Dynamic Pricing Strategies for Social Networks in the Presence of Externalities

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Abstract

We propose a dynamic pricing strategy for maximizing the revenue of a seller who wishes to sell a divisible service (good) to buyers (agents) embedded in a social network. We investigate the case where the seller can perfectly price discriminate the socially interconnected buyers with positive social influences on each other. We assume that each buyer purchases a non-negative quantity of the service depending both on her internal valuations and on the purchases of her neighbors in the social network. We relax the usual game theoretic assumption of strategic agents and assume a more realistic model of myopic buyers who choose actions that maximize their present utility and do not take into account the effect of their current actions to their future payoffs. We explore the space of dynamic pricing strategies where the seller approaches the buyers sequentially and repeatedly in multiple rounds of a selected order. We propose a dynamic policy for networks with reciprocating agents and compare its performance against an existing static policy that achieves the equilibrium consumption of the buyers. We show that our sequential dynamic pricing policy yields a greater revenue value for the seller and also drives the aggregate social welfare to a higher level than what is achievable by even the best equilibrium. Further, the computational complexity of each new round is cubic in the size of the social network. Our results indicate that dynamics provide a means to improve both seller and buyer objectives even though the strategy is designed specifically to maximize the seller's revenue.

I. Introduction

The proliferation of on-line social networks such as MySpace, Facebook, and Google Plus has enabled public and private institutions to customize their interactions with potential consumers by taking advantage of the information about their social neighborhood. For profit-seeking private institutions, these developments imply a lucrative new market for providing new and personalized services to a large public. For public institutions, they may be utilized as new means of increasing social welfare or of influencing public opinion, etc. This has rekindled interest in the classical study of dynamics of information spread [1] and technology adoption [2] in social networks.

In this work, we study optimal dynamic pricing strategies for a seller who wishes to sell a divisible service/goods to a set of buyers (agents) embedded in a social network. We assume that the seller has complete knowledge of the social network and can perfectly price discriminate the buyers. The seller enforces a price (payment) vector onto the social network and each buyer reacts according to her own preferences as well the social influences from her neighbors. We assume that the influence among the buyers is positive leading to positive externalities, i.e., each sale of the product induces further sales and revenue for the seller. Such positive externalities can occur indirectly through word-of-mouth advertising or directly through an increase in the perceived utilities of the buyers. In this work, we primarily model the latter case that occurs commonly in online games, services, and applications where the utility of each user increases with the number of friends that use the service.

We explore the space of dynamic pricing strategies where the seller approaches the buyers sequentially in some selected order and repeatedly in multiple rounds. Further, we consider a pricing structure where the pricing and buyer responses operate in the same timescale. The main contributions of this work are:

- We relax the rationality and complete knowledge assumptions on the agents that are common in a game-theoretic framework and consider behavioral models that better capture the reality of their interaction. We assume that the agents best respond to past observations of their peers' behavior. In particular, agents are choosing actions that

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maximize their present utility, i.e., they are myopic, and do not take into account the effect of their current actions to their future payoffs. For networks with myopic and reciprocating buyers, we propose a sequential dynamic pricing policy by taking advantage of the structural properties of the social network.

- We show that our sequential dynamic pricing policy yields a greater revenue value for the seller and also drives the aggregate social welfare to a higher level than what is achievable by even the best equilibrium. Further, the computational complexity of each round of our sequential dynamic policy is polynomial, more precisely $O(n^3)$, where n is the number of buyers.
- We propose modifications to our dynamic pricing strategy with the objective of ensuring a max-min fairness among the buyers in terms of their individual utilities.

The rest of the paper is organized as follows. In Section II, we formulate the revenue maximization problem and define the seller and the consumer network objectives. In Section III, we discuss some of the most relevant recent work that study the problem of revenue maximization in social networks. Next, we propose a sequential dynamic strategy in Section IV and compare its performance against static policies that achieve the consumption equilibrium. In Section V, we propose modifications to our dynamic strategy with the objective of ensuring fairness among the buyers in terms of their individual utilities. We propose some interesting extensions to our work in Section VI.

II. Model

In this section, we introduce the basic formulation of the revenue maximization problem considered in this paper. Consider the setting of a seller who wishes to sell a divisible service (or good) with the objective of long-term revenue maximization. The service is purchased at the offered price by socially interconnected *buyers* with positive social influences on each other.

The buyers, denoted by the set $\mathcal{N} = \{1, \ldots, n\}$, are embedded in a social network represented by the adjacency matrix $G = [g_{i,j}]_{i,j \in \mathcal{N}}$, where $g_{i,j}$ measures the influence of buyer j on i. We assume that $g_{i,j} \in [0, 1]$ and $g_{i,i} = 0$.

The seller can perfectly price discriminate the buyers. Let $\mathbf{p}(t) = [p_1(t), \ldots, p_n(t)]^T$ be the instantaneous pricing vector, where $p_i(t)$ is the price offered to the i^{th} buyer at time (t). Let $x_i(t)$ be the quantity purchased by buyer i at the offered price $p_i(t)$. Let $\mathcal{I}_i(t)$ be information available to buyer i at t. In its most general form, this information is a collection of any accurate or inaccurate knowledge about neighbors' purchases, prices offered to the neighbors, as well as one's own past history.

In order to model the behavior of the buyers, we use a real-valued *utility function* $u_i\left(x_i(t), \mathcal{I}_i(t), p_i(t)\right)$ to capture the collective effect of the internal preferences, the price vector, and the social influences under the observed/available information. Given the price $p_i(t)$, the information $\mathcal{I}_i(t)$, buyer i purchases a quantity $x_i(t)$ in order to maximize her utility function assuming that the other buyers remain stationary.

The revenue maximization problem can be expressed as

$$\max_{\{\mathbf{p}(t)\}_{t \in \mathcal{T}}} \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{N}} p_i(t) x_i(t), \tag{1}$$

where $\mathcal{T}$ is the time-horizon of interest. We assume that the cost of production is zero.

While we later extend this model to the dynamic setting in Section IV, here we consider the following static formulation of the revenue maximization. Consider a static pricing vector $\mathbf{p} = (p_i)_i$ selected by the seller, where p_i is the price of unit service for agent i. Then, $\mathbf{x} = (x_i)_i$ is the non-negative-valued vector of purchased amount of service by the buyers. The objective of the seller is to maximize the revenue, $f(\mathbf{p}, \mathbf{x}) = \sum_{i \in \mathcal{N}} x_i p_i$.

