{x}$. However, as mentioned before, they can easily be extended to the case of moving formations (swarms) with the same motion term of the individuals.

Finally, we would like to mention that the results on the cohesiveness of the swarm obtained in the preceding sections will still hold for the more general model considered in this section, i.e. for the model with the pair dependent attraction/repulsion functions $g(\cdot)$, as described in (15), as long as each of the attraction/repulsion functions $g_{ij}(\cdot)$ are from the class $\mathcal{G}$ defined earlier. The only difference would be that the bounds on the swarm size will be given in terms of the *minimum attraction parameter* $a_{\min}$ and the maximum repulsion parameter $b_{\max}$. For example, for the linearly bounded from below attraction and unbounded repulsion case we will have $a_{\min} = \min_{1\leq i,j\leq M}\{a_{ij}\}$ and $b_{\max} = \max_{1\leq i,j\leq M}\{b_{ij}\}$ and the bound ε_{rms} in (13) still holds with the parameters a and b interchanged with $a_{\min}$ and $b_{\max}$, respectively.

6. Simulation examples

In this section we will provide some simulation examples in order to illustrate the operation of the swarm model. We chose either $n=2$ or $n=3$ for the simulations for easy visualization. However, note that the results hold for any n. We first choose the case of almost constant attraction and unbounded repulsion with $g_a(\|x^i - x^j\|) = a/\|x^i - x^j\|$, and $g_r(\|x^i - x^j\|) = b/\|x^i - x^j\|^2$ with parameters $a = b = 0.2$. The plot in figure 2(*a*) shows the behaviour of $M = 31$ swarm members with initial positions chosen at random. As one can see, the individuals form a cohesive cluster (around the centre) as predicted by the theory. For this case we have the bound $\varepsilon_{\text{avg}} = b/a = 1$ as the ultimate size of the swarm. Figure 2(*b*) shows the average e_{avg} of the distances of the individual positions to the swarm centre. Note that the average converges to a value smaller than ε_{avg}, confirming the analytical derivations. The behaviour of the swarm for the other two cases is similar.

Now, consider the case of hard-limiting repulsion

$$g_r(\|x^i - x^j\|) = \frac{b}{(\|x^i - x^j\| - 2\eta)^2}$$

with $\eta = 1$. In other words, we would like the individuals to have a safety area of radius $\eta = 1$ and therefore keep a distance of at least $2\eta = 2$ units apart from each other. Figure 3(*a*) shows the behaviour of the swarm for this case. As is seen from the figure the behaviour