Each buyer is assumed to be unaware of the price offered to the others. This scenario occurs in cases where consumers receive personalized offers from the seller in the form of coupons, discounts, etc.. However, each buyer has accurate knowledge of the purchases of his neighbors. Hence, $\mathcal{I}_i = \mathbf{x_{-i}}$, where $-i$ denotes the neighborhood of buyer i. The utility of buyer i is determined by her preferences, the actions of the rest of the buyers and the influence matrix. We use the well-studied *linear-quadratic utility function* $\{u_i(\cdot)\}_i$ with parameters $\{(a_i, b_i)\}_i$ to capture these three components:

$$u_i(x_i, \mathbf{x_{-i}}, p_i) = a_i x_i - b_i x_i^2 + x_i \sum_{j \in \mathcal{N}} g_{i,j} x_j - p_i x_i,$$

where the first two terms model the internal preferences of buyer i, the third term captures the positive social network effect, and the final term is the cost of usage. The linear-quadratic utility function allows for a tractable analysis and also serves as a good second order approximation of the broader class of concave utility functions. We note that such a utility function is commonly used in economic models [3], [4], [5], [6], [7]. To ensure the boundedness of the consumption level of each buyer, we assume that

ASSUMPTION *1:* For all $i \in \mathcal{N}$, $b_i > \sum_{j \in \mathcal{N}} g_{i,j}$.

In what follows, I is the $n \times n$ identity matrix and $\mathbf{1}$ is the column vector of all $1'$s. Let $\Lambda = Diag[2b_1, \ldots, 2b_n]$, and $\mathbf{a} = [a_1, \ldots, a_n]^T > \mathbf{0}$. In Section III, we briefly describe the optimal static pricing policy proposed in [8] for the above static formulation of the revenue maximization problem. In Section IV, we propose an alternate dynamic strategy and compare its performance against the optimal static policy.

III. Characterizing Equilibrium and Optimal Static Pricing

In this section, we briefly describe the results of [8] which studies the revenue maximization problem for a divisible service/good in a game-theoretic framework by modeling a network game among the consumers. They consider the static formulation of the revenue maximization problem with linear-quadratic utility functions described in Section II and characterize the consumption equilibrium given the prices. They find optimal static pricing strategies of the seller that achieve the consumption equilibrium while maximizing the seller revenue.

DEFINITION *1 (Consumption Equilibrium):* For a fixed vector of prices $\mathbf{p}$ chosen by the seller, a vector $\bar{\mathbf{x}}$ is a consumption equilibrium if, for all $i \in \mathcal{N}$,

$$\bar{x}_i \in \arg \max_{z \in [0,\infty)} u_i(z, \bar{\mathbf{x}}_{-\mathbf{i}}, p_i).$$

In [8], the authors show that the consumption equilibrium is unique and that under the optimal pricing strategy of the seller with perfect price discrimination, all consumers purchase a positive amount of the good. Specifically, $\bar{x} = [\Lambda - G]^{-1}(\mathbf{a} - \bar{\mathbf{p}})$, where the optimal prices $\bar{p}$ are given as

$$\bar{\mathbf{p}} = \frac{\mathbf{a}}{2} + G\Lambda^{-1}\mathcal{K}\left(\tilde{G}, \Lambda^{-1}, \frac{\mathbf{a}}{2}\right) - G^T\Lambda^{-1}\mathcal{K}\left(\tilde{G}, \Lambda^{-1}, \frac{\mathbf{a}}{2}\right).$$

In the above expression, $\tilde{G} = \frac{G+G^T}{2}$, and $\mathcal{K}\left(\tilde{G}, \Lambda^{-1}, \frac{\mathbf{a}}{2}\right) = (I - \tilde{G}\Lambda^{-1})^{-1}\frac{\mathbf{a}}{2}$ is the weighted Bonacich centrality measure. This centrality measures how central each agent is with respect to the average network interaction measured by $\tilde{G}$. This characterization of the optimal prices in terms of the Bonacich centrality measure indicates that a higher discount is given to those buyers that can influence more central agents.

The consumption subgame is supermodular owing to the concave utility function and hence, the equilibrium solution can be implemented using a *static pricing policy* based on the greedy algorithm for computing equilibrium given in [9]: if the prices are fixed at the optimal and the buyers purchase iteratively, the amounts purchased converge to the consumption equilibrium. This static pricing policy is practical since the buyers need only local information and past history to converge to the global equilibrium.

The above result derived in [8] focuses on understanding/computing the equilibrium conditions of the social network for a *static* pricing vector. While these studies are crucial as a basis for guiding the operation of a social network, they do not exploit the full dynamic capabilities of the interactions that actually exist in today's systems. In particular, the seller does not utilize his ability to strategically modify his input into the social network in response to the history and the observed network state. This motivates us to explore the strategies of the seller outside of the limited space of static schemes and compare dynamic with static policies with respect to the seller's objective and the buyers' aggregate welfare.

However, the problem of finding optimal dynamic pricing strategies in a setting where the adoption of the new technology/product depends on the price offered often leads to computational intractability. For instance, in [10], the authors consider the problem of maximizing the revenue of a seller who wishes to sell an indivisible good and study marketing strategies within the class of sequential strategies with perfect price discrimination. They show that the problem of finding optimal pricing is NP-hard even for simple settings and investigate greedy pricing strategies. In the next section, we propose a sequential dynamic strategy for networks with reciprocating buyers that *yields a greater revenue for the seller and also drives the aggregate social welfare to a higher level than what is achievable by the static policy proposed in [8].*

IV. Sequential dynamic pricing

We now propose an alternative dynamic pricing strategy for the revenue maximization problem formulated in Section II that improves both the seller's revenue and the total utility of the buyers. We assume that buyers only have local information about their neighborhood and past history of what their neighbors have purchased. This assumption models myopic and non-strategic buyers.

The seller adopts a discrete-time dynamic pricing strategy where the price vector is changed once every *round*. In each round, the seller approaches the buyers in some order $\eta^{(k)} := \{\eta_1^{(k)}, \ldots, \eta_n^{(k)}\}$, where $\eta_i^{(k)} \in \mathcal{N}$ and $\eta_i^{(k)} \neq \eta_j^{(k)}$ for any $i \neq j$. Let $p_i^{(k)}$ denote the price per unit offered by the seller to the buyer i in the k^{th} round. Let $x_i^{(k)}$ denote the amount purchased by buyer i in round k.

Let $\mathcal{I}_i^{(k)}$ represent the information available to buyer i in round k. We assume that each buyer has accurate knowledge of the quantities purchased by all her neighbors prior to her own purchase in round k. This information can be compactly represented by the pair $(\mathbf{x_{-i}}^{(1:k-1)}, [x_j^{(k)}]_{j\in\mathcal{N}_i^{(k)}})$, where $-\mathbf{i}$ is the set of neighbors of i, and $\mathcal{N}_i^{(k)} \subset \mathcal{N}$ is the set of buyers that purchase before buyer i in the current round. Then $\mathcal{I}_i^{(k)} = \left(x_i^{(1:k-1)}, p_i^{(1:k-1)}, \mathbf{x_{-i}}^{(1:k-1)}, [x_j^{(k)}]_{j\in\mathcal{N}_i^{(k)}}\right)$. Let $y_i^{(k)} = \sum_{j=1}^{k} x_i^{(k)}$ be the cumulative quantity purchased by buyer i till round k. Buyer i wishes to maximize her present utility in round k, which is given by the following linear-quadratic function:

$$
\begin{aligned}
u_i\left(x_i^{(k)}, \mathcal{I}_i^{(k)}, p_i^{(k)}\right) &= a_i\left(y_i^{(k-1)} + x_i^{(k)}\right) \\
&- b_i\left(y_i^{(k-1)} + x_i^{(k)}\right)^2 + \left(y_i^{(k-1)} + x_i^{(k)}\right)\sum_{j\in\mathcal{N}} g_{i,j} y_j^{(k-1)} + \\
&+ \left(y_i^{(k-1)} + x_i^{(k)}\right)\sum_{j\in\mathcal{N}_i^{(k)}} g_{i,j} x_j^{(k)} - \sum_{j=1}^{k} p_i^{(j)} x_i^{(j)}.
\end{aligned} \tag{2}
$$

In each round k, the seller chooses $p_i^{(k)}$ in order to maximize his profit given by $\sum_{i\in\mathcal{N}} p_i^{(k)} x_i^{(k)}$. We note that this price offered by the seller can act as a price or a reward depending on whether it is positive or negative valued.

We make the following assumption on the reciprocity between any two neighbors in the social network:

Assumption *2:* For all $i \in \mathcal{N}$, $g_{i,j} = g_{j,i}$.

This assumption models the important case of undirected networks such as a co-authorship network where the weight of each edge represents the strength of ties between the nodes. It also models the case of commodities such as online messaging services where the mutual benefit obtained by two users due to each others' actions is equal.

Let $\tilde{x}_i^{(k)}$ and $\tilde{p}_i^{(k)}$ be the optimal quantity purchased and optimal price of buyer i in round k. Fix an ordering of buyers $\eta := \{\eta_1, \ldots, \eta_n\}$ and consider the following problem,

Problem *1:*

$$
V^{(k)}(\eta) = \max \sum_{i=1}^{n} p_{\eta_i}^{(k)} x_{\eta_i}^{(k)}
$$
$$
\text{subject to } x_{\eta_i}^{(k)} = \arg\max_{z\geq 0} u_{\eta_i}\left(z, \tilde{I}_{\eta_i}^{(k)}, p_{\eta_i}^{(k)}\right) \quad \forall i,
$$

where $\tilde{\mathcal{I}}_{\eta_i}^{(k)} = \left(\tilde{x}_{\eta_i}^{(1:k-1)}, \tilde{p}_{\eta_i}^{(1:k-1)}, \tilde{\mathbf{x}}_{-\eta_\mathbf{i}}^{(1:k-1)}, [\tilde{x}_{\eta_j}^{(k)}]_{j<i}\right)$ is the information available to buyer η_i in round k. The revenue maximization problem for round k can be formally stated as follows,

Problem *2:*

$$
\max V^{(k)}(\eta)
$$
$$
\text{subject to } \eta \in Sym(\mathcal{N}),
$$

where $Sym(\mathcal{N})$ is the set of all permutations of the elements of $\mathcal{N}$, i.e., it is the set of all possible orderings of buyers.

The seller has to choose the ordering of buyers and the prices in order to maximize his profit. For each round, PROBLEM 2 can be formulated as a dynamic program, but the large size of the state space makes its analysis intractable. In each round, we need to consider $n!$ orderings to find the optimal order - this lends high complexity to any potential dynamic strategy. However, under Assumptions 1, and 2, the optimal solution of PROBLEM 1 has an interesting structure as characterized by the following proposition.

PROPOSITION *1:* Under Assumptions 1, and 2, the optimal value of PROBLEM 1, $V^{(k)}(\eta)$, and the unique optimal solution, $\tilde{\mathbf{x}}^{(\mathbf{k})}(\eta)$, are both independent of the ordering η. ◇

Proof: Please see Appendix I. ■

In light of Proposition 1, the complexity of PROBLEM 2 reduces dramatically. PROBLEM 1 for any ordering η gives an optimal pricing solution to PROBLEM 2. The following proposition characterizes the optimal solution of PROBLEM 2.

PROPOSITION *2:* Under Assumptions 1, and 2, the unique optimal solution of PROBLEM 2 for round k is

$$\tilde{\mathbf{x}}^{(\mathbf{k})} = [2\Lambda - G]^{-1}[I - (\Lambda - G)(2\Lambda - G)^{-1}]^{k-1}\mathbf{a}.$$

Also, the total optimal consumption, $\tilde{y}_i = \lim_{k\to\infty} \sum_{j=1}^{k} \tilde{x}_i^{(j)}$ converges to $[\Lambda - G]^{-1}\mathbf{a}$ as the number of rounds, k goes to ∞. ◇

Proof: Please see Appendix I. ■

Algorithm 1 : Sequential dynamic pricing strategy

for each Round k **do**

STEP 1. Let $\tilde{\mathbf{x}}^{(\mathbf{k})} = [\tilde{x}_1^{(k)}, \ldots, \tilde{x}_n^{(k)}]^T$.