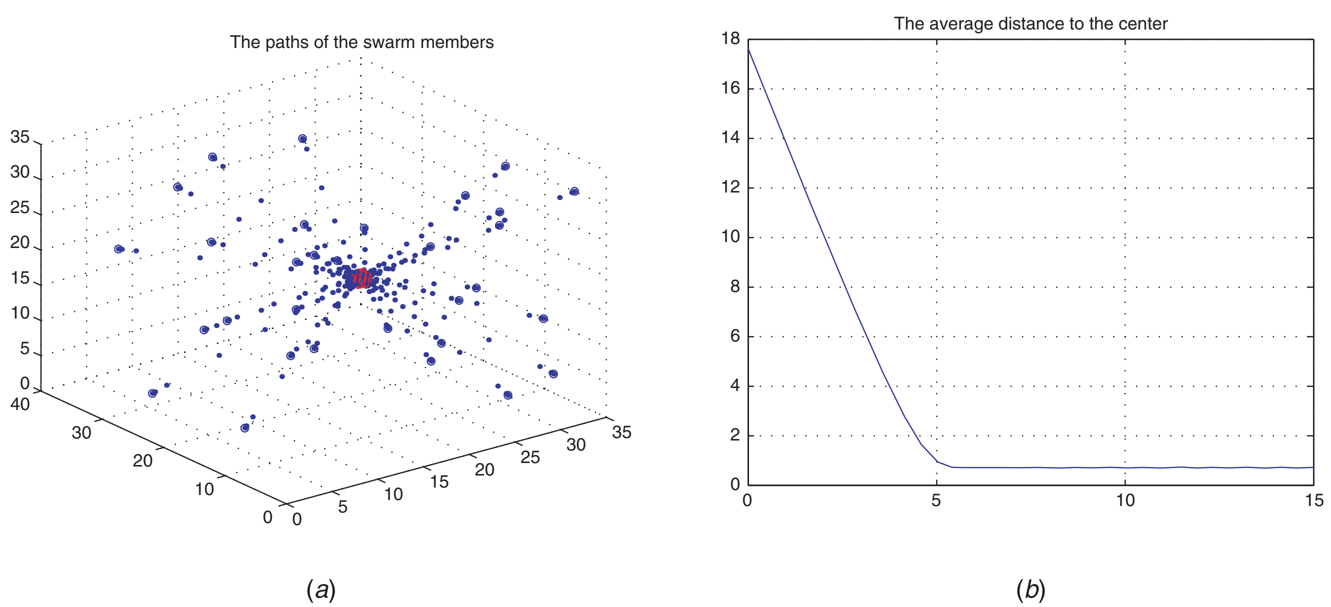

Figure 2. Swarm simulation (almost constant attraction and unbounded repulsion, $M = 31$). (*a*) Aggregation of the swarm members. (*b*) The average distance of the individuals to the centre.

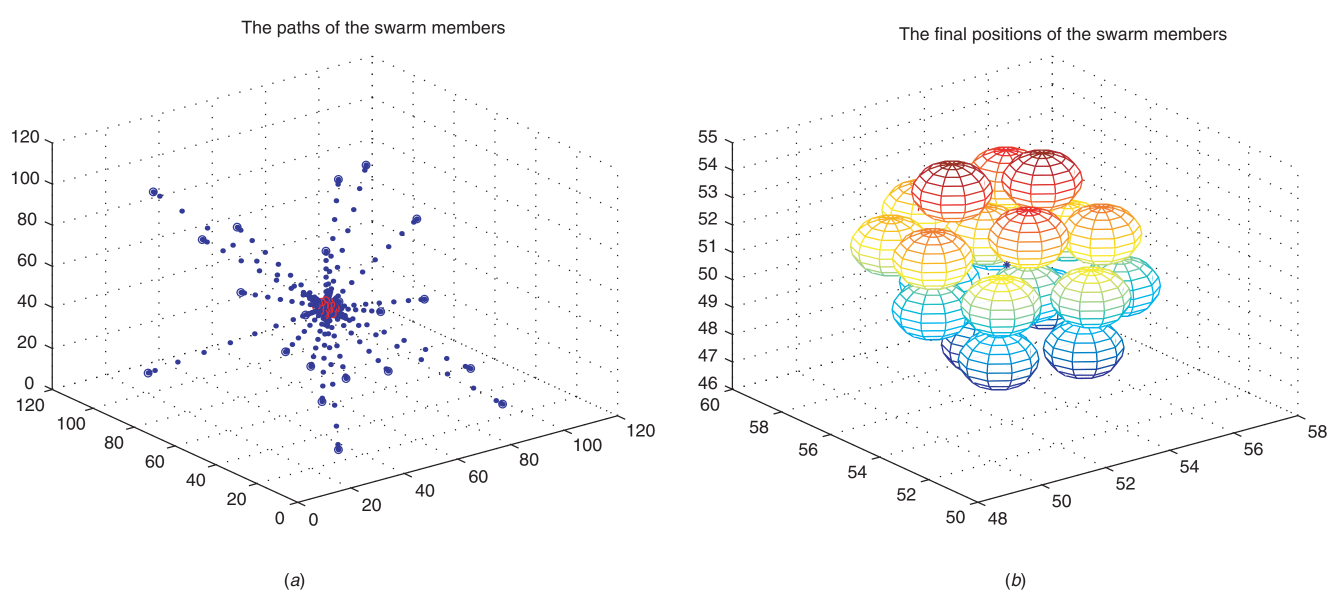

Figure 3. Swarm behaviour for hard-limiting repulsion for $n = 3$ and $M = 21$. (*a*) Swarm motion. (*b*) Final swarm arrangement.