STEP 2. Consumption in Round k :

$$\tilde{\mathbf{x}}^{(\mathbf{k})} = [2\Lambda - \mathbb{G}]^{-1}[I - (\Lambda - \mathbb{G})(2\Lambda - \mathbb{G})^{-1}]^{k-1}\mathbf{a}.$$

STEP 3. Total consumption until Round k :

$$\tilde{y}_i^{(k-1)} = \sum_{m=1}^{k-1} \tilde{x}_i^{(m)}.$$

STEP 4. Fix the ordering of buyers as $\eta = \{1, 2, \ldots, n\}$.

STEP 5. Offered prices for Round k :

$$\tilde{p}_i^{(k)} = a_i - 2b_i\tilde{y}_i^{(k-1)} + \sum_{j\in\mathcal{N}} g_{i,j}\tilde{y}_j^{(k-1)} - 2b_i\tilde{x}_i^{(k)} + \sum_{j<i} g_{j,i}\tilde{x}_j^{(k)}.$$

Output: $\{\tilde{p}_1^{(k)}, \ldots, \tilde{p}_n^{(k)}\}$.

end for

Based on Propositions 1, and 2, we propose the sequential dynamic pricing strategy given in Algorithm 1 which has significantly lower computational complexity.

The complexity of running our pricing strategy in each round is polynomial in n, more precisely, round k incurs an additional complexity of $O(n^3)$ computations associated with matrix multiplication.

The revenue under our pricing strategy is given by

$$R_D = \sum_{k=1}^{\infty} \sum_{i\in\mathcal{N}} \tilde{p}_i^{(k)} \tilde{x}_i^{(k)}.$$

Let us define the total utility gained by the buyers as

$$U_D = \sum_{i\in\mathcal{N}} \left(a_i\tilde{y}_i - b_i\tilde{y}_i^2 + \tilde{y}_i \sum_{j\in\mathcal{N}} g_{i,j}\tilde{y}_j \right) - R_D.$$

For comparison, let $R_S = \sum_{i\in\mathcal{N}} \bar{p}_i\bar{x}_i$ and $U_S = \sum_{i\in\mathcal{N}} u_i(\bar{x}_i, \bar{\mathbf{x}}_{-\mathbf{i}}, \bar{p}_i)$ denote the seller's revenue and the buyers' total utility respectively under the optimal static pricing strategy described in Section III that converges to the revenue maximizing equilibrium.

Proposition 3 compares the revenue and total utility of the static pricing strategy with perfect price discrimination [8] with that of our pricing strategy.

PROPOSITION *3:* The performance of the sequential dynamic strategy strictly dominates the optimal static policy both in total revenue and in total utility. More precisely, under Assumptions 1, and 2, $R_D > R_S$ and $U_D > U_S$. $\diamond$

Proof: Please see Appendix I. ■

Hence, under similar assumptions of myopic and non-strategic buyers, our alternative pricing strategy achieves a strictly higher revenue and total utility than that obtained by the static pricing policy in [8].

Figure 1 compares the revenue obtained by our dynamic pricing strategy with that of the greedy implementation

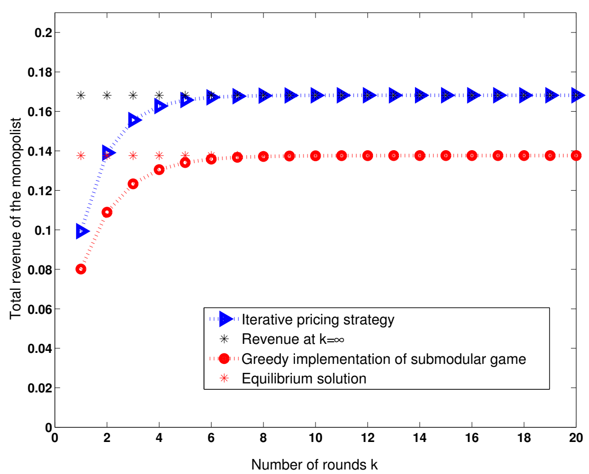

Fig. 1: Total revenue obtained by the seller in a random network of 500 buyers.

to attain the equilibrium solution. We consider a random network of $n = 500$ buyers and a_i is a uniform random variable in $(0, 1]$ for all i. We can see that the rates of convergence of both strategies are roughly equal and that the total revenue obtained by our scheme is higher than that obtained by the static policy.

Figure 2 compares the revenue and total utility obtained by our dynamic pricing strategy at the end of each round with that of the greedy implementation to attain the consumption equilibrium solution. We consider a random network of $n = 500$ buyers, a_i is a uniform random variable in $(0, 10]$ for all i, and we implement the dynamic strategy for $k = 20$ rounds. The results show more than 20% and 160% improvement in the total revenue and utility levels.

V. REORDERING FOR FAIRNESS

In Section IV, we showed in Proposition 2 that Algorithm 1 is optimal *independent of the ordering in which the buyers are approached*. We can explore this invariance to further improve *each buyer*'s individual utility by selecting a different ordering of buyers in each round to distribute the wealth fairly over all buyers over time. This flexibility offers an exciting extension that refines the optimal method to maximize individual buyer utilities. We

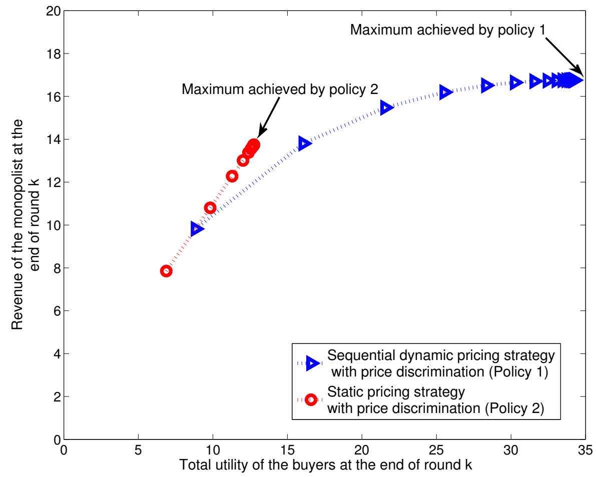

Fig. 2: The sequential dynamic pricing strategy takes advantage of the dynamic pricing capability to improve both the seller's revenue and the buyers' utility.

propose a modification to our dynamic strategy in Algorithm 2 that chooses an arbitrary ordering in the first round and uses a max-min fairness criteria to find the order in subsequent rounds.

Algorithm 2 : Sequential dynamic pricing strategy with reordering

for each Round k **do**

STEP 1. Let $\tilde{\mathbf{x}}^{(\mathbf{k})} = [\tilde{x}_1^{(k)}, \ldots, \tilde{x}_n^{(k)}]^T$.

STEP 2. Consumption in Round k :

$$\tilde{\mathbf{x}}^{(\mathbf{k})} = [2\Lambda - \mathbb{G}]^{-1}[I - (\Lambda - \mathbb{G})(2\Lambda - \mathbb{G})^{-1}]^{k-1}\mathbf{a}.$$

STEP 3. Total consumption until Round k :

$$\tilde{y}_i^{(k-1)} = \sum_{m=1}^{k-1} \tilde{x}_i^{(m)}.$$

STEP 4. Fix the ordering $\eta := \{\eta_1, \ldots, \eta_n\}$ such that

$$\eta_i = \arg \min_{i \in \mathcal{N} \setminus \{\eta_1, \ldots, \eta_{i-1}\}} u_i\left(x_i^{(k)}, I_i^{(k)}, p_i^{(k)}\right),$$

where $u_i(.)$ is defined in Equation 2.

STEP 5. Offered prices for Round k :

$$\tilde{p}_{\eta_i}^{(k)} = a_{\eta_i} - 2b_{\eta_i}\tilde{y}_{\eta_i}^{(k-1)} + \sum_{j \in \mathcal{N}} g_{\eta_i,j}\tilde{y}_j^{(k-1)} - 2b_{\eta_i}\tilde{x}_{\eta_i}^{(k)} + \\ + \sum_{j<i} g_{\eta_j,\eta_i}\tilde{x}_{\eta_j}^{(k)}.$$

Output: $\{\tilde{p}_1^{(k)}, \ldots, \tilde{p}_n^{(k)}\}$.

end for

Consider the case of a completely symmetric graph with $a_i = a, b_i = b$, and $g_{i,j} = g$ for all $i, j \in \mathcal{N}$. The optimal quantity purchased by each buyer in round k of the sequential dynamic strategy in Algorithm 1 is given as

$$\tilde{x}_i^{(k)} = \alpha^{k-1}\frac{a}{4b-(n-1)g} \forall i \in \mathcal{N},$$

where $\alpha = \frac{2b}{4b-(n-1)g} < 1$. The price offered to the m^{th} buyer in round k of repeating the same sequence

$\{1, 2, \dots, n\}$ is given as,

$$\tilde{p}_i^{(k)} = \alpha^{k-1} \frac{a(2b - (n-m)g)}{4b - (n-1)g}.$$

Hence, the prices increase linearly in the buyer index m. This generates a wide disparity in the individual buyer utility with the n^{th} buyer receiving the least utility.

For the symmetric graph case, Algorithm 2 reduces to the simple scheme of choosing the order $\eta := \{1, 2, \dots, n\}$ for round $k = 1$ and choosing $\eta := \{n, n-1, \dots, 1\}$ for round $k > 1$. For this simple scheme, the total asymptotic utility of buyer m denoted by u_m obtained in Algorithm 2 is given as,

$$u_m = \frac{a^2}{(4b - (n-1)g)^2} \left((2b - (n-m)g) + \frac{(2b - mg)\alpha^2}{1 - \alpha^2} \right).$$

It can be seen that $u_n < u_{n-1} < \dots < u_1$. This implies that any other order in round $k > 1$ in which buyer n is not scheduled in the beginning will lead to a lower utility compared to orderings in which she is scheduled in the beginning. This argument holds for any buyer m in position $(n - m + 1)$ for $k > 1$. Hence, Algorithm 2 achieves max-min fairness as summarized below:

FACT *1:* For a completely symmetric graph with $a_i = a, b_i = b$, and $g_{i,j} = g$ for all $i, j \in \mathcal{N}$, Algorithm 2 is optimal in terms of fairly distributing the net utility among the buyers under a max-min fairness criterion.

Figure 3 compares the individual buyer utilities of buyer 1 and buyer n obtained by Algorithms 1 and 2 for a symmetric network of 50 buyers. Note the individual buyer utilities of both Algorithms 1 and 2 are greater than that of the static policy described in Section III for the case of a symmetric graph.

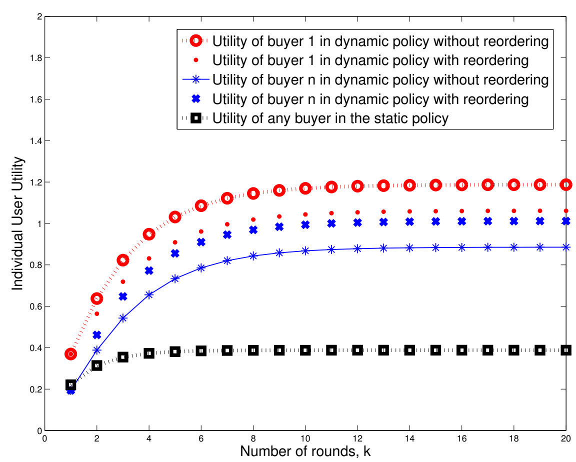

Fig. 3: The sequential dynamic pricing strategy with reordering reduces the difference between the individual buyer utilities.

VI. CONCLUSION

We developed a dynamic pricing strategy for maximizing the revenue of a seller who wishes to sell divisible goods to buyers embedded in a social network. We showed that our sequential dynamic pricing policy yields a greater revenue value for the seller and also drives the aggregate social welfare to a higher level than what is achievable by the optimal static pricing policy in [8]. Further, we proposed a modified sequential dynamic pricing policy that ensures max-min fairness among buyers in terms of their individual utilities for a completely symmetric graph. Our results demonstrate that dynamics provide a means to improve both seller and buyer objectives even though the strategy is designed specifically to maximize the seller's revenue. This motivates us to pursue the following research questions in future:

- An avenue of immediate work is the extension of our sequential dynamic algorithm to the case of asymmetric social influences with $g_{i,j} \neq g_{j,i}$. In this case, the invariance of ordering no longer holds. Yet, we can still utilize the connection unearthed in Section III between the optimal static pricing strategy and Bonacich centrality measure to propose and investigate the performance of low-complexity dynamic pricing strategies that determine the ordering of buyers according to this and other centrality measures for each round.

- In this work, we have considered a particular *sequential* pricing structure where the pricing and user responses operate in the same timescale. This structured setup was sufficient to reap significant gains from dynamic pricing and motivates us to consider more general dynamics in which the parties may operate at different timescales and in a less orderly fashion. This opens up a rich space of dynamic interactions, and call for new designs that are amenable to scalable and efficient operation.