of the swarm is similar to the case in which no hard-limiting repulsion is used. Figure 3(*b*), on the other hand, shows the positions of the individuals denoted as spheres after the swarm has been formed and the individuals have almost stopped. As can be seen from the figure the individuals keep distance $\|x^i(t) - x^j(t)\| > 2\eta$ for all pairs (i,j) as desired. We used $M = 21$ individuals for this simulation.

Figure 4(*a*) shows the final positions of the swarm members for a swarm with 11 individuals in a two-dimensional space. As can be seen from the figure the swarm members do not collide with each other and are distributed in almost a grid-like arrangement. Figure 4(*b*), on the other hand shows the final positions of a swarm with 31 individuals. As can be seen from the figure, once more there are no collisions. Moreover, the swarm size has scaled significantly with the number of individuals as expected from the discussions in the earlier sections.

Figure 5(*a*) and (*b*) show the minimum, the average, and the maximum distances between the individuals in the swarm for the cases with $M = 11$ and $M = 31$ agents, respectively. As we can see, while the minimum distance between pairs is still greater than $2\eta = 2$ for

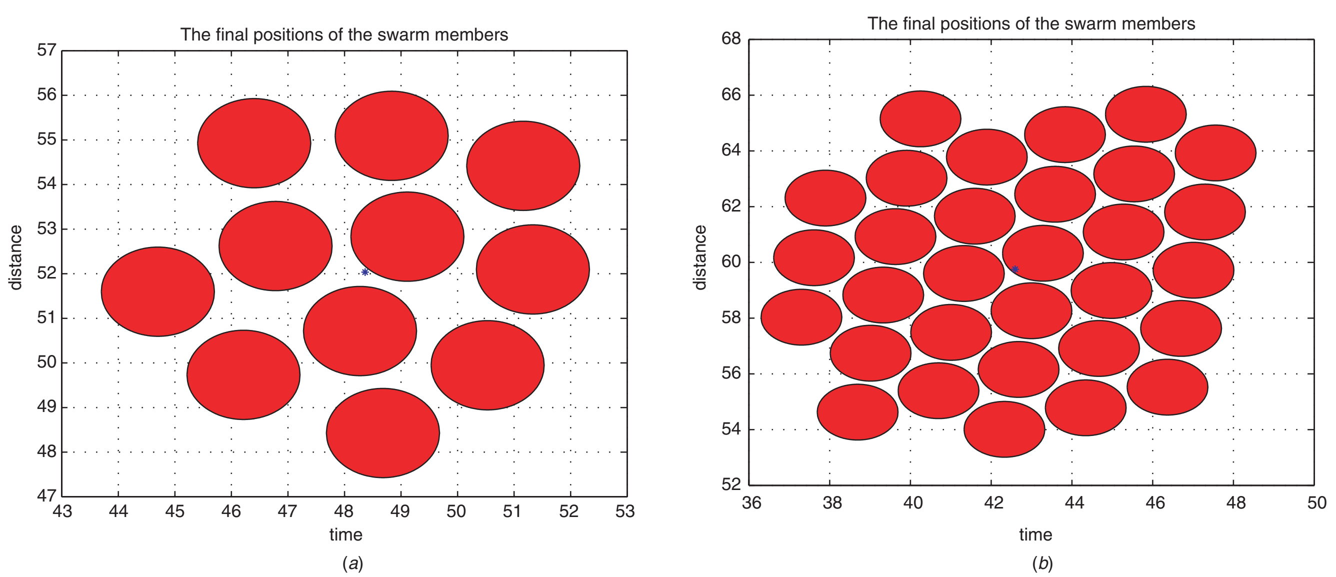

Figure 4. Final swarm positions: hard-limiting repulsion, $n = 2$, and different M. (*a*) For $M = 11$. (*b*) For $M = 31$.