VII. Appendix I

Lemma *1:* Under Assumption 1, the matrix $(r\Lambda - G)$ is invertible and has non-negative entries for all $r \geq 1$.

Proof: This proof is similar to proof of Lemma 4 in [8]. Let v be an eigenvector of $(r\Lambda)^{-1}G$ with λ being the corresponding eigenvalue. Let v_i be the largest entry of v in absolute value, i.e., $|v_i| \geq |v_j|$ for all $j \in \mathcal{N}$. Hence, $|\lambda|$ can be bounded as follows,

$$
\begin{aligned}
|\lambda v_i| = |((r\Lambda)^{-1}G)_i v| &\leq \sum_{j\in\mathcal{N}} ((r\Lambda)^{-1}G)_{ij}|v_j| \\
&\leq \frac{1}{2rb_i}|v_i| \sum_{j\in\mathcal{N}} g_{ij} \\
&< \frac{|v_i|}{2r},
\end{aligned}
$$

where $((r\Lambda)^{-1}G)_i$ denotes the ith row of $(r\Lambda)^{-1}G$. The first and second inequalities use the fact that $((r\Lambda)^{-1}G)_{ij} = \frac{g_{ij}}{2rb_i} \geq 0$, and the last inequality follows from Assumption 1. Hence, every eigenvalue of $(r\Lambda)^{-1}G$ is strictly less than 1 for $r \geq 1$.

Note that each eigenvalue of $I - (r\Lambda)^{-1}G$ can be written as $1 - \lambda$ where λ is an eigenvalue of $(r\Lambda)^{-1}G$. Since every eigenvalue of $(r\Lambda)^{-1}G$ is strictly smaller than 1, it follows that none of the eigenvalues of $I - (r\Lambda)^{-1}G$ is zero, hence the matrix is invertible. This also implies that $(r\Lambda - G)$ is invertible for all $r \geq 1$ and furthermore, we have,

$$
\begin{aligned}
(r\Lambda - G)^{-1} &= \left(I - (r\Lambda)^{-1}G\right)^{-1}(r\Lambda)^{-1} \\
&= \sum_{j=0}^{\infty} \left((r\Lambda)^{-1}G\right)^j (r\Lambda)^{-1},
\end{aligned}
$$

where the last equality follows since the spectral radius of $(r\Lambda)^{-1}G$ is strictly smaller than 1. Since the entries of Λ and G are non-negative, it follows that all the entries of $(r\Lambda - G)^{-1}$ are non-negative. ■

Lemma *2:* Under Assumptions 1, and 2, the spectral radius of $[\Lambda(2\Lambda - G)^{-1}]$ is less than 1.

Proof: From Lemma 1, we have that every eigenvalue of $(r\Lambda)^{-1}G$ is less than $\frac{1}{2r}$. Let $\lambda_{max}(A)$ and $\lambda_{min}(A)$ denote the maximum and minimum eigenvalue of a matrix A. Since Λ and G are symmetric matrices, we have that $\left((2\Lambda)^{-1}G\right)^T = G(2\Lambda)^{-1}$ and hence, the matrices $(2\Lambda)^{-1}G$ and $G(2\Lambda)^{-1}$ have the same set of eigenvalues. Now, $[\Lambda(2\Lambda - G)^{-1}] = \frac{1}{2}(I - G(2\Lambda)^{-1})$. Therefore, we have

$$
\begin{aligned}
|\lambda_{max}\left([\Lambda(2\Lambda - G)^{-1}]\right)| &= \left|\lambda_{max}\left(\frac{1}{2}(I - G(2\Lambda)^{-1})^{-1}\right)\right| \\
&= \frac{1}{2}\frac{1}{|\lambda_{min}\left(I - G(2\Lambda)^{-1}\right)|} \\
&\leq \frac{1}{2}\frac{1}{1 - |\lambda\left(G(2\Lambda)^{-1}\right)|} < \frac{2}{3},
\end{aligned}
$$

where the last inequality follows from the fact that $|\lambda(G(2\Lambda)^{-1})| < \frac{1}{4}$. ■

A. Proof of Propositions 1, and 2

Consider the first round. We drop the superscript (k) for this case. Fix an order $\eta := (\eta_1, \ldots, \eta_n)$. Firstly, note that the optimal quantity purchased is positive for all buyers. If the optimal quantity is zero for some buyer j, then the seller can reduce the price p_j to make the quantity purchased positive and this can only result in a greater revenue extracted from buyer j as well as the other buyers in the sequence after j. The revenue extracted from the buyers that purchase before j in the sequence remain unaffected by this change. Hence under the assumption that x_i is positive for all i, the revenue is given as,

$$\begin{aligned} V(\eta, \mathbf{x}) &= \sum_{i=1}^{n} p_{\eta_i} x_{\eta_i} \\ &= \sum_{i=1}^{n} (a_{\eta_i} - 2b_{\eta_i} x_{\eta_i} + \sum_{j<i} g_{\eta_i,\eta_j} x_{\eta_j}) x_{\eta_i}. \end{aligned}$$

Let $\tilde{x}_{\eta_i}$ be the optimal quantity purchased by buyer i. Then, $\frac{\partial V(\eta,\mathbf{x})}{\partial z} = 0\Big|_{z=\tilde{x}_{\eta_i}}$ which can be simplified as follows:

$$a_{\eta_i} - 4b_{\eta_i}\tilde{x}_{\eta_i} + \sum_{j<i} g_{\eta_i,\eta_j}\tilde{x}_{\eta_j} + \sum_{j>i} g_{\eta_j,\eta_i}\tilde{x}_{\eta_j} = 0.$$

However, since $g_{i,j} = g_{j,i}$, the above expression reduces to,

$$a_i - 4b_i\tilde{x}_i + \sum_{j} g_{i,j}\tilde{x}_j = 0,$$

which is independent of the order η. Hence, $\tilde{\mathbf{x}} = (2\Lambda - G)^{-1}\mathbf{a}$. Also, the optimal revenue for first round is independent of η and is given as,

$$\begin{aligned} V(\eta) = V(\eta, \tilde{\mathbf{x}}) &= \sum_{i=1}^{n} (a_i - 2b_i x_i + \sum_{j<i} g_{i,j} x_j) x_i \\ &= \frac{1}{2}\mathbf{a}^T (2\Lambda - G)^{-1}\mathbf{a}. \end{aligned}$$

We prove the results in Propositions 1, and 2 inductively. The results are true for round $j = 1$. Suppose the results hold for rounds $j = 2, 3, \ldots, k-1$. Let $\tilde{x}_i^{(k)}$ be the optimal quantity purchased by buyer i in round k. Define $\tilde{y}_i^{(k)} = \sum_{j=1}^{k} \tilde{x}_i^{(j)}$. Then, by a similar argument as in the case of a single round, we have that the optimal quantity purchased is positive for all buyers. Hence under the assumption that $x_i^{(k)}$ is positive for all i, the revenue is given as,

$$\begin{aligned} V^{(k)}(\eta, \mathbf{x}^{(k)}) &= \sum_{i=1}^{n} p_{\eta_i}^{(k)} x_{\eta_i}^{(k)} \\ &= \sum_{i=1}^{n} \Big(a_{\eta_i} - 2b_{\eta_i}\tilde{y}_{\eta_i}^{(k-1)} + \sum_{j\in\mathcal{N}} g_{\eta_i\eta_j}\tilde{y}_{\eta_j}^{(k-1)} - \\ &\qquad - 2b_{\eta_i} x_{\eta_i}^{(k)} + \sum_{j<i} g_{\eta_i,\eta_j} x_{\eta_j}^{(k)} \Big) x_{\eta_i}^{(k)}. \end{aligned}$$

Let $a_i^{(k)} = a_i - 2b_i\tilde{y}_i^{(k-1)} + \sum_{j\in\mathcal{N}} g_{i,j}\tilde{y}_j^{(k-1)}$. Since the spectral radius of the matrix $[I - (\Lambda - G)(2\Lambda - G)^{-1}]$ is less than 1, we have that,

$$\begin{aligned} \mathbf{a}^{(k)} &= \mathbf{a} - \Lambda\tilde{y}_i^{(k-1)} + G\tilde{y}_i^{(k-1)} \\ &= [I - (\Lambda - G)(2\Lambda - G)^{-1}]^{(k-1)}\mathbf{a}. \end{aligned}$$

The optimal quantity, $\tilde{x}_{\eta_i}^{(k)}$ should satisfy $\frac{\partial V^{(k)}(\eta,\mathbf{x}^{(k)})}{\partial z} = 0\Big|_{z=\tilde{x}_{\eta_i}^{(k)}}$ resulting in the following expression:

$$a_i^{(k)} - 4b_{\eta_i}\tilde{x}_{\eta_i}^{(k)} + \sum_{j<i} g_{\eta_i,\eta_j}\tilde{x}_{\eta_j}^{(k)} + \sum_{j>i} g_{\eta_j,\eta_i}\tilde{x}_{\eta_j}^{(k)} = 0.$$

which is independent of the order η since $g_{i,j} = g_{j,i}$. Hence, the optimal quantity $\tilde{\mathbf{x}}^{(k)} = (2\Lambda - G)^{-1}[I - (\Lambda - G)(2\Lambda - G)^{-1}]^{(k-1)}\mathbf{a}$. Also, the total optimal consumption as the number of rounds, k goes to ∞ is

$$\begin{aligned}\lim_{k\to\infty}\tilde{\mathbf{y}}^{(k)} &= \sum_{k=1}^{\infty}(2\Lambda - G)^{-1}[I - (\Lambda - G)(2\Lambda - G)^{-1}]^{(k-1)}\mathbf{a},\\ &= (2\Lambda - G)^{-1}[I - \big(I - (\Lambda - G)(2\Lambda - G)^{-1}\big)]^{-1}\mathbf{a},\\ &= (2\Lambda - G)^{-1}(2\Lambda - G)(\Lambda - G)^{-1}\mathbf{a},\\ &= (\Lambda - G)^{-1}\mathbf{a}.\end{aligned}$$

This proves the inductive step and completes the proof of Propositions 1, and 2.

B. Proof of Proposition 3

We first show that our dynamic strategy outperforms the static policy in terms of revenue.

Let $V^{(k)}(\mathbf{x}^{(k)})$ be maximum revenue obtained by our sequential dynamic policy in the $k-$th round. Then $V^{(k)}(\mathbf{x}^{(k)}) = \frac{1}{2}\mathbf{a}^{(k)T}(2\Lambda - G)^{-1}\mathbf{a}^{(k)}$. The asymptotic revenue, R_D under our pricing strategy is given by

$$\begin{aligned}R_D &= \sum_{k=1}^{\infty}V^{(k)}(\mathbf{x}^{(k)})\\ &= \frac{1}{2}\sum_{k=1}^{\infty}\mathbf{a}^T\left([I - (\Lambda - G)(2\Lambda - G)^{-1}]^{(k-1)}\right)^T\\ &\qquad (2\Lambda - G)^{-1}[I - (\Lambda - G)(2\Lambda - G)^{-1}]^{(k-1)}\mathbf{a}\\ &= \frac{1}{2}\sum_{k=1}^{\infty}\mathbf{a}^T\left([\Lambda(2\Lambda - G)^{-1}]^{(k-1)}\right)^T\\ &\qquad (2\Lambda - G)^{-1}[\Lambda(2\Lambda - G)^{-1}]^{(k-1)}\mathbf{a}\\ &= \frac{1}{2}\sum_{k=1}^{\infty}\mathbf{a}^T[(2\Lambda - G)^{-1}\Lambda]^{(k-1)}(2\Lambda - G)^{-1}\\ &\qquad [\Lambda(2\Lambda - G)^{-1}]^{(k-1)}\mathbf{a}\\ &= \frac{1}{2}\sum_{k=1}^{\infty}\mathbf{a}^T\Lambda^{-1}[\Lambda(2\Lambda - G)^{-1}]^{(2k-1)}\mathbf{a}.\end{aligned}$$

The sum $\sum_{k=1}^{\infty}\Lambda^{-1}[\Lambda(2\Lambda - G)^{-1}]^{(2k-1)}$ converges since the spectral radius of $[\Lambda(2\Lambda - G)^{-1}]^2$ is less than 1. Specifically,

$$\begin{aligned}&\sum_{k=1}^{\infty}[\Lambda(2\Lambda - G)^{-1}]^{(2k-1)} =\\ &\quad = [\Lambda(2\Lambda - G)^{-1}]\left(I - [\Lambda(2\Lambda - G)^{-1}]^2\right)^{-1}\\ &\quad = \Lambda(2\Lambda - G)^{-1}\left(I - [I - (\Lambda - G)(2\Lambda - G)^{-1}]^2\right)^{-1}\\ &\quad = \Lambda(2\Lambda - G)^{-1}(2\Lambda - G)(\Lambda - G)^{-1}\\ &\qquad\qquad \left(2I - (\Lambda - G)(2\Lambda - G)^{-1}\right)^{-1}.\end{aligned}$$