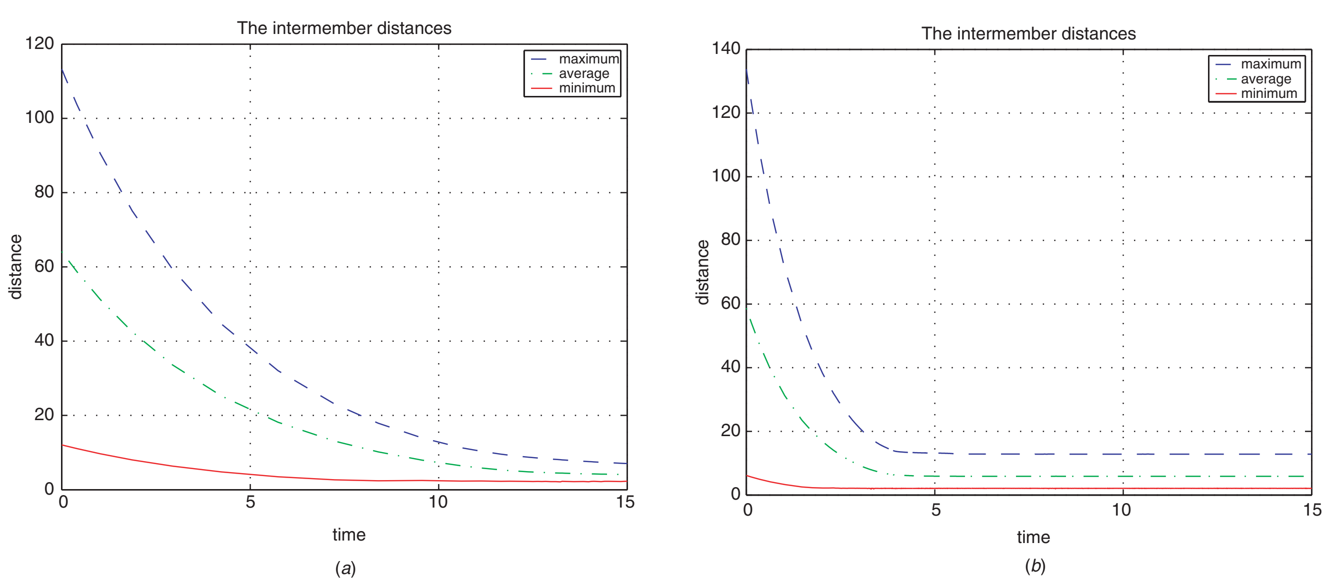

Figure 5. Swarm inter-individual distances: hard-limiting repulsion, $n = 2$, and different M. (*a*) For $M = 11$. (*b*) For $M = 31$.