Therefore, the optimal revenue obtained by Algorithm 1 is given by

$$\begin{aligned}R_D &= \frac{1}{2}\sum_{k=1}^{\infty}\mathbf{a}^T\Lambda^{-1}[\Lambda(2\Lambda - G)^{-1}]^{(2k-1)}\mathbf{a}\\ &= \frac{1}{2}\mathbf{a}^T\Lambda^{-1}\Lambda(\Lambda - G)^{-1}\left(2I - (\Lambda - G)(2\Lambda - G)^{-1}\right)^{-1}\mathbf{a}\\ &= \frac{1}{2}\mathbf{a}^T(\Lambda - G)^{-1}\left(2I - (\Lambda - G)(2\Lambda - G)^{-1}\right)^{-1}\mathbf{a}\end{aligned}$$

The optimal revenue obtained by the static policy is given by $R_S = \frac{1}{4}\mathbf{a}^T(\Lambda - G)^{-1}\mathbf{a}$. Comparing the revenue obtained by the two policies,

$$\begin{aligned}
R_D - R_S = & \\
&= \frac{1}{2}\mathbf{a}^T(\Lambda - G)^{-1}\left(\left(2I - (\Lambda - G)(2\Lambda - G)^{-1}\right)^{-1} - \frac{I}{2}\right)\mathbf{a} \\
&= \frac{1}{2}\mathbf{a}^T(\Lambda - G)^{-1}\left((2\Lambda - G)\,(3\Lambda - G)^{-1} - \frac{I}{2}\right)\mathbf{a} \\
&= \frac{1}{2}\mathbf{a}^T(\Lambda - G)^{-1}\frac{1}{2}(\Lambda - G)\,(3\Lambda - G)^{-1}\,\mathbf{a} \\
&= \frac{1}{4}\mathbf{a}^T\,(3\Lambda - G)^{-1}\,\mathbf{a}.
\end{aligned}$$

Since all the entries of $(3\Lambda - G)^{-1}$ are non-negative, we have that $R_D - R_S > 0$.
Next, we compare the total utilities obtained by the two strategies. Let $\mathbf{z}_D$ and $\mathbf{z}_U$ be the total optimal consumption in the sequential dynamic strategy and the static strategy respectively. Then $\mathbf{z}_D = (\Lambda - G)^{-1}\mathbf{a}$ and $\mathbf{z}_U = (\Lambda - G)^{-1}\frac{\mathbf{a}}{2} = \frac{\mathbf{z}_D}{2}$. Then the asymptotic total utilities compare as follows,

$$\begin{aligned}
U_D - U_S = & \\
&= \frac{\mathbf{a}^T\mathbf{z}_D}{2} - \frac{3}{8}\mathbf{z}_D^T\Lambda\mathbf{z}_D + \frac{3}{4}\mathbf{z}_D^T G\mathbf{z}_D - R_D + R_S \\
&= \frac{3}{8}\mathbf{z}_D^T\Lambda\mathbf{z}_D - R_D \\
&= \frac{\mathbf{a}^T}{2}(\Lambda - G)^{-1} \\
&\quad \left[\frac{3}{4}\Lambda(\Lambda - G)^{-1} - \left(2I - (\Lambda - G)(2\Lambda - G)^{-1}\right)^{-1}\right]\mathbf{a} \\
&= \frac{\mathbf{a}^T}{2}(\Lambda - G)^{-1} \\
&\quad \left[\frac{3}{4}\Lambda(\Lambda - G)^{-1} - (2\Lambda - G)(3\Lambda - G)^{-1}\right]\mathbf{a}.
\end{aligned}$$

Consider the matrix $(2\Lambda - G)(3\Lambda - G)^{-1}$,

$$\begin{aligned}
(2\Lambda - G)(3\Lambda - G)^{-1} = & \\
&= (2\Lambda - G)\left(I - \frac{\Lambda^{-1}G}{3}\right)^{-1}\frac{\Lambda^{-1}}{3} \\
&= (2\Lambda - G)\sum_{j=0}^{\infty}\left(\frac{\Lambda^{-1}G}{3}\right)^j\frac{\Lambda^{-1}}{3} \\
&= 2\Lambda\sum_{j=0}^{\infty}\left(\frac{\Lambda^{-1}G}{3}\right)^j\frac{\Lambda^{-1}}{3} - 3\Lambda\sum_{j=1}^{\infty}\left(\frac{\Lambda^{-1}G}{3}\right)^j\frac{\Lambda^{-1}}{3} \\
&= \frac{2I}{3} - \Lambda\sum_{j=1}^{\infty}\left(\frac{\Lambda^{-1}G}{3}\right)^j\frac{\Lambda^{-1}}{3}.
\end{aligned}$$

Therefore, the two utilities compare as follows,

$$\begin{aligned}
&U_D - U_S = \\
&= \frac{\mathbf{a}^T}{2}(\Lambda - G)^{-1}\left[\frac{3}{4}\Lambda\sum_{j=0}^{\infty}\left(\Lambda^{-1}G\right)^j\Lambda^{-1} - \frac{2I}{3} + \right. \\
&\qquad\qquad \left. + \Lambda\sum_{j=1}^{\infty}\left(\frac{\Lambda^{-1}G}{3}\right)^j\frac{\Lambda^{-1}}{3}\right]\mathbf{a} \\
&= \frac{\mathbf{a}^T}{2}(\Lambda - G)^{-1}\left[\frac{I}{12} + \frac{3}{4}\Lambda\sum_{j=1}^{\infty}\left(\Lambda^{-1}G\right)^j\Lambda^{-1} + \right. \\
&\qquad\qquad \left. + \Lambda\sum_{j=1}^{\infty}\left(\frac{\Lambda^{-1}G}{3}\right)^j\frac{\Lambda^{-1}}{3}\right]\mathbf{a} \\
&\geq 0.
\end{aligned}$$

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