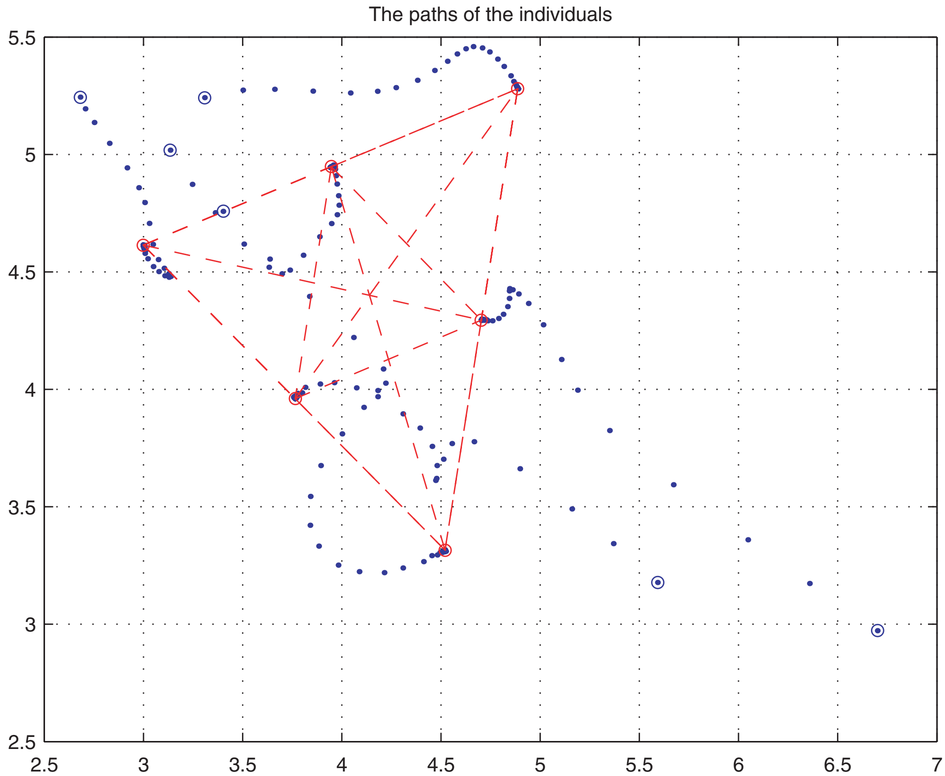

Figure 6. Equilateral triangle formation of six agents.

both of the cases, the maximum distance is larger for the case with more individuals, implying that the size of the swarm scaled with the number of individuals while the density of the swarm remained relatively constant which, on the other hand, conforms with our expectations. Having the swarm density nearly constant is an important feature of the real biological swarms and this shows that our model can describe that fact.

Now, consider the model in (15). Assume that we have six agents that are required to form a formation of an equilateral triangle with three of the agents in the middle of each edge and distances between two neighbouring agents equal to 1. For this case we design the attraction/repulsion functions for each pair of individuals such that the generalized Lyapunov function achieves a unique minimum at the desired formation. This is done by choosing $g_{ij}(\cdot)$s such that the equilibrium distances are one of $\delta_{ij} = 1$, $\delta_{ij} = 2$, or $\delta_{ij} = \sqrt{3}$ for different pairs (i, j) of individuals depending on their relative location in the desired formation. Figure 6 shows the trajectories of the agents with initial positions chosen at random. As one can see the agents move and form the required formation while avoiding collisions in accordance with the expectations since we used unbounded repulsion.

7. Concluding remarks

In this article we considered a class attraction/repulsion functions and extended our earlier results on swarm aggregations in Gazi and Passino (2003). We derived bounds on the swarm size for three different cases of attraction/repulsion functions. A disadvantage of the swarm model considered is that the agents need to know (or sense) the (relative) position of all the other agents. This results in the fact that as the number of the swarm members grows, the computation needed by each agent also grow linearly. This does not happen in approaches based on nearest neighbour rules, since an individual can have only a limited number of neighbours. Also, in crowded biological swarms the individuals may not necessarily know the position of all the other agents although in engineering applications this could be overcome with technologies such as the global positioning system.

The model considered has the advantage that it allows for global stability results to be obtained, whereas with nearest neighbour rules only local stability is possible. Moreover, it can be used not only for achieving swarm aggregations, but also for formation stabilization. The introduction of a 'private area' for the individuals prevents occurrence of collisions, an issue overlooked by many authors. Moreover, it leads to a swarms size which scales with the number of individuals and therefore to a 'more uniform' swarm density, which is an important feature of real biological swarms.

The attraction/repulsion functions discussed here are not limited to only the swarm model considered. They can be used in other models such as anisotropic swarms with reciprocal and non-reciprocal interactions Chu *et al*. (2003) as well as swarms moving in a noisy environment (Passino 2003, Liu and Passino 2004). In engineering applications with agents/vehicles with predefined dynamics, the model in this article can serve as a virtual system generating the trajectories for the real agents. Then, using methods like sliding mode control the agents can be 'forced' to obey the model (Gazi 2003). Communication or sensing delays (in obtaining the positions of the other agents in the swarm) could be also incorporated in the model in future work.

